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The Maxwell-Burr X Distribution: Its Properties and Applications to the COVID-19 Mortality Rate in Thailand

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Abstract

A novel distribution, the Maxwell-Burr X (M-BX) distribution, was proposed. This distribution was an extension of the Burr X distribution by applying the Maxwell generalized family of distributions. The cumulative distribution function, probability density function, survival function, hazard function and quantile function of the M-BX distribution were defined. Some important properties and the parameters of its estimates were discussed. A simulation study was conducted from the basis of quantile function to ascertain the performance of maximum likelihood estimators. The M-BX distribution were also applied to model two lifetime data sets relating to the COVID-19 mortality rate in Thailand during different periods to express the flexibility of the distribution against other competing distributions. According to information criteria, AIC, CAIC, BIC, and HQIC, the M-BX distribution gave the best fit among all chosen distributions.

Keywords: Maxwell generalized family, hazard function, quantile function, model selection criteria, maximum likelihood estimation.

1. Introduction

In statistical modeling, probability distributions play a crucial role in describing data behavior especially lifetime datasets in many fields such as medical research, engineering, industry, agriculture, and so on. With a wide variety of applications, lifetime distribution has received great interest from researchers over the past decades. Moreover, the recent advancement of technology allows us to collect and to access a wider variety of data that many datasets cannot be explained by existing probabilistic distributions. Therefore, researchers have attempted to develop new distributions either by extending well-known distributions by introducing new parameters, or by generating a new family of distribution to provide more flexibility in modeling real datasets.

The cumulative distribution function (cdf) of the Burr X (BX) distribution was originally introduced by Burr (1942) for modeling non-negative data. The cdf of the BX distribution is given.

$$Q(x;r)=\left[1-e^{-x^2}\right]^r,\ r>0;\ \ x>0. \tag{1}$$

The corresponding probability density function (pdf) to (1) is

$$q(x;r)=2rxe^{-x^2}\left[1-e^{-x^2}\right]^{r-1},\ \ r>0;\ \ x>0, \tag{2}$$

where r is a shape parameter. The BX distribution is related to some well-known standard theoretical distributions such as the Weibull distribution and the Gamma distribution in terms of its hazard function behavior which can be constant, bathtub, and increasing function. Several aspects of the BX distribution were studied by Sartawi and Abu-Salih (1991), Jaheen (1995), Ahmad et al. (1998), Raqab (1998), and Surles and Padgett (1998). Recently, Surles and Padgett (2001) observed that the BX distribution can be used quite effectively in modelling not only strength data, but also in general lifetime data. In addition, the BX distribution had been extended by some researchers. For instance, the beta Burr type X distribution (Merovci et al. 2016), the beta Burr type X distribution (Khalee et al. 2017), the Marshall-Olkin extended Burr type X distribution (Al-Saiari et al. 2014), the Weibull Burr X distribution (Ibrahim et al. 2017), the exponentiated generalized Burr type distribution (Khalee et al. 2017), and the new version of the exponentiated Burr X distribution (Ahmed et al. 2021). All above extensions of the BX distribution shown that they could fit real datasets better than original BX distribution.

The Maxwell Generalized (M-G) family of distributions introduced by Ishaq and Abiodun (2020) originated from the Maxwell-Boltzmann distribution (Maxwell 1860) was extensively used to explain the data relating to molecular behavior in physics, chemistry, and biology. However, it could fit only to the right skewed datasets. Later, in 2020, Ishaq and Abiodun applied the odd ratio technique developed by Alzaatreh et al. (2013) to generate a new family of distributions. The cdf and pdf of the M-G family are given by

$$F(x;r,\beta)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\beta^2}\left(\frac{Q(x;\varepsilon)}{1-Q(x;\varepsilon)}\right)^2\right),\ \beta>0,x\in R \tag{3}$$

and

$$f\left(x;\beta,\varepsilon\right)=\frac{2q\left(x;\varepsilon\right)}{\beta^3\sqrt{2\pi}\left\{1-Q\left(x;\varepsilon\right)\right\}^2}\left(\frac{Q\left(x;\varepsilon\right)}{1-Q\left(x;\varepsilon\right)}\right)^2\exp\left\{-\frac{1}{2\beta^2}\left(\frac{Q\left(x;\varepsilon\right)}{1-Q\left(x;\varepsilon\right)}\right)^2\right\},\ \beta>0,x\in R \tag{4}$$

respectively, where β is a scale parameter, $Q\left(x;\varepsilon\right)$ and $q\left(x;\varepsilon\right)$ are the cdf and pdf of the baseline distribution with parameter ε, respectively, and $\gamma\left(a,b\right)=\int_a^b t^{a-1}e^{-t}dt$ is an incomplete gamma function. The M-G family was used to extend different baseline distributions to yield new compound distributions with different properties and applications. Many recent distributions were proposed based on this extension, for example, the Maxwell-Weibull distribution (Ishaq and Abiodun 2020) extended from the Weibull distribution to model the exchange rates from Naira (Nigeria) to Yen (Japan), the Maxwell-Exponential distribution (Abdullahi et al. 2021) generalized from the exponential distribution to model datasets with both left and right skewed distribution, and the

Maxwell-Mukherjee Islam distribution (Ishaq et al. 2021) extended from the Mukherjee Islam distribution to study its parameter estimation by using the Bayesian method.

The aim of this research was to study the Maxwell-Burr X (M-BX) distribution extended from the BX distribution by adding the scale parameter from the M-G family. The M-BX distribution was more flexible for datasets with skewness and long tail than the BX distribution. As a result, the M-BX can be efficiently used to describe and model new lifetime datasets. Some properties of the proposed distribution as well as the simulation study, and the application to the real data set were also provided.

2. The Maxwell-Burr X distribution

The Maxwell-Burr X (M-BX) distribution is an extended distribution from the BX distribution by adding extra scale parameter. The cdf of this distribution can be derived by substituting (1) into (3) as

$$F(x;\beta,r)=\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\beta^2}\left(\frac{\left[1-e^{-x^2}\right]^r}{1-\left[1-e^{-x^2}\right]^r}\right)^2\right),\quad \text{for } x>0, \tag{5}$$

where $\beta>0$ and $r>0$ are the scale and shape parameters, respectively. The corresponding pdf is given by

$$f(x;\beta,r)=\frac{4xre^{-x^2}\left(1-e^{-x^2}\right)^{3r-1}}{\beta^3\sqrt{2\pi}\left[1-\left(1-e^{-x^2}\right)^r\right]^4}\exp\left\{-\frac{1}{2\beta^2}\left[\frac{\left(1-e^{-x^2}\right)^r}{1-\left(1-e^{-x^2}\right)^r}\right]^2\right\},\quad \text{for } x>0. \tag{6}$$

Therefore, a random variable with the pdf given in (6) is denoted by $X\sim$ M-BX(β,r). For simplicity the parameters on the cdf and pdf are omitted by writing $F(x;\beta,r)=F(x)$ and $f(x;\beta,r)=f(x)$. The pdf plots of the M-BX distribution with different values of β and r are displayed in Figure 1. The pdf can be right-skewed (with a long tail on the right side), left-skewed (with a long tail on the left side), symmetric, or increasing.

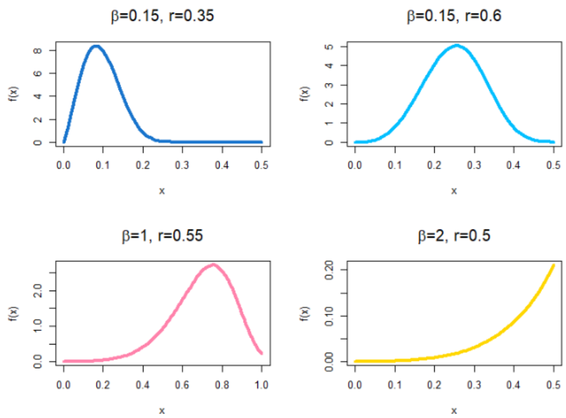

Figure 1 The pdf plots of M-BX distribution with different values of β and r

2.1. Linear representation

By considering the power series expansion of the exponential function, the pdf in (6) can be expressed as

$$f(x)=\frac{4xre^{-x^2}}{\sqrt{2\pi}}\sum_{k=0}^{\infty}\frac{(-1)^k\left(1-e^{-x^2}\right)^{3r+2rk-1}}{k!\beta^{3+2k}2^k\left[1-\left(1-e^{-x^2}\right)^r\right]^{4+2k}}. \tag{7}$$

It is known that if $|m|<1$ and $c>0,$ then the power series expansion of this holds

$$(1-m)^{-c}=\sum_{j=0}^{\infty}\frac{\Gamma(j+c)}{j!\Gamma(c)}m^j. \tag{8}$$

Applying (8) to the dominator in (7) we get

$$f(x)=\frac{4xre^{-x^2}}{\sqrt{\pi}}\sum_{k,j=0}^{\infty}\frac{(-1)^k\Gamma(4+2k+j)}{k!j!\beta^{3+2k}2^{k+\frac{1}{2}}\Gamma(4+2k)}\left(1-e^{-x^2}\right)^{3r+2rk+rj-1}. \tag{9}$$

It is known that for $|m|<1$ and c is not natural number, the power series expansion of this holds

$$(1-m)^c=\sum_{l=0}^{\infty}(-1)^l\binom{c}{l}m^l. \tag{10}$$

By applying (10), (9) can be rewritten as

$$f(x)=\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}xe^{-(1+l)x^2}, \tag{11}$$

which is the pdf of the M-BX distribution expressed in terms of linear representation, where

$$\Phi_{k,j,l}=\frac{2r(-1)^{k+l}\Gamma(4+2k+j)}{k!j!2^k\beta^{3+2k}\Gamma(4+2k)}\binom{r(3+2k+j)-1}{l}\sqrt{\frac{2}{\pi}}. \tag{12}$$

2.2. Survival, Hazard, and Quantile Functions of the Maxwell-Burr X Distribution

2.2.1. Survival Function

The survival function of the M-BX distribution is obtained from (5) as

$$S(x)=1-\frac{2}{\sqrt{\pi}}\gamma\left(\frac{3}{2},\frac{1}{2\beta^2}\left(\frac{\left[1-e^{-x^2}\right]^r}{1-\left[1-e^{-x^2}\right]^r}\right)^2\right),\ \text{for } x>0, \tag{13}$$

where $\beta>0$ and $r>0$ are the scale and shape parameters, respectively.

2.2.2. Hazard Function

The hazard function of the M-BX distribution is obtained from (5) and (6) as

$$h(x)=\frac{\dfrac{4xre^{-x^2}\left(1-e^{-x^2}\right)^{3r-1}}{\beta^3\sqrt{2\pi}\left[1-\left(1-e^{-x^2}\right)^r\right]^4}\exp\left\{-\dfrac{1}{2\beta^2}\left[\dfrac{\left(1-e^{-x^2}\right)^r}{1-\left(1-e^{-x^2}\right)^r}\right]^2\right\}}{1-\dfrac{2}{\sqrt{\pi}}\gamma\left(\dfrac{3}{2},\dfrac{1}{2\beta^2}\left(\dfrac{\left[1-e^{-x^2}\right]^r}{1-\left[1-e^{-x^2}\right]^r}\right)^2\right)},\ \text{for } x>0, \tag{14}$$

where $\beta > 0$ and $r > 0$ are the scale and shape parameters, respectively. The hazard function plots of the M-BX distribution with different values of β and r are presented in Figure 2 .The hazard function could be increasing gradually to the peak then decreasing rapidly or increasing.

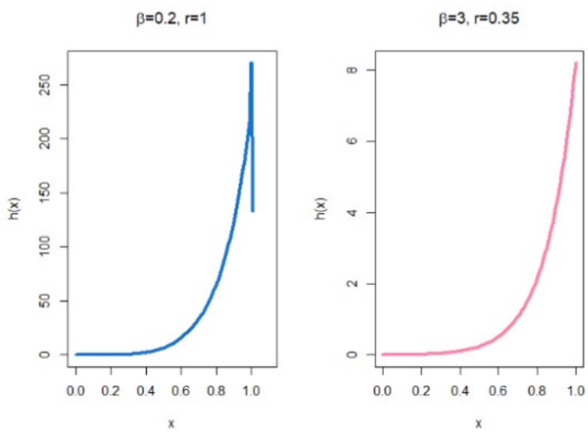

Figure 2 The hazard function plots of the M-BX distribution with different values of β and r

2.2.3. Quantile Function

The quantile function of M-BX distribution can be derived from an inverse function of the cdf defined in (3) as

$$x_q = \left\{ -\log\left[1 - \left[\frac{\left(2\beta^2\gamma^1\left(\frac{3}{2}, u\Gamma\left(\frac{3}{2}\right)\right)\right)^{1/2}}{1 + \left(2\beta^2\gamma^1\left(\frac{3}{2}, u\Gamma\left(\frac{3}{2}\right)\right)\right)^{1/2}}\right]^{1/r}\right]\right\}^{1/2}, \tag{15}$$

where u has a uniform distribution on interval 0 to 1.

3. Properties of the Maxwell-Burr X Distribution

In this section, we explore some properties of the M-BX distribution including moments, moment generating function, and order statistics.

3.1. Moments

By the definition of moments, the moments of the M-BX distribution can be derived from linear representation of the pdf of the M-BX distribution defined in (11) as

$$E(X^r) = \int_{-\infty}^{\infty} x^r f(x)dx = \int_0^{\infty} x^r \sum_{k,j,l=0}^{\infty} \Phi_{k,j,l} x e^{-(1+l)x^2} dx = \sum_{k,j,l=0}^{\infty} \Phi_{k,j,l} \int_0^{\infty} x^{r+1} e^{-(1+l)x^2} dx, \tag{16}$$

where $\Phi_{k,j,l} = \frac{2r(-1)^{k+l}\Gamma(4+2k+j)}{k!j!2^k\beta^{3+2k}\Gamma(4+2k)}\binom{r(3+2k+j)-1}{l}\sqrt{\frac{2}{\pi}}$. Let $y = (1+l)x^2$, then we get

$$x = \left(\frac{y}{1+l}\right)^{1/2} \text{ and } dx = \frac{dy}{2(1+l)x}. \tag{17}$$

If we insert (17) in (16), we obtain

$$E(X^r)=\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}\int_0^{\infty}\frac{x^{r+1}e^{-y}}{2(1+l)x}dy=\frac{\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}}{2(1+l)^{1+\frac{r}{2}}}\int_0^{\infty}y^{\frac{r}{2}}e^{-y}dy, \tag{18}$$

this can be rewritten as

$$E(X^r)=\frac{\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}}{2(1+l)^{1+\frac{r}{2}}}\Gamma\left(\frac{r}{2}+1\right), \tag{19}$$

which is the moment of the M-BX distribution. If we set $r=1,$ the mean of the M-BX distribution is obtained from (19) as

$$Mean=E(X)=\frac{\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}}{2(1+l)^{\frac{3}{2}}}\Gamma\left(\frac{3}{2}\right)=\frac{\sqrt{\pi}\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}}{4(1+l)^{\frac{3}{2}}} \tag{20}$$

and the variance of the M-BX distribution is received by setting $r=2$ in (19) so that

$$Variance=E(X^2)-\left[E(X)\right]^2=\frac{8(1+l)\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}-\pi\left(\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}\right)^2}{16(1+l)^3}. \tag{21}$$

3.2. Moment Generating Function

By the definition, the moment generating function of the random variable X is defined by

$$M_x(t)=E(e^{tx})=\int_{-\infty}^{\infty}e^{tx}f(x)dx=\int_{-\infty}^{\infty}\sum_{n=0}^{\infty}\frac{(tx)^n}{n!}f(x)dx=\sum_{n=0}^{\infty}\frac{t^n}{n!}\int_{-\infty}^{\infty}x^nf(x)dx=\sum_{n=0}^{\infty}\frac{t^n}{n!}E(x^n). \tag{22}$$

If we set $r=n$ in (19) and insert it in (22), we obtain

$$M_x(t)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\left\{\frac{\sum_{k,j,l=0}^{\infty}\Phi_{k,j,l}}{2(1+l)^{1+\frac{n}{2}}}\Gamma\left(\frac{n}{2}+1\right)\right\}, \tag{23}$$

which is the moment generating function of the M-BX distribution.

3.3. Order Statistics

Let $X_1,X_2,...,X_n$ be a random sample of size n from the M-BX distribution and let $X_{(i)}$ denote the order statistics of the sample. Then, the probability density function of the i^{th} order statistics denoted $f_{i,n}(x)$ can be expressed as

$$f_{i,n}(x)=\frac{n!f(x)}{(i-1)!(n-i)!}F(x)^{i-1}\left[1-F(x)\right]^{n-i}=\frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{\infty}(-1)^m\binom{n-i}{m}f(x)F(x)^{i+m-1}, \tag{24}$$

where $F(x)$ and $f(x)$ are defined in (5) and (6), respectively. Consider the expression of term $f(x)F(x)^q$ where $q=i+m-1.$ It can be simplified as

$$f(x)F(x)^q = f(x)\left[\frac{\gamma(a,w)}{\Gamma(a)}\right]^q, \tag{25}$$

where $a=\frac{3}{2}$ and $w=\frac{1}{2\beta^2}\left[\frac{\left(1-e^{-x^2}\right)^r}{1-\left(1-e^{-x^2}\right)^r}\right]^2$. Then we obtain

$$f(x)F(x)^q = f(x)\left[1-\left(1-\gamma_1(a,w)\right)\right]^q, \tag{26}$$

where $\gamma_1(a,w)=\frac{\gamma(a,w)}{\Gamma(a)}$. By using the expansion of binomial defined in (10) to (25), we get

$$\begin{aligned} f(x)F(x)^q &= f(x)\sum_{i=0}^{\infty}(-1)^i\binom{q}{i}\left[1-\gamma_1(a,w)\right]^i \\ &= f(x)\sum_{i=0}^{\infty}\sum_{j=0}^{i}(-1)^{i+j}\binom{q}{i}\binom{i}{j}\gamma_1^j(a,w) \\ &= f(x)\sum_{j=0}^{\infty}A_i(q)\gamma_1^j(a,w), \text{ where } A_i(q)=\sum_{i=j}^{\infty}(-1)^{i+j}\binom{q}{i}\binom{i}{j} \\ &= \frac{f(x)}{\left[\Gamma(a)\right]^j}\sum_{j=0}^{\infty}A_i(q)\gamma^j(a,w). \end{aligned} \tag{27}$$

By applying the series representation of the incomplete gamma function and the power series raised to power expansion given in Gradshteyn and Ryzhik (2007), we obtain

$$\gamma^j(a,w) = w^{aj}\sum_{k=0}^{\infty}B_{k,m}w^k, \tag{28}$$

where $B_{k,0}=b_0^n, B_{k,m}=\frac{1}{mb_0}\sum_{k=1}^{m}(kj-m+k)b_kB_{m-k}$ with $b_n=(-1)^n/n!(a+n)$. Insert (28) into (27), it becomes

$$f(x)F(x)^q = \sum_{j,k,s=0}^{\infty}\frac{C_{j,k,s}xe^{-x^2}\left(1-e^{-x^2}\right)^{3r-1+2k+3j+2s}}{\left[1-\left(1-e^{-x^2}\right)\right]^{4+2k+3j+2s}}, \tag{29}$$

where $C_{j,k,s}=\frac{4rA_i(q)B_k}{\left[\Gamma\left(\frac{3}{2}\right)\right]^j\left(2\beta^2\right)^{\frac{2k+3j}{2}}}\sum_{s=0}^{\infty}\frac{(-1)^s}{s!}\left(\frac{1}{2\beta^2}\right)$. By applying (8) and (10) to (29), we get

$$f(x)F(x)^q = D_{j,k,s,t,v}xe^{-x^{2(1+v)}}, \tag{30}$$

where $D_{j,k,s,t,v}=\sum_{j,k,s,t,v=0}^{\infty}C_{j,k,s}\frac{\Gamma(4+2k+3j+2s+t)}{t!\Gamma(4+2k+3j+2s)}(-1)^v\binom{3r-1+2k+3j+2s+rt}{v}$.

Substituting (30) into (24) for $q=i+m-1$ gives

$$f_{i,n}(x)=\frac{n!}{(i-1)!(n-i)!}\sum_{m=0}^{\infty}(-1)^m\binom{n-i}{m}D_{j,k,s,t,v}xe^{-x^{2(1+v)}}, \tag{31}$$

which are the order statistics of the M-BX distribution for $i=1,2,3,...,n$.

4. Parameter Estimation

In this section, the estimates of parameters of the M-BX distribution using the maximum likelihood method were provided. Let $X_1, X_2, ..., X_n$ be a random variable of sample size n from the M-BX distribution with parameters β and r, then the likelihood function of the M-BX distribution is obtained from (6) as

$$L=\left[\frac{4r}{\beta^3\sqrt{2\pi}}\right]^n\prod_{i=1}^{n}\frac{x_ie^{-x_i^2}\left(1-e^{-x_i^2}\right)^{3r-1}}{\left[1-\left(1-e^{-x_i^2}\right)^r\right]^4}\prod_{i=1}^{n}\exp\left\{-\frac{1}{2\beta^2}\left[\frac{\left(1-e^{-x_i^2}\right)^r}{1-\left(1-e^{-x_i^2}\right)^r}\right]^2\right\}. \tag{32}$$

The log-likelihood function of (32) denoted by l is

$$l=n\log(4)+n\log(r)-3n\log(\beta)-\frac{n}{2}\log(2\pi)+\sum_{i=1}^{n}\log(x_i)-\sum_{i=1}^{n}x_i^2+(3r-1)\sum_{i=1}^{n}\log(1-e^{-x_i^2})$$

$$-4\sum_{i=1}^{n}\log\left[1-(1-e^{-x_i^2})^r\right]-\frac{1}{2\beta^2}\sum_{i=1}^{n}\left[\frac{\left(1-e^{-x_i^2}\right)^r}{1-\left(1-e^{-x_i^2}\right)^r}\right]^2. \tag{33}$$

To get the maximum likelihood estimators of the M-BX distribution, we must maximize the log-likelihood function. To achieve this purpose, we take the first partial derivative of (33) with respect to parameters β and r, then we obtain

$$\frac{\partial l}{\partial\beta}=-\frac{3n}{\beta}+\frac{1}{\beta^3}\sum_{i=1}^{n}\left[\frac{\left(1-e^{-x_i^2}\right)^r}{1-\left(1-e^{-x_i^2}\right)^r}\right]^2 \tag{34}$$

and

$$\frac{\partial l}{\partial r}=\frac{n}{r}+3\sum_{i=1}^{n}\log(1-e^{-x_i^2})+4\sum_{i=1}^{n}\left[\frac{\left(1-e^{-x_i^2}\right)^r}{1-\left(1-e^{-x_i^2}\right)^r}\right]\log\left(1-e^{-x_i^2}\right)-\frac{1}{\beta^2}\sum_{i=1}^{n}\left[\frac{\left(1-e^{-x_i^2}\right)^r}{1-\left(1-e^{-x_i^2}\right)^r}\right]^2\frac{\log\left(1-e^{-x_i^2}\right)}{\left[1-\left(1-e^{-x_i^2}\right)^r\right]}. \tag{35}$$

Setting (32) to zero, we obtained $\hat{\beta}$ which is closed form of the maximum likelihood estimator for the parameter β as

$$\hat{\beta}=\sqrt{\frac{1}{3n}\sum_{i=1}^{n}\left[\frac{\left(1-e^{-x_i^2}\right)^r}{1-\left(1-e^{-x_i^2}\right)^r}\right]^2}. \tag{36}$$

Obviously, it is impossible to solve a closed form of $\hat{r}$ which is the maximum likelihood estimator for the parameter r. Therefore, we can use statistical software likes R program to solve (36) by using numerical technique. The Newton-Raphson method was used in this study.

5. Simulation Study

In this section, we evaluated the performance of the maximum likelihood estimators of parameters of the M-BX distribution through simulation study based on a quantile function which is defined in (14). A data was generated for three different sample sizes (n), 25, 70, and 150, which are referred to small, medium, and large sample sizes of data, respectively. The simulation was conducted on 9 cases depending on the combinations of exact parameter values β and r taken as (0.15, 0.5, 1) and (0.35, 0.7, 1), respectively. These different values give the different curves of pdf plots. The repetition of the simulation is 1,000 times for each sample size to obtain the mean, bias, variance and mean square error (MSE). The results of simulation provided in Tables 1, 2, and 3.

Table 1 The mean, variance, bias and MSE of the M-BX distribution for β =0.15 and r =0.35, 0.7, 1

Parameter value	Sample size		Mean	Bias	Variance	MSE
$\beta = 0.15$ $r = 0.35$	25	$\hat{\beta} =$	0.14563	–0.00437	0.00602	0.00604
		$\hat{r} =$	0.38284	0.03284	0.00934	0.01042
	70	$\hat{\beta} =$	0.14752	–0.00248	0.00185	0.00185
		$\hat{r} =$	0.36161	0.01161	0.00285	0.00299
	150	$\hat{\beta} =$	0.14891	–0.00109	0.00089	0.00089
		$\hat{r} =$	0.35545	0.00545	0.00134	0.00137
$\beta = 0.15$ $r = 0.7$	25	$\hat{\beta} =$	0.14563	–0.00437	0.00602	0.00604
		$\hat{r} =$	0.76567	0.06567	0.03737	0.04168
	70	$\hat{\beta} =$	0.14752	–0.00248	0.00185	0.00185
		$\hat{r} =$	0.72321	0.02321	0.01141	0.01195
	150	$\hat{\beta} =$	0.14891	–0.00109	0.00089	0.00089
		$\hat{r} =$	0.71091	0.01091	0.00535	0.00547
$\beta = 0.15$ $r = 1$	25	$\hat{\beta} =$	0.14563	–0.00437	0.00602	0.00604
		$\hat{r} =$	1.09382	0.09382	0.07626	0.08506
	70	$\hat{\beta} =$	0.14752	–0.00248	0.00185	0.00185
		$\hat{r} =$	1.03316	0.03316	0.02329	0.02439
	150	$\hat{\beta} =$	0.14891	–0.00109	0.00089	0.00089
		$\hat{r} =$	1.01558	0.01558	0.01091	0.01115

As observed the simulation results from Tables 1, 2, and 3, we noticed that

- In most cases, the estimators were overestimated except when $\beta = 0.15$, the estimators $\hat{\beta}$ in every value of r were underestimated.

- By focusing on the sample size for a fixed parameter value, the mean values of estimators tended to be the exact values of parameters.

- In almost all cases, the estimators performed very well and provided small MSE, but when $\beta = 1$ at small sample size, $\hat{\beta}$ could not estimate close to the exact values of parameter β.

6. Application to Real Data Set

In this section, we illustrated the potentiality of the M-BX distribution through applying to real data sets. As the COVID-19 pandemic had affected people around the world over the past few years, the information about COVID-19 had attracted the attention of researchers in various fields. In this research, we provided two data sets relating to COVID-19 mortality rate in Thailand. According to the simulation study, the estimators of the proposed distribution performed very well when the sample size was more than 25 observations therefore, we showed two real data sets for different sample sizes as follows.

- The first data set comprised of 52 observations of a COVID-19 mortality rate data in Thailand between 4 September to 25 October 2021.

- The second data set comprised of 170 observations of a COVID-19 mortality rate data in Thailand between 1 November 2021 to 18 April 2022.

The data sets were obtained from the Ministry of Public Health of Thailand website. See the link (DDC COVID-19 Interactive Dashboard | 1-dash-tiles-w (moph.go.th)) for more details.

Table 2 The mean, variance, bias and MSE of the M-BX distribution for β =0.5 and r = 0.35, 0.7, 1

Parameter value	Sample size		Mean	Bias	Variance	MSE
$\beta=0.5$, $r=0.35$	25	$\hat{\beta}=$	0.53594	0.03594	0.23374	0.23503
		$\hat{r}=$	0.40670	0.05670	0.02547	0.02869
	70	$\hat{\beta}=$	0.50206	0.00206	0.03290	0.03291
		$\hat{r}=$	0.37021	0.02021	0.00773	0.00814
	150	$\hat{\beta}=$	0.50149	0.00149	0.01519	0.01519
		$\hat{r}=$	0.35961	0.00961	0.00367	0.00376
$\beta=0.5$, $r=0.7$	25	$\hat{\beta}=$	0.53589	0.03589	0.23296	0.23425
		$\hat{r}=$	0.81340	0.11340	0.10189	0.11475
	70	$\hat{\beta}=$	0.50206	0.00206	0.03290	0.03291
		$\hat{r}=$	0.74042	0.04042	0.03092	0.03255
	150	$\hat{\beta}=$	0.50149	0.00149	0.01519	0.01519
		$\hat{r}=$	0.71922	0.01922	0.01467	0.01504
$\beta=0.5$, $r=1$	25	$\hat{\beta}=$	0.53591	0.03591	0.23324	0.23453
		$\hat{r}=$	1.16200	0.16200	0.20793	0.23418
	70	$\hat{\beta}=$	0.50206	0.00206	0.03290	0.03291
		$\hat{r}=$	1.05774	0.05774	0.06310	0.06643
	150	$\hat{\beta}=$	0.50149	0.00149	0.01519	0.01519
		$\hat{r}=$	1.02746	0.02746	0.02993	0.03068

We applied the M-BX distribution to fit the two data sets .The results were compared with the fitting from baseline distributions and some existing usual survival distributions including Burr X (BX (distribution, two parameter Burr X (BX2) distribution, Weibull (W) distribution and Normal (N) distribution .Akaike Information Criterion (AIC), Consistent Akaike Information Criterion (CAIC), Bayesian Information Criterion (BIC) and Hannan Quinn Information Criterion (HQIC) were used to compare the performance of the distribution .Based on these criteria, the lowest value indicates the best fit of the model. The pdf of comparative models is stated as follows:

- Burr X distribution (Burr, 1942): $f(x;r)=2rxe^{-x^2}\left[1-e^{-x^2}\right]^{r-1}$, $r>0;\ x>0.$

- Two parameter Burr X distribution (Surles and Padgett 2001):

$$f(x;r,\lambda)=2r\lambda^2 xe^{-(\lambda x)^2}\left[1-e^{-(\lambda x)^2}\right]^{r-1}, r>0, \lambda>0;\ x>0.$$

Table 3 The mean, variance, bias and MSE of the M-BX distribution for $\beta = 1$ and $r = 0.35, 0.7, 1$

Parameter value	Sample size		Mean	Bias	Variance	MSE
$\beta=1$ $r=0.35$	25	$\hat{\beta}=$	2.18545	1.18545	92.55300	93.95830
		$\hat{r}=$	0.44186	0.09186	0.05794	0.06638
	70	$\hat{\beta}=$	1.06716	0.06716	0.40370	0.40821
		$\hat{r}=$	0.38280	0.03280	0.01753	0.01861
	150	$\hat{\beta}=$	1.02898	0.02898	0.12475	0.12559
		$\hat{r}=$	0.36580	0.01580	0.00834	0.00859
$\beta=1$ $r=0.7$	25	$\hat{\beta}=$	2.23342	1.23342	98.68449	100.20580
		$\hat{r}=$	0.88370	0.18370	0.23179	0.26554
	70	$\hat{\beta}=$	1.06717	0.06716	0.40374	0.40825
		$\hat{r}=$	0.76561	0.06561	0.07013	0.07443
	150	$\hat{\beta}=$	1.02898	0.02898	0.12475	0.12559
		$\hat{r}=$	0.73159	0.03159	0.03335	0.03434
$\beta=1$ $r=1$	25	$\hat{\beta}=$	2.19542	1.19542	91.47051	92.89953
		$\hat{r}=$	1.26245	0.26245	0.47302	0.54189
	70	$\hat{\beta}=$	1.06715	0.06715	0.40361	0.40812
		$\hat{r}=$	1.09372	0.09372	0.14311	0.15190
	150	$\hat{\beta}=$	1.02898	0.02898	0.12475	0.12559
		$\hat{r}=$	1.04513	0.04513	0.06805	0.07009

- Weibull distribution (Weibull 1939): $f(x;\lambda,k)=\frac{k}{\lambda}\left(\frac{x}{\lambda}\right)^{k-1}e^{-(x/\lambda)^k}, \lambda>0, k>0; x>0.$

- Normal distribution (Gauss 1809): $f(x;\lambda,k)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ \mu\in R, \sigma^2>0;\ x\in R$

Table 4 The maximum likelihood estimators and information criterion values of the first data set

Comparative models	ML estimators		AIC	CAIC	BIC	HQIC
M-BX	$\hat{\beta}=$	0.2819	−112.6298	−112.3849	−108.7274	−111.1337
	$\hat{r}=$	0.3521				
BX	$\hat{r}=$	0.2814	−15.78212	−15.70212	−13.83088	−15.03406
BX2	$\hat{r}=$	1.9905	−54.72123	−54.47633	−50.81874	−53.22511
	$\hat{\lambda}=$	0.3992				
W	$\hat{\lambda}=$	1.9925	−110.1764	−109.9315	−51.6584	−108.6803
	$\hat{k}=$	0.2056				
N	$\hat{\mu}=$	0.1855	−109.5755	−109.3306	−105.673	−108.0794
	$\hat{\sigma}=$	0.0832				

Burr X (BX) distribution, two parameter Burr X (BX2) distribution, Weibull (W) distribution and Normal (N) distribution.

Table 5 The maximum likelihood estimators and information criterion values of the second data set

Comparative models	ML estimators		AIC	CAIC	BIC	HQIC
M-BX	$\hat{\beta}=$	0.2165	−611.5211	−611.4492	−605.2495	−608.9761
	$\hat{r}=$	0.2552				
BX	$\hat{r}=$	0.1741	−273.8227	−273.7989	−270.6869	−272.5503
BX2	$\hat{r}=$	1.1799	968.5556	968.6275	974.8272	971.1005
	$\hat{\lambda}=$	1.2177				
W	$\hat{\lambda}=$	1.3841	−597.8955	−597.8236	−591.6239	−595.3505
	$\hat{k}=$	0.0647				
N	$\hat{\mu}=$	0.0732	−563.1739	−563.1021	−556.9023	−560.629
	$\hat{\sigma}=$	0.0495				

Burr X (BX) distribution, two parameter Burr X (BX2) distribution, Weibull (W) distribution and Normal (N) distribution.

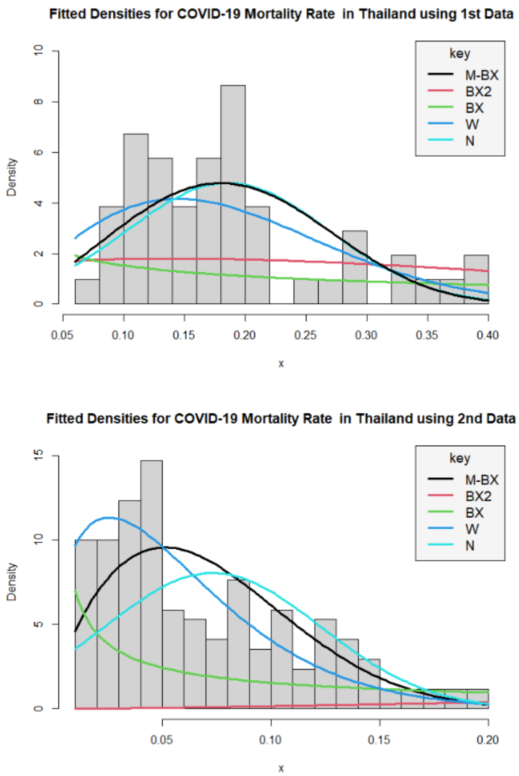

Figure 3 The density plots for the M-BX distribution and other comparative distributions using two real data sets relating to COVID-19 mortality rate data in Thailand

From Tables 4 and 5, it was obvious that the M-BX distribution provided the minimum value of all information criteria .Thus, it gave the best fit for both data sets relating to the COVID-19 mortality rate in Thailand comparing to other distributions. Moreover, Figure 3 also shown that the M-BX distribution was the best model for fitting both datasets. Ultimately, COVID-19 is the latest emerging infectious disease globally; hence, not only the treatment, but also the mortality rate is very necessary for medical researchers to working on their research. Consequently, our proposed distribution is able to support their effort properly.

6. Conclusion

We proposed a novel distribution, the Maxwell-Burr X (M-BX) distribution. This distribution was an extension of the Burr X distribution (Burr, 1942) by applying the Maxwell generalized family of distributions (Ishaq and Abiodun 2020). The probability density function of the M-BX distribution could be right-skewed with a long right tail, left-skewed with a long-left tail, symmetric, or increasing . Some basic properties of this distribution were explored. The simulation study based on a quantile function was conducted to evaluate the performance of the M-BX distribution with the maximum likelihood estimators .The results shown that, for the M-BX distribution, the larger the sample size, the closer the value of estimators to the exact value of parameters .We also applied the M-BX distribution to model two lifetime data sets relating to the COVID-19 mortality rate in Thailand during different periods. The results shown that it performed the best among the Burr X distribution, the two parameters Burr X distribution, the Weibull distribution, and the normal distribution in terms of information criteria, AIC, CAIC, BIC, and HQIC.

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