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The Fraction Nonconforming m-of-m Control Chart with Warning Limits

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Abstract

This article addresses the development of the fraction nonconforming m-of-m control chart with warning limits to increase the sensitivity of the fraction nonconforming control chart. When the process runs in an in-control state for a long time, it is appropriate to investigate the steady-state average run length. We use average run length, standard deviation of run length and quartiles to measure the performance of the proposed chart under steady-state mode. The Markov chain technique is employed to compute the average run length and standard deviation of run length. The study shows that the proposed runs rule schemes with warning limits performs better than the runs rule schemes without warning limits to detect small to moderate shifts. Usefulness of the proposed control chart is explained by giving the numerical example.

Keywords: fraction nonconforming, steady-state, Markov chain, runs rules, warning limits

1. Introduction

The Shewharts control charts are able to detect large shifts in the process, while they are relatively insensitive to detect small shifts. To increase the sensitivity of the Shewharts control charts to detect small to moderate shifts in the process various runs rule schemes have been suggested and studied in the literature (see e.g. Page (1955), Roberts (1958), Bissel (1978), Champ and Woodall (1987), Hurwitz and Mathur (1992), Klein (2000) and references therein). Motivated by Klein (2000), Khoo (2003) developed 2/4, 3/3 and 3/4 control charts. Antzoulakos and Rakitzis (2008) proposed a modified version of the r/m (r-out of-m) control chart and denoted it by M: r/m. Zhang and Wu (2005) computed steady-state average run length of control charts with supplementary runs rules. Lim and Cho (2009) investigated the economic-statistical properties of the Shewhart $\overline{X}$ control chart supplemented with m-of-m runs rules and reported that the steady-state average run length is much more affected by rule length 'm' than the control limit. Sheperds et al. (2012) measured the performance of an attribute control chart for a Markov process using runs rules. Rakitzis and Antzoulakos (2014) proposed the control charts for monitoring process variation using sensitizing runs rules. Amdouni et al. (2016) studied the performance of one-sided run rules control charts for monitoring the coefficient of variation. Tran (2017) proposed run rules median control charts for monitoring process in manufacturing. Tran (2018) presented the designing of run rules t control charts for monitoring changes in the process mean.

In the literature review on the fraction nonconforming control charts using runs rule schemes, Khoo (2003) proposed the control chart to increase the sensitivity of control chart for fraction nonconforming. Khilare and Shirke (2015) proposed the fraction nonconforming m-of-m control chart and studied the zero-state and steady-state properties in term of average run length, standard deviation of run length and quartiles. Their study reveals that the proposed fraction nonconforming m-of-m control chart performs significantly better than the standard p-chart to detect small to moderate shifts. They also reported that the zero-state average run length and steady-state average run length does not differ significantly. Now first we define zero-state and steady-state modes. In zero-state mode it is assumes that there is nonconforming sample at time zero whereas in steady state mode one assumes that the process starts and stays in-control for a long time before a process shift occurs at some random time. There is no work reported on the fraction nonconforming control charts with warning limits using runs rule schemes in the literature. Therefore, we are interested in proposing a fraction nonconforming m-of-m control chart with warning limits and knowning how fraction nonconforming m-of-m control with warning lines performs under steady-state mode. Our main aim is to compare the performance of the proposed control chart with fraction nonconforming m-of-m control chart.

The sample fraction nonconforming is the ratio of the number of nonconforming units (X) in the sample to the sample size (n). That is,

$$\hat{p} = \frac{X}{n}. \tag{1}$$

Here, X follows a binomial distribution with parameters n and p. When the process is in the in-control state, the mean and variance of $\hat{p}$ are $\mu_0 = p_0$ and $\sigma_0^2 = \frac{p_0(1-p_0)}{n}$ respectively, where p_0 is the in-control fraction nonconforming. If p_0 is unknown, it will be estimated from the observed data. The control chart is based on standardized statistic Z, where Z is defined as follows

$$Z_j = \frac{\hat{p}_j - p_0}{\sqrt{p_0(1-p_0)/n}}, \; j = 1,2,3,\ldots. \tag{2}$$

Here, Z_j is approximately distributed as a standard normal variate.

Rest of the paper is an organized as: In Section 2, we describes the m-of-m runs rules schemes with warning limits.The steady-state average run length of the proposed control chart is given in Section 3. Section 4 gives the steady-state performance study of the m-of-m control chart with warning limits. In Section 5 numerical example is given. Section 6 gives concluding remarks.

2. The m-of-m Runs Rue Schemes with Warning Limits

The m-of-m runs rule control chart with warning limits signals an out-of-control status, if a point exceeds the above upper control limit (below the lower control limit) or m points lie between the upper warning limit and the upper control limit (the lower warning limit and the lower control limit). Now, we first define some notations for fraction nonconforming m-of-m control chart with warning limits as follows:

UCL and LCL denote the upper control limit and the lower control limit, respectively. UWL and LWL denote the upper warning limit and the lower warning limit, respectively, p_2 denotes the probability of a single point falling above UCL. p_3 denotes the probability of a single point falling below LCL. pc denotes the probability of a single point falling between both the warning limits. pu denotes the probability of a single point falling between the upper control limit and the upper warning limit. pl denotes the probability of a single point falling between the lower control limit and the lower warning limit.

Consider a control chart based on standardized statistic Z_j defined in Equation (2), with two control limits $(UCL = k, LCL = -k)$ and two warning limits $(UWL = w, LWL = -w)$.

In m-of-m control chart with warning limits five regions are defined:

- Region 1: The region between the upper warning limit and the lower warning limit.
- Region 2: The region between the upper control limit and the upper warning limit.
- Region 3: The region between the lower control limit and the lower warning limit.
- Region 4: The regions above upper control limit.
- Region 5: The regions below lower control limit.

The probabilities of a single point fall in Regions 1, 2, 3, 4 and 5 are given by

$$pc = \Phi\left[\frac{UWL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right] - \Phi\left[\frac{LWL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right],$$

$$pu = \Phi\left[\frac{UCL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right] - \Phi\left[\frac{UWL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right],$$

$$pl = \Phi\left[\frac{LWL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right] - \Phi\left[\frac{LCL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right],$$

$$p_2 = 1-\Phi\left[\frac{UCL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right], \text{ and } p_3 = \Phi\left[\frac{LCL\sqrt{p_0(1-p_0)/n}+p_0-p}{\sqrt{p(1-p)/n}}\right].$$

The process is in the in-control state if $p = p_0$, otherwise the process is out-of-control. Let $\{X_t, t \geq 1\}$ be is a sequence of independent and identically distributed (i.i.d.) random variables taking values in the set $\mathbf{A} = \{1,2,3,4,5\}$ and let $P(X_t = 1) = pc$, $P(X_t = 2) = pu$, $P(X_t = 3) = pl$, $P(X_t = 4) = p_2$ and $P(X_t = 5) = p_3$. Consider the compound pattern $\xi = \left\{\underbrace{222,...,2}_{m}, \underbrace{333,...,3}_{m}, 4, 5\right\}$.

Note that the elements in the compound pattern ξ show all the possible ways of obtaining an out-of-control signal using m-of-m rule with warning limits. For example, elements 222,...,2 indicates m consecutive sample points falling in Region 2. So, that an out-of-control signal is produced. Let T be the waiting time for the first occurrence of ξ. The run length distribution of the m-of-m control chart is coinciding with the waiting time distribution T of the compound pattern ξ. Decomposing the pattern ξ into $2m$ blocks as follows:

$$"1"=1,"2"=2,"3"=22,...,"m"=\underbrace{222,...,2}_{m-1},"m+1"=3,"m+2"=33,...,"2m-1"=\underbrace{333,...,3}_{m-1},$$

$$"2m"=\left\{\underbrace{222,...,2}_{m},\underbrace{333,...,3}_{m},4,5\right\}$$

The Markov chain representation of the fraction nonconforming m-of-m control chart with warning limits consists of $2m$ states with the first $(2m-1)$ of them being transient. Then, the $2m\times 2m$ transition probability matrix can be partitioned as

$$P=\begin{bmatrix} Q & (I-Q)J \\ 0 & 1 \end{bmatrix},$$

where Q is the $(2m-1)\times(2m-1)$ transition probability matrix for the transient states, I is the $(2m-1)\times(2m-1)$ identity matrix and J is the column vector of average run lengths of an order $(2m-1).$ The probability mass function, the expected value and the variance of the run length random variable T are respectively given by

$$\Pr\left[T=i\right]=eQ^{i-1}\left(I-Q\right)J,\ i=1,2,3,...,\qquad (3)$$

$$E\left[T\right]=e\left(I-Q\right)^{-1}J,\qquad (4)$$

and

$$Var\left(T\right)=e\left(I-Q\right)\left(I-Q\right)^{-2}J-\left[E\left(T\right)\right]^{2},\qquad (5)$$

where $e_{1\times 2m-1}=\left(1,0,0,...,0\right)$ is the initial distribution.

Let $M=\left(M_1,M_2,...,M_{2m-1}\right)$ be the vector of average run lengths and can be obtained by solving the following linear system of equations corresponding to $\left(I-Q\right)J=1$ where 1 is the column vector of ones,

$$\begin{aligned}
M_1 &= 1+pc.M_1+pu.M_2+pl.M_3,\\
M_2 &= 1+pc.M_1+pu.M_3+pl.M_4,\\
M_3 &= 1+pc.M_1+pu.M_2+pl.M_5,\\
M_4 &= 1+pc.M_1+pu.M_3+pl.M_6,\\
M_5 &= 1+pc.M_1+pu.M_2+pl.M_7,\\
&\vdots\\
M_{2m-4} &= 1+pc.M_1+pl.M_3+....+pu.M_{2m-2},\\
M_{2m-3} &= 1+pc.M_1+pu.M_2+...+pl.M_{2m-1},\\
M_{2m-2} &= 1+pc.M_1+pl.M_3,\\
M_{2m-1} &= 1+pc.M_1+pu.M_2.
\end{aligned}$$

By solving the above linear system of equations, the ARL M_1 for a chart with m-of-m runs rule $\left(m>1\right)$ is given by

$$M_1=\frac{\left(1-pu^{m}\right)\left(1-pl^{m}\right)}{\left(1-pu\right)\left(1-pl\right)-pu.pl\left(1-pu^{m-1}\right)\left(1-pl^{m-1}\right)-pc\left(1-pu^{m}\right)\left(1-pl^{m}\right)}.\qquad (6)$$

Equation (6) gives the zero-state ARL. The following section gives steady-state ARL.

3. Steady-State Average Run Length

A technique of an obtaining the steady-state ARL was suggested by Crosier (1986). Since then many researchers has reported the study of steady-state properties of a control chart in the literature. If the process is in an in-control state for a long period of time, it will reach in the steady-state mode. In order to study the long term properties of a control chart, it is appropriate to investigate the steady-state average run length (SSARL).

Let Q_0 be a square matrix obtained from Q by imposing the condition that no signal occurs. Let $\pi^T = [\pi_1, \pi_2, ..., \pi_{2m-1}]$ be the vector of steady-state probabilities for the in-control transient states. The steady-state probabilities can be obtained by solving the following equations:

$$\pi^T Q_0 = \pi^T \text{ and } \pi^T 1_{2m-1} = 1. \tag{7}$$

Under the in-control situation $p = p_0$, let $p_1 = pc$ and $u = pc = pl$.

To find the warning limits of control chart under steady-state mode, suppose that the desired steady-state average run length (SSARL(0)) is approximately 370.4. We may note that the π and ARL(0) are functions of m, control limit (k) and warning limit (w) only. Therefore, for a given combination of m and control limit (k), the warning limits can be determined from Equation (7). The performance of the proposed control chart is investigated in the following section. In the following subsection SSARL of the 3-of-3 control chart is explained in detail.

3.1. Steady-state average run length of the 3-of-3 control chart

We now present the Markov chain approach to obtain SSARL of the fraction nonconforming 3-of-3 control chart with warning limits. Consider an absorbing Markov chain with five transient states, which are defined below. The fraction nonconforming m-of-m control chart with the warning limits signals an out-of-control status if a point falls above (below) upper (lower) control limit or m-consecutive points fall between UWL and UCL or LCL and LWL. The states of the Markov chain are defined as follows:

State 1: A point falls between UWL and LWL.

State 2: A point falls between UWL and UCL.

State 3: A point between LCL and LWL.

State 4: Two consecutive points between UWL and UCL.

State 5: Two consecutive points between LCL and LWL.

State 6: An absorbing state (out-of-control state).

Then the 6×6 one step transition probability matrix of the Markov chain can be expressed as follows:

$$\begin{bmatrix} pc & pu & pl & 0 & 0 & 0 \\ pc & 0 & pl & pu & 0 & 0 \\ pc & pu & 0 & 0 & pl & 0 \\ pc & 0 & pl & 0 & 0 & pu \\ pc & pu & 0 & 0 & 0 & pl \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

Let Q be matrix obtained from the matrix defined in (8) by removing the last row and column. ARL of the 3-of-3 control chart is obtained by solving the linear system corresponding to $(I - Q)J = 1$, where I is 5×5 identity matrix and J is column vector of ones

$$\left.\begin{aligned} M_1 &= 1 + pc.M_1 + pu.M_2 + pl.M_3 \\ M_2 &= 1 + pc.M_1 + pl.M_3 + pu.M_4 \\ M_3 &= 1 + pc.M_1 + pu.M_2 + plM_5 \\ M_4 &= 1 + pc.M_1 + pl.M_3 \\ M_5 &= 1 + pc.M_1 + pu.M_2 \end{aligned}\right\}, \tag{9}$$

where M_j, $j = 1,2,3,4,5$ denotes the expected value of the waiting time from state j to until the first occurrence of an out-of-control signal. Let $M = (M_1, M_2, M_3, M_4, M_5)$ be the vector of ARL's.

The conditional transition probability matrix for $m = 3$ under the in-control situation $p = p_0$ and let $p_1 = pc$ and $u = pc = pl$ is

$$Q_0 = \begin{bmatrix} C_0 & C_1 & C_1 & 0 & 0 \\ C_0 & 0 & C_1 & C_1 & 0 \\ C_0 & C_1 & 0 & 0 & C_1 \\ C_2 & 0 & C_3 & 0 & 0 \\ C_2 & C_3 & 0 & 0 & 0 \end{bmatrix},$$

where $C_0 = \dfrac{p_1}{p_1 + 2u}$, $C_1 = \dfrac{u}{p_1 + 2u}$, $C_2 = \dfrac{p_1}{p_1 + u}$ and $C_3 = \dfrac{u}{p_1 + u}$. The stationary probabilities using Equation (7) are given below,

$$\pi_1 = \frac{C_0}{1 - 2C_1^3}, \pi_2 = \frac{C_1\pi_1}{C_2}, \pi_3 = \frac{C_1\pi_1}{C_2}, \pi_4 = \frac{C_1^2\pi_1}{C_2} \text{ and } \pi_5 = \frac{C_1^2\pi_1}{C_2}.$$

The SSARL can be obtained by

$$SSARL = \pi^T ARL,$$

where

$$ARL = (I - Q)^{-1} J = [M_1, M_2, ..., M_{2m-1}]^T.$$

The in-control SSARL can be obtained as

$$SSARL(0) = \pi^T ARL(0).$$

For 3-of-3 chart, the ARL vector can be obtained by solving linear system of equations given in Equation (9) as

$$M_1 = \frac{(1 + pl + pl^2)(1 + pu + pu^2)}{D}, \ M_2 = \frac{(1 + pl + pl^2)(1 + pu)}{D},$$

$$M_3 = \frac{(1 + pu + pu^2)(1 + pl)}{D}, \ M_4 = \frac{(1 + pl + pl^2)}{D}, \ M_5 = \frac{(1 + pu + pu^2)}{D},$$

where $D = 1 - pu.pl(1 + pu)(1 + pl) - pc(1 + pu + pu^2)(1 + pl + pl^2)$.

The SSARL of 3-of-3 chart can be obtained by

$$SSARL = \frac{\pi_1.M_1 + \pi_2.M_2 + \pi_3.M_3 + \pi_4.M_4 + \pi_5.M_5}{D}. \tag{10}$$

The in-control SSARL of the 3-of-3 control chart can be obtained by substituting $u = pu = pl$ and $pc + pu + pl = 1$ in Equation (10) as,

$$SSARL(0)=\frac{\left(1+u+u^2\right).\pi_1+2.\pi_2\left(1+u+u^2\right)\left(1+u\right)+2.\pi_3\left(1+u+u^2\right)}{1-u^2\left(1+u\right)^2-\left(1-2.u\right)\left(1+u+u^2\right)^2}. \tag{11}$$

Suppose that the desired in-control SSARL is approximately 370.4. Solving Equation (11) for u gives $u=0.900953$ The control limits for 3-of-3 control chart are ±1.287 using standard normal distribution. The following section gives the performance study of the m-of-m control chart with warning limits.

4. Performance of the m-of-m Control Chart with Warning Limits

In the present section, we proceed to detailed study of the proposed fraction nonconforming m-of-m control chart with warning limits. We computed steady-state average run length (SSARL) of the 2-of-2 and the 3-of-3 control charts with warning limits and compared with the SSARL of the 2-of-2 and the 3-of-3 control charts without warning limits. The standard deviation of run length (SDRL) is also given in parentheses along with the SSARL, since SSARL alone does not reveal important information regarding the performance of a control chart. The steady-state SDRL values are computed using Equation (5). The SSARL and SDRL of the 1-of-1, the 2-of-2 and the 3-of-3 charts are given in Table 1 for standardized control limit $k\left(k_0\right)=\pm3$ and the control limit $k=\pm3.3$.

Table 1 SSARL and SDRL profile of the 1-of-1, the 2-of-2 and the 3-of-3 control charts with $p_0=0.1, n=100$

p	The 1-of-1 chart	$k=\pm3$			$k=\pm3.3$
		The 2-of-2 chart $w=\pm2.899$	The 3-of-3 chart $w=\pm2.472$	The 2-of-2 chart $w=\pm1.876$	The 3-of-3 chart $w=\pm1.287$
0.10	370.4 (369.90)	370.33 (369.83)	370.36 (369.85)	370.40 (369.90)	370.40 (369.90)
0.11	167.27 (166.77)	167.20 (166.70)	167.17 (166.66)	142.64 (141.85)	135.79 (135.29)
0.12	62.61 (62.11)	62.55 (62.05)	62.48 (61.97)	46.28 (45.35)	42.75 (41.25)
0.13	26.75 (26.25)	26.71 (26.20)	26.62 (26.10)	18.90 (17.95)	17.70 (17.29)
0.14	13.35 (12.84)	13.33 (12.82)	13.24 (12.71)	9.57 (8.63)	9.26 (8.75)
0.15	7.61 (7.09)	7.60 (7.08)	7.53 (6.98)	5.72 (4.80)	5.74 (5.22)
0.17	3.36 (2.82)	3.36 (2.81)	3.33 (2.75)	2.87 (2.00)	3.05 (2.50)
0.20	1.67 (1.06)	1.67 (1.05)	1.66 (1.04)	1.65 (0.85)	1.77 (1.17)

From the Table 1 it is observed that for $k=\pm3,$ the SSARL and SDRL of the 1-of-1, the 2-of-2 and the 3-of-3 control charts are approximately same. For instance, consider a shift of 0.11, for this the SSARL of the 1-of-1, the 2-of-2 and the 3-of-3 control charts are 167.27, 167.20 and 167.17

respectively. These SSARL values do not significantly differ. However, for the choice of control limit $k = \pm 3.3,$ the SSARL and SDRL values of the 1-of-1 control chart and the m-of-m control chart with m =2, 3 are significantly different. Therefore, the value of control limit $k = \pm 3.3$ is chosen to study the steady-state properties of the m-of-m control chart with warning limits. The m-of-m control chart with m =2, 3 performs significantly better than the 1-of-1 control chart to detect all shifts in the fraction nonconforming. Except few large shifts, the 3-of-3 control chart is more efficient than the 2-of-2 control chart.

Quartiles give more information about the performance of a control chart. Therefore, the quartiles of the run length distribution are computed under steady-state mode. Table 2 gives the three quartiles and inter quartile range (IQR) of the run length distribution associated with m-of-m control chart with warning limits when SSARL(0)= 370.4.

Table 2 Quartiles and IQR of the 1-of-1, the 2-of-2 and the 3-of-3 control charts with warning limits under steady-state mode when $p_0 = 0.1, n = 100$

p	The 1-of-1 chart				The 2-of-2 chart				The 3-of-3 chart			
	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR	Q_1	Q_2	Q_3	IQR
0.10	106	256	513	406	106	256	513	407	104	554	511	407
0.11	48	116	231	183	42	99	197	155	38	93	186	148
0.12	18	43	86	68	14	32	64	50	12	29	58	46
0.13	8	18	36	29	6	13	26	20	6	12	27	21
0.14	4	9	18	14	3	7	13	10	3	7	12	9
0.15	2	5	10	8	2	4	8	6	3	4	7	4
0.16	1	3	6	5	2	3	5	3	2	3	5	3
0.17	1	2	4	3	2	2	4	2	1	3	4	3

We have seen that from Table 2, the quartile values and IQR values of the 3-of-3 control chart are less than that of the 2-of-2 and the 1-of-1 control charts. Therefore, the 3-of-3 control chart performs significantly better than the 2-of-2 and the 1-of-1 control charts. Also, the performance of the 2-of-2 control chart is better as compared to the 1-of-1 control chart with warning limits.

For a comparison study of the m-of-m control chart with warning limits and without warning limits, SSARL and SDRL of the 1-of-1, the 2-of-2 and the 3-of-3 control charts are given in Table 3.

Control chart without warning limits means control chart with control limits. From Table 3, it is observed that the SSARL and SDRL values of the 2-of-2 and 3-of-3 control charts with warning limits are significantly less than the 2-of-2 as well as 3-of-3 control charts without warning limits. It is clear that the 2-of-2 and 3-of-3 control charts with warning limits perform significantly better than that of the 2-of-2 and 3-of-3 control charts without warning limits. The m-of-m control chart with warning limits for m =2, 3 requires the less average number of samples to produce an out-of-control signal as compared to the control charts without warning limits.

The following Section presents the construction of the proposed control chart using data set on the frozen orange juice.

Table 3 SSARL and SDRL of the 1-of-1, the 2-of-2 and the 3-of-3 control charts when $n = 100$ and $SSARL(0) = 370.4$

p	The 1-of-1 chart	The control charts with warning limits		The control charts with warning limits	
		The 2-of-2 chart	The 3-of-3 chart	The 2-of-2 chart	The 3-of-3 chart
0.15	370.4 (369.90)	370.40 (369.90)	370.40 (367.24)	370.36 (369.86)	370.37 (169.87)
0.16	211.24 (210.74)	184.67 (183.93)	175.67 (172.48)	188.44 (187.60)	178.58 (178.08)
0.17	94.83 (94.33)	70.89 (69.98)	64.34 (61.82)	73.31 (72.23)	66.37 (65.87)
0.18	44.38 (43.88)	30.70 (29.74)	27.73 (25.62)	32.49 (31.32)	29.60 (29.10)
0.19	22.92 (22.41)	15.54 (14.57)	14.32 (12.45)	16.89 (15.69)	15.93 (15.42)
0.20	13.04 (12.53)	9.00 (8.04)	8.57 (6.87)	10.07 (8.85)	9.97 (9.46)
0.21	8.09 (7.57)	5.82 (4.89)	5.76 (4.18)	6.71 (5.49)	7.02 (6.50)
0.22	5.4 (4.87)	4.12 (3.20)	4.22 (2.75)	4.89 (3.65)	5.41 (4.88)
0.23	3.85 (3.31)	3.12 (2.22)	3.29 (1.93)	3.84 (2.57)	4.47 (3.94)
0.24	2.9 (2.35)	2.50 (1.62)	2.69 (1.44)	3.19 (1.88)	3.89 (3.35)
0.25	2.3 (1.73)	2.10 (1.24)	2.27 (1.13)	2.77 (1.42)	3.52 (2.98)

5. Example

To construct the proposed m-of-m control chart with warning limits, the data set is taken from Montgomery (2005) related to frozen orange juice packed in 6-oz cardboard cans by machine. The data set contain 30 samples of size $n = 50$ cans each. The data is shown in Table 4. The construction of the fraction nonconforming m-of-m control chart with warning limits will be divided into two stages.

Stage 1: The preliminary control chart is constructed to see whether the process is in-control.

Stage 2: After an in-control preliminary control chart is set up, it will be used for monitoring the fraction of nonconforming cans when additional samples are taken.

From Table 4, the estimate of p_0 is

$$\overline{p}_0 = \frac{\sum_{j=1}^{m} X_j}{mn} = \frac{347}{(30 \times 50)} = 0.233133.$$

The standardized Z_j statistic is computed for each of the 30 samples using the estimated value of p_0.

Table 4 Data for constructing a preliminary fraction nonconforming m-of-m control chart with warning limits

Sample number, j	Number of nonconforming cans, X_j	Sample fraction nonconforming, $\hat{p}_j = \frac{X_j}{50}$	$Z_j = \frac{(\hat{p}_j - \bar{p}_0)}{\sqrt{\bar{p}_0(1-\bar{p}_0)/n}}$
1	12	0.24	0.1453
2	15	0.3	1.1515
3	8	0.16	−1.1962
4	10	0.2	−0.5254
5	4	0.08	−2.5376
6	7	0.14	−1.5315
7	16	0.32	1.4868
8	9	0.18	−0.8608
9	14	0.28	0.8161
10	10	0.20	−0.5254
11	5	0.10	−2.2023
12	6	0.12	−1.8669
13	17	0.34	1.8222
14	12	0.24	0.1453
15	22	0.44	3.4991
16	8	0.16	−1.1962
17	10	0.20	−0.5254
18	5	0.10	−2.2023
19	13	0.26	0.4807
20	11	0.22	−0.1900
21	20	0.40	2.8283
22	18	0.36	2.1576
23	24	0.48	4.1698
24	15	0.30	1.1515
25	9	0.18	−0.8608
26	12	0.24	0.1453
27	7	0.14	−1.5315
28	13	0.26	0.4807
29	9	0.18	−0.8608
30	6	0.12	−1.8669

The warning control limits of the proposed control chart are determined under steady-state mode by setting in-control ARL equal to 370.4. Thus the control limits (upper control limit and lower control limit) of the proposed control chart are set to be ± 3.3. The warning limits of the 2-of-2 and 3-of-3 control charts under steady-state mode are ±1.876 and ±1.287 respectively. In figure 1, the Z_j statistics for 30 samples in the preliminary stage are plotted on the fraction nonconforming m-of-m control chart with warning limits.

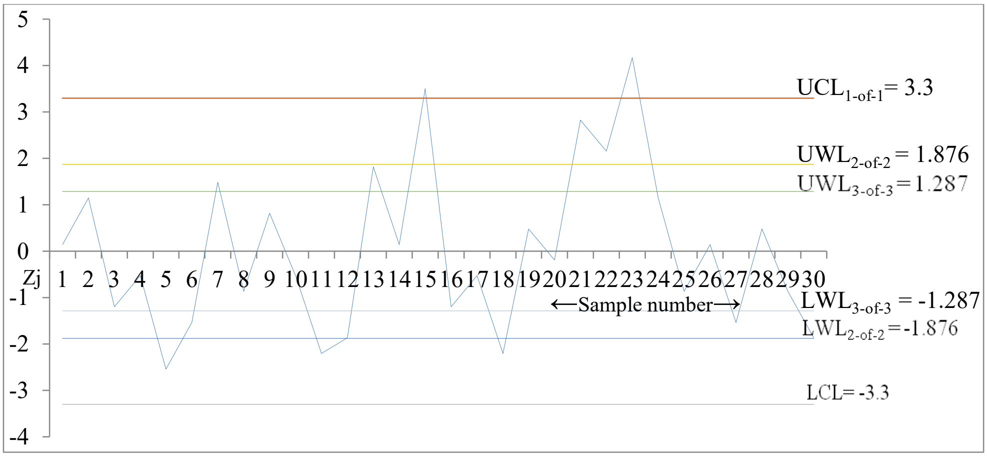

Figure 1 Preliminary fraction nonconforming m-of-m chart with warning limits for data in Table 4

The 2-of-2 control chart signals an out-of-control status at sample 12. The points 11 and 12 fall below the lower warning limit of the 2-of-2 control chart. Therefore, assume that an analysis of the data does not produce any reasonable or logical assignable cause for this. Thus, the points 11 and 12 does not eliminated at this instant. The 1-of-1 control chart signals at point 15 and a point falls above upper control limit. Assume that an assignable cause can be found at point 15. Thus a point 15 must be eliminated from the estimation of p_0 in Stage 2. Another out-of-control signal observed at samples 22 (for the first time) and 23 (for the second time) using the 2-of-2 control chart, while the 1-of-1 control chart signals at sample 23. Upon investigation assumes that an assignable cause is found at points 21, 22 and 23. Therefore, these three samples must be eliminated in the computation of the estimate of p_0 in Stage 2.

The estimate of p_0 after eliminating samples 15, 21, 22 and 23 is computed as follows:

$$\overline{p}_0 = \frac{\sum_{j=1}^{m} X_j - X_{15} - X_{21} - X_{22} - X_{23}}{(m-4)n} = \frac{263}{(26\times 50)} = 0.202308.$$

The Z_j statistic for the remaining 26 samples is computed using the above revised estimate of p_0 and these Z_j values are given in Table 7. The fraction nonconforming m-of-m control chart with warning limits for the remaining 26 samples is given in Figure 2. Figure 2 shows that none of the control charts signals out-of-control status. That is, now the process is in-control state. We can safely conclude that the process is stable and the revised estimate of p_0 can be used in the computation of the Z_j statistics for additional samples in Stage 2.

Table 5 Data for constructing a preliminary fraction nonconforming m-of-m control chart with warning limits for discarded samples 15, 21, 22 and 23

Sample number, j	Number of nonconforming cans, X_j	Sample fraction nonconforming, $\hat{p}_j = \frac{X_j}{50}$	$Z_j = \frac{(\hat{p}_j - \overline{p}_0)}{\sqrt{\overline{p}_0(1-\overline{p}_0)/n}}$
1	12	0.24	0.6635
2	15	0.30	1.7196
3	8	0.16	–0.7447
4	10	0.20	–0.0406
5	4	0.08	–2.1529
6	7	0.14	–1.0967
7	16	0.32	2.0716
8	9	0.18	–0.3927
9	14	0.28	1.3675
10	10	0.20	–0.0406
11	5	0.10	–1.8008
12	6	0.12	–1.4488
13	17	0.34	2.4236
14	12	0.24	0.6635
16	8	0.16	–0.7447
17	10	0.20	–0.0406
18	5	0.10	–1.8008
19	13	0.26	1.0155
20	11	0.22	0.3114
24	15	0.30	1.7196
25	9	0.18	–0.3927
26	12	0.24	0.6635
27	7	0.14	–1.0967
28	13	0.26	1.0155
29	9	0.18	–0.3927
30	6	0.12	–1.4488

As noted in Montgomery (2005) plant manager feels that although the process is in control, the fraction nonconforming is much too high and it is needs to reduce the level of fraction nonconforming cans. To improve the performance of the process, several adjustments can be made to the machine and then an additional 24 samples of size 50 are taken. Table 6 gives an additional 24 samples along with the computed Z_j statistics based on $\overline{p}_0 = 0.202308.$ In Figure 3, the Z_j statistics for additional 24 samples are plotted on the fraction nonconforming m-of-m control chart with warning limits.

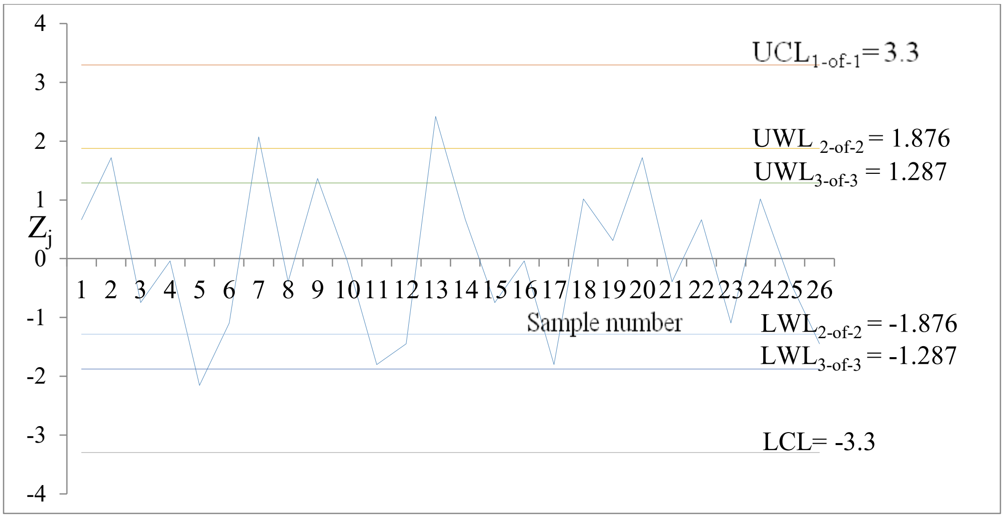

Figure 2 Preliminary fraction nonconforming m-of-m chart with warning limits for data in Table 5.

Table 6 Data for the additional samples in Stage 2

Sample number, j	Number of nonconforming cans, X_j	Sample fraction nonconforming, $\hat{p}_j = \frac{X_j}{50}$	$Z_j = \frac{(\hat{p}_j - \bar{p}_0)}{\sqrt{\bar{p}_0(1-\bar{p}_0)/n}}$
31	9	0.18	−0.3927
32	6	0.12	−1.4488
33	12	0.24	0.66345
34	5	0.1	−1.8008
35	6	0.12	−1.4488
36	4	0.08	−2.1529
37	6	0.12	−1.4488
38	3	0.06	−2.5049
39	7	0.14	−1.0967
40	6	0.12	−1.4488
41	2	0.04	−2.8569
42	4	0.08	−2.1529
43	3	0.06	−2.5049
44	6	0.12	−1.4488
45	5	0.10	−1.8008
46	4	0.08	−2.1529
47	8	0.16	−0.7447
48	5	0.10	−1.8008
49	6	0.12	−1.4488
50	7	0.14	−1.0967
51	5	0.10	−1.8008

Table 6 (Continued)

Sample number, j	Number of nonconforming cans, X_j	Sample fraction nonconforming, $\hat{p}_j = \frac{X_j}{50}$	$Z_j = \frac{(\hat{p}_j - \bar{p}_0)}{\sqrt{\bar{p}_0(1-\bar{p}_0)/n}}$
52	6	0.12	–1.4488
53	3	0.06	–2.5049
54	3	0.06	–2.5049

Figure 3 show that the 3-of-3 control chart signals a process improvement at samples 36 (for the first time) and 38 (for the second time). It is also observed that the Z_j statistics for all samples plot below the center line except sample 33. However, the 1-of-1 control chart does not indicate an improvement in the process, although we have strong evidence that the process has an actually improved based on the significant reduction in the number of nonconforming cans in Table 5 compared to that in Table 4. In general, the fraction nonconforming m-of-m control chart with warning limits gives a better indication of either improvement or deterioration if there is a systematic upward or downward trend in the process.

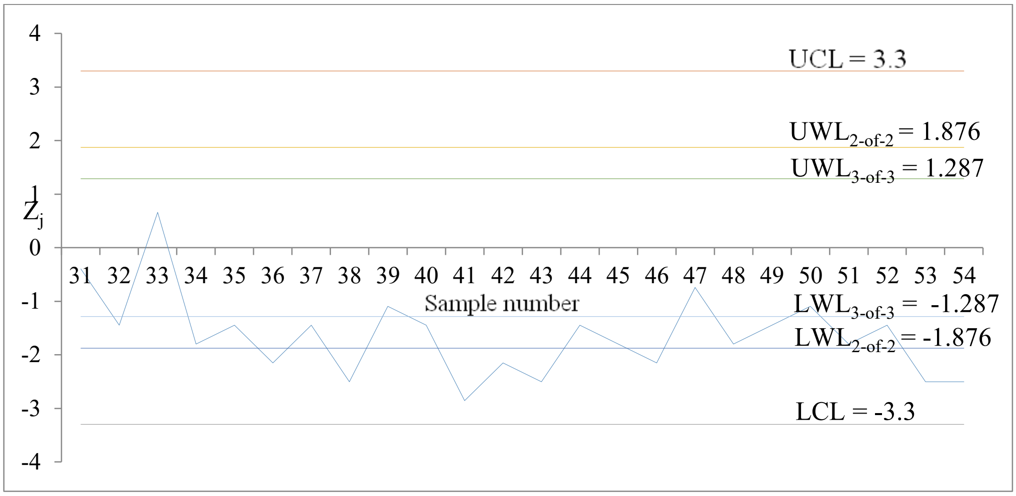

Figure 3 The fraction nonconforming m-of-m chart with warning limits for additional samples (Data in Table 6)

6. Conclusions

In the present article, the m-of-m control charts with warning limits is proposed and studied to detect shifts in the fraction nonconforming. Performance of the proposed control chart is measured using ARL, SDRL and quartiles under the steady-state mode. The study revealed that the m-of-m control chart with warning limit is more sensitive to detect all shifts in the fraction nonconforming than the m-of-m control chart without warning limits. The m-of-m control chart with warning limits has a higher power to detect an out-of-control signal in a process. For small to moderate shifts in the p, we suggest the use of the 3-of-3 runs rule with $k = 3.3$ and $w = 1.2874$, otherwise we suggest the use of the 2-of-2 runs rule with $k = 3.3$ and $w = 1.876$.

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