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Parameter Estimation for H-self-similar Stable Fields

Dang Thi To Nhu*
University of Economics, The University of Danang, Danang city, Vietnam
*Corresponding author; e-mail: nhudtt@due.edu.vn

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Abstract

In this work, we are interested in H-sssi α- stable fields, that is, in stable random fields that are self-similar with parameter H and have stationary increments. We give two estimators of the stability and the self-similar indices based on β-negative power variations with $-1/2 < \beta < 0$. The consistency of those two estimators are also proved. We illustrate these convergences with some examples: Lévy fractional Brownian field, well-balanced linear fractional stable field and Takenaka random field.

Keywords: H-sssi fields; stable fields, self-similarity parameter estimator, stability parameter estimator, negative power variations.

1. Introduction

In this paper we will consider random fields $\{X(t), t \in \mathbb{R}^d\}, X : \mathbb{R}^d \to \mathbb{R}$ whose parameter space is the Euclidean space $\mathbb{R}^d$, where $d \in \mathbb{N}$. More precisely, we will be interested in symmetric α-stable ($S\alpha S$) random fields that are self-similar with parameter H (ss) and have stationary increments (si). Such fields are widely used as models for real data, see e.g. Bonami and Estrade (2003), Peitgen and Saupe (1988), Wilson (2000), Dubuc et al. (1989), Taylor and Taylor (1991), Taylor and Taylor (1999), Thomas (1982), Thomas and Thomas (1988).

In the statistical literature, the estimation of various indices of H-sssi $S\alpha S$-stable random fields, especially for Lévy fractional Brownian fields, has been studied by many authors. Among these, to estimate the parameter H, one can mention the generalized quadratic variations with the use of the spectral density, e.g. Bierme et al. (2011), Cohen and Istas (2013), the emperical variogram based on a finite number of observations of X on a regular grid, see e.g. Constantine and Hall (1994), Kent and Wood (1997), Taylor and Taylor (1999). Other references, for example Hall and Wood (1993) proposed box-counting estimators, Feuerverger et al. (1994) recommended estimators based on the level crossings, Istas and Lang (1997) considered the use of higher-order increments and generalized least squared.

The main goal of this work is to develop a method using β-negative power variations with $-1/2 < \beta < 0$ to identify consistent estimators of the stability index α and the self-similarity index H of H-sssi $S\alpha S$-stable random fields. This method has been introduced recently in e.g. Dang (2020), Dang and Istas (2017). More precisely, in Dang and Istas (2017), the authors used β-negative power variations to estimate the Hurst and the stability indices of a H-self-similar stable process, in the context H and α are constants, based on the fact that β-negative power variations have expectations and covariances for $-1/2 < \beta < 0$. In this paper, using this approach gives a consistent estimator of H without a priori knowledge on α and vice versa, the consistent estimator of α can be obtained without assumptions on H. In this context, we deal with the multi-indices by the use of the transformation from Cartesian coordinates to spherical coordinates to prove some inequalities for covariances of β-negative power variations. Moreover, the rate of convergence of our estimates is given. All the results for the case $d = 1$ is coincident to those in Dang and Istas (2017).

The remainder of this paper is organized as follows: in the next sections, we present the setting and some general results for the estimation of H and α under some given assumptions. We get the consistent estimator for H and α. Then the obtained results will be applied to some particular examples presented in Section 3: Lévy fractional Brownian field, well-balanced linear fractional stable field and Takenaka random field. In Section 4, we gather all the proofs of the main results and the illustrated examples.

2. Settings and main results

We first recall the definition of ss and si properties of random fields.

Definition 1 *(self-similarity, see e.g., Samorodnitsky and Taqqu (1988))* A random field $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ is self-similar with index $H > 0$ ($H - ss$) if $\{X(a\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\} \overset{(d)}{=} \{a^H X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^n\}$ for all $a > 0$, where $\overset{(d)}{=}$ denotes equality of the finite-dimensional distributions.

Definition 2 *(stationary increments, see e.g., Samorodnitsky and Taqqu (1988))* A random field $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ has stationary increments (si) if

$$\{X(\mathbf{t} + \mathbf{s}) - X(\mathbf{s}), \mathbf{t} \in \mathbb{R}^d\} \overset{(d)}{=} \{X(\mathbf{t}) - X(\mathbf{0}), \mathbf{t} \in \mathbb{R}^d\}$$

for all $\mathbf{s} \in \mathbb{R}^d$.

Let X be a $H-$sssi symmetric $\alpha-$ stable random fields. Let $K \in \mathbb{N}, L \in \mathbb{N}$ be fixed integers, $a = (a_0, \ldots, a_K)$ be a finite sequence with exactly L vanishing first moments, that is for all $q \in \{0, \ldots, L\}$, one has

$$\sum_{k=0}^{K} k^q a_k = 0, \sum_{k=0}^{K} k^{L+1} a_k \neq 0 \tag{1}$$

with convention $0^0 = 1$. For example, here we can choose $K = L + 1$ and

$$a_k = (-1)^{L+1-k} \frac{(L+1)!}{k!(L+1-k)!}. \tag{2}$$

For each $\mathbf{p} = (p_1, \ldots, p_d) \in \{0, \ldots, K\}^d$, let $a_{\mathbf{p}}$ be defined by

$$a_{\mathbf{p}} = a_{p_1} a_{p_2} \ldots a_{p_d}. \tag{3}$$

For each $\mathbf{k} = (k_1, \ldots, k_d) \in \mathbb{N}^d, n \in \mathbb{N}$, we denote by $\triangle_{\mathbf{k},n} X$ the increment of the H-sssi $S\alpha S$-stable random field X with respect to $a_{\mathbf{p}}$, i.e.

$$\triangle_{\mathbf{k},n} X = \sum_{\mathbf{p} \in \{0,\ldots,K\}^d} a_{\mathbf{p}} X(\frac{\mathbf{k} + \mathbf{p}}{n}). \tag{4}$$

Fix $-1/2 < \beta < 0$, let us consider $V_n(\beta)$, the empirical mean of order β, defined by

$$V_n(\beta) = \frac{1}{(n - K + 1)^d} \sum_{\mathbf{k} \in \{0,\ldots,n-K\}^d} |\triangle_{\mathbf{k},n} X|^{\beta}, \tag{5}$$

and the statistic

$$W_n(\beta) = n^{\beta H} V_n(\beta). \tag{6}$$

We will prove later that $W_n(\beta)$ converges to $\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta$. We define an estimator of H as follows

$$\widehat{H}_n(\beta) = \frac{1}{\beta}\log_2 \frac{V_{n/2}(\beta)}{V_n(\beta)}. \tag{7}$$

We can see that the estimator of H is presented as a function of two empirical means of order β. Let $u > v > 0$, to define an estimator of α, we first introduce auxiliary functions $\psi_{u,v}, h_{u,v}, \varphi_{u,v}$. Let $\psi_{u,v}$: $\mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\psi_{u,v}(x, y) = -v \ln x + u \ln y + C(u, v), \tag{8}$$

where $C(u, v)$ is a constant depending on u, v

$$\begin{aligned} C(u,v) =& \frac{u-v}{2}\ln\pi + u\ln\left(\Gamma(1+\frac{v}{2})\right) + v\ln\left(\Gamma(\frac{1-u}{2})\right) \\ & - v\ln\left(\Gamma(1+\frac{u}{2})\right) - u\ln\left(\Gamma(\frac{1-v}{2})\right), \end{aligned}$$

here $\Gamma(.)$ is the Gamma function. Let $h_{u,v} : (0, +\infty) \to (-\infty, 0)$ be given by

$$h_{u,v}(x) = u\ln\left(\Gamma(1+\frac{v}{x})\right) - v\ln\left(\Gamma(1+\frac{u}{x})\right). \tag{9}$$

From Lemma 4.11 in Dang and Istas (2017), one gets that $h_{u,v}$ is a strictly increasing function and there exists the inverse function $h^{-1}_{u,v}$. Therefore, we can define a function $\varphi_{u,v} : \mathbb{R} \to [0, +\infty)$ as follows

$$\varphi_{u,v}(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ h^{-1}_{u,v}(x) & \text{if } x < 0 \end{cases} \tag{10}$$

where $h_{u,v}$ is defined by (9).
Let β_1, β_2 be in $\mathbb{R}$ such that $-1/2 < \beta_1 < \beta_2 < 0$, then $-\beta_1 > -\beta_2 > 0$. The estimator of α is defined by

$$\widehat{\alpha}_n = \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(V_n(\beta_1), V_n(\beta_2))\right), \tag{11}$$

where $\psi_{u,v}, \varphi_{u,v}$ are defined as in (8) and (10), respectively. We make the following assumptions with $\beta \in (-1/2, 0)$ fixed:
There exists a sequence $\{b_n, n \in \mathbb{N}\}$ and a constant C such that $\lim_{n\to+\infty} b_n = 0, b_{n/2} = O(b_n)$ and

$$\lim_{n\to+\infty} \frac{1}{n^d b_n^2} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, |k_i|\leq n} \left|cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)\right| \leq C^2. \tag{12}$$

Based on the assumption (12), we present the estimators of H and α as follows.

Theorem 1 *Let X be a H-sssi, $S\alpha S$ random field, $X : \mathbb{R}^d \to \mathbb{R}$. Also, let $\beta, \beta_1, \beta_2 \in \mathbb{R}, -1/2 < \beta < 0, -1/2 < \beta_1 < \beta_2 < 0$ and $\widehat{H}_n(\beta), \widehat{\alpha}_n$ be defined as in* (7), (11)*, respectively. Assume (12), then*

$$W_n(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta = O_{\mathbb{P}}(b_n). \tag{13}$$

$$\widehat{H}_n(\beta) - H = O_{\mathbb{P}}(b_n), \hat{\alpha}_n - \alpha = O_{\mathbb{P}}(b_n), \tag{14}$$

where $O_{\mathbb{P}}(b_n)$ is defined by:
- $X_n = O_{\mathbb{P}}(1)$ *iff for all $\epsilon > 0$, there exists $M > 0$ such that* $\sup_n \mathbb{P}(|X_n| > M < \epsilon)$,
- $Y_n = O_{\mathbb{P}}(a_n)$ *means $Y_n = a_n X_n$ with $X_n = O_{\mathbb{P}}(1)$.*

Proof: See Subsection 4.1 for the Proof of Theorem 1.

3. Examples

We are in position to give some examples to illustrate the results presented in the latter section for estimating H and α. n this section, let $\beta, \beta_1, \beta_2 \in \mathbb{R}, -1/2 < \beta < 0, -1/2 < \beta_1 < \beta_2 < 0$.

3.1. Lévy fractional Brownian field

We will present here estimators for the self-similarity index H and the stability index α of the Lévy fractional Brownian field.

Definition 3 *Lévy fractional Brownian field (see e.g., Samorodnitsky and Taqqu (1988)).*
Let $0 < H < 1$ and $\sigma > 0$. The Gaussian field $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ with mean 0 and autocovariance function

$$\mathbb{E}X(\mathbf{t})X(\mathbf{s}) = \frac{\sigma^2}{2}\{||\mathbf{t}||^{2H} + ||\mathbf{s}||^{2H} - ||\mathbf{t}-\mathbf{s}||^{2H}\}, \mathbf{t}, \mathbf{s} \in \mathbb{R}^d, \tag{15}$$

is called the Lévy fractional Brownian field. When $d = 1$, it is the fractional Brownian motion.

One can also construct X as the integral $X(\mathbf{t}) = \sigma_0 \int\limits_{\mathbb{R}^d} (||\mathbf{t}-\mathbf{x}||^{H-\frac{d}{2}} - ||\mathbf{x}||^{H-\frac{d}{2}}) M(d\mathbf{x})$, where M is a Gaussian random measure on $\mathbb{R}^d$ with Lebesgue control measure and σ_0 is a constant proportional to σ. Lévy fractional Brownian field is an extension of fractional Brownian motion to $\mathbb{R}^d$. It is H-sssi 2-stable random field (see Chapter 8 in (Samorodnitsky and Taqqu , 1988) for more details). Now we will just consider the case $d \geq 2$. Let

$$b_n = n^{-\frac{d}{2}}. \tag{16}$$

Theorem 2 *Let X be a Lévy fractional Brownian field in $\mathbb{R}^d (d \geq 2)$, defined by* (15). *Then $\widehat{H}_n(\beta), \widehat{\alpha}_n$, defined by (7), (11) are consistent estimators of H and α, respectively. More precisely, one obtains the results in Theorem 1 with b_n as in (16).*

Proof: See Subsection 4.2.

3.2. Well-balanced linear fractional stable field

We now apply the results on estimating H and α for another random field which is the generalization of the well-balanced linear fractional stable motion.

Definition 4 *Well-balanced linear fractional stable field.*
Let $H \in (0,1), \alpha \in (0,2], H \neq 1/\alpha$. Let $X = \{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ be a random field defined by

$$X(\mathbf{t}) = \int\limits_{\mathbb{R}^d} \left(||\mathbf{t}-\mathbf{s}||^{H-\frac{d}{\alpha}} - ||\mathbf{s}||^{H-\frac{d}{\alpha}}\right) M_\alpha(d\mathbf{s}), \tag{17}$$

where M_α is a $S\alpha S$ random measure on $\mathbb{R}^d$ with Lebesgue control measure. Then $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ is called *well-balanced linear fractional stable field.*

The field X is H-sssi (see the Notes to Chapter 8 in Samorodnitsky and Taqqu (1988) for more details). If $d = 1$, it reduces to the well-balanced linear fractional stable motion. Let

$$b_n = \begin{cases} n^{-\frac{d}{2}} & \text{if } \frac{\alpha H-(L+1)\alpha d}{2} < -d, \\ n^{\frac{\alpha H-(L+1)\alpha d}{4}} & \text{if } -d < \frac{\alpha H-(L+1)\alpha d}{2} < 0, \\ \sqrt{\frac{\ln n}{n^d}} & \text{if } \frac{\alpha H-(L+1)\alpha d}{2} = -d. \end{cases} \tag{18}$$

It is clear that $\lim\limits_{n\to+\infty} b_n = 0$ as $n \to +\infty$ and $b_{n/2} = O(b_n)$. We will prove that the assumption (12) is satisfied with b_n defined by (18), then the results in Theorem 1 are obtained.
Remark The case $\alpha = 2$ corresponds to the Lévy fractional Brownian field with $b_n = n^{-d/2}$ as presented in the subsection 3.1. Now we deal with the case $0 < \alpha < 2$.

Theorem 3 *Let* $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ *be a well-balanced linear fractional stable field defined by (17), where* $0 < \alpha < 2$*. Then the results in Theorem 1 occur with* b_n *as in (18).*

Proof: See Subsection 4.2.

3.3. Takenaka random field

We now apply our results to Takenaka random field which is an extension of Takenaka process to $\mathbb{R}^d$. For $\mathbf{t} \in \mathbb{R}^d$, let $C_{\mathbf{t}}, V_{\mathbf{t}}$ be defined by

$$C_{\mathbf{t}} = \{(\mathbf{x}, r) \in \mathbb{R}^d \times \mathbb{R}^+ : ||\mathbf{x} - \mathbf{t}|| \le r\} \tag{19}$$

$$V_{\mathbf{t}} = C_{\mathbf{t}} \triangle C_{\mathbf{0}} = \{(\mathbf{x}, r) \in \mathbb{R}^d \times \mathbb{R}^+ : ||\mathbf{x} - \mathbf{t}|| \le r\}. \tag{20}$$

Definition 5 *Takenaka random field (see e.g., Samorodnitsky and Taqqu (1988))*
Let $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ be defined by

$$X(\mathbf{t}) = \int_{\mathbb{R}^d \times \mathbb{R}^+} \mathbf{1}_{V_{\mathbf{t}}}(\mathbf{x}, r) M_\alpha(d\mathbf{x}, dr), \tag{21}$$

where $V_{\mathbf{t}}$ is defined by (20) and M_α is a $S\alpha S$ random measure with control measure m defined by

$$m(d\mathbf{x}, dr) = r^{\nu-d-1} d\mathbf{x} dr, 0 < \nu < 1. \tag{22}$$

$\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ is called an $(\alpha, H)-$ Takenaka random field, where $0 < \alpha < 2$ and $H = \nu/\alpha$.

Following Theorem 8.4.4 in Samorodnitsky and Taqqu (1988), $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ is H-sssi. Let b_n be defined by

$$b_n = n^{\frac{\nu-1}{2}}. \tag{23}$$

The following result is a corollary of Theorem 1.

Theorem 4 *Let* $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ *be a Takenaka random field defined by (21). Then one gets the conclusion of Theorem 1 with* b_n *as in (23).*

Proof: See Subsection 4.2

4. Proofs

In this section, we gather all the proofs of the main results and the examples presented in Section 3. The key point in these proofs is the use of the Jacobian's transformation from Cartesian coordinates to spherical coordinates which is presented in Appendix.

4.1. Proof of Theorem 1

Proof: The proof consists of the following steps:

1. applying the assumption (12) and $sssi$ properties of $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ to show that $W_n(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1} X|^\beta = O_{\mathbb{P}}(b_n)$.
2. combining the above result with Lemma 4.10 and Lemma 4.11 in Dang and Istas (2017) to get that $\hat{H}_n(\beta)$ and $\hat{\alpha}_n$ are consistent estimators of H and α, respectively.

We first prove that $W_n(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta = O_{\mathbb{P}}(b_n)$.
Since $\{X(\mathbf{t}), \mathbf{t} \in \mathbb{R}^d\}$ is H-ss and $\sum_{\mathbf{p}\in\{0,\dots,K\}^d} a_{\mathbf{p}} = 0$, then for $\mathbf{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $\triangle_{\mathbf{k},n}X \overset{(d)}{=} \frac{1}{n^H}\triangle_{\mathbf{0},1}X$. It follows that $\mathbb{E}|\triangle_{\mathbf{k},n}X|^\beta = \mathbb{E}\left|\frac{\triangle_{\mathbf{0},1}X}{n^H}\right|^\beta = \frac{\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta}{n^{\beta H}}$ and

$$\mathbb{E}W_n(\beta) = \frac{n^{\beta H}}{(n-K+1)^d}\sum_{\mathbf{k}\in\{0,\dots,n-K\}^d}\mathbb{E}|\triangle_{\mathbf{k},n}X|^\beta = \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta. \tag{24}$$

One gets

$$\mathbb{E}W_n(\beta)^2 = \frac{n^{2\beta H}}{(n-K+1)^{2d}}\sum_{\mathbf{k}^{(1)},\mathbf{k}^{(2)}\in\{0,\dots,n-K\}^d}\mathbb{E}|\triangle_{\mathbf{k}^{(1)},n}X|^\beta|\triangle_{\mathbf{k}^{(2)},n}X|^\beta. \tag{25}$$

For $\mathbf{k}^{(1)}, \mathbf{k}^{(2)} \in \{0, \dots, n-K\}^d$, one has $|\triangle_{\mathbf{k}^{(1)},n}X|^\beta|\triangle_{\mathbf{k}^{(2)},n}X|^\beta \overset{(d)}{=} |\triangle_{\mathbf{k}^{(1)}-\mathbf{k}^{(2)},1}X|^\beta|\triangle_{\mathbf{0},1}X|^\beta$. Then it follows that

$$\mathbb{E}|\triangle_{\mathbf{k}^{(1)},n}X|^\beta|\triangle_{\mathbf{k}^{(2)},n}X|^\beta = n^{-2\beta H}\mathbb{E}|\triangle_{\mathbf{k}^{(1)}-\mathbf{k}^{(2)},1}X|^\beta|\triangle_{\mathbf{0},1}X|^\beta. \tag{26}$$

Combining (25) with (26), it deduces that

$$\begin{aligned}\mathbb{E}W_n(\beta)^2 = \frac{1}{(n-K+1)^d}&\sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d,|k_i|\le n-K}(1-\frac{|k_1|}{n-K+1})\dots(1-\frac{|k_d|}{n-K+1})\\ &\times \mathbb{E}|\triangle_{\mathbf{k},1}X|^\beta|\triangle_{\mathbf{0},1}X|^\beta.\end{aligned} \tag{27}$$

Moreover,by induction, we can deduce that

$$\frac{1}{(n-K+1)^d}\sum_{\substack{i=1\\ k_i\in\mathbb{Z},|k_i|\le n-K}}^{d}(1-\frac{|k_1|}{n-K+1})\dots(1-\frac{|k_d|}{n-K+1}) = 1.$$

Together with (24), (25), (27), one can derive

$$\begin{aligned}\mathbb{E}|W_n(\beta)-\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta|^2 &= \frac{1}{(n-K+1)^d}\sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d,|k_i|\le n-K}(1-\frac{|k_1|}{n-K+1})\dots(1-\frac{|k_d|}{n-K+1})\\ &\times\left(\mathbb{E}|\triangle_{\mathbf{k},1}X|^\beta|\triangle_{\mathbf{0},1}X|^\beta - (\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta)^2\right)\\ &\le \frac{1}{(n-K+1)^d}\sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d,|k_i|\le n-K}|cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)|.\end{aligned} \tag{28}$$

Using (28) and the assumptions (12), one gets

$$\limsup_{n\to+\infty}\frac{1}{b_n^2}\mathbb{E}|W_n(\beta)-\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta|^2 \le C^2.$$

Then for all $\epsilon > 0$, applying Markov's inequality, one deduces

$$\sup_n \mathbb{P}(|W_n(\beta)-\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta| > b_n\frac{C}{\sqrt{\epsilon}}) \le \limsup_n \frac{\mathbb{E}|W_n(\beta)-\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta|^2}{b_n^2\frac{C^2}{\epsilon}} \le \epsilon.$$

It follows that $W_n(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta = O_{\mathbb{P}}(b_n)$. Since $b_{n/2} = O(b_n)$, one also has

$$W_{n/2}(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta = O_{\mathbb{P}}(b_n).$$

We deduce that $\widehat{H}_n(\beta) - H = \phi(W_{n/2}(\beta), W_n(\beta))$. Combining to the fact that ϕ is differentiable at $(\mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta, \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta)$ and

$$W_n(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta = O_{\mathbb{P}}(b_n), W_{n/2}(\beta) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^\beta = O_{\mathbb{P}}(b_n),$$

applying Lemma 4.10 in (Dang and Istas , 2017), one gets $\widehat{H}_n(\beta) - H = O_{\mathbb{P}}(b_n)$. Moreover

$$\widehat{\alpha}_n - \alpha = \varphi_{-\beta_1,-\beta_2}\left(\psi_{-\beta_1,-\beta_2}(W_n(\beta_1), W_n(\beta_2))\right) - \varphi_{-\beta_1,-\beta_2}(\psi_{-\beta_1,-\beta_2}(\mathbb{E}|\triangle_{\mathbf{0},1}X|^{\beta_1}, \mathbb{E}|\triangle_{\mathbf{0},1}X|^{\beta_2})).$$

From Lemma 4.11 in Dang and Istas (2017), it follows that $\varphi_{-\beta_1,-\beta_2} \circ \psi_{-\beta_1,-\beta_2}$ is differentiable at $x_0 = (\mathbb{E}|\triangle_{\mathbf{0},1}X|^{\beta_1}, \mathbb{E}|\triangle_{\mathbf{0},1}X|^{\beta_2})$. Combining with the assumption (12) and the fact that $W_n(\beta_1) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^{\beta_1} = O_{\mathbb{P}}(b_n), W_n(\beta_2) - \mathbb{E}|\triangle_{\mathbf{0},1}X|^{\beta_2} = O_{\mathbb{P}}(b_n)$ and applying Lemma 4.10 in Dang and Istas (2017), we obtain that $\widehat{\alpha}_n - \alpha = O_{\mathbb{P}}(b_n)$.

4.2. Proofs related to Section 3

In this subsection, we give the proofs of theorems presented in Section 3 for the case $d \geq 2$. For the one-dimensional case ($d = 1$), we refer to Dang and Istas (2017) for more details.
A strategy for proving these theorems is as follows:

1. to find a bound for $|cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X)|^\beta|$,
2. to show that the assumption (12) is satisfied for the mentioned fields, then apply Theorem 1 to get the conclusion.

Proof of Theorem 2. We just consider the case $d \geq 2$. One has

$$\triangle_{\mathbf{k},1}X = \sum_{\mathbf{p}\in\{0,\ldots,K\}^d}^{K} a_{\mathbf{p}} X(\mathbf{k}+\mathbf{p}), \triangle_{\mathbf{0},1}X = \sum_{\mathbf{p}'\in\{0,\ldots,K\}^d} a_{\mathbf{p}'} X(\mathbf{p}').$$

From (1), one can derive $\sum_{\mathbf{p}\in\{0,\ldots,K\}^d} a_{\mathbf{p}} = 0$. Then for $\mathbf{k} \in \mathbb{Z}^d$ fixed, $||\mathbf{k}|| > 0$, we have

$$cov(\triangle_{\mathbf{k},1}X, \triangle_{\mathbf{0},1}X) = -\sum_{\mathbf{p},\mathbf{p}'\in\{0,\ldots,K\}^d} \frac{\sigma^2}{2} a_{\mathbf{p}} a_{\mathbf{p}'} ||\mathbf{k}||^{2H} \left\|\frac{\mathbf{k}}{||\mathbf{k}||} + \frac{\mathbf{p}-\mathbf{p}'}{||\mathbf{k}||}\right\|^{2H}. \tag{29}$$

We consider the function $f(\mathbf{x}) = ||\mathbf{x}||^{2H}$ where $\mathbf{x} \in \mathbb{R}^d, ||\mathbf{x}|| \neq 0$. One can choose $k_0 \in \mathbb{N}, k_0 > 2$ such that for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| \geq k_0$, then $\frac{\mathbf{k}+\mathbf{p}-\mathbf{p}'}{||\mathbf{k}||} \in \{\mathbf{x} \in \mathbb{R}^d, 1/2 \leq ||\mathbf{x}|| \leq 3/2\}$ for all $\mathbf{p}, \mathbf{p}' \in \{0, \ldots, K\}^d$. We can see that $f(\mathbf{x})$ is infinite differentiable for all $\mathbf{x} \in \mathbb{R}^d, ||\mathbf{x}|| \neq 0$.
For each $\mathbf{p} = (p_1, \ldots, p_d), \mathbf{p}' = (p_1', \ldots, p_d'), \mathbf{p}, \mathbf{p}' \in \{0, \ldots, K\}^d$ and $||\mathbf{k}|| \geq k_0$, applying Taylor expansion to the function $f(\mathbf{x})$ to order d at $x_0 = \frac{\mathbf{k}}{||\mathbf{k}||}$, one gets

$$\begin{aligned} f\left(\frac{\mathbf{k}+\mathbf{p}-\mathbf{p}'}{||\mathbf{k}||}\right) &= 1 + \sum_{r=1}^{d} \sum_{r_1+\ldots r_d = r} \frac{\partial^r f(\frac{\mathbf{k}}{||\mathbf{k}||})}{r_1!\ldots r_d!} \frac{(p_1-p_1')^{r_1}\ldots(p_d-p_d')^{r_d}}{||\mathbf{k}||^r} \\ &+ \sum_{r_1+\ldots+r_d=d+1} \frac{\partial^{d+1} f(\xi_{\mathbf{p}-\mathbf{p}'})}{r_1!\ldots r_d!} \frac{(p_1-p_1')^{r_1}\ldots(p_d-p_d')^{r_d}}{||\mathbf{k}||^{d+1}} \end{aligned}$$

where $\xi_{\mathbf{p}-\mathbf{p}'}$ is on a line segment connecting $\frac{\mathbf{k}}{||\mathbf{k}||}$ and $\frac{\mathbf{k}+\mathbf{p}-\mathbf{p}'}{||\mathbf{k}||}$. Obviously, $\xi_{\mathbf{p}-\mathbf{p}'} \in \{\mathbf{x} \in \mathbb{R}^d, 1/2 \leq ||\mathbf{x}|| \leq 3/2\}$- a compact set in $\mathbb{R}^d$.

From (1), it follows that $\sum\limits_{\mathbf{p}=(p_1,\ldots,p_d)\in\{0,\ldots,K\}^d} a_{\mathbf{p}} p_1^{r_1}\ldots p_d^{r_d} = 0$ for $r_1,\ldots,r_d \in \mathbb{N}$ fixed and $r_1 + \ldots + r_d \leq d$. Then one can derive

$$\sum_{r=1}^{d} \sum_{r_1+\ldots+r_d=r} \sum_{\substack{\mathbf{p},\mathbf{p}'\in\{0,\ldots,K\}^d \\ \mathbf{p}=(p_1,\ldots,p_d),\mathbf{p}'=(p_1',\ldots,p_d')}} a_{\mathbf{p}} a_{\mathbf{p}'} \frac{\partial^r f(\frac{\mathbf{k}}{||\mathbf{k}||})}{r_1!\ldots r_d!} \frac{(p_1-p_1')^{r_1}\ldots(p_d-p_d')^{r_d}}{||\mathbf{k}||^r} = 0$$

Combining with (29), it follows that there is a constant C and $k_0^* \geq k_0 > 2$ such that for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| \geq k_0^*$, one has

$$|cov(\triangle_{\mathbf{k},1}X, \triangle_{\mathbf{0},1}X)| \leq C||\mathbf{k}||^{2H-d-1} \leq 1. \tag{30}$$

Moreover, since $\sum\limits_{\mathbf{p}\in\{0,\ldots,K\}^d} a_{\mathbf{p}} = 0$, one has $var\triangle_{\mathbf{k},1}X = var\triangle_{\mathbf{0},1}X$. On the other hand, since X is a Gaussian field, together with (30), then for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| \geq k_0^* > 2$, apply Lemma A.1 in Dang and Istas (2017), one has $|cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X)|^\beta| \leq C||\mathbf{k}||^{4H-2d-2}$, where C is a running constant which may change from an occurrence to another. Thus

$$\begin{aligned}
&\frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d,|k_i|\leq n} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\
&\leq \frac{C}{n^d}\left(1 + \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d,|k_i|\leq n,||\mathbf{k}||\leq k_0^*} ||\mathbf{k}||^{4H-2d-2}\right).
\end{aligned}$$

Since $d \geq 2, 0 < H < 1$, it follows $4H - 2d - 2 < -d$. Then applying Lemma 3, there exists a constant $\Sigma > 0$ such that

$$\frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d,|k_i|\leq n} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \leq \frac{\Sigma}{n^d}.$$

The condition (12) is followed with $b_n = n^{-d/2}$ and one gets the conclusion. □

Proof of Theorem 3. The following lemma is used to prove Theorem 3.

Lemma 1 *There exist $K_0 > 0$, $C > 0$ such that for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| > K_0$, we have*

$$I_{\mathbf{k}} = \int_{\mathbb{R}^d} |f(\mathbf{s})f(\mathbf{s}-\mathbf{k})|^{\alpha/2} d\mathbf{s} \leq C||\mathbf{k}||^{\frac{\alpha H-(L+1)\alpha d}{2}}, \tag{31}$$

where $f(\mathbf{s})$ is defined by

$$f(\mathbf{s}) = \sum_{\mathbf{p}\in\{0,\ldots,K\}^d} a_{\mathbf{p}} ||\mathbf{p}-\mathbf{s}||^{H-\frac{d}{\alpha}}, \mathbf{s} \in \mathbb{R}^d, \mathbf{s} \neq \mathbf{0}. \tag{32}$$

Proof: For $\mathrm{s} \in \mathbb{R}^d, ||\mathrm{s}|| > 0$, one has

$$f(\mathbf{s}) = ||\mathbf{s}||^{H-\frac{d}{\alpha}} \sum_{\mathbf{p}\in\{0,\ldots,K\}^d} a_{\mathbf{p}} ||\frac{\mathbf{s}-\mathbf{p}}{||\mathbf{s}||}||^{H-\frac{d}{\alpha}}. \tag{33}$$

We set

$$g(\mathbf{x}) = ||\mathbf{x}||^{H-\frac{d}{\alpha}}, \tag{34}$$

where $\mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0}$. One can choose $k_0 \in \mathbb{N}$ such that for all $\mathbf{s} \in \mathbb{R}^d, ||\mathbf{s}|| \geq k_0, \mathbf{p} \in \{0, \ldots, K\}^d$, $\frac{\mathbf{s}-\mathbf{p}}{||\mathbf{s}||} \in \{\mathbf{x} \in \mathbb{R}^d, 1/2 \leq ||\mathbf{x}|| \leq 3/2\}$. It is clear that $g(\mathbf{x})$ is infinite differentiable for all $\mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0}$. For each $\mathbf{p} = (p_1, \ldots, p_d) \in \{0, \ldots, K\}^d$, applying Taylor expansion to the function $g(\mathbf{x})$ to order $(L+1)d-1$ at $x_0 = \frac{\mathbf{s}}{||\mathbf{s}||}$, one gets

$$g(\frac{\mathbf{s}-\mathbf{p}}{||\mathbf{s}||}) = 1 + \sum_{r=1}^{(L+1)d-1} \sum_{r_1+\ldots r_d=r} \frac{\partial^r f(\frac{\mathbf{s}}{||\mathbf{s}||})}{r_1! \ldots r_d!} \frac{(-p_1)^{r_1} \ldots (-p_d)^{r_d}}{||\mathbf{s}||^r}$$
$$+ \sum_{r_1+\ldots+r_d=(L+1)d} \frac{\partial^{d+1} f(\xi_{\mathbf{p}})}{r_1! \ldots r_d!} \frac{(-p_1)^{r_1} \ldots (-p_d)^{r_d}}{||\mathbf{s}||^{(L+1)d}}$$

where $\xi_{\mathbf{p}}$ is on a line segment connecting $\frac{\mathbf{s}}{||\mathbf{s}||}$ and $\frac{\mathbf{s}-\mathbf{p}}{||\mathbf{s}||}$. Then $\xi_{\mathbf{p}} \in \{\mathbf{x} \in \mathbb{R}^d, 1/2 \leq ||\mathbf{x}|| \leq 3/2\}$- a compact set in $\mathbb{R}^d$. From (1), it follows that

$$\sum_{\mathbf{p}=(p_1,\ldots,p_d)\in\{0,\ldots,K\}^d} a_{\mathbf{p}} p_1^{r_1} \ldots p_d^{r_d} = 0$$

for $r_1, \ldots, r_d \in \mathbb{N}$ fixed and $r_1 + \ldots + r_d \leq (L+1)d-1$. Then one obtains

$$\sum_{r=1}^{(L+1)d-1} \sum_{r_1+\ldots+r_d=r} \sum_{\mathbf{p}=(p_1,\ldots,p_d)\in\{0,\ldots,K\}^d} a_{\mathbf{p}} \frac{\partial^r f(\frac{\mathbf{s}}{||\mathbf{s}||})}{r_1! \ldots r_d!} \frac{(-p_1)^{r_1} \ldots (-p_d)^{r_d}}{||\mathbf{s}||^r} = 0$$

Combining with (33), it follows that there is a constant C such that for all $\mathbf{s} \in \mathbb{R}^d, ||\mathbf{s}|| \geq k_0$, one has

$$|f(\mathbf{s})| \leq C||\mathbf{s}||^{H-\frac{d}{\alpha}-(L+1)d}. \tag{35}$$

Let $K_0 = 2k_0$, for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| \geq K_0$, by changing variable then

$$I_{\mathbf{k}} = \int\limits_{||\mathbf{s}||\geq\frac{||\mathbf{k}||}{2}} |f(\mathbf{s})f(\mathbf{s}+\mathbf{k})|^{\alpha/2} d\mathbf{s} + \int\limits_{||\mathbf{s}||\leq\frac{||\mathbf{k}||}{2}} |f(\mathbf{s})f(\mathbf{s}+\mathbf{k})|^{\alpha/2} d\mathbf{s} := I_{1\mathbf{k}} + I_{2\mathbf{k}},$$

where $I_{1\mathbf{k}} = \int\limits_{||\mathbf{s}||\geq\frac{||\mathbf{k}||}{2}} |f(\mathbf{s})f(\mathbf{s}+\mathbf{k})|^{\alpha/2} d\mathbf{s}, I_{2\mathbf{k}} = \int\limits_{||\mathbf{s}||\leq\frac{||\mathbf{k}||}{2}} |f(\mathbf{s})f(\mathbf{s}+\mathbf{k})|^{\alpha/2} d\mathbf{s}$. Applying Cauchy-Schwartz's inequality, since $||\mathbf{k}||/2 \geq k_0$, one can use (35) and then make the transformation from Cartesian coordinates to spherical coordinates to get

$$I_{1\mathbf{k}} \leq \left(\int\limits_{||\mathbf{s}||\geq\frac{||\mathbf{k}||}{2}} |f(\mathbf{s})|^{\alpha} d\mathbf{s} \int\limits_{||\mathbf{s}||\geq\frac{||\mathbf{k}||}{2}} |f(\mathbf{s}+\mathbf{k})|^{\alpha} d\mathbf{s} \right)^{1/2}$$
$$\leq C \left(\int\limits_{||\mathbf{s}||\geq\frac{||\mathbf{k}||}{2}} ||\mathbf{s}||^{\alpha H - d - (L+1)\alpha d} d\mathbf{s} \right)^{1/2} \leq C \left(\int\limits_{\rho\geq\frac{||\mathbf{k}||}{2}} \rho^{\alpha H - d - (L+1)\alpha d} d\rho \right)^{1/2} \leq C||\mathbf{k}||^{\frac{\alpha H - (L+1)\alpha d}{2}} \tag{36}$$

where C is a running constant which may change from an occurrence to another. The latter inequality comes from the fact that $\alpha H - (L+1)\alpha d < 0$. Similarly, one gets

$$I_{2\mathbf{k}} \leq C||\mathbf{k}||^{\frac{\alpha H - (L+1)\alpha d}{2}}.$$

Together with (36), (36) and since $\frac{\alpha H-(L+1)\alpha d}{2} < 0$, one can choose k_0 such that for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| > K_0$, one has $I_{\mathbf{k}} \leq C||\mathbf{k}||^{\frac{\alpha H-(L+1)\alpha d}{2}}$.

We are back to the proof of Theorem 3 by proving that condition (12) is satisfied. Since $\sum\limits_{\mathbf{p}\in\{0,\ldots,K\}^d} a_{\mathbf{p}} = 0$, one has

$$\triangle_{\mathbf{k},1}X = \int\limits_{\mathbb{R}^d} \left(\sum_{\mathbf{p}\in\{0,\ldots,K\}^d} a_{\mathbf{p}} ||\mathbf{p}-(\mathbf{s}-\mathbf{k})||^{H-\frac{d}{\alpha}} \right) M_\alpha(d\mathbf{s}) = \int\limits_{\mathbb{R}^d} f(\mathbf{s}-\mathbf{k}) M_\alpha(d\mathbf{s})$$

where $f(\mathbf{s})$ is defined by (32). It follows that

$$||\triangle_{\mathbf{k},1}X||_\alpha = \left(\int\limits_{\mathbb{R}^d} |f(\mathbf{s}-\mathbf{k})|^\alpha d\mathbf{s} \right)^{1/\alpha} = \left(\int\limits_{\mathbb{R}^d} |f(\mathbf{s})|^\alpha d\mathbf{s} \right)^{1/\alpha} = ||\triangle_{\mathbf{0},1}X||_\alpha. \tag{37}$$

From Lemma 1, there exist $K_0 > 0, 0 < \eta < 1, C > 0$ such that for $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| > K_0$, we have

$$\left[\frac{\triangle_{\mathbf{k},1}X}{||\triangle_{\mathbf{k},1}X||_\alpha}, \frac{\triangle_{\mathbf{0},1}X}{||\triangle_{\mathbf{0},1}X||_\alpha} \right]_2 = \frac{1}{||\triangle_{\mathbf{0},1}X||_\alpha^\alpha} \int\limits_{\mathbb{R}^d} |f(\mathbf{s})f(\mathbf{s}-\mathbf{k})|^{\alpha/2} d\mathbf{s} \leq C||\mathbf{k}||^{\frac{\alpha H-(L+1)\alpha d}{2}} \leq \eta < 1$$

where $[.,.]_2$ is defined by $\left[\int\limits_{\mathbb{R}^d} f(\mathbf{s})M_\alpha(d\mathbf{s}), \int\limits_{\mathbb{R}^d} g(\mathbf{s})M_\alpha(d\mathbf{s}) \right]_2 = \int\limits_{\mathbb{R}^d} |f(\mathbf{s})g(\mathbf{s})|^{\alpha/2} d\mathbf{s}$.
From (37) and applying Theorem 4.2 in Dang and Istas (2017), for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| > K_0$, there exists a constant C such that

$$\left| cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta) \right| = ||\triangle_{\mathbf{0},1}X||_\alpha^{2\beta} \left| cov\left(\frac{|\triangle_{\mathbf{k},1}X|^\beta}{||\triangle_{\mathbf{k},1}X||_\alpha^\beta}, \frac{|\triangle_{\mathbf{0},1}X|^\beta}{||\triangle_{\mathbf{0},1}X||_\alpha^\beta}\right) \right| \leq C||\mathbf{k}||^{\frac{\alpha H-(L+1)\alpha d}{2}}.$$

Then one has

$$\begin{aligned}
&\frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, |k_i|\leq n} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\
&= \frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||\leq K_0} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\
&+ \frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||> K_0} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\
&\leq \frac{C}{n^d} \left(1 + \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||> K_0} ||\mathbf{k}||^{\frac{\alpha H-(L+1)\alpha d}{2}} \right) \leq C b_n^2,
\end{aligned}$$

where b_n is defined by (18). The latter inequality comes from Lemma 3. Thus the condition (12) is satisfied. Then one gets the conclusion from Theorem 1. □

Proof of Theorem 4. The following lemma is used to prove Theorem 4.

Lemma 2 *Let X be a Takenaka random field defined by (21). For $\mathbf{k} \in \mathbb{R}^d$, let*

$$I_{\mathbf{k}} = [\triangle_{\mathbf{k},1}X, \triangle_{\mathbf{0},1}X]_2 = \int\limits_0^{+\infty} \int\limits_{\mathbb{R}^d} r^{\nu-d-1} |f_{\mathbf{k}}(\mathbf{x},r) f_{\mathbf{0}}(\mathbf{x},r)|^{\alpha/2} d\mathbf{x} dr \tag{38}$$

where $f_{\mathbf{k}}(\mathbf{x}, r)$ is defined by

$$f_{\mathbf{k}}(\mathbf{x}, r) = \sum_{\mathbf{p} \in \{0,\dots,K\}^d} a_{\mathbf{p}} \mathbf{1}_{V_{\mathbf{k}+\mathbf{p}}}(\mathbf{x}, r). \tag{39}$$

Then there exist $K_1 > 0$ and a constant $C > 0$ such that for all $\mathbf{k} \in \mathbb{R}^d, ||\mathbf{k}|| > K_1$, we have $I_{\mathbf{k}} \leq C||\mathbf{k}||^{\nu-1}$.

Proof: Since $\mathbf{1}_{A\triangle B} = (\mathbf{1}_A - \mathbf{1}_B)^2$, then $f_{\mathbf{k}}(\mathbf{x}, r) = (1 - 2 \times \mathbf{1}_{C_{\mathbf{0}}}(\mathbf{x}, r)) \sum_{\mathbf{p} \in \{0,\dots,K\}^d} a_{\mathbf{p}} \mathbf{1}_{C_{\mathbf{k}+\mathbf{p}}}(\mathbf{x}, r)$. It induces that $|f_{\mathbf{k}}(\mathbf{x}, r)| = |\sum_{\mathbf{p} \in \{0,\dots,K\}^d} a_{\mathbf{p}} \mathbf{1}_{C_{\mathbf{k}+\mathbf{p}}}(\mathbf{x}, r)|$. Let $K_0 = K\sqrt{d}$, now we will consider $I_{\mathbf{k}}$ for $\mathbf{k} \in \mathbb{R}^d, ||\mathbf{k}|| > 4K_0$.
If $||\mathbf{x}|| > K_0 + r$, then for all $\mathbf{p} \in \{0, \dots, K\}^d$, $||\mathbf{x} - \mathbf{p}|| \geq ||\mathbf{x}|| - ||\mathbf{p}|| > K_0 + r - ||\mathbf{p}|| \geq r$, it follows $\mathbf{1}_{C_{\mathbf{p}}}(\mathbf{x}, r) = 0$. Thus $f_{\mathbf{0}}(\mathbf{x}, r) = 0$ for $||\mathbf{x}|| > K_0 + r$. One can derive

$$I_{\mathbf{k}} = \int_0^{+\infty} r^{\nu-d-1} \int_{||\mathbf{x}|| \leq K_0 + r} |f_{\mathbf{0}}(\mathbf{x}, r) f_{\mathbf{k}}(\mathbf{x}, r)|^{\alpha/2} d\mathbf{x} dr := I_{1\mathbf{k}} + I_{2\mathbf{k}}, \tag{40}$$

where

$$I_{1\mathbf{k}} = \int_0^{\frac{||\mathbf{k}||}{2} - K_0} r^{\nu-d-1} \int_{||\mathbf{x}|| \leq K_0 + r} |f_{\mathbf{0}}(\mathbf{x}, r) f_{\mathbf{k}}(\mathbf{x}, r)|^{\alpha/2} d\mathbf{x} dr,$$

$$I_{2\mathbf{k}} = \int_{\frac{||\mathbf{k}||}{2} - K_0}^{+\infty} r^{\nu-d-1} \int_{||\mathbf{x}|| \leq K_0 + r} |f_{\mathbf{0}}(\mathbf{x}, r) f_{\mathbf{k}}(\mathbf{x}, r)|^{\alpha/2} d\mathbf{x} dr.$$

We consider $I_{1\mathbf{k}}$. For $0 \leq r \leq \frac{||\mathbf{k}||}{2} - K_0$, then $0 < K_0 + r \leq ||\mathbf{k}|| - K_0 - r$. If $||\mathbf{x}|| < ||\mathbf{k}|| - K_0 - r$, one has

$$||\mathbf{x} - \mathbf{p} - \mathbf{k}|| \geq ||\mathbf{k}|| - ||\mathbf{x}|| - ||\mathbf{p}|| > ||\mathbf{k}|| - ||\mathbf{p}|| - (||\mathbf{k}|| - K_0 - r) = r + K_0 - ||\mathbf{p}|| \geq r$$

for all $\mathbf{p} \in \{0, \dots, K\}^d$, it follows that $\mathbf{1}_{C_{\mathbf{k}+\mathbf{p}}} = 0$.
Thus $f_{\mathbf{k}}(\mathbf{x}, r) = 0$ for $||\mathbf{x}|| < ||\mathbf{k}|| - K_0 - r$. Then one gets $I_{1\mathbf{k}} = 0$ and $I_{\mathbf{k}} = I_{2\mathbf{k}}$.
For $r \geq \frac{||\mathbf{k}||}{2} - K_0$, one has $r - K_0 \geq \frac{||\mathbf{k}||}{2} - 2K_0 > 0$. If $||\mathbf{x}|| < r - K_0$ then $||\mathbf{x} - \mathbf{p}|| \leq ||\mathbf{x}|| + ||\mathbf{p}|| < r - K_0 + ||\mathbf{p}|| \leq r$ for all $\mathbf{p} \in \{0, \dots, K\}^d$, it induces that $\mathbf{1}_{C_{\mathbf{p}}}(\mathbf{x}, r) = 1$. Thus one can derive that $f_{\mathbf{0}}(\mathbf{x}, r) = 0$ for $||\mathbf{x}|| < r - K_0$. It induces

$$I_{\mathbf{k}} = I_{2\mathbf{k}} = \int_{\frac{||\mathbf{k}||}{2} - K_0}^{+\infty} r^{\nu-d-1} \int_{r - K_0 \leq ||\mathbf{x}|| \leq K_0 + r} |f_{\mathbf{0}}(\mathbf{x}, r) f_{\mathbf{k}}(\mathbf{x}, r)|^{\alpha/2} d\mathbf{x} dr.$$

Moreover $|f_{\mathbf{0}}(\mathbf{x}, r) f_{\mathbf{k}}(\mathbf{x}, r)|^{\alpha/2} \leq \left(\sum_{\mathbf{p} \in \{0,\dots,K\}^d} |a_{\mathbf{p}}| \right)^{\alpha} < +\infty$. It follows that

$$I_{\mathbf{k}} \leq C \int_{\frac{||\mathbf{k}||}{2} - K_0}^{+\infty} r^{\nu-d-1} \int_{r - K_0 \leq ||\mathbf{x}|| \leq K_0 + r} d\mathbf{x} dr.$$

We make the transformation from Cartesian coordinates to spherical coordinates to get

$$I_{\mathbf{k}} \leq C \int_{\frac{||\mathbf{k}||}{2}-K_0}^{+\infty} r^{\nu-d-1} \int_{r-K_0\leq\rho\leq K_0+r} \rho^{d-1} d\rho dr \leq 2K_0 C \int_{\frac{||\mathbf{k}||}{2}-K_0}^{+\infty} r^{\nu-d-1}(2r)^{d-1} dr. \quad (41)$$

The latter inequality comes from the fact that for $r \geq \frac{||\mathbf{k}||}{2} - K_0$, then $r - K_0 \leq \rho \leq r + K_0 \leq 2r$. From (41), one gets $I_{\mathbf{k}} \leq C\left(\frac{||\mathbf{k}||}{2} - K_0\right)^{\nu-1} \leq C\left(\frac{||\mathbf{k}||}{4}\right)^{\nu-1}$ since $\nu - 1 < 0$. Thus there exists a constant $C > 0$ such that for all $\mathbf{k} \in \mathbb{R}^d, ||\mathbf{k}|| > 4K_0$, where $K_0 = K\sqrt{d}$, one has $I_{\mathbf{k}} \leq C||\mathbf{k}||^{\nu-1}$.

We come back to the proof of Theorem 4. One has

$$\triangle_{\mathbf{k},1}X = \int_{\mathbb{R}^d\times\mathbb{R}^+} \left(\sum_{\mathbf{p}\in\{0,\dots,K\}^d} a_{\mathbf{p}} \mathbf{1}_{V_{\mathbf{k+p}}}(\mathbf{x}, r)\right) M(d\mathbf{x}, dr) = \int_{\mathbb{R}^d\times\mathbb{R}^+} f_{\mathbf{k}}(\mathbf{x}, r) M(d\mathbf{x}, dr),$$

where $f_{\mathbf{k}}(\mathbf{x}, r)$ is defined by (39). We write

$$\begin{aligned} ||\triangle_{\mathbf{k},1}X||_\alpha^\alpha &= \int_0^{+\infty}\int_{\mathbb{R}^d} r^{\nu-d-1} \Big|\sum_{\mathbf{p}\in\{0,\dots,K\}^d} a_{\mathbf{p}} \mathbf{1}_{C_{\mathbf{k+p}}}(\mathbf{x}, r)\Big|^\alpha d\mathbf{x}dr \\ &= \int_0^{+\infty}\int_{\mathbb{R}^d} r^{\nu-d-1} \Big|\sum_{\mathbf{p}\in\{0,\dots,K\}^d} a_{\mathbf{p}} \mathbf{1}_{C_{\mathbf{p}}}(\mathbf{x}-\mathbf{k}, r)\Big|^\alpha d(\mathbf{x}-\mathbf{k})dr = ||\triangle_{\mathbf{0},1}X||_\alpha^\alpha. \quad (42) \end{aligned}$$

From Lemma 2, there exist $K_1 > 0, 0 < \eta < 1$ such that for $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| > K_1$, we have

$$\begin{aligned} [\frac{\triangle_{\mathbf{k},1}X}{||\triangle_{\mathbf{k},1}X||_\alpha}, \frac{\triangle_{\mathbf{0},1}X}{||\triangle_{\mathbf{0},1}X||_\alpha}]_2 &= \frac{1}{||\triangle_{\mathbf{0},1}X||_\alpha^\alpha} \int_0^{+\infty}\int_{\mathbb{R}^d} r^{\nu-d-1} |f_{\mathbf{k}}(\mathbf{x}, r) f_{\mathbf{0}}(\mathbf{x}, r)|^{\alpha/2} d\mathbf{x}dr \\ &\leq C||\mathbf{k}||^{\nu-1} \leq \eta < 1. \end{aligned}$$

From (42) and applying Theorem 4.2 in Dang and Istas (2017), there exists a constant $C > 0$ such that for all $\mathbf{k} \in \mathbb{Z}^d, ||\mathbf{k}|| > K_1$, one can derive

$$|cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| = ||\triangle_{\mathbf{0},1}X||_\alpha^{2\beta} \left|cov\left(\frac{|\triangle_{\mathbf{k},1}X|^\beta}{||\triangle_{\mathbf{k},1}X||_\alpha^\beta}, \frac{|\triangle_{\mathbf{0},1}X|^\beta}{||\triangle_{\mathbf{0},1}X||_\alpha^\beta}\right)\right| \leq C||\mathbf{k}||^{\nu-1}.$$

Then one has

$$\begin{aligned} &\frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d, |k_i|\leq n} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\ &= \frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||\leq K_1} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\ &+ \frac{1}{n^d} \sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||> K_1} |cov(|\triangle_{\mathbf{k},1}X|^\beta, |\triangle_{\mathbf{0},1}X|^\beta)| \\ &\leq \frac{C}{n^d}\left(1 + \sum_{\mathbf{k}=(k_1,\dots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||> K_0} ||\mathbf{k}||^{\nu-1}\right) \leq Cn^{\nu-1}. \end{aligned}$$

where b_n is defined by (23). The latter inequality comes from Lemma 3. Then (12) is satisfied. Together with Theorem 1, one gets the conclusion. □

5. Conclusions

In this paper, we have extended the results in Dang and Istas (2017) to estimate the Hurst index H and the stability index α for H-sssi, $S\alpha S$-stable random fields, whose parameter space is in high dimensions $\mathbb{R}^d$. From the statistical literature, there are several methods used to estimate H and α, however, in many situations, one needs an a priori knowledge on α to estimate H and vice versa. In this framework, there always exists the moments of $\beta-$ negative-power variations $(-1/2 < \beta < 0)$ of underlying fields without any assumptions. The estimations are based on this variations. One of the key point of proofs is to find the inequalities for their covariances. In this context, the difficulty is the appearance of multi-indices. This problem has been solved by using the transformation from Cartesian coordinates to spherical coordinates. In this work, some examples has also been presented to illustrate obtained results.

Appendix

In this part, we will define a spherical coordinate system in $\mathbb{R}^d, d \geq 2$ where the coordinates consist of a radial coordinate $\rho, \rho \geq 0$ and $d-1$ angular coordinates $\phi_1, \ldots, \phi_{d-1}$ where $\phi_{n-1} \in [0, 2\pi]$ and $\phi_i \in [0, \pi]$ for $i = 1, \ldots, d-2$ (in case $d = 2$, then we have just one angular coordinate $\phi \in [0, 2\pi]$). We will find the Jacobian of the transformation from Cartesian coordinates to spherical coordinates.
We define the spherical coordinates by

$$\begin{aligned}
x_1 &= \rho\cos(\phi_1)\\
x_2 &= \rho\sin(\phi_1)\cos(\phi_2)\\
x_3 &= \rho\sin(\phi_1)\sin(\phi_2)\cos(\phi_3))\\
&\ldots\\
x_{n-1} &= \rho\sin(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-3})\sin(\phi_{n-2})\cos(\phi_{n-1})\\
x_n &= \rho\sin(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-3})\sin(\phi_{n-2})\sin(\phi_{n-1})
\end{aligned} \tag{43}$$

Then the Jacobian of the transformation from Cartesian coordinates to spherical coordinates is

$$J = \frac{\partial(x_1x_2\ldots x_d)}{\partial(\rho\phi_1\phi_2\ldots\phi_{n-1})} = \rho^{d-1}\times$$
$$\begin{vmatrix}
\cos(\phi_1) & -\sin(\phi_1) & \ldots & 0\\
\sin(\phi_1)\cos(\phi_2) & \cos(\phi_1)\cos(\phi_2) & \ldots & 0\\
\ldots & \ldots & \ldots & \ldots\\
\sin(\phi_1)\ldots\cos(\phi_{n-1}) & \cos(\phi_1)\sin(\phi_2)\ldots\cos(\phi_{n-1}) & \ldots & -\sin(\phi_1)\ldots\sin(\phi_{n-1})\\
\sin(\phi_1)\ldots\sin(\phi_{n-1}) & \cos(\phi_1)\sin(\phi_2)\ldots\sin(\phi_{n-1}) & \ldots & \sin(\phi_1)\ldots\cos(\phi_{n-1})
\end{vmatrix}. \tag{44}$$

$$J = \rho^{n-1}\prod_{j=1}^{d-2}\sin^{d-1-j}(\phi_j).$$

Lemma 3 *For $p < 0, K_0, d \in \mathbb{N}, K_0, d \geq 2$, one has*

$$S_p = \begin{cases} O(n^{-d}) & \text{if } p < -d,\\ O(n^p) & \text{if } -d < p < 0,\\ O(\frac{\ln n}{n^d}) & \text{if } p = -d. \end{cases} \tag{45}$$

where

$$S_p = \frac{1}{n^d}\sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, |k_i|\leq n, ||\mathbf{k}||\geq K_0 > 2} ||\mathbf{k}||^p. \tag{46}$$

Proof: Since $p < 0$, we have

$$S_n = \frac{2^d}{n^d} \sum_{\mathbf{k}=(k_1,\ldots,k_d)\in\mathbb{Z}^d, 0\le k_i\le n, ||\mathbf{k}||\ge K_0>2} ||\mathbf{k}||^p$$
$$\le \frac{2^d}{n^d} \int_{K_0-1\le||\mathbf{x}||\le n\sqrt{d}} |\mathbf{x}||^p dx \le \frac{C}{n^d} \int_{K_0-1\le\rho\le n\sqrt{d}} \rho^{d+p-1} dx. \quad (47)$$

The latter inequality comes from the transformation from Cartesian coordinates to spherical coordinates.
If $p < -d$, then $d + p < 0$. From (47), one gets $S_n = O(n^{-d})$.
If $-d < p < 0$, then $d + p > 0$. Thus one has $S_n = O(n^p)$.
If $p = -d$, one gets $S_n = O(\frac{\ln n}{n^d})$. Then we get the conclusion.

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