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On Flexible Weibull-Logistic Distribution with Properties and Applications

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Abstract

In this paper we propose a new three-parameter flexible Weibull-logistic (FW-L) distribution to increase the level of flexibility of logistic distribution without increasing the number of parameters. We obtain some of fundamental mathematical properties including asymptotes, moments, quantile function, entropy, and order statistics. The goodness-of-fit test for the proposed distribution is also studied. Then, we derived estimation of model parameters for both complete and right censored data sets. Further, a simulation study is conducted to observe the asymptotic behavior of maximum likehood estimations. The flexibility and importance of the proposed models are illustrated by means of the real data set. The results of the study shows that the main advantage of the new distribution is that it has increasing, decreasing or bathtub curve failure rate depending upon the shape parameter. This property makes FW-L is very useful in data analysis.

Keywords: Order statistics, maximum likelihood estimation, quantile function, generating function, moments.

1. Introduction

Real world phenomena can be described and predicted by statistical distributions. Many classical distributions have been used in recent years and proposed to model the data in several areas such as reliability, survival analysis, demography, actuarial science and others. Furthermore, introduction of new families that extend well-known distributions provides great ï¬,exibility in modeling real data. Among the recent families of distributions having a high impact in statistical modelling, there are generalized odd generalized exponential family by Alizadeh et al. (2017), exponentiated generalized-G Poisson by Aryal and Yousof (2017), Topp-Leone odd log-logistic by Brito et al. (2017), odd log-logistic Topp Leone G family by Alizadeh et al. (2018a), exponentiated Weibull-H family by Cordeiro et al. (2017), generalized two-sided family by Korkmaz and Genc (2017), exponentiated transmuted-G family by Merovci et al. (2017), transmuted exponentiated generalized-G family by Yousuf et al. (2017), complementary generalized transmuted Poisson-G family by Alizadeh et al.

(2018b), odd log-logistic Poisson-G family by Alizadeh et al. (2018c), Weibull generalized-G family by Yousuf et al. (2018), exponential Lindley odd log-logistic-G family by Korkmaz et al. (2018), type II general exponential class by Hamedani et al. (2019), and flexible Weibull-G (FW-G) family by Alizadeh et al. (2020).

The logistic distribution is very useful for modeling lifetime data in medicine, biology, finance and engineering. It is used for growth models and in logistic regression with the longer tails and a higher kurtosis than the normal distribution. The logistic distribution may be the good choice for researchers to model data following monotonic failure rates. However, it is inappropriate for modeling data following non-monotonic failure rates. In this study, we introduce a generalization of the logistic distribution motivated from the flexible Weibull generated (FW-G) family of Alizadeh et al. (2020) and called as flexible Weibull logistic (FW-L) distribution. Before going further, let us briefly describe the FW-G family of Alizadeh et al. (2020). The cumulative distribution function (cdf) of the FW-G family is given by

$$F_{\alpha,\beta,\psi}(x) = 1 - \exp\left\{-\exp\left[\frac{\alpha\, G(x;\psi)}{\bar{G}(x;\psi)} - \frac{\beta\, \bar{G}(x;\psi)}{G(x;\psi)}\right]\right\}, \tag{1}$$

where $\alpha, \beta > 0$ are two shape parameters. Here, $\bar{G}(x;\psi) = 1 - G(x;\psi)$ and ψ is the vector of parameters for the baseline cdf $G(.;\psi)$.

Corresponding probability density function (pdf) and hazard rate function (hrf) are, respectively, given by

$$f_{\alpha,\beta,\psi}(x) = g(x;\psi)\left[\frac{\alpha}{\bar{G}(x;\psi)^2} + \frac{\beta}{G(x;\psi)^2}\right]\exp\left\{\begin{array}{c}\frac{\alpha\, G(x;\psi)}{\bar{G}(x;\psi)} - \frac{\beta\, \bar{G}(x;\psi)}{G(x;\psi)} \\ -\exp\left[\frac{\alpha\, G(x;\psi)}{\bar{G}(x;\psi)} - \frac{\beta\, \bar{G}(x;\psi)}{G(x;\psi)}\right]\end{array}\right\}, \tag{2}$$

$$\text{and } h_{\alpha,\beta,\psi}(x) = g(x;\psi)\left[\frac{\alpha}{\bar{G}(x;\psi)^2} + \frac{\beta}{G(x;\psi)^2}\right]\exp\left\{\frac{\alpha\, G(x;\psi)}{\bar{G}(x;\psi)} - \frac{\beta\, \bar{G}(x;\psi)}{G(x;\psi)}\right\}. \tag{3}$$

The aim of this study is to gain flexibility the logistic distribution enough for modelling different types of lifetime data important in reliability, engineering, marketing and in other areas via FW-G family. The paper is organized as follows: In Section 2, we present the main functions and properties of the FW-L distribution. In Section 3, the goodness of fit test is studied for the proposed distribution and in Section 4, the FW-L model parameters are estimated by the maximum likelihood method and a simulation study is performed in Section 5. Then, the flexibility of the FW-L model is illustrated by means of practical data set in Section 6. Finally, Section 7 offers some concluding remarks.

2. Main Properties

Consider the cdf $G(x;\lambda) = [1+\exp(-\lambda x)]^{-1}$ of the logistic distribution with shape parameter $\lambda > 0$. Inserting the cdf $G(x;\lambda) = [1+\exp(-\lambda x)]^{-1}$ in (1.1), we obtain the cdf of the FW-L distribution for $x \in \mathbf{R}$ as

$$F_{\alpha,\beta,\lambda}(x) = 1 - \exp\left\{-\exp\left[\frac{\alpha\,[1+\exp(-\lambda x)]^{-1}}{1-[1+\exp(-\lambda x)]^{-1}} - \frac{\beta\left(1-[1+\exp(-\lambda x)]^{-1}\right)}{[1+\exp(-\lambda x)]^{-1}}\right]\right\}$$

or it can be written as

$$F_{\alpha,\beta,\lambda}(x) = 1 - \exp\left\{-e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}}\right\}.$$

Then, the density function of the FW-L distribution is obtained by

$$f_{\alpha,\beta,\lambda}(x) = \left(\alpha\lambda\, e^{\lambda x} + \lambda\beta e^{-\lambda x}\right) e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}} \exp\left\{-e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}}\right\}$$

and the survival function for the FW-L distribution is given by

$$S_{\alpha,\beta,\lambda}(x) = \exp\left\{-e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}}\right\}.$$

Then, the hrf of the FW-L distribution is obtained as

$$h_{\alpha,\beta,\lambda}(x) = \left(\alpha\lambda\, e^{\lambda x} + \lambda\beta e^{-\lambda x}\right) e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}}$$

and corresponding cumulative hrf of the distribution is given by

$$H_{\alpha,\beta,\lambda}(x) = e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}}.$$

Some plots of the pdf and the hrf of the FW-L distribution for selected parameter values are displayed in Figure 1. Figure 1 reveals that the FW-L density can be bimodal and unimodal. The plots for the hrf of the FW-L model can be decreasing, increasing, constant, unimodal then bathtub or unimodal. This shows that the flexibility of the proposed distribution.

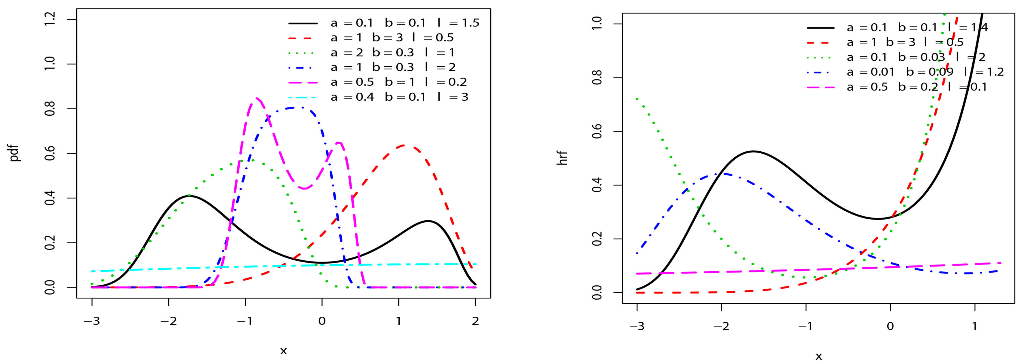

Figure 1 Plots of the pdf and the hrf of the FW-L distribution for several values parameters

2.1. Asymptotes

Corollary 1 *Let* $a = \inf\{x|F_{\alpha,\beta,\lambda}(x) > 0\}$, *the asymptotics of the cdf, pdf and hrf for* $x \to a$ *are, respectively, given by*

$$F_{\alpha,\beta,\lambda}(x) \sim \exp\left[\frac{-\beta}{[1+\exp(-\lambda x)]^{-1}}\right] \quad as\ \ x \to a,$$

$$f_{\alpha,\beta,\lambda}(x) \sim \frac{\beta\, g(x;\lambda)}{[1+\exp(-\lambda x)]^{-2}} \exp\left[\frac{-\beta}{[1+\exp(-\lambda x)]^{-1}}\right] \quad as\ \ x \to a,$$

and

$$h_{\alpha,\beta,\lambda}(x) \sim \frac{\beta\, g(x;\lambda)}{[1+\exp(-\lambda x)]^{-2}} \exp\left[\frac{-\beta}{[1+\exp(-\lambda x)]^{-1}}\right] \quad as\ \ x \to a.$$

where $g(x;\lambda)$ *is the pdf for the logistic distribution.*

Corollary 2 *The asymptotics of cdf, pdf and hrf for $x \to \infty$ are, respectively, given by*

$$\begin{aligned}
1 - F_{\alpha,\beta,\lambda}(x) &\sim \exp\left\{-\exp\left[\frac{\alpha}{1-[1+\exp(-\lambda x)]^{-1}}\right]\right\} \quad as \;\; x \to \infty, \\
f_{\alpha,\beta,\lambda}(x) &\sim \frac{\alpha\, g(x;\lambda)}{\left[1-[1+\exp(-\lambda x)]^{-1}\right]^2} \exp\left[\frac{\alpha}{1-[1+\exp(-\lambda x)]^{-1}}\right] \\
&\quad \exp\left\{-\exp\left[\frac{\alpha}{1-[1+\exp(-\lambda x)]^{-1}}\right]\right\} \quad as \;\; x \to \infty, \\
and \;\; h_{\alpha,\beta,\lambda}(x) &\sim \frac{\alpha\, g(x;\lambda)}{\left[1-[1+\exp(-\lambda x)]^{-1}\right]^2} \exp\left[\frac{\alpha}{1-[1+\exp(-\lambda x)]^{-1}}\right] \quad as \;\; x \to \infty.
\end{aligned}$$

2.2. Useful expansion

We introduce some representations for the pdf and cdf of the FW-L distribution. For the following cdf of FW-L distribution

$$F_{\alpha,\beta,\lambda}(x) = 1 - \exp\left\{-e^{\alpha\, e^{\lambda x} - \beta e^{-\lambda x}}\right\}.$$

A useful expansion can be obtained from power series as $e^{\theta} = \sum_{i=0}^{\infty} \frac{\theta^i}{i!}$

$$\begin{aligned}
F_{\alpha,\beta,\lambda}(x) &= \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i!} e^{i\alpha\, e^{\lambda x} - i\beta e^{-\lambda x}} \\
&= \sum_{i=1}^{\infty} \sum_{j=k=0}^{\infty} \frac{(-1)^{i+k+1}(i)^{j+k}\alpha^j \beta^k e^{-\lambda(k-j)x}}{i!j!k!} \\
&= \sum_{i=1}^{\infty} \sum_{j=k=0}^{\infty} \sum_{l=0}^{\infty} \frac{(-1)^{i+k+l+1}(i)^{j+k}\alpha^i \beta^k \binom{\lambda}{l}}{i!j!k!} \left(1 - e^{-(k-j)x}\right)^l \\
&= \sum_{l=0}^{\infty} S_l \left(1 - e^{-(k-j)x}\right)^l,
\end{aligned}$$

where

$$S_l = \sum_{i=1}^{\infty} \sum_{j=k=0}^{\infty} \frac{(-1)^{i+k+l+1}(i)^{j+k}\alpha^i \beta^k \binom{\lambda}{l}}{i!j!k!}.$$

Then, we can write the cdf of the FW-L by means of the expansion as $F(x) = \sum_{l=0}^{\infty} S_l H_l(x)$ where $H_l(x) = [G(x)]^l$; $G(x) = 1 - e^{-(k-j)x}$. Here, $H_l(x)$ is the exponentiated exponential (EE) distribution with power parameter l. Now, we can write the FW-L distribution as a mixture of the EE distribution densities as

$$f(x)_{\alpha,\beta,\lambda} = \sum_{l=0}^{\infty} S_{l+1} h_{l+1}(x)$$

where $h_{l+1}(x) = (l+1)g(x)G^l(x)$ is the density function of the EE with power parameter $(l+1)$ and

$$g(x) = (k-j)e^{-(k-j)x}, \qquad x \geq 0.$$

Zubair et al. (2018) discussed the basic properties of the EE distribution. Let $z \rightsquigarrow EE(\alpha, \lambda)$ distribution the rth moment about origin is

$$E(z^r) = \frac{\alpha\Gamma(r+1)}{\lambda^r} A_r(\alpha)$$

where

$$A_r(\alpha) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^{r+1}} \left(\begin{array}{c} \alpha - 1 \\ m \end{array} \right), \qquad r = 1, 2, 3, ...$$

Then, the mean and variance of z are, respectively, given by

$$E(z) = \frac{\alpha}{\lambda} A_1(\alpha)$$
$$Var(z) = \frac{\alpha}{\lambda^2} \left[A_2(\alpha) - \alpha A_1^2(\alpha) \right].$$

The moment generating function (mgf) of z is given by

$$M_z(t) = \alpha B \left(1 - \frac{t}{\lambda}, \alpha \right),$$

where

$$B(P, z) = \int_0^1 t^{P-1} (1-t)^{z-1} dt = \frac{\Gamma(P)\Gamma(z)}{\Gamma(P+z)}$$

is the beta function and $\Gamma(p) = \int_0^\infty w^{P-1} e^{-w} dw$ (for $p > 0$) is the gamma function. Using the mgf, the rth incomplete moment of z is obtained as

$$\mu'_{(r,z)} = \frac{\alpha}{\lambda^r} A_r^*(\alpha),$$

where

$$A_r^*(\alpha) = \sum_{P=0}^{\infty} \frac{(-1)^P}{(P+1)^{r+1}} \left(\begin{array}{c} \alpha - 1 \\ P \end{array} \right) \gamma(r+1, (P+1)\lambda z), \qquad r = 1, 2, 3, ...$$

and $\gamma(P, z) = \int_0^z w^{P-1} e^{-w} dw$ for $P > 0$ is the incomplete gamma function.

2.3. Moments and moment generating function

The rth moment of the FW-L distribution can be expressed as

$$\mu'_r = E(X^r) = \int_0^\infty X^r f(x) dx = \sum_{l=0}^{\infty} S_l \frac{l\Gamma(r+1)}{(k-j)^r} A_r(l),$$

where

$$A_r(l) = 1 + \sum_{m=0}^{\infty} \frac{(-1)^m}{(m+1)^{r+1}} \left(\begin{array}{c} k - j - 1 \\ m \end{array} \right), \qquad r = 1, 2, 3, ...$$

The rth incomplete moment about origin of the FW-L distribution is given as

$$\mu_{(r,x)} = \sum_{l=0}^{\infty} S_l \frac{l}{(k-j)^r} A_r^*(l)$$

where

$$A_r^*(l) = \sum_{P=0}^{\infty} \frac{(-1)^P}{(P+1)^{r+1}} \left(\begin{array}{c} l - 1 \\ P \end{array} \right) \gamma(r+1, (P+1)(k-j)x), \qquad r = 1, 2, 3, ...$$

and $\gamma(P, z) = \int_0^z w^{P-1} e^{-w} dw$ for $P > 0$ is the incomplete gamma function.

The mgf of the FW-L distribution is obtained as

$$\begin{aligned} M_x(t) &= E(e^{tx}) = \int_0^{\infty} e^{tx} f(x)dx \\ &= \sum_{l=0}^{\infty} l S_l B\left(1 - \frac{t}{k-j}, l\right), \end{aligned}$$

where

$$\begin{aligned} B(P,q) &= \int_0^1 t^{P-1}(1-t)^{q-1}dt \\ &= \frac{\Gamma(P)\Gamma(q)}{\Gamma(P+q)} \end{aligned}$$

is the beta function and $\Gamma(P) = \int_0^{\infty} w^{P-1}e^{-w}dw$ is the gamma function for $P > 0$.

2.4. Entropy

The notion of entropy is of fundamental importance in different areas such as physics, probability and statistics, communication theory, and economics. The residual entropy of X is given by

$$\mathcal{E}(X) = -\int_0^{\infty} F(x)\log(F(x))dx$$

and the cumulative residual entropy of X is given by

$$\mathcal{CE}(X) = -\int_0^{\infty} \bar{F}(x)\log(\bar{F}(x))dx.$$

After some simple algebra using geometric expansion and generalized binomial expansion for the FW-L distribution, we can obtain residual entropy as

$$\mathcal{E}(X) = \sum_{i,j,k,r=0}^{\infty}\sum_{l=0}^{k} \frac{(-1)^{i+r+k-l}\alpha^l\,\beta^{k-l}\left[(i+1)^j - (i+2)^j\right]\binom{2l-k}{r} I_{2l-k+r}}{j!k!(i+1)}$$

where $I_{2l-k+r} = \int_{-\infty}^{\infty}\left[[1+\exp(-\lambda x)]^{-1}\right]^{2l-k+r} dx$ and the cumulative entropy as

$$\mathcal{CE}(X) = \sum_{i,j,l=0}^{\infty}\sum_{k=0}^{j} \frac{(-1)^{i+j-k+l}\alpha^k\,\beta^{j-k}(i+1)^j\binom{j}{k}\binom{2k-j}{l} I_{2k-j+r}}{i!j!}$$

where $I_{2k-j+r} = \int_{-\infty}^{\infty}\left[[1+\exp(-\lambda x)]^{-1}\right]^{2k-j+r} dx.$

3. Goodness-of-fit Test

In the presence of censorship, classical goodness-of-fit statistics cannot be applied to fit data to the model chosen in the analysis, so the researchers proposed some modifications of those statistics. In this section, we construct a modified chi-square goodness-of-fit test statistic Y^2 which enable us to check the validity of this new model. We use the approach proposed by Bagdonavicius and Nikulin (2011) based on the maximum likelihood method on non-grouped data.

Let $X_1, ..., X_n$ be i.i.d. random variables grouped into k classes I_j where $I_j = (a_{j-1}, a_j]$ with $a_0 = 0$, $a_k = \tau$ and τ is a finite time. Consider the null hypothesis H_0

$$P(X_i \le x \mid H_0) = F_0(x;\theta), x \ge 0, \;\; \theta = (\theta_1, ..., \theta_s)^T \in \Theta \subset R^s$$

and consider the vector

$$Z_j = \frac{1}{\sqrt{n}}(U_j - e_j)\,, \quad\; j = 1, 2, ..., k \;\;, \text{ with } k \succ s.$$

where U_j and e_j are the observed and expected numbers of failures to fall into the grouping intervals I_j. The statistic Y^2 is defined by

$$Y^2 = Z^T \widehat{\Sigma}^- Z$$

where $\widehat{\Sigma}^-$ is a generalized inverse of the covariance matrix $\widehat{\Sigma}$. Now, we write this statistic as the sum of a chi-square statistic and a quadratic form Q as

$$Y^2 = \sum_{j=1}^{k} \frac{(U_j - e_j)^2}{U_j} + Q$$

with following equations

$$\begin{aligned}
Q &= W^T \widehat{G}^- W, \qquad \widehat{A}_j = U_j/n, \qquad U_j = \sum_{i:X_i \in I_j} \delta_i, \\
W &= (W_1, ..., W_s)^T, \quad \widehat{G} = [\widehat{g}_{ll'}]_{sxs}, \quad \widehat{g}_{ll'} = \widehat{i}_{ll'} - \sum_{j=1}^{k} \widehat{C}_{lj} \widehat{C}_{l'j} \hat{A}_j^{-1}, \\
\widehat{C}_{lj} &= \frac{1}{n} \sum_{i:X_i \in I_j} \delta_i \frac{\partial}{\partial\theta} \ln h(x_i, \widehat{\theta}), \qquad \widehat{i}_{ll'} = \frac{1}{n} \sum_{i=1}^{n} \delta_i \frac{\partial \ln h(x_i, \widehat{\theta})}{\partial \theta_l} \frac{\partial \ln h(x_i, \widehat{\theta})}{\partial \theta_{l'}}, \\
\widehat{W_l} &= \sum_{j=1}^{k} \widehat{C}_{lj} \widehat{A}_j^{-1} Z_j, \quad l, l' = 1, ..., s.
\end{aligned}$$

Here, $\widehat{\theta}$ represents the maximum likelihood estimator (MLE) of θ on initial non-grouped data. Under the null hypothesis H_0, the limit distribution of the statistic Y^2 is a chi-square with $k = rank(\Sigma)$ degrees of freedom. For more details, see Voinov and Shugart et al. (2013).

To apply this test statistic, the expected failure times e_j to fall into the grouping intervals I_j must be the same for any j, so the estimated interval limits a_j are equal to

$$\hat{a}_j = H^{-1}\left(\frac{E_j - \sum_{l=1}^{i-1} H\left(x_l, \theta\right)}{n - i + 1}, \hat{\theta}\right), \qquad \hat{a}_k = \max\left(X_{(n)}, \tau\right)$$

with $E_k = \sum_{i=1}^{n} H\left(x_i, \hat{\theta}\right)$. So, for the FW-L distribution, we have

$$E_j = \frac{j}{k-1} \sum_{i=1}^{n} e^{\alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}}, \quad j = 1, ..., k-1.$$

The components of the estimated matrix $\hat{W}$ are derived from the estimated matrix $\hat{C}$ which is given by

$$\hat{C}_{1j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{\lambda\, e^{\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} + e^{\lambda x_i}\right],$$

$$\hat{C}_{2j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{\lambda\, e^{-\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} - e^{-\lambda x_i}\right],$$

$$\hat{C}_{3j} = \frac{1}{n} \sum_{i:x_i \in I_j}^{n} \delta_i \left[\frac{\alpha\,(1+\lambda x_i)\, e^{\lambda x_i} + \beta\,(1-\lambda x_i)\, e^{-\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} + x_i \left(\alpha e^{\lambda x_i} + \beta e^{-\lambda x_i} \right) \right]$$

and

$$\hat{W}_l = \sum_{j=1}^{k} \hat{C}_{lj} A_j^{-1} Z_j, \quad l = 1,..,m \qquad j = 1,..,k.$$

Therefore, the test statistic can be obtained easily from

$$Y_n^2 \left(\widehat{\theta} \right) = \sum_{j=1}^{k} \frac{\left(U_j - e_j \right)^2}{U_j} + \hat{W}^T \left[\hat{\imath}_{ll'} - \sum_{j=1}^{k} \hat{C}_{lj} \hat{C}_{l'j} \hat{A}_j^{-1} \right]^{-1} \hat{W}.$$

4. Estimation

4.1. Maximum likelihood estimation with the complete data

In this section, the parameters of the FW-L distribution are estimated using the method of maximum likelihood. Let $x_1, x_2, ...x_n$ be random samples distributed according to the distribution, the likelihood function is obtained by the relationship

$$L_n(\alpha, \beta, \lambda) = \prod_{i=1}^{n} f(x_i, \alpha, \beta, \lambda).$$

By taking the natural logarithm, the log-likelihood function is obtained as

$$\log L_n = \sum_{i=1}^{n} \ln \left(\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i} \right) + \alpha \sum_{i=1}^{n} e^{\lambda x_i} - \beta \sum_{i=1}^{n} e^{-\lambda x_i} - \sum_{i=1}^{n} e^{\alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}}.$$

The MLEs $\widehat{\alpha}$,$\hat{\beta}$ and $\widehat{\lambda}$ of the unknown parameters α, β and λ are derived from the nonlinear following score equations

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \frac{\lambda\, e^{\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} + \sum_{i=1}^{n} e^{\lambda x_i} - \sum_{i=1}^{n} e^{\lambda x_i + \alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}},$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \frac{\lambda\, e^{-\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} - \sum_{i=1}^{n} e^{-\lambda x_i} + \sum_{i=1}^{n} e^{-\lambda x_i + \alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}},$$

$$\begin{aligned} \frac{\partial L}{\partial \lambda} \quad = \quad & \sum_{i=1}^{n} \frac{\alpha\,(1+\lambda x_i)\, e^{\lambda x_i} + \beta\,(1-\lambda x_i)\, e^{-\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} + \alpha \sum_{i=1}^{n} x_i e^{\lambda x_i} + \beta \sum_{i=1}^{n} x_i e^{-\lambda x_i} \\ & - \sum_{i=1}^{n} \left(\alpha x_i\, e^{\lambda x_i} + \beta x_i e^{-\lambda x_i} \right) e^{\alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}}. \end{aligned}$$

4.2. Maximum likelihood estimation with the right censored data

Let us consider $X = (X_1, X_2, ..., X_n)^T$ a sample from the FW-L distribution with the parameter vector $\theta = (\alpha, \beta, \lambda)^T$ which can contain right censored data with the fixed censoring time τ. Each X_i can be written as $X_i = (x_i, \delta_i)$ where

$$\delta_i = \begin{cases} 0, & \text{is a censoring time,} \\ 1, & \text{is a failure time.} \end{cases}$$

The right censoring is assumed to be non informative, so the log-likelihood function can be written as:

$$\begin{aligned} L_n(\theta) &= \sum_{i=1}^{n} \delta_i \ln h(x_i,\theta) + \sum_{i=1}^{n} \ln S(x_i,\theta) \\ &= \sum_{i=1}^{n} \delta_i \left[\ln\left(\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}\right) + \alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}\right] - \sum_{i=1}^{n} e^{\alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}}. \end{aligned}$$

The MLEs $\widehat{\alpha}$, $\hat{\beta}$ and $\widehat{\lambda}$ of the unknown parameters α, β and λ are derived from the nonlinear following score equations

$$\frac{\partial L}{\partial \alpha} = \sum_{i=1}^{n} \delta_i \left[\frac{\lambda\, e^{\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} + e^{\lambda x_i}\right] - \sum_{i=1}^{n} e^{\lambda x_i + \alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}},$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{n} \delta_i \left[\frac{\lambda\, e^{-\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} - e^{-\lambda x_i}\right] + \sum_{i=1}^{n} e^{-\lambda x_i + \alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}},$$

$$\begin{aligned} \frac{\partial L}{\partial \lambda} &= \sum_{i=1}^{n} \delta_i \left[\frac{\alpha\,(1+\lambda x_i)\, e^{\lambda x_i} + \beta\,(1-\lambda x_i)\, e^{-\lambda x_i}}{\alpha\lambda\, e^{\lambda x_i} + \lambda\beta e^{-\lambda x_i}} + x_i\left(\alpha e^{\lambda x_i} + \beta e^{-\lambda x_i}\right)\right] \\ &\quad - \sum_{i=1}^{n} \left(\alpha x_i\, e^{\lambda x_i} + \beta x_i e^{-\lambda x_i}\right) e^{\alpha\, e^{\lambda x_i} - \beta e^{-\lambda x_i}}. \end{aligned}$$

Let us point out that the explicit form of $\widehat{\alpha}, \hat{\beta}$ and $\widehat{\lambda}$ cannot be obtained, so numerical methods are required.

5. Simulation

To show the praticability of this test, we carry out a simulation study by generating $N = 10,000$ right censored samples with different percentage (15% and 30%) of right censoring and different sizes ($n = 25,\ 50,\ 130,\ 350,\ 500$) from the FL-W model with parameters $\alpha = 2, \beta = 0.8$ and $\lambda = 1.5$. Using R statistical software and the Barzilai-Borwein (BB) algorithm (Varadhan and Gilbert, 2009), we calculate the MLEs of the unknown parameters and their mean squared errors ($MSEs$). The results are presented in Table 1.

Table 1 Mean simulated values of MLEs of $\hat{\alpha}, \hat{\beta}$ and $\hat{\lambda}$ and their corresponding MSEs

$N = 10.000$	$n_1 = 25$	$n_2 = 50$	$n_3 = 130$	$n_4 = 350$	$n_5 = 500$
$\hat{\alpha}$	1.8394	1.8986	1.9544	1.9873	1.9936
MSE	0.0091	0.0085	0.0056	0.0033	0.0022
$\hat{\beta}$	0.9346	0.9238	0.8934	0.8275	0.8083
MSE	0.0097	0.0079	0.0059	0.0043	0.0025
$\hat{\lambda}$	1.5533	1.5446	1.5379	1.5103	1.5018
MSE	0.0059	0.0055	0.0048	0.0032	0.0012

The MLEs of parameter values, presented in Table 1, agree closely with the true parameter values. Then, we calculate the test statistic Y^2 for each sample with respect to the FW-L model and we compare the obtained values with the theoretical levels of significance ($\varepsilon = 0.01$, 0.05 , 0.1). The results are summarized in Tables 2 and 3.

Table 2 Simulated levels of significance for Y^2 against their theoretical values for 15% of censorship

$N = 10,000$	$n_1 = 25$	$n_2 = 50$	$n_3 = 130$	$n_4 = 350$	$n_5 = 500$
$\varepsilon = 1\%$	0.0065	0.0075	0.0082	0.0089	0.0097
$\varepsilon = 5\%$	0.0435	0.0445	0.0476	0.0485	0.0496
$\varepsilon = 10\%$	0.0926	0.0932	0.0984	0.0992	0.1003

Table 3 Simulated levels of significance for Y^2 against their theoretical values for 30% of censorship

$N = 10,000$	$n_1 = 25$	$n_2 = 50$	$n_3 = 130$	$n_4 = 350$	$n_5 = 500$
$\varepsilon = 1\%$	0.0075	0.0085	0.0091	0.0095	0.0101
$\varepsilon = 5\%$	0.0412	0.0442	0.0462	0.0482	0.0495
$\varepsilon = 10\%$	0.0889	0.0935	0.0974	0.0982	0.0994

As seen in Tables 2 and 3, empirical proportions of rejection of the null hypothesis H_0 for $\varepsilon = 1\%, 5\%$ and 10% levels of significance for all sample sizes and for different percentage of censorship are very close to the theoretical ones. Therefore, the test statistic Y^2, proposed in this work, can be applied to fit data to FL-W distribution.

6. Applications

6.1. First application with the complete data

We now provide applications to show empirically the potentiality of the new model. We compare the FWL distribution with those of the Logistic (L), beta Logistic (BL), exponentiated Logistic (EL), Kumaraswamy Logistic (KwL) and transmuted Weibull Logistic (TWL). We use the data set recently used by Cankaya (2018) which consists of 118 observations. The data are as follows:

0.029, 0.062, 0.011, 0.009, 0.065, -0.128, 0.133, 0.116, 0.184, 0.111, -0.066, -0.049 0.05, 0.137, 0.162, 0.173, 0.033, 0.107, 0.11, 0.147, 0.118 , 0.172, 0.284, -0.137 0.038, -0.145, -0.181, -0.155, 0.198, 0.024, 0.079, -0.252, 0.062, 0.097, 0.032 0.026, 0.195, 0.019, 0.138, -0.3, -0.105, -0.11, -0.168, -0.173, -0.15, 0.078, 0.113 -0.047, 0.024, 0.001, -0.075, 0.014, 0.058, -0.083, -0.339, -0.177, -0.073, -0.044 -0.106, -0.159, -0.101, -0.074, -0.126, -0.131, -0.22, -0.184, -0.105, 0.173, 0.151 0.064, -0.007, -0.005, -0.189, -0.219, -0.301, -0.212, -0.088, 0.157, 0.042, 0.184 0.114 , 0.102 , 0.119 , -0.064, -0.075, 0.073, 0.038, 0.017, -0.134, -0.118, -0.097 0.059, 0.025, -0.102, -0.096, -0.035, 0.057 , -0.055 , 0.015, -0.23, -0.115, 0.255 0.034, 0.078, 0.129, 0.081, 0.032, 0.047, -0.145, 0.012 , -0.224, 0.074, -0.06 -0.137, 0.034, 0.009, -0.139, -0.141

The MLEs and some statistics of the models for the data set are presented in Tables 4 and 5, respectively. We consider the Cramér-von Mises and the Anderson-Darling $(W^*,\ A^*)$ and the Kolmogorov-Smirnov (K-S) statistic with its p-value. W^* and A^* statistics are, respectively, given by

$$W^* = (1 + 1/2n)\left[\frac{1}{12n} + \sum\nolimits_{j=1}^{n} w_j\right],$$

and

$$A^* = a^{(n)}\left(n + n^{-1}\sum\nolimits_{j=1}^{n} a_j\right),$$

where

$$w_j = \left[z_i - (2j-1)\,/\,(2n)\right]^2,$$

$$a^{(n)} = \left(1 + \frac{9}{4}n^{-2} + \frac{3}{4}n^{-1}\right),$$

and

$$a_j = (2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right],$$

Here, $z_i = F(y_j)$ and the y_j's values are the ordered observations. Note that the smaller these statistics are, the better the fit is.

Table 4 The MLEs for the data set

Distribution	Estimates with standard error in parenthesis			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{t}$
FWL	0.5463838	0.9242085	5.0754028	-
	(0.1863241)	(0.2315904)	(1.0611709)	
BL	508.7562790	512.0486055	0.4864484	-
	(556.1311682)	(557.9322)	(0.2675502)	
KL	7.416539	118.610140	2.408877	-
	(9.623443)	(782.922193)	(3.154974)	
GL	76.163092	89.143183	1.175464	-
	(142.723560)	(185.595702)	(1.193495)	
TWL	0.06357792	5.24083845	1.52840070	0.29944251
	(0.01640107)	(55.20841571)	(16.10059)	(0.21395963)
EL	8.363962	5.158805	-	-
	(0.4220599)	(0.4354993)		

Table 5 Some statistics for the models fitted to data set

Distribution	Goodness of fit criteria					
	A^*	W^*	L	KS	P-value	AIC
FWL	0.4273275	0.07126977	-75.24204	0.061852	0.7574	-144.4841
BL	0.8036802	0.1621459	-74.45516	0.10258	0.1669	-142.9103
KL	0.7021602	0.1272965	-74.22824	0.077346	0.4803	-142.4565
GL	2.210552	0.3016586	-72.2051	0.059891	0.4699	-138.4101
TWL	3.030841	0.44206	-67.05935	0.078908	0.1656	-126.1187
EL	2.99168	0.4312295	-65.96237	0.075193	0.2081	-117.9247

Based on Tables the FW-L model provides adequate fits as compared to other models with small values for A^*, W^*, KS and largest P-values among all fitted model. The FW-L model is better than the BL, KL, GL, TML and EL models in modeling the data set. The estimated pdfs and cdfs plots are displayed in Figure 2 It is clear from Figure 2 shows that the FWL distribution provides the best fits to all data sets.

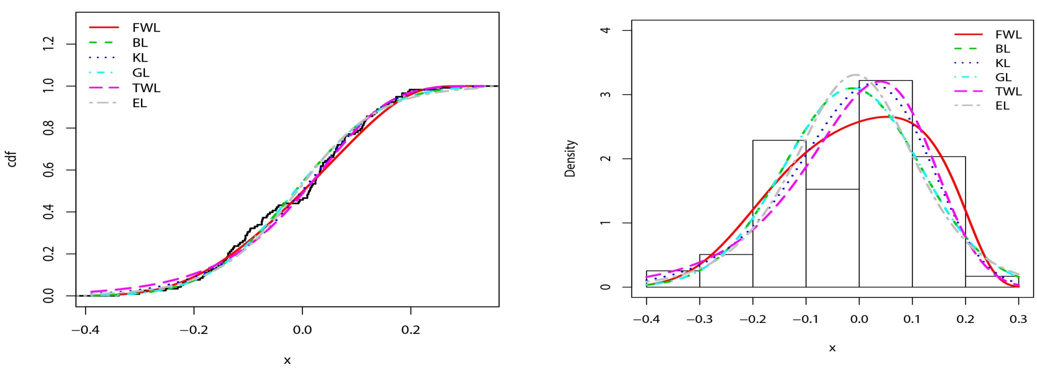

Figure 2 Estimated CDF and PDF for data set

6.2. Second application with the right censored data

Consider data of times to infection of kidney dialysis patients (Nahman et al. 1992).

Percutaneous Placed Catbeter

Infection Times: 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 2.5, 2.5, 3.5, 6.5, 9.5, 15.5

Censored observations: 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 0.5, 1.5, 1.5, 1.5, 1.5, 2.5, 2.5, 2.5, 2.5, 2.5, 3.5, 3.5, 3.5, 3.5, 3.5, 4.5, 4.5, 4.5, 5.5, 5.5, 5.5, 5.5, 5.5, 6.5, 7.5, 7.5, 7.5, 8.5, 8.5, 8.5, 9.5, 9.5, 10.5, 10.5, 10.5, 11.5, 11.5, 12.5, 12.5, 12.5, 12.5, 14.5, 14.5, 16.5, 16.5, 18.5, 19.5, 19.5, 19.5, 20.5, 22.5, 24.5, 25.5, 26.5, 26.5, 28.5.

We use the statistic test provided above to verify if these data are modeled by the FW-L. distribution, At that end, we first calculate the MLEs of the unknown parameters as

$$\hat{\gamma} = \left(\hat{\alpha}, \hat{\lambda}, \hat{\beta}\right)^T = (1.235, 0.8196, 1.8469)^T .$$

Data are grouped into $k = 6$ intervals I_j. We give the necessary values in Table 6.

Table 6 Values of $\widehat{a}_j, e_j, U_j, \hat{C}_{1j}, \hat{C}_{2j}, \hat{C}_{3j}$

$\hat{a}_j$	0.56	3.12	5.26	9.15	15.25	28.5
U_J	16	11	09	13	14	14
$\hat{C}_{1j}$	0.5531	0.3625	0.2412	0.4223	0.5315	0.5748
$\hat{C}_{2j}$	0.1364	0.2152	0.1352	0.2073	0.2561	0.1245
$\hat{C}_{3j}$	0.1485	0.2581	0.1346	0.3964	0.1236	0.2241
e_j	10.826	10.826	10.826	10.826	10.826	10.826

Then, we obtain the value of the statistic test Y_n^2:

$$Y_n^2 = X^2 + Q = 3.1849 + 2.1684 = 5.3533$$

For significance level $\varepsilon = 0.05$, the critical value $\chi_5^2 = 12,5916$ is superior than the value of $Y_n^2 = 5.3533$, so we can conclude that the proposed FL-W model fit the data.

7. Conclusion

In this paper, we proposed a new distribution, called the flexible Weibull logistic (FW-L) distribution which extends the logistic distribution. Several properties of the new distribution were investigated, including moments, moment generating function, asymptotes, entropy. The model parameters both comple and censored data sets tare estimated by maximum likelihood approach. Then, Monte Carlo simulation results indicate that the performance of the MLE are quite satisfactory. Applications with completed and censored data sets indicate that the FW-L distribution provide the best fit among all the sub-models. From the plots of the fitted densities and histogram, clearly, the FW-L distribution yields a closer fit to the histogram than the other other models. Therefore, the new FW-L model can be used quite effectively in analyzing data. We hope that the proposed model can be used effectively as a competitive model to fit real data.

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