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An Efficient Approach for Estimating Population Mean in Simple Random Sampling Using an Auxiliary Attribute

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Abstract

This research article addresses the problem of estimating the population mean of the study variable y using auxiliary attribute. Under large sample approximation, we proposed a class of estimators and their properties. We observe that Shabbir and Gupta (2007)'s mean squared error of his estimator is incorrect; hence we have obtained the correct mean squared error of Shabbir and Gupta's estimator (2007). It is also identified that the class of estimators due to Grover and Kaur (2011), Zaman (2019), Singh et al. (2007) and Solanki and Singh (2013) are members of the suggested class of estimators. It is found that the proposed classes of estimators are more efficient than Shabbir and Gupta (2007), Grover and Kaur (2011), Zaman (2019), Singh et al. (2007) and Solanki and Singh (2013). In support of the current work, a numerical illustration is provided.

Keywords: SRSWOR, study variable, bias, auxiliary attribute, MSE

1. Introduction

1.1. Background

It is a well-known fact that including auxiliary data during the estimation stage improves the precision of an estimate of the population parameter such as population mean (PM) or total. Additional information is provided in the form of an attribute in a number of situations where it would not be directly available but is of a qualitative attribute. Some examples are:

1. The yield of a wheat harvest may be varied by the wheat variety used.

2. In agricultural engineering, the planting area and the proportion of excellent seeds are two essential auxiliary factors to consider when determining average cotton yield.

3. When determining average milk yield, the cow's breed is an important auxiliary property in animal husbandry engineering.

4. A person's height can vary depending on whether he or she is male or female, and so on.

See, Grover and Kaur (2011) and Zaman and Toksoy (2019).

The above examples show that the present study can be employed in different areas of interest such as environmental science, medical researches, education, various engineering fields, agricultural researches etc. Using the point bi-serial correlation between (b/w) the research variable y and the auxiliary attribute ϕ, as well as prior knowledge of the auxiliary attribute's population parameters, various authors including Naik and Gupta (1996), Shabbir and Gupta (2007), Jhajj et al. (2006), Singh et al. (2008), Grover and Kaur (2011), Sharma and Singh (2015), Malik and Singh (2013), Abd-Elfatlah et al. (2010) and Zaman and Toksoy (2019) etc. have formulated the estimators for PM and studied their properties.

1.2. Notations

Let $(\Omega = \Omega_1, \Omega_2, ..., \Omega_N)$ be a finite population of N units. Let y_i and ϕ_i be the observations on the study variable (SV) y and auxiliary attribute ϕ respectively for the i^{th} population unit $\Omega_i, i = 1,2,...,N.$ Let there be complete dichotomy in the population with respect to presence or absence of an attribute ϕ and it is supposed that attribute ϕ takes only two values 1 and 0 accordingly,

$$\phi_i = \begin{cases} 1, \text{ if } i^{\text{th}} \text{ unit of the population posseses attribute } \phi \\ 0, \text{ otherwise.} \end{cases}$$

Let $A = \sum_{i=1}^{N} \phi_i$ and $a = \sum_{i \in s} \phi_i$ be the sum of all units in the population and in the sample possessing attribute $\phi,$ respectively. Let $P = \frac{A}{N}$ and $p = \frac{a}{n}$ denote the proportion of units in the population and in the sample, possessing attribute $\phi,$ respectively. Let $\bar{Y} = \frac{1}{N}\sum_{i=1}^{N} y_i$ and $\bar{y} = \frac{1}{n}\sum_{i \in S} y_i$ be the unknown PM and the sample mean of the SV $y,$ respectively.

$b_{y\phi} = \frac{S_{y\phi}}{S_\phi^2} = \frac{\sum_{i \in S}(y_i - \bar{y})(\phi_i - p)}{\sum_{i \in S}(\phi_i - p)^2}$: the sample regression coefficient of y on $\phi.$

$\rho_{pb} = \frac{S_{y\phi}}{S_y S_\phi}$: the correlation coefficient between the variables y and $\phi.$

$S_{y\phi} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(\phi_i - P)$: The covariance between y and $\phi.$

$S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})^2$: The population mean square of $y.$

$S_\phi^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(\phi_i - P)^2$: The population mean square of $x.$

$C_y = \frac{S_y}{\bar{Y}}$ and $C_p = \frac{S_\varphi}{P}$: The coefficients of variation of y and $\phi.$

$k_p = \rho_{pb}\frac{C_y}{C_p}, \ \lambda = \left(\frac{1}{n} - \frac{1}{N}\right).$

1.3. Review of some existing estimators

Different estimators of PM $\bar{Y}$ on the basis of known population proportion P of the units having the attribute ϕ are listed in Table 1.

Table 1 List of existing estimators of population mean and their mean squared errors

	Estimators of $\bar{Y}$	MSE/Minimum MSE
1	$t_0 = \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$	$MSE(t_0) = \lambda \bar{Y}^2 C_y^2$
2	$t_R = \bar{y}\left(\frac{P}{p}\right)$, $t_P = \bar{y}\left(\frac{p}{P}\right)$, $t_{lr} = \bar{y} + b_{y\phi}(P - p)$ (Naik and Gupta 1996)	$MSE(t_R) = \lambda \bar{Y}^2\left(C_y^2 + C_p^2 - 2\rho_{pb}C_yC_p\right)$ $MSE(t_P) = \lambda \bar{Y}^2\left(C_y^2 + C_p^2 + 2\rho_{pb}C_yC_p\right)$ $MSE(t_{lr}) = \lambda \bar{Y}^2 C_y^2\left(1-\rho_{pb}^2\right)$
3	$t_g = g(\bar{y}, v)$, where $v = \frac{p}{P}$ and $g(\bar{y}, v)$ is a parametric function of $\bar{y}$ and v such that $g(\bar{Y}, 1) = \bar{Y}\ \forall \bar{Y}$ (Jhajj et al. 2006)	$MSE_{\min}(t_g) = \lambda \bar{Y}^2 C_y^2\left(1-\rho_{pb}^2\right)$
4	$t_{sg} = \bar{y}\left[\omega_1 + \omega_2(P-p)\right]\left(\frac{P}{p}\right)$, where ω_1 and ω_2 are suitable chosen constants (Shabbir and Gupta 2007)	$MSE_{\min}(t_{sg})_C = \bar{Y}^2\left[1 - \frac{\left(A_{2(1)}A_{4(1)}^2 - 2A_{3(1)}A_{4(1)}A_{5(1)} + A_{1(1)}A_{5(1)}^2\right)}{\left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right)}\right]$, where $A_{1(1)} = \left[1 + \lambda\left(C_y^2 - 4\rho_{pb}C_yC_p + 3C_p^2\right)\right]$, $A_{2(1)} = \lambda P^2 C_p^2$, $A_{3(1)} = 2\lambda P\left(C_P^2 - \rho_{pb}C_yC_p\right)$, $A_{4(1)} = \left[1 + \lambda\left(C_p^2 - \rho_{pb}C_yC_p\right)\right]$, $A_{5(1)} = \lambda P\left(C_p^2 - \rho_{pb}C_yC_p\right)$.
5	$t_{s1} = t_{lr}\frac{P}{p}$, $t_{s2} = t_{lr}\left(\frac{P+\beta_{2(\phi)}}{p+\beta_{2(\phi)}}\right)$, $t_{s3} = t_{lr}\left(\frac{P+C_p}{p+C_p}\right)$, $t_{s4} = t_{lr}\left(\frac{P\beta_{2(\phi)}+C_p}{p\beta_{2(\phi)}+C_p}\right)$,	$MSE(t_{si}) = \lambda \bar{Y}^2\left[R_i^2 C_p^2 + C_y^2\left(1-\rho_{pb}^2\right)\right]$, where t_{si}, $i = 1$ to 10, $R_1 = 1$, $R_2 = \frac{P}{\left(P+\beta_{2(\phi)}\right)}$, $R_3 = \frac{P}{\left(P+C_p\right)}$, $R_4 = \frac{P\beta_{2(\phi)}}{\left(P\beta_{2(\phi)}+C_p\right)}$, $R_5 = \frac{PC_p}{\left(PC_p+\beta_{2(\phi)}\right)}$, $R_6 = \frac{P}{\left(P+\rho_{pb}\right)}$, $R_7 = \frac{PC_p}{\left(PC_p+\rho_{pb}\right)}$,

Table 1 (Continued)

	Estimators of $\bar{Y}$	MSE/Minimum MSE
	$t_{s5}=t_{lr}\left(\frac{PC_p+\beta_{2(\phi)}}{pC_p+\beta_{2(\phi)}}\right)$, $t_{s6}=t_{lr}\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)$, $t_{s7}=t_{lr}\left(\frac{PC_p+\rho_{pb}}{pC_p+\rho_{pb}}\right)$, $t_{s8}=t_{lr}\left(\frac{P\rho_{pb}+C_p}{p\rho_{pb}+C_p}\right)$, $t_{s9}=t_{lr}\left(\frac{P\beta_{2(\phi)}+\rho_{pb}}{p\beta_{2(\phi)}+\rho_{pb}}\right)$, $t_{s10}=t_{lr}\left(\frac{P\rho_{pb}+\beta_{2(\phi)}}{p\rho_{pb}+\beta_{2(\phi)}}\right)$ (Singh et al. 2008)	$R_8=\frac{P\rho_{pb}}{\left(P\rho_{pb}+C_p\right)}$, $R_9=\frac{P\beta_{2(\phi)}}{\left(P\beta_{2(\phi)}+\rho_{pb}\right)}$, $R_{10}=\frac{P\rho_{pb}}{\left(P\rho_{pb}+\beta_{2(\phi)}\right)}$, $\beta_{2(\phi)}=\frac{\frac{1}{N}\sum_{i=1}^{N}(\phi_i-P)^4}{\left[\frac{1}{N}\sum_{i=1}^{N}(\phi_i-P)^2\right]^2}$ and $t_{lr}=\bar{y}+b_{y\phi}(P-p)$
6	$t_{\mathrm{Re}}=\bar{y}\exp\left(\frac{P-p}{P+p}\right)$, $t_{P\mathrm{e}}=\bar{y}\exp\left(\frac{p-P}{p+P}\right)$ (Singh et al. 2007)	$MSE(t_{\mathrm{Re}})=\lambda\bar{Y}^2\left[C_y^2+\frac{C_p^2}{4}-\rho_{pb}C_yC_p\right]$, $MSE(t_{P\mathrm{e}})=\lambda\bar{Y}^2\left[C_y^2+\frac{C_p^2}{4}+\rho_{pb}C_yC_p\right]$
7	$t_{RPe}=\bar{y}\left[\alpha\exp\left(\frac{P-p}{P+p}\right)+(1-\alpha)\exp\left(\frac{p-P}{p+P}\right)\right]$ α is a suitable chosen constant (Singh et al. 2007)	$MSE_{\min}(t_{RPe})=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$
8	$t_{A2}=\bar{y}\left(\frac{P+\beta_{2(\phi)}}{p+\beta_{2(\phi)}}\right)$, $t_{A3}=\bar{y}\left(\frac{P+C_p}{p+C_p}\right)$, $t_{A4}=\bar{y}\left(\frac{P\beta_{2(\phi)}+C_p}{p\beta_{2(\phi)}+C_p}\right)$, $t_{A5}=\bar{y}\left(\frac{PC_p+\beta_{2(\phi)}}{pC_p+\beta_{2(\phi)}}\right)$, $t_{A6}=\bar{y}\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)$ (Abd-Elfattah et al. 2010)	$MSE(t_{Ai})=\lambda\bar{Y}^2\left[C_y^2+R_i^2C_p^2-2R_i\rho_{pb}C_yC_p\right]$, where $i=2, 3, 4, 5, 6$
9	$t_{A(12)}=mt_{lr}\frac{P}{p}+(1-m)t_{lr}\left(\frac{P+\beta_{2(\phi)}}{p+\beta_{2(\phi)}}\right)$, $t_{A(13)}=mt_{lr}\frac{P}{p}+(1-m)t_{lr}\left(\frac{P+C_p}{p+C_p}\right)$,	$MSE_{\min}\left(t_{A(i)}\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$, where $i=12$, 13, 14, 15

Table 1 (Continued)

	Estimators of $\bar{Y}$	MSE/Minimum MSE
	$t_{A(14)} = mt_{lr}\dfrac{P}{p} + (1-m)t_{lr}\left(\dfrac{P\beta_{2(\phi)}+C_p}{p\beta_{2(\phi)}+C_p}\right)$, $t_{A(15)} = mt_{lr}\dfrac{P}{p} + (1-m)t_{lr}\left(\dfrac{PC_p+\beta_{2(\phi)}}{pC_p+\beta_{2(\phi)}}\right)$, where m is a constant (Abd-Elfattah et al. 2010)	
10	$t_{GK1} = \omega_1\bar{y} + \omega_2(P-p)$ (Grover and Kaur 2011)	$MSE_{\min}(t_{GK1}) = \dfrac{\lambda\bar{Y}^2C_y^2(1-\rho_{pb}^2)}{\left[1+\lambda C_y^2(1-\rho_{pb}^2)\right]}$
11	$t_{GK2} = \left[\omega_1\bar{y} + \omega_2(P-p)\right]\exp\left(\dfrac{P-p}{P+p}\right)$	$MSE_{min}(t_{GK2}) = \bar{Y}^2\left[1-\dfrac{\left(A_{2(1)}^*A_{4(1)}^{*\,2} - 2A_{3(1)}^*A_{4(1)}^*A_{5(1)}^* + A_{1(1)}^*A_{5(1)}^{*\,2}\right)}{\left(A_{1(1)}^*A_{2(1)}^* - A_{3(1)}^{*\,2}\right)}\right]$ where $A_{1(1)}^* = \left[1+\lambda\left\{C_y^2 + C_p^2(1-2k_p)\right\}\right]$, $A_{2(1)}^* = \dfrac{\lambda C_p^2}{R^2}$, $A_{3(1)}^* = \dfrac{\lambda C_p^2}{R}(1-k_p)$, $A_{4(1)}^* = \left[1+\dfrac{\lambda}{8}C_p^2(3-4k_p)\right]$, $A_{5(1)}^* = \dfrac{\lambda C_p^2}{2R}$ and $R = \dfrac{\bar{Y}}{P}$
12	$t_{ss} = \bar{y}\left[\omega_1 + \omega_2(P-p)\right]\left(\dfrac{\psi P+\delta\eta}{\psi p+\delta\eta}\right)^{\alpha}$, where ψ and η are either real numbers or function of known parameters of the auxiliary attribute ϕ, α and δ which are constants possessing values $+1$ and -1 in such a way that for a product estimator α is -1 and for a ratio estimator α is $+1$ and for designing the estimators such that $(\psi p+\delta\eta)$ and $(\psi P+\delta\eta)$ are non-negative δ is $+1$ and -1. (ω_1, ω_2) are constants i.e. $(\omega_1+\omega_2 \neq 1)$. (Singh and Solanki 2012)	$MSE_{\min}(t_{ss}) = \bar{Y}^2\left[1-\dfrac{\left(A_{2(\alpha)}A_{4(\alpha)}^2 - 2A_{3(\alpha)}A_{4(\alpha)}A_{5(\alpha)} + A_{1(\alpha)}A_{5(\alpha)}^2\right)}{\left(A_{1(\alpha)}A_{2(\alpha)} - A_{3(\alpha)}^2\right)}\right]$ where $A_{1(\alpha)} = \left[1+\lambda\left\{C_y^2 + \alpha\upsilon C_p^2\left((2\alpha+1)\upsilon - 4k_p\right)\right\}\right]$, $A_{2(\alpha)} = \lambda P^2C_p^2$, $A_{3(\alpha)} = 2\lambda PC_p^2(\alpha\upsilon - k_p)$, $A_{4(\alpha)} = \left[1+\lambda\alpha\upsilon C_p^2\left(\dfrac{(\alpha+1)\upsilon}{2} - k_p\right)\right]$, $A_{5(\alpha)} = \lambda PC_p^2(\alpha\upsilon - k_p)$ and $\upsilon = \dfrac{\psi P}{(\psi P+\delta\eta)}$

Table 1 (Continued)

	Estimators of $\bar{Y}$	MSE/Minimum MSE
13	$t_1=\left[\bar{y}+b_{y\phi}\left(P-p\right)\right]\left(\dfrac{\psi P+\eta}{\psi p+\eta}\right)$ (Singh and Solanki 2012)	$MSE\left(t_1\right)=\lambda\bar{Y}^2\left[R^{*2}C_{\phi}^2+C_y^2\left(1-\rho_{pb}^2\right)\right]$ $MSE_{\min}\left(t_1\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$, where $R^*=\dfrac{\bar{Y}\psi}{\psi P+\eta}$
14	$t_{1e}=\left[\bar{y}+b_{y\phi}\left(P-p\right)\right]\times\exp\left(\dfrac{\psi\left(P-p\right)}{\psi\left(P+p\right)+2\eta}\right)$ (Authors defined)	$MSE\left(t_{1e}\right)=\lambda\bar{Y}^2\left[\dfrac{R^{*2}C_{\phi}^2}{4}+C_y^2\left(1-\rho_{pb}^2\right)\right]$ $MSE_{\min}\left(t_{1e}\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$
15	$t_2=\bar{y}\left(\dfrac{\psi P+\eta}{\psi p+\eta}\right)$ (Singh and Solanki 2012)	$MSE_{\min}\left(t_2\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$
16	$t_{sse}=\bar{y}\exp\left\{\dfrac{\gamma\left(P-p\right)}{P+p}\right\}$, where γ is a suitable chosen constant (Solanki and Singh 2013)	$MSE_{\min}\left(t_{sse}\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$
17	$\bar{y}_{pr1}=\bar{y}\exp\left(\dfrac{p_1}{P_1}\right)^{\alpha_1}\left(\dfrac{p_2}{P_2}\right)^{\alpha_2}$, $\bar{y}_{pr2}=w\bar{y}\left(\dfrac{p_1}{P_1}\right)^{\alpha_1}+\left(1-w\right)\bar{y}\left(\dfrac{p_2}{P_2}\right)^{\alpha_2}$, where α_1 and α_2 are real numbers and w denotes the weight (Zaman and Toksoy 2019)	$MSE_{min}\left(\bar{y}_{pr1}\right)=\dfrac{1-f}{n}\bar{Y}^2C_y^2\left[1-\dfrac{\rho_{yp_2}^2+\rho_{yp_1}^2-2\rho_{yp_1}\rho_{yp_2}\rho_{p_1p_2}}{1-\rho_{p_1p_2}^2}\right]$ $MSE_{\min}\left(\bar{y}_{pr2}\right)=\dfrac{1-f}{n}\bar{Y}^2$ $\times\left[\begin{array}{l}C_y^2+w^{*2}\left(\alpha_1^2C_{p_1}^2+\alpha_2^2C_{p_2}^2-2\alpha_1\alpha_2\rho_{p_1p_2}C_{p_1}C_{p_2}\right)\\+2w^*\left(\begin{array}{l}-\alpha_2^2C_{p_2}^2+\alpha_1\rho_{yp_1}C_yC_{p_1}-\alpha_2\rho_{yp_2}C_yC_{p_2}\\+\alpha_1\alpha_2\rho_{p_1p_2}C_{p_1}C_{p_2}\end{array}\right)\\+\alpha_2^2C_{p_2}^2+2\alpha_2\rho_{yp_2}C_yC_{p_2}\end{array}\right]$
18	$t_{2e}=\bar{y}\exp\left(\dfrac{\psi\left(P-p\right)}{\psi\left(P+p\right)+2\eta}\right)$ (Zaman and Kadilar 2019)	$MSE_{\min}\left(t_{2e}\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$
19	$t_{yz(1)}=K\exp\left(\dfrac{\psi\left(P-p\right)}{\psi\left(P+p\right)+2\eta}\right)$ where K is the characterizing scalars such that MSE of the estimator is minimum, $t_{yz(2)}=k_1\bar{y}+k_2\bar{y}\exp\left(\dfrac{\psi\left(P-p\right)}{\psi\left(P+p\right)+2\eta}\right)$	$MSE_{\min}\left(t_{yz(1)}\right)=\bar{Y}^2\left(1-\dfrac{A^2}{B^2}\right)$ where $A=\left[1+\lambda C_p^2\dfrac{\upsilon}{8}\left(3\upsilon-4k_p\right)\right]$ $B=\left[1+\lambda C_p^2\upsilon\left(\upsilon-2k_p\right)\right]$ $MSE_{\min}\left(t_{yz(2)}\right)=\lambda\bar{Y}^2C_y^2\left(1-\rho_{pb}^2\right)$

Table 1 (Continued)

Estimators of $\bar{Y}$	MSE/Minimum MSE
where k_1 and k_2 are scalar constants such that $k_1 + k_2 = 1$.	

The MSE of the estimator t_{sg} obtained by Shabbir and Gupta (2007) is not correct. The correct expressions of the MSE and hence optimum MSE of the estimator t_{sg} is given in Table 1 of this paper. We propose two classes of estimators for estimating the PM $\bar{Y}$ of the SV y in this work when auxiliary information is qualitative in nature. We have calculated the biases and mean squared error (MSEs) up to first order of approximation. Optimum conditions are calculated at which the MSEs of the classes of estimators are minimum. Preference regions are obtained under which the suggested classes of estimators are further efficient than the all other considered estimators. In support of current work, a numerical illustration is provided.

2. Suggested Estimator

2.1. Improved version of Singh and Solanki (2012) class of estimators

We have suggested the following class of estimators for the PM $\bar{Y}$ as

$$T_\alpha^\gamma = \bar{y}\left[\omega_1 + \omega_2\left(P - p\right)\right]\left[d\left(\frac{\psi P + \delta\eta}{\psi p + \delta\eta}\right)^\alpha + \left(1 - d\right)\exp\left\{\frac{\gamma\psi\left(P - p\right)}{\psi\left(P + p\right) + 2\delta\eta}\right\}\right], \tag{1}$$

where $\left(\omega_1, \omega_2, \psi, \delta, \eta, \alpha\right)$ are same as defined before and $\left(d, \gamma\right)$ are also suitable real numbers. The class of T_α^γ boils down to the following set of estimators/classes of estimators for various values of constants $\left(\omega_1, \omega_2, \psi, \delta, \eta, \alpha, d, \gamma\right)$ and listed in Table 2.

Table 2 Some members of the class of T_α^γ for $\left(\alpha, \gamma\right) = \left(1,1\right)$ and $\delta = +1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$T_{1(1)}^1 = \omega_d\left[d\left(\frac{P}{p}\right) + \left(1 - d\right)\exp\left\{\frac{\left(P - p\right)}{\left(P + p\right)}\right\}\right]$, where $\omega_d = \bar{y}\left[\omega_1 + \omega_2\left(P - p\right)\right]$
2	1	β_2	$T_{1(2)}^1 = \omega_d\left[d\left(\frac{P + \beta_2}{p + \beta_2}\right) + \left(1 - d\right)\exp\left\{\frac{\left(P - p\right)}{\left(P + p\right) + 2\beta_2}\right\}\right]$
3	1	C_p	$T_{1(3)}^1 = \omega_d\left[d\left(\frac{P + C_p}{p + C_p}\right) + \left(1 - d\right)\exp\left\{\frac{\left(P - p\right)}{\left(P + p\right) + 2C_p}\right\}\right]$
4	1	ρ_{pb}	$T_{1(4)}^1 = \omega_d\left[d\left(\frac{P + \rho_{pb}}{p + \rho_{pb}}\right) + \left(1 - d\right)\exp\left\{\frac{\left(P - p\right)}{\left(P + p\right) + 2\rho_{pb}}\right\}\right]$

Table 2 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
5	β_2	C_p	$T_{1(5)}^{1}=\omega_d\left[d\left(\frac{\beta_2 P+C_p}{\beta_2 p+C_p}\right)+(1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$T_{1(6)}^{1}=\omega_d\left[d\left(\frac{C_p P+\beta_2}{C_p p+\beta_2}\right)+(1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$T_{1(7)}^{1}=\omega_d\left[d\left(\frac{C_p P+\rho_{pb}}{C_p p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$T_{1(8)}^{1}=\omega_d\left[d\left(\frac{\rho_{pb} P+C_p}{\rho_{pb} p+C_p}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$T_{1(9)}^{1}=\omega_d\left[d\left(\frac{\beta_2 P+\rho_{pb}}{\beta_2 p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$T_{1(10)}^{1}=\omega_d\left[d\left(\frac{\rho_{pb} P+\beta_2}{\rho_{pb} p+\beta_2}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Further some more members of the class of T_α^γ are given in Tables 3 to 7.

Table 3 Some members of the class of T_α^γ for $(\alpha,\gamma)=(-1,-1)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$T_{-1(1)}^{-1}=\omega_d\left[d\left(\frac{p}{P}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)}\right\}\right]$
2	1	β_2	$T_{-1(2)}^{-1}=\omega_d\left[d\left(\frac{p+\beta_2}{P+\beta_2}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$T_{-1(3)}^{-1}=\omega_d\left[d\left(\frac{p+C_p}{P+C_p}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$T_{-1(4)}^{-1}=\omega_d\left[d\left(\frac{p+\rho_{pb}}{P+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$T_{-1(5)}^{-1}=\omega_d\left[d\left(\frac{\beta_2 p+C_p}{\beta_2 P+C_p}\right)+(1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2C_p}\right\}\right]$

Table 3 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
6	C_p	β_2	$T_{-1(6)}^{-1}=\omega_d\left[d\left(\frac{C_p p+\beta_2}{C_p P+\beta_2}\right)+(1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$T_{-1(7)}^{-1}=\omega_d\left[d\left(\frac{C_p p+\rho_{pb}}{C_p P+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$T_{-1(8)}^{-1}=\omega_d\left[d\left(\frac{\rho_{pb} p+C_p}{\rho_{pb} P+C_p}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$T_{-1(9)}^{-1}=\omega_d\left[d\left(\frac{\beta_2 p+\rho_{pb}}{\beta_2 P+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$T_{-1(10)}^{-1}=\omega_d\left[d\left(\frac{\rho_{pb} p+\beta_2}{\rho_{pb} P+\beta_2}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 4 Some members of the class of T_α^γ for $(\alpha,\gamma)=(1,-1)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$T_{-1(1)}^{1}=\omega_d\left[d\left(\frac{P}{p}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)}\right\}\right]$
2	1	β_2	$T_{-1(2)}^{1}=\omega_d\left[d\left(\frac{P+\beta_2}{p+\beta_2}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$T_{-1(3)}^{1}=\omega_d\left[d\left(\frac{P+C_p}{p+C_p}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$T_{-1(4)}^{1}=\omega_d\left[d\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$T_{-1(5)}^{1}=\omega_d\left[d\left(\frac{\beta_2 P+C_p}{\beta_2 p+C_p}\right)+(1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$T_{-1(6)}^{1}=\omega_d\left[d\left(\frac{C_p P+\beta_2}{C_p p+\beta_2}\right)+(1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$T_{-1(7)}^{1}=\omega_d\left[d\left(\frac{C_p P+\rho_{pb}}{C_p p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$

Table 4 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
8	ρ_{pb}	C_p	$T_{-1(8)}^{1}=\omega_d\left[d\left(\dfrac{\rho_{pb}P+C_p}{\rho_{pb}p+C_p}\right)+(1-d)\exp\left\{\dfrac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$T_{-1(9)}^{1}=\omega_d\left[d\left(\dfrac{\beta_2P+\rho_{pb}}{\beta_2p+\rho_{pb}}\right)+(1-d)\exp\left\{\dfrac{\beta_2(p-P)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$T_{-1(10)}^{1}=\omega_d\left[d\left(\dfrac{\rho_{pb}P+\beta_2}{\rho_{pb}p+\beta_2}\right)+(1-d)\exp\left\{\dfrac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 5 Some members of the class of T_{α}^{γ} for $(\alpha,\gamma)=(-1,1)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$T_{1(1)}^{-1}=\omega_d\left[d\left(\dfrac{p}{P}\right)+(1-d)\exp\left\{\dfrac{(P-p)}{(P+p)}\right\}\right]$
2	1	β_2	$T_{1(2)}^{-1}=\omega_d\left[d\left(\dfrac{p+\beta_2}{P+\beta_2}\right)+(1-d)\exp\left\{\dfrac{(P-p)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$T_{1(3)}^{-1}=\omega_d\left[d\left(\dfrac{p+C_p}{P+C_p}\right)+(1-d)\exp\left\{\dfrac{(P-p)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$T_{1(4)}^{-1}=\omega_d\left[d\left(\dfrac{p+\rho_{pb}}{P+\rho_{pb}}\right)+(1-d)\exp\left\{\dfrac{(P-p)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$T_{1(5)}^{-1}=\omega_d\left[d\left(\dfrac{\beta_2p+C_p}{\beta_2P+C_p}\right)+(1-d)\exp\left\{\dfrac{\beta_2(P-p)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$T_{1(6)}^{-1}=\omega_d\left[d\left(\dfrac{C_pp+\beta_2}{C_pP+\beta_2}\right)+(1-d)\exp\left\{\dfrac{C_p(P-p)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$T_{1(7)}^{-1}=\omega_d\left[d\left(\dfrac{C_pp+\rho_{pb}}{C_pP+\rho_{pb}}\right)+(1-d)\exp\left\{\dfrac{C_p(P-p)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$T_{1(8)}^{-1}=\omega_d\left[d\left(\dfrac{\rho_{pb}p+C_p}{\rho_{pb}P+C_p}\right)+(1-d)\exp\left\{\dfrac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$T_{-1(9)}^{-1}=\omega_d\left[d\left(\dfrac{\beta_2p+\rho_{pb}}{\beta_2P+\rho_{pb}}\right)+(1-d)\exp\left\{\dfrac{\beta_2(P-p)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$

Table 5 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
10	ρ_{pb}	β_2	$T_{-1(10)}^{-1} = \omega_d \left[d\left(\frac{\rho_{pb}p+\beta_2}{\rho_{pb}P+\beta_2}\right) + (1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 6 Some members of the class of T_α^γ for $(\alpha,\gamma)=(0,1)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$T_{1(1)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{(P-p)}{(P+p)}\right\}\right]$
2	1	β_2	$T_{1(2)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{(P-p)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$T_{1(3)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{(P-p)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$T_{1(4)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{(P-p)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$T_{1(5)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$T_{1(6)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$T_{1(7)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$T_{1(8)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$T_{1(9)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$T_{1(10)}^{0} = \omega_d \left[d + (1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 7 Some members of the class of T_α^γ for $(\alpha,\gamma)=(1,0)$ and $\delta=+1$

S. No.	Values of Constants	Estimator

	ψ	η	
1	1	0	$T_{0(1)}^{1}=\omega_d\left[d\left(\frac{P}{p}\right)+(1-d)\right]$
2	1	β_2	$T_{0(2)}^{1}=\omega_d\left[d\left(\frac{P+\beta_2}{p+\beta_2}\right)+(1-d)\right]$
3	1	C_p	$T_{0(3)}^{1}=\omega_d\left[d\left(\frac{P+C_p}{p+C_p}\right)+(1-d)\right]$
4	1	ρ_{pb}	$T_{0(4)}^{1}=\omega_d\left[d\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)+(1-d)\right]$
5	β_2	C_p	$T_{0(5)}^{1}=\omega_d\left[d\left(\frac{\beta_2 P+C_p}{\beta_2 p+C_p}\right)+(1-d)\right]$
6	C_p	β_2	$T_{0(6)}^{1}=\omega_d\left[d\left(\frac{C_p P+\beta_2}{C_p p+\beta_2}\right)+(1-d)\right]$
7	C_p	ρ_{pb}	$T_{0(7)}^{1}=\omega_d\left[d\left(\frac{C_p P+\rho_{pb}}{C_p p+\rho_{pb}}\right)+(1-d)\right]$
8	ρ_{pb}	C_p	$T_{0(8)}^{1}=\omega_d\left[d\left(\frac{\rho_{pb} P+C_p}{\rho_{pb} p+C_p}\right)+(1-d)\right]$
9	β_2	ρ_{pb}	$T_{0(9)}^{1}=\omega_d\left[d\left(\frac{\beta_2 P+\rho_{pb}}{\beta_2 p+\rho_{pb}}\right)+(1-d)\right]$
10	ρ_{pb}	β_2	$T_{0(10)}^{1}=\omega_d\left[d\left(\frac{\rho_{pb} P+\beta_2}{\rho_{pb} p+\beta_2}\right)+(1-d)\right]$

2.2. Bias and MSE of the class of estimators T_α^γ

Expressing T_α^γ at (1) in terms of $e_0=\frac{(\bar{y}-\bar{Y})}{\bar{Y}}$ and $e_1=\frac{(p-P)}{P}$, we have

$$T_\alpha^\gamma=\bar{Y}(1+e_0)[\omega_1-\omega_2 Pe_1]\left[d(1+\upsilon e_1)^{-\alpha}+(1-d)\exp\left\{-\frac{\gamma\upsilon e_1}{2}\left(1+\frac{\upsilon}{2}e_1\right)^{-1}\right\}\right]$$

$$=\bar{Y}(1+e_0)[\omega_1-\omega_2 Pe_1]\left[d\left(1-\alpha\upsilon e_1+\frac{\alpha(\alpha+1)}{2}\upsilon^2e_1^2-...\right)+(1-d)\left\{1-\frac{\gamma\upsilon e_1}{2}\left(1+\frac{\upsilon}{2}e_1\right)^{-1}+\frac{\gamma^2\upsilon^2e_1^2}{8}\left(1+\frac{\upsilon}{2}e_1\right)^{-2}-...\right\}\right]$$

$$=\bar{Y}(1+e_0)[\omega_1-\omega_2 Pe_1]\left[d\left(1-\alpha\upsilon e_1+\frac{\alpha(\alpha+1)}{2}\upsilon^2e_1^2-...\right)+(1-d)\left\{1-\frac{\gamma\upsilon e_1}{2}\left(1-\frac{\upsilon}{2}e_1+...\right)+\frac{\gamma^2\upsilon^2e_1^2}{8}-...\right\}\right]$$

$$= \bar{Y}(1+e_0)[\omega_1 - \omega_2 P e_1]\left[d\left(1-\alpha\upsilon e_1+\frac{\alpha(\alpha+1)}{2}\upsilon^2 e_1^2-...\right)+(1-d)\left\{1-\frac{\gamma\upsilon e_1}{2}+\frac{\gamma\upsilon^2 e_1^2}{4}+\frac{\gamma^2\upsilon^2 e_1^2}{8}-...\right\}\right]$$

$$= \bar{Y}(1+e_0)[\omega_1 - \omega_2 P e_1]\left[d\left(1-\alpha\upsilon e_1+\frac{\alpha(\alpha+1)}{2}\upsilon^2 e_1^2-...\right)+(1-d)\left\{1-\frac{\gamma\upsilon e_1}{2}+\frac{\gamma(\gamma+2)\upsilon^2 e_1^2}{8}-...\right\}\right]$$

$$= \bar{Y}(1+e_0)[\omega_1 - \omega_2 P e_1]\left[1-\left(\alpha d+\frac{\gamma(1-d)}{2}\right)\upsilon e_1+\left\{d\alpha(\alpha+1)+\frac{(1-d)\gamma(\gamma+2)}{4}\right\}\frac{\upsilon^2 e_1^2}{2}-...\right]$$

$$= \bar{Y}(1+e_0)[\omega_1 - \omega_2 P e_1]\left[1-\theta_0\upsilon e_1+\theta_1\frac{\upsilon^2 e_1^2}{2}-...\right]$$

$$= \bar{Y}\left[\omega_1(1+e_0)\left\{1-\theta_0\upsilon e_1+\theta_1\frac{\upsilon^2 e_1^2}{2}-...\right\}-\omega_2 P(e_1+e_0 e_1)\left\{1-\theta_0\upsilon e_1+\theta_1\frac{\upsilon^2 e_1^2}{2}-...\right\}\right], \quad (2)$$

where $\theta_0 = \left(\alpha d+\frac{\gamma(1-d)}{2}\right)$, $\theta_1 = \left\{d\alpha(\alpha+1)+\frac{(1-d)\gamma(\gamma+2)}{4}\right\}$.

Skipping terms of *e*'s (having power more than two), we have

$$T_\alpha^\gamma \cong \bar{Y}\left[\omega_1\left\{1+e_0-\theta_0\upsilon e_1-\theta_0\upsilon e_0 e_1+\theta_1\frac{\upsilon^2 e_1^2}{2}\right\}-\omega_2 P\left(e_1+e_0 e_1-\theta_0\upsilon e_1^2\right)\right] \text{ or}$$

$$\left(T_\alpha^\gamma-\bar{Y}\right)=\bar{Y}\left[\omega_1\left\{1+e_0-\theta_0\upsilon e_1-\theta_0\upsilon e_0 e_1+\theta_1\frac{\upsilon^2 e_1^2}{2}\right\}+\omega_2 P\left(\theta_0\upsilon e_1^2-e_1-e_0 e_1\right)-1\right]. \quad (3)$$

The bias of T_α^γ to the *fda* as

$$B\left(T_\alpha^\gamma\right)=\bar{Y}\left[\omega_1\left\{1+\frac{\lambda\upsilon}{2}C_p^2\left(\theta_1\upsilon-2\theta_0 k_p\right)\right\}+\omega_2\lambda P C_p^2\left(\theta_0\upsilon-k_p\right)-1\right]=\bar{Y}\left[\omega_1 B_{4(\alpha)}+\omega_2 B_{5(\alpha)}-1\right], \quad (4)$$

where $B_{4(\alpha)}=\left[1+\frac{\lambda\upsilon}{2}C_p^2\left(\theta_1\upsilon-2\theta_0 k_p\right)\right]$, $B_{5(\alpha)}=\lambda P C_p^2\left(\theta_0\upsilon-k_p\right)$. Square of (3) is approximated as

$$\left(T_\alpha^\gamma-\bar{Y}\right)^2=\bar{Y}^2\left[1+\omega_1^2\left\{1+2e_0-2\theta_0\upsilon e_1+e_0^2-4\theta_0\upsilon e_0 e_1+\left(\theta_0^2+\theta_1\right)\upsilon^2 e_1^2\right\}\right.$$
$$+\omega_2^2 P^2 e_1^2+2\omega_1\omega_2 P\left\{2\theta_0\upsilon e_1^2-2e_0 e_1-e_1\right\}-2\omega_1\left\{1+e_0-\theta_0\upsilon e_1-\theta_0\upsilon e_0 e_1+\frac{\theta_1\upsilon^2 e_1^2}{2}\right\}$$
$$\left.-2\omega_2 P\left\{\theta_0\upsilon e_1^2-e_0 e_1-e_1\right\}\right]. \quad (5)$$

The MSE of T_α^γ to the *fda* as

$$MSE\left(T_\alpha^\gamma\right)=\bar{Y}^2\left[1+\omega_1^2 B_{1(\alpha)}+\omega_2^2 B_{2(\alpha)}+2\omega_1\omega_2 B_{3(\alpha)}-2\omega_1 B_{4(\alpha)}-2\omega_2 B_{5(\alpha)}\right], \quad (6)$$

where $B_{1(\alpha)}=\left[1+\lambda\left\{C_y^2+\upsilon C_p^2\left(\left(\theta_0^2+\theta_1\right)\upsilon-4\theta_0 k_p\right)\right\}\right]$, $B_{2(\alpha)}=\lambda P^2 C_p^2$, $B_{3(\alpha)}=\left[2PC_p^2\lambda\left(\theta_0\upsilon-k_p\right)\right]$, $B_{4(\alpha)}=\left[1+\dfrac{\lambda C_p^2\upsilon}{2}\left(\theta_1\upsilon-2\theta_0 k_p\right)\right]$, $B_{5(\alpha)}=\lambda PC_p^2\left(\theta_0\upsilon-k_p\right)$. Setting $\dfrac{\partial MSE\left(T_\alpha^\gamma\right)}{\partial\omega_1}=0$ and $\dfrac{\partial MSE\left(T_\alpha^\gamma\right)}{\partial\omega_2}=0$, we have

$$\begin{bmatrix} B_{1(\alpha)} & B_{3(\alpha)} \\ B_{3(\alpha)} & B_{2(\alpha)} \end{bmatrix}\begin{bmatrix} \omega_1 \\ \omega_2 \end{bmatrix}=\begin{bmatrix} B_{4(\alpha)} \\ B_{5(\alpha)} \end{bmatrix}. \tag{7}$$

We determine the optimum values of $\left(\omega_1,\omega_2\right)$ by simplifying (7) i.e.

$$\left.\begin{aligned} \omega_1 &= \frac{\left(B_{2(\alpha)}B_{4(\alpha)}-B_{3(\alpha)}B_{5(\alpha)}\right)}{\left(B_{1(\alpha)}B_{2(\alpha)}-B_{3(\alpha)}^2\right)}=\omega_{1(0)} \\ \omega_2 &= \frac{\left(B_{1(\alpha)}B_{5(\alpha)}-B_{3(\alpha)}B_{4(\alpha)}\right)}{\left(B_{1(\alpha)}B_{2(\alpha)}-B_{3(\alpha)}^2\right)}=\omega_{2(0)} \end{aligned}\right\}, \tag{8}$$

Minimum MSE of T_α^γ is given by

$$MSE_{\min}\left(T_\alpha^\gamma\right)=\bar{Y}^2\left[1-\frac{\left(B_{2(\alpha)}B_{4(\alpha)}^2-2B_{3(\alpha)}B_{4(\alpha)}B_{5(\alpha)}+B_{1(\alpha)}B_{5(\alpha)}^2\right)}{\left(B_{1(\alpha)}B_{2(\alpha)}-B_{3(\alpha)}^2\right)}\right] \tag{9}$$

Now we state the following theorem.

Theorem 1 *Up to first degree of approximation,*

$$MSE\left(T_\alpha^\gamma\right)\geq MSE_{\min}\left(T_\alpha^\gamma\right)=\bar{Y}^2\left[1-\frac{\left(B_{2(\alpha)}B_{4(\alpha)}^2-2B_{3(\alpha)}B_{4(\alpha)}B_{5(\alpha)}+B_{1(\alpha)}B_{5(\alpha)}^2\right)}{\left(B_{1(\alpha)}B_{2(\alpha)}-B_{3(\alpha)}^2\right)}\right]$$

with equality holding if $\omega_1=\omega_{1(0)}$, $\omega_2=\omega_{2(0)}$ *where* $\omega_i=\omega_{i(0)}$'s, $i=1,2...$ *are given by* (7).

2.3. Efficiency comparison

For the purpose of comparisons of MSE of different estimators, we proceed as follows:

(i) $MSE_{\min}\left(T_\alpha^\gamma\right)<MSE\left(\bar{y}\right)$ if $\left(1-\lambda C_y^2\right)<B_{(\alpha)}$, (10)

(ii) $MSE_{\min}\left(T_\alpha^\gamma\right)<MSE\left(t_R\right)$ if $\left(1-\lambda C_y^2\right)<\left[B_{(\alpha)}+\lambda C_p^2\left(2k_p-1\right)\right]$, (11)

(iii) $MSE_{\min}\left(T_\alpha^\gamma\right)<MSE\left(t_p\right)$ if $\left(1-\lambda C_y^2\right)<\left[B_{(\alpha)}-\lambda C_p^2\left(2k_p+1\right)\right]$, (12)

(iv) $MSE_{\min}\left(T_\alpha^\gamma\right)<MSE\left(t_{lr}\right)=MSE_{\min}\left(t_g\right)=MSE_{\min}\left(t_{RPe}\right)=MSE_{\min}\left(t_{sse}\right)=MSE_{\min}\left(t_{A(i)}\right)$,
$i=12,13,14,15$, if $\left(1-\lambda C_y^2\right)<\left[B_{(\alpha)}+\lambda C_p^2 k_p^2\right]$, (13)

(v) $MSE_{\min}\left(T_\alpha^\gamma\right)<MSE_{\min}\left(t_{sg}\right)_C$ if $A_1<B_{(\alpha)}$, (14)

(vi) $MSE_{\min}\left(T_{\alpha}^{\gamma}\right) < MSE_{\min}\left(t_{GK1}\right)$ if $\left[1+\lambda\left(C_y^2 - C_p^2 k_p^2\right)\right] < B_{(\alpha)}$, (15)

(vii) $MSE_{\min}\left(T_{\alpha}^{\gamma}\right) < MSE_{\min}\left(t_{GK2}\right)$ if $A_1^* < B_{(\alpha)}$, (16)

(viii) $MSE_{\min}\left(T_{\alpha}^{\gamma}\right) < MSE\left(t_{si}\right)$ if $\left(1-B_{(\alpha)}\right) < \lambda\left[C_y^2 + C_p^2\left(R_i - k_p^2\right)\right]$, $i = 1,...,10$, (17)

(ix) $MSE_{\min}\left(T_{\alpha}^{\gamma}\right) < MSE\left(t_{Ai}\right)$ if $\left(1-B_{(\alpha)}\right) < \lambda\left[C_y^2 + C_p^2\left(R_i^2 - 2R_i k_p\right)\right]$,
$i = 2,3,4,5,6,$ (18)

(x) $MSE_{\min}\left(T_{\alpha}^{\gamma}\right) < MSE\left(t_{\text{Re}}\right)$ if $\left(1-B_{(\alpha)}\right) < \lambda\left[C_y^2 + \frac{C_p^2}{4}\left(1-4k_p\right)\right]$, (19)

(xi) $MSE_{\min}\left(T_{\alpha}^{\gamma}\right) < MSE\left(t_{Pe}\right)$ if $\left(1-B_{(\alpha)}\right) < \lambda\left[C_y^2 + \frac{C_p^2}{4}\left(1+4k_p\right)\right]$, (20)

where $A_1 = \frac{\left(A_{2(1)}A_{4(1)}^2 - 2A_{3(1)}A_{4(1)}A_{5(1)} + A_{1(1)}A_{5(1)}^2\right)}{\left(A_{1(1)}A_{2(1)} - A_{3(1)}^2\right)}$, $B_{(\alpha)} = \frac{\left(B_{2(\alpha)}B_{4(\alpha)}^2 - 2B_{3(\alpha)}B_{4(\alpha)}B_{5(\alpha)} + B_{1(\alpha)}B_{5(\alpha)}^2\right)}{\left(B_{1(\alpha)}B_{2(\alpha)} - B_{3(\alpha)}^2\right)}$,

$$A_1^* = \frac{\left(A_{2(1)}^* {A_{4(1)}^*}^2 - 2A_{3(1)}^* A_{4(1)}^* A_{5(1)}^* + A_{1(1)}^* {A_{5(1)}^*}^2\right)}{\left(A_{1(1)}^* A_{2(1)}^* - {A_{3(1)}^*}^2\right)}.$$

Thus, the suggested class of estimators T_{α}^{γ} (at its optimum conditions) is highly efficient than the estimators $\bar{y}, t_R, t_P$, $\left(t_{lr}, t_g, t_{RPe}, t_{sse}, t_{A(i)}, \left(i = 12,13,14,15\right)\right)$, $t_{sg}, t_{GR}, t_{si}\left(i = 1 \text{ to } 10\right)$, t_{Ai}, t_{Re} and t_{Pe} as long as the conditions (10), (11), (12), (13), (14), (15), (16), (17), (18), and (19), are respectively satisfied.

3. Proposed Improved Version of Grover and Kaur (2011) Class of Estimators

In the situation, where population proportion P possessing an attribute ϕ is known, the purpose of searching improved version and generalization of Grover and Kaur (2011) class of estimators led authors to propose the following class of estimators for estimating the PM $\bar{Y}$ as

$$\dot{T}_{\alpha}^{\gamma} = \left[\omega_1 \bar{y} + \omega_2\left(P-p\right)\right]\left[d\left(\frac{\psi P + \delta\eta}{\psi p + \delta\eta}\right)^{\alpha} + \left(1-d\right)\exp\left\{\frac{\gamma\psi\left(P-p\right)}{\psi\left(P+p\right)+2\delta\eta}\right\}\right], \quad (21)$$

where $\left(\omega_1, \omega_2, d, \alpha, \psi, \delta, \eta, \gamma\right)$ are same as defined for the class of estimators T_{α}^{γ} defined in Equation (1). The class of estimators $\dot{T}_{\alpha}^{\gamma}$ includes a large number of estimators. Some known members of the class of estimators are given in Table 8 where as some unknown members of this are shown in Table 9.

Table 8 Some members of the class of $\dot{T}_{\alpha}^{\gamma}$ for $\left(\alpha, \gamma\right) = \left(1,1\right)$ and $\delta = +1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$\dot{T}_{1(1)}^{1}=\omega_d^*\left[d\left(\frac{P}{p}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)}\right\}\right]$ where $\omega_d^*=\left[\omega_1\bar{y}+\omega_2(P-p)\right]$
2	1	β_2	$\dot{T}_{1(2)}^{1}=\omega_d^*\left[d\left(\frac{P+\beta_2}{p+\beta_2}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$\dot{T}_{1(3)}^{1}=\omega_d^*\left[d\left(\frac{P+C_p}{p+C_p}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$\dot{T}_{1(4)}^{1}=\omega_d^*\left[d\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$\dot{T}_{1(5)}^{1}=\omega_d^*\left[d\left(\frac{\beta_2P+C_p}{\beta_2p+C_p}\right)+(1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$\dot{T}_{1(6)}^{1}=\omega_d^*\left[d\left(\frac{C_pP+\beta_2}{C_pp+\beta_2}\right)+(1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$\dot{T}_{1(7)}^{1}=\omega_d^*\left[d\left(\frac{C_pP+\rho_{pb}}{C_pp+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$\dot{T}_{1(8)}^{1}=\omega_d^*\left[d\left(\frac{\rho_{pb}P+C_p}{\rho_{pb}p+C_p}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$\dot{T}_{1(9)}^{1}=\omega_d^*\left[d\left(\frac{\beta_2P+\rho_{pb}}{\beta_2p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$\dot{T}_{1(10)}^{1}=\omega_d^*\left[d\left(\frac{\rho_{pb}P+\beta_2}{\rho_{pb}p+\beta_2}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Further some more members of the suggested class of estimators $\dot{T}_{\alpha}^{\gamma}$ are given in Tables 9 to 13.

Table 9 Some members of the class of $\dot{T}_{\alpha}^{\gamma}$ for $(\alpha,\gamma)=(-1,-1)$ and $\delta=+1$

S. No.	Values of Constants ψ	η	Estimator
1	1	0	$\dot{T}_{-1(1)}^{-1} = \omega_d^* \left[d\left(\frac{p}{P}\right) + (1-d)\exp\left\{\frac{(p-P)}{(P+p)}\right\}\right]$
2	1	β_2	$\dot{T}_{-1(2)}^{-1} = \omega_d^* \left[d\left(\frac{p+\beta_2}{P+\beta_2}\right) + (1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$\dot{T}_{-1(3)}^{-1} = \omega_d^* \left[d\left(\frac{p+C_p}{P+C_p}\right) + (1-d)\exp\left\{\frac{(p-P)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$\dot{T}_{-1(4)}^{-1} = \omega_d^* \left[d\left(\frac{p+\rho_{pb}}{P+\rho_{pb}}\right) + (1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$\dot{T}_{-1(5)}^{-1} = \omega_d^* \left[d\left(\frac{\beta_2 p+C_p}{\beta_2 P+C_p}\right) + (1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$\dot{T}_{-1(6)}^{-1} = \omega_d^* \left[d\left(\frac{C_p p+\beta_2}{C_p P+\beta_2}\right) + (1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$\dot{T}_{-1(7)}^{-1} = \omega_d^* \left[d\left(\frac{C_p p+\rho_{pb}}{C_p P+\rho_{pb}}\right) + (1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$\dot{T}_{-1(8)}^{-1} = \omega_d^* \left[d\left(\frac{\rho_{pb} p+C_p}{\rho_{pb} P+C_p}\right) + (1-d)\exp\left\{\frac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$\dot{T}_{-1(9)}^{-1} = \omega_d^* \left[d\left(\frac{\beta_2 p+\rho_{pb}}{\beta_2 P+\rho_{pb}}\right) + (1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$\dot{T}_{-1(10)}^{-1} = \omega_d^* \left[d\left(\frac{\rho_{pb} p+\beta_2}{\rho_{pb} P+\beta_2}\right) + (1-d)\exp\left\{\frac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 10 Some members of the of estimators $\dot{T}_{\alpha}^{\gamma}$ for $(\alpha,\gamma)=(1,-1)$ and $\delta=+1$

S. No.	Values of Constants ψ	η	Estimator
1	1	0	$\dot{T}_{-1(1)}^{-1} = \omega_d^* \left[d\left(\frac{P}{p}\right) + (1-d)\exp\left\{\frac{(p-P)}{(P+p)}\right\}\right]$
2	1	β_2	$\dot{T}_{-1(2)}^{-1} = \omega_d^* \left[d\left(\frac{P+\beta_2}{p+\beta_2}\right) + (1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\beta_2}\right\}\right]$

Table 10 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
3	1	C_p	$\dot{T}_{-1(3)}^{-1}=\omega_d^*\left[d\left(\frac{P+C_p}{p+C_p}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$\dot{T}_{-1(4)}^{-1}=\omega_d^*\left[d\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{(p-P)}{(P+p)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$\dot{T}_{-1(5)}^{-1}=\omega_d^*\left[d\left(\frac{\beta_2 P+C_p}{\beta_2 p+C_p}\right)+(1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2C_p}\right\}\right]$
6	C_p	β_2	$\dot{T}_{-1(6)}^{-1}=\omega_d^*\left[d\left(\frac{C_p P+\beta_2}{C_p p+\beta_2}\right)+(1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$\dot{T}_{-1(7)}^{-1}=\omega_d^*\left[d\left(\frac{C_p P+\rho_{pb}}{C_p p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{C_p(p-P)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$\dot{T}_{-1(8)}^{-1}=\omega_d^*\left[d\left(\frac{\rho_{pb} P+C_p}{\rho_{pb} p+C_p}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$\dot{T}_{-1(9)}^{-1}=\omega_d^*\left[d\left(\frac{\beta_2 P+\rho_{pb}}{\beta_2 p+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{\beta_2(p-P)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$\dot{T}_{-1(10)}^{-1}=\omega_d^*\left[d\left(\frac{\rho_{pb} P+\beta_2}{\rho_{pb} p+\beta_2}\right)+(1-d)\exp\left\{\frac{\rho_{pb}(p-P)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 11 Some members of the class of $\dot{T}_{\alpha}^{\gamma}$ for $(\alpha,\gamma)=(-1,1)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$\dot{T}_{1(1)}^{-1}=\omega_d^*\left[d\left(\frac{p}{P}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)}\right\}\right]$
2	1	β_2	$\dot{T}_{1(2)}^{-1}=\omega_d^*\left[d\left(\frac{p+\beta_2}{P+\beta_2}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)+2\beta_2}\right\}\right]$
3	1	C_p	$\dot{T}_{1(3)}^{-1}=\omega_d^*\left[d\left(\frac{p+C_p}{P+C_p}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$\dot{T}_{1(4)}^{-1}=\omega_d^*\left[d\left(\frac{p+\rho_{pb}}{P+\rho_{pb}}\right)+(1-d)\exp\left\{\frac{(P-p)}{(P+p)+2\rho_{pb}}\right\}\right]$

Table 11 (Continued)

S. No.	Values of Constants		Estimator

	ψ	η	
5	β_2	C_p	$\dot{T}_{1(5)}^{-1}=\omega_d^*\left[d\left(\frac{\beta_2 p+C_p}{\beta_2 P+C_p}\right)+\left(1-d\right)\exp\left\{\frac{\beta_2\left(P-p\right)}{\beta_2\left(P+p\right)+2C_p}\right\}\right]$
6	C_p	β_2	$\dot{T}_{1(6)}^{-1}=\omega_d^*\left[d\left(\frac{C_p p+\beta_2}{C_p P+\beta_2}\right)+\left(1-d\right)\exp\left\{\frac{C_p\left(P-p\right)}{C_p\left(P+p\right)+2\beta_2}\right\}\right]$
7	C_p	ρ_{pb}	$\dot{T}_{1(7)}^{-1}=\omega_d^*\left[d\left(\frac{C_p p+\rho_{pb}}{C_p P+\rho_{pb}}\right)+\left(1-d\right)\exp\left\{\frac{C_p\left(P-p\right)}{C_p\left(P+p\right)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$\dot{T}_{1(8)}^{-1}=\omega_d^*\left[d\left(\frac{\rho_{pb} p+C_p}{\rho_{pb} P+C_p}\right)+\left(1-d\right)\exp\left\{\frac{\rho_{pb}\left(P-p\right)}{\rho_{pb}\left(P+p\right)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$\dot{T}_{1(9)}^{-1}=\omega_d^*\left[d\left(\frac{\beta_2 p+\rho_{pb}}{\beta_2 P+\rho_{pb}}\right)+\left(1-d\right)\exp\left\{\frac{\beta_2\left(P-p\right)}{\beta_2\left(P+p\right)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$\dot{T}_{1(10)}^{-1}=\omega_d^*\left[d\left(\frac{\rho_{pb} p+\beta_2}{\rho_{pb} P+\beta_2}\right)+\left(1-d\right)\exp\left\{\frac{\rho_{pb}\left(P-p\right)}{\rho_{pb}\left(P+p\right)+2\beta_2}\right\}\right]$

Table 12 Some members of the class of $\dot{T}_\alpha^\gamma$ for $\left(\alpha,\gamma\right)=\left(0,1\right)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$\dot{T}_{1(1)}^{0}=\omega_d^*\left[d+\left(1-d\right)\exp\left\{\frac{\left(P-p\right)}{\left(P+p\right)}\right\}\right]$
2	1	β_2	$\dot{T}_{1(2)}^{0}=\omega_d^*\left[d+\left(1-d\right)\exp\left\{\frac{\left(P-p\right)}{\left(P+p\right)+2\beta_2}\right\}\right]$
3	1	C_p	$\dot{T}_{1(3)}^{0}=\omega_d^*\left[d+\left(1-d\right)\exp\left\{\frac{\left(P-p\right)}{\left(P+p\right)+2C_p}\right\}\right]$
4	1	ρ_{pb}	$\dot{T}_{1(4)}^{0}=\omega_d^*\left[d+\left(1-d\right)\exp\left\{\frac{\left(P-p\right)}{\left(P+p\right)+2\rho_{pb}}\right\}\right]$
5	β_2	C_p	$\dot{T}_{1(5)}^{0}=\omega_d^*\left[d+\left(1-d\right)\exp\left\{\frac{\beta_2\left(P-p\right)}{\beta_2\left(P+p\right)+2C_p}\right\}\right]$
6	C_p	β_2	$\dot{T}_{1(6)}^{0}=\omega_d^*\left[d+\left(1-d\right)\exp\left\{\frac{C_p\left(P-p\right)}{C_p\left(P+p\right)+2\beta_2}\right\}\right]$

Table 12 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
7	C_p	ρ_{pb}	$\dot{T}_{1(7)}^{0}=\omega_d^*\left[d+(1-d)\exp\left\{\frac{C_p(P-p)}{C_p(P+p)+2\rho_{pb}}\right\}\right]$
8	ρ_{pb}	C_p	$\dot{T}_{1(8)}^{0}=\omega_d^*\left[d+(1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2C_p}\right\}\right]$
9	β_2	ρ_{pb}	$\dot{T}_{1(9)}^{0}=\omega_d^*\left[d+(1-d)\exp\left\{\frac{\beta_2(P-p)}{\beta_2(P+p)+2\rho_{pb}}\right\}\right]$
10	ρ_{pb}	β_2	$\dot{T}_{1(10)}^{0}=\omega_d^*\left[d+(1-d)\exp\left\{\frac{\rho_{pb}(P-p)}{\rho_{pb}(P+p)+2\beta_2}\right\}\right]$

Table 13 Some members of the class of $\dot{T}_{\alpha}^{\gamma}$ for $(\alpha,\gamma)=(1,0)$ and $\delta=+1$

S. No.	Values of Constants		Estimator
	ψ	η	
1	1	0	$\dot{T}_{0(1)}^{1}=\omega_d^*\left[d\left(\frac{P}{p}\right)+(1-d)\right]$
2	1	β_2	$\dot{T}_{0(2)}^{1}=\omega_d^*\left[d\left(\frac{P+\beta_2}{p+\beta_2}\right)+(1-d)\right]$
3	1	C_p	$\dot{T}_{0(3)}^{1}=\omega_d^*\left[d\left(\frac{P+C_p}{p+C_p}\right)+(1-d)\right]$
4	1	ρ_{pb}	$\dot{T}_{0(4)}^{1}=\omega_d^*\left[d\left(\frac{P+\rho_{pb}}{p+\rho_{pb}}\right)+(1-d)\right]$
5	β_2	C_p	$\dot{T}_{0(5)}^{1}=\omega_d^*\left[d\left(\frac{\beta_2 P+C_p}{\beta_2 p+C_p}\right)+(1-d)\right]$
6	C_p	β_2	$\dot{T}_{0(6)}^{1}=\omega_d^*\left[d\left(\frac{C_p P+\beta_2}{C_p p+\beta_2}\right)+(1-d)\right]$
7	C_p	ρ_{pb}	$\dot{T}_{0(7)}^{1}=\omega_d^*\left[d\left(\frac{C_p P+\rho_{pb}}{C_p p+\rho_{pb}}\right)+(1-d)\right]$
8	ρ_{pb}	C_p	$\dot{T}_{0(8)}^{1}=\omega_d^*\left[d\left(\frac{\rho_{pb} P+C_p}{\rho_{pb} p+C_p}\right)+(1-d)\right]$

Table 13 (Continued)

S. No.	Values of Constants	Estimator

	ψ	η	
9	β_2	ρ_{pb}	$\dot{T}^1_{0(9)} = \omega^*_d \left[d\left(\frac{\beta_2 P + \rho_{pb}}{\beta_2 p + \rho_{pb}}\right) + (1-d) \right]$
10	ρ_{pb}	β_2	$\dot{T}^1_{0(10)} = \omega^*_d \left[d\left(\frac{\rho_{pb} P + \beta_2}{\rho_{pb} p + \beta_2}\right) + (1-d) \right]$

3.1. Bias and MSE of the class of estimators $\dot{T}^\gamma_\alpha$

Expressing $\dot{T}^\gamma_\alpha$ at (21) in terms of e_0 and e_1 as

$$\dot{T}^\gamma_\alpha = \bar{Y}\left[\omega_1(1+e_0) - \frac{\omega_2}{R}e_1\right]\left[d(1+\upsilon e_1)^{-\alpha} + (1-d)\exp\left\{-\frac{\gamma\upsilon e_1}{2}\left(1+\frac{\upsilon}{2}e_1\right)^{-1}\right\}\right]$$

$$= \bar{Y}\left[\omega_1(1+e_0) - \frac{\omega_2}{R}e_1\right]\left[1 - \theta_0\upsilon e_1 + \theta_1\frac{\upsilon^2 e_1^2}{2} - \dots\right]$$

$$= \bar{Y}\left[\omega_1\left\{1 + e_0 - \theta_0\upsilon e_1 - \theta_0\upsilon e_0 e_1 + \theta_1\frac{\upsilon^2 e_1^2}{2} - \dots\right\} - \frac{\omega_2}{R}\left\{e_1 - \theta_0\upsilon e_1^2 + \dots\right\}\right]$$

where $\theta_0 = \left(\alpha d + \frac{\gamma(1-d)}{2}\right)$, $\theta_1 = \left\{d\alpha(\alpha+1) + \frac{(1-d)\gamma(\gamma+2)}{4}\right\}$, $R = \frac{\bar{Y}}{\bar{P}}$.

Skipping terms of e's (having power more than two) we have

$$\dot{T}^\gamma_\alpha \cong \bar{Y}\left[\omega_1\left\{1 + e_0 - \theta_0\upsilon(e_1 + e_0 e_1) + \theta_1\frac{\upsilon^2 e_1^2}{2}\right\} + \frac{\omega_2}{R}\left(\theta_0\upsilon e_1^2 - e_1\right)\right]$$

or

$$\left(\dot{T}^\gamma_\alpha - \bar{Y}\right) = \bar{Y}\left[\omega_1\left\{1 + e_0 - \theta_0\upsilon(e_1 + e_0 e_1) + \theta_1\frac{\upsilon^2 e_1^2}{2}\right\} + \frac{\omega_2}{R}\left(\theta_0\upsilon e_1^2 - e_1\right) - 1\right]. \quad (22)$$

The bias of $\dot{T}^\gamma_\alpha$ to the fda as

$$B\left(\dot{T}^\gamma_\alpha\right) = \bar{Y}\left[\omega_1\left\{1 + \frac{\lambda\upsilon}{2}C_p^2\left(\theta_1\upsilon - 2\theta_0 k_p\right)\right\} + \frac{\omega_2\lambda\theta_0\upsilon C_p^2}{R} - 1\right] = \bar{Y}\left[\omega_1 C_{4(\alpha)} + \omega_2 C_{5(\alpha)} - 1\right], \quad (23)$$

where $C_{4(\alpha)} = \left[1 + \frac{\lambda\upsilon}{2}C_p^2\left(\theta_1\upsilon - 2\theta_0 k_p\right)\right]$ and $C_{5(\alpha)} = \frac{\lambda\theta_0\upsilon C_p^2}{R}$.

Square of (22) and skipping terms of e's having power more than two we have

$$\left(\dot{T}^\gamma_\alpha - \bar{Y}\right)^2 = \bar{Y}^2\left[1 + \omega_1^2\left\{1 + 2e_0 - 2\theta_0\upsilon e_1 + e_0^2 - 4\theta_0\upsilon e_0 e_1 + \left(\theta_0^2 + \theta_1\right)\upsilon^2 e_1^2\right\}\right.$$
$$+ \frac{\omega_2^2}{R^2}e_1^2 + \frac{2\omega_1\omega_2}{R}\left\{2\theta_0\upsilon e_1^2 - 2e_0 e_1 - e_1\right\} - 2\omega_1\left\{1 + e_0 - \theta_0\upsilon(e_1 + e_0 e_1) + \frac{\theta_1\upsilon^2 e_1^2}{2}\right\}$$
$$\left. - \frac{2\omega_2}{R}\left\{\theta_0\upsilon e_1^2 - e_1\right\}\right], \quad (24)$$

The MSE of the estimator $\dot{T}^\gamma_\alpha$ to the fda as

$$MSE\left(\dot{T}_{\alpha}^{\gamma}\right)=\bar{Y}^{2}\left[1+\omega_{1}^{2}C_{1(\alpha)}+\omega_{2}^{2}C_{2(\alpha)}+2\omega_{1}\omega_{2}C_{3(\alpha)}-2\omega_{1}C_{4(\alpha)}-2\omega_{2}C_{5(\alpha)}\right], \tag{25}$$

where $C_{1(\alpha)}=\left[1+\lambda\left\{C_{y}^{2}+\upsilon C_{p}^{2}\left(\left(\theta_{0}^{2}+\theta_{1}\right)\upsilon-4\theta_{0}k_{p}\right)\right\}\right]$, $C_{2(\alpha)}=\dfrac{\lambda}{R^{2}}C_{p}^{2}$,

$C_{3(\alpha)}=\left[\dfrac{\lambda C_{p}^{2}}{R}\left(2\theta_{0}\upsilon-k_{p}\right)\right]$, $C_{4(\alpha)}=\left[1+\dfrac{\lambda\upsilon C_{p}^{2}}{2}\left(\theta_{1}\upsilon-2\theta_{0}k_{p}\right)\right]$, $C_{5(\alpha)}=\dfrac{\lambda\theta_{0}\upsilon C_{p}^{2}}{R}$.

The MSE of $\dot{T}_{\alpha}^{\gamma}$ is minimized for

$$\begin{bmatrix} C_{1(\alpha)} & C_{3(\alpha)} \\ C_{3(\alpha)} & C_{2(\alpha)} \end{bmatrix}\begin{bmatrix} \omega_{1} \\ \omega_{2} \end{bmatrix}=\begin{bmatrix} C_{4(\alpha)} \\ C_{5(\alpha)} \end{bmatrix}, \tag{26}$$

We determine the optimum values of $\left(\omega_{1},\omega_{2}\right)$ by solving the Equation (26), i.e.

$$\left.\begin{aligned} \omega_{1}&=\frac{\left(C_{2(\alpha)}C_{4(\alpha)}-C_{3(\alpha)}C_{5(\alpha)}\right)}{\left(C_{1(\alpha)}C_{2(\alpha)}-C_{3(\alpha)}^{2}\right)}=\omega_{1(1)} \\ \omega_{2}&=\frac{\left(C_{1(\alpha)}C_{5(\alpha)}-C_{3(\alpha)}C_{4(\alpha)}\right)}{\left(C_{1(\alpha)}C_{2(\alpha)}-C_{3(\alpha)}^{2}\right)}=\omega_{2(1)} \end{aligned}\right\}. \tag{27}$$

Putting (27) in (25) give the minimum MSE of (25) as

$$MSE_{\min}\left(\dot{T}_{\alpha}^{\gamma}\right)=\bar{Y}^{2}\left[1-\frac{\left(C_{2(\alpha)}C_{4(\alpha)}^{2}-2C_{3(\alpha)}C_{4(\alpha)}C_{5(\alpha)}+C_{1(\alpha)}C_{5(\alpha)}^{2}\right)}{\left(C_{1(\alpha)}C_{2(\alpha)}-C_{3(\alpha)}^{2}\right)}\right], \tag{28}$$

Thus, we state the following theorem.

Theorem 2 *Up to the first degree of approximation,*

$$MSE\left(\dot{T}_{\alpha}^{\gamma}\right)\geq MSE_{\min}\left(\dot{T}_{\alpha}^{\gamma}\right)=\bar{Y}^{2}\left[1-\frac{\left(C_{2(\alpha)}C_{4(\alpha)}^{2}-2C_{3(\alpha)}C_{4(\alpha)}C_{5(\alpha)}+C_{1(\alpha)}C_{5(\alpha)}^{2}\right)}{\left(C_{1(\alpha)}C_{2(\alpha)}-C_{3(\alpha)}^{2}\right)}\right]$$

with equality holding if $\omega_{1}=\omega_{1(1)}$, $\omega_{2}=\omega_{2(1)}$ *where* $\omega_{1(1)}=\omega_{2(1)}$ *are defined in* (27).

3.2. Efficiency comparison

In this part, we determined the conditions under which the proposed class of estimators outperforms other current estimators.

(i) $MSE_{\min}\left(\dot{T}_{\alpha}^{\gamma}\right)<MSE\left(\bar{y}\right)$ if $\left(1-\lambda C_{y}^{2}\right)<C_{(\alpha)}$, (29)

(ii) $MSE_{\min}\left(\dot{T}_{\alpha}^{\gamma}\right)<MSE\left(t_{R}\right)$ if $\left(1-\lambda C_{y}^{2}\right)<\left[C_{(\alpha)}+\lambda C_{p}^{2}\left(1-2k_{p}\right)\right]$, (30)

(iii) $MSE_{min}\left(\dot{T}_{\alpha}^{\gamma}\right)<MSE\left(t_{p}\right)$ if $\left(1-\lambda C_{y}^{2}\right)<\left[C_{(\alpha)}+\lambda C_{p}^{2}\left(1+2k_{p}\right)\right]$, (31)

(iv) $MSE_{\min}\left(\dot{T}_{\alpha}^{\gamma}\right)<MSE\left(t_{lr}\right)=MSE_{\min}\left(t_{g}\right)=MSE_{\min}\left(t_{RPe}\right)=MSE_{\min}\left(t_{sse}\right)=MSE_{\min}\left(t_{A(i)}\right)$,

$\left(i=12,13,14,15\right)$ if $\left(1-\lambda C_{y}^{2}\right)<\left[C_{(\alpha)}-\lambda C_{p}^{2}k_{p}^{2}\right]$, (32)

(v) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE_{\min}\left(t_{sg}\right)_C$ if $A_1 < C_{(\alpha)}$, (33)

(vi) $MSE_{min}\left(\dot{T}_\alpha^\gamma\right) < MSE_{min}\left(t_{GK1}\right)$ if $\left[1+\lambda\left(C_y^2 - C_p^2 k_p^2\right)\right] < C_{(\alpha)}$, (34)

(vii) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE_{\min}\left(t_{GK2}\right)$ if $A_1^* < C_{(\alpha)}$, (35)

(viii) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE\left(t_{si}\right)$ if $\left(1-\lambda C_y^2\right) < \left[C_{(\alpha)} + \lambda C_p^2\left(R_I^2 - k_p^2\right)\right]$, $i = 1,\ldots,10$, (36)

(ix) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE\left(t_{Ai}\right)$ if $\left(1-\lambda C_y^2\right) < \left[\lambda C_p^2\left(R_i^2 - 2k_p\right) + C_{(\alpha)}\right]$, $i = 2,\ldots,6$, (37)

(x) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE\left(t_{\mathrm{Re}}\right)$ if $\left(1-\lambda C_y^2\right) < \left[\left\{C_{(\alpha)} + \lambda C_p^2\left(\frac{1}{4} - k_p\right)\right\} + C_{(\alpha)}\right]$, (38)

(xi) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE_{\min}\left(t_{Pe}\right)$ if $\left(1-\lambda C_y^2\right) < \left[\lambda C_p^2\left(\frac{1}{4} + k_p\right) + C_{(\alpha)}\right]$, (39)

(xii) $MSE_{\min}\left(\dot{T}_\alpha^\gamma\right) < MSE_{\min}\left(t_{ss}\right)$ if $A_{(\alpha)} < C_{(\alpha)}$, (40)

(xiii) $MSE_{min}\left(\dot{T}_\alpha^\gamma\right) < MSE_{min}\left(T_\alpha^\gamma\right)$ if $B_{(\alpha)} < C_{(\alpha)}$, (41)

where $C_{(\alpha)} = \dfrac{\left(C_{2(\alpha)}C_{4(\alpha)}^2 - 2C_{3(\alpha)}C_{4(\alpha)}C_{5(\alpha)} + C_{1(\alpha)}C_{5(\alpha)}^2\right)}{\left(C_{1(\alpha)}C_{2(\alpha)} - C_{3(\alpha)}^2\right)}$.

Thus the suggested class of estimators $\dot{T}_\alpha^\gamma$ is more efficient than the estimators $\bar{y}, t_R, t_P$, $\left(t_{lr}, t_g, t_{RPe}, t_{sse}, t_{A(i)}\left(i = 12,13,14,15\right)\right)$, $t_{sg}, t_{GK1}, t_{si}\left(i = 1 \text{ to } 10\right)$, $t_{Ai}, t_{Re}, t_{Pe}, t_{ss}, t_{GK2}$ and T_α^γ as long as the conditions (39) to (41) holds, respectively.

Remark We attempted to estimate the population mean $\bar{Y}$ of the SV y in Sections 2 and 3 assuming that information regarding the proportion P of population units carrying the auxiliary attribute ϕ, , which is significantly linked with $SV\ y$, is known ahead of time. In many real-world situations, however, the value of population proportion P is unknown before the survey begins. In such circumstances we generally apply two- phase (or double) sampling procedure to obtain an efficient estimator of the PM $\bar{Y}$. Under a two-phase sampling design, let p' denote the proportion of units with auxiliary attribute ϕ in the first phase (preliminary large) sample of size n', p denote the proportion of units with auxiliary attribute ϕ in the second phase sample of size $n\left(< n'\right)$, and $\bar{y}$ denote the sample mean of the SV y in the second phase sample of size n. SRSWOR sampling design is used to choose samples in both phases.

Thus double sampling versions of the proposed estimators T_α^γ and $\dot{T}_\alpha^\gamma$ are respectively defined by

$$\hat{T}_\alpha^\gamma = \bar{y}\left[\omega_1^* + \omega_2^*\left(p' - p\right)\right]\left[d^*\left(\frac{\psi p' + \delta\eta}{\psi p + \delta\eta}\right)^\alpha + \left(1-d^*\right)\exp\left\{\frac{\gamma\psi\left(p' - p\right)}{\psi\left(p' + p\right) + 2\delta\eta}\right\}\right], \quad (42)$$

$$\hat{\dot{T}}_{\alpha}^{\gamma}=\left[\omega_1^{\circ}\bar{y}+\omega_2^{\circ}\left(p'-p\right)\right]\left[d^{\circ}\left(\frac{\psi p'+\delta\eta}{\psi p+\delta\eta}\right)^{\alpha}+\left(1-d^{\circ}\right)\exp\left\{\frac{\gamma\psi\left(p'-p\right)}{\psi\left(p'+p\right)+2\delta\eta}\right\}\right], \qquad (43)$$

where $(\alpha,\psi,\eta,\delta,\gamma)$ are same as defined earlier; (d^{*},d°) are suitable chosen real numbers; (w_1^{*},w_2^{*}) and $(w_1^{\circ},w_2^{\circ})$ are constants to be determined such that the MSEs of $\hat{T}_{\alpha}^{\gamma}$ and $\hat{\dot{T}}_{\alpha}^{\gamma}$, respectively are minimum. The properties of the estimators $\hat{T}_{\alpha}^{\gamma}$ and $\hat{\dot{T}}_{\alpha}^{\gamma}$ can be studied. Recommendations in the favour of the proposed classes of estimators $\hat{T}_{\alpha}^{\gamma}$ and $\hat{\dot{T}}_{\alpha}^{\gamma}$ can be made on the basis of numerical illustrations.

4. Numerical Illustrations

To compare efficiency of proposed classes of estimators with other existing estimators, we have considered two natural populations taken from Zaman and Kadilar (2019) for the empirical justification. The parameters of the populations are given in Table 14.

Table 14 Parameters of the two natural populations

Populations	N	n	$\bar{Y}$	P	C_y	C_P	ρ_{pb}	β_2
Population I	89	20	3.360	0.124	0.601	2.678	0.766	3.492
Population II	111	30	29.279	0.117	0.872	2.758	0.797	3.898

Table 15 PREs of the different estimators with respect to $\bar{y}$

Estimators	Population I	Population II
PRE t_R	7.12	16.77
PRE t_P	3.61	6.23
PRE t_{lr} = PRE t_g =PRE t_{RPe} = PRE$_{\min}$ t_{Ai} (i=2,…,6) =PRE t_2 =PRE t_{sse} =PRE t_{2e} =PRE $t_{yz(2)}$	241.99	274.13
PRE$\left(t_{sg}\right)_C$	401.74	372.41
PRE t_{s1}	4.93	9.64
PRE t_{s2}	229.12	267.89
PRE t_{s3}	221.28	262.22
PRE t_{s4}	125.51	189.14
PRE t_{s5}	125.63	176.61
PRE t_{s6}	177.88	236.26

Table 15 (Cont.)

Estimators	Population 1	Population 2
PRE t_{s7}	45.01	83.64

PRE t_{s8}	229.14	266.32
PRE t_{s9}	33.40	59.17
PRE t_{s10}	234.14	270.09
PRE t_{Re}	39.18	102.03
PRE t_{A2}	126.62	116.07
PRE t_{A3}	135.61	123.24
PRE t_{A4}	230.01	192.84
PRE t_{A5}	229.91	205.75
PRE t_{A6}	179.30	148.58
PRE t_{Pe}	10.66	16.61
PRE t_{GK1}	243.39	275.98
PRE t_{GK2}	4.74	8.80
PRE t_{ss}	401.74	372.41

Table 16 PREs of the estimator T_α^γ and $\dot{T}_\alpha^\gamma$ with respect to t_{2e} for $(\psi,\eta)=(1,0)$ and $\delta=+1$

α	γ	d	PRE $\left(T_\alpha^\gamma,\bar{y}\right)$		d	PRE $\left(\dot{T}_\alpha^\gamma,\bar{y}\right)$	
			Population I	Population II		Population I	Population II
1	1	–1.00	*	492.17	–1.00	1273.21	367.00
		–0.99	803750.00	470.90	–0.99	1023.20	359.71
		–0.90	533.40	357.41	–0.90	421.11	314.29
		–0.75	279.03	291.22	–0.75	272.34	282.03
		–0.50	245.79	275.93	–0.50	243.41	276.00
		–0.25	312.82	305.60	–0.25	277.98	293.50
		0.00	491.28	360.23	0.00	337.25	315.60
		0.25	849.41	421.21	0.25	376.36	326.85
		0.50	1071.26	454.29	0.50	350.28	318.53
		0.75	710.29	432.52	0.75	286.84	296.20
		0.90	499.45	398.51	0.90	255.73	282.72
		1.00	401.74	372.41	1.00	243.93	276.40

Table 16 (Continued)

α	γ	d	PRE $\left(T_\alpha^\gamma,\bar{y}\right)$		d	PRE $\left(\dot{T}_\alpha^\gamma,\bar{y}\right)$	
			Population I	Population II		Population I	Population II
1		0.00	*	*	0.00	*	937.68

	–1	0.10	780.49	442.94	0.10	300.37	289.39
		0.11	509.38	393.48	0.11	276.85	283.16
		0.12	395.23	359.05	0.12	261.32	278.77
		0.22	252.52	274.94	0.22	290.36	298.89
		0.25	287.40	283.86	0.25	357.21	322.77
		0.26	305.97	288.89	0.26	390.98	332.63
		0.27	329.30	294.92	0.27	433.64	343.52
		0.30	444.94	319.38	0.30	662.21	383.04
		0.35	1335.66	385.45	0.35	7152.33	477.31
		0.99	457.38	389.78	0.99	248.88	279.27
		1.00	401.74	372.41	1.00	243.93	276.40
–1	1	–0.28	1962.92	324.69	–0.28	*	1207.42
		–0.27	800.09	310.91	–0.27	*	791.37
		–0.26	532.60	300.27	–0.26	*	606.80
		–0.25	415.78	292.08	–0.25	*	503.53
		–0.22	286.37	277.88	–0.22	1785.87	362.09
		–0.20	257.15	274.49	–0.20	514.57	321.07
		0.00	491.28	360.23	0.00	337.25	315.60
		0.10	1482.97	424.23	0.10	758.41	392.54
		0.20	4467.64	435.94	0.20	3225.53	462.32
		0.25	2464.32	415.04	0.25	5273.54	472.74
		0.30	1182.75	382.97	0.30	2901.19	460.33
		0.50	261.01	275.91	0.50	330.83	314.02
0	1	–1.00	279.29	275.12	–1.00	*	401.52
		–0.99	273.97	274.77	–0.99	*	390.13
		–0.90	247.14	274.89	–0.90	589.16	324.58
		–0.75	245.90	284.86	–0.75	282.43	283.40
		–0.50	302.22	315.86	–0.50	243.65	276.19
		–0.25	405.64	347.58	–0.25	281.07	294.50
		0.00	491.28	360.23	0.00	337.25	315.60
		0.10	494.02	357.32	0.10	354.11	321.13
		0.50	352.46	313.24	0.50	330.83	314.02
		0.75	270.59	283.75	0.75	276.61	293.03
		0.90	245.94	274.72	0.90	252.23	281.28
		1.00	242.24	275.18	1.00	243.39	275.98

Table 16 (Continued)

α	γ	d	PRE $\left(T_\alpha^\gamma, \bar{y}\right)$		d	PRE $\left(T_\alpha^\gamma, \bar{y}\right)$	
			Population I	Population II		Population I	Population II
1	0	0.00	242.24	275.18	0.00	243.39	275.98

0.10	276.13	285.14	0.10	286.65	297.37
0.20	417.04	326.68	0.20	405.44	336.36
0.25	588.57	358.57	0.25	515.62	360.26
0.27	704.36	373.49	0.27	575.39	370.25
0.29	873.77	389.67	0.29	647.31	380.33
0.30	990.98	398.23	0.30	688.72	385.36
0.90	1011.05	469.81	0.90	301.55	301.63
0.95	578.07	417.16	0.95	262.97	286.07
0.99	427.62	380.69	0.99	246.18	277.78
1.00	401.74	372.41	1.00	243.93	276.40

Table 15 shows that the estimators $\left(t_{sg}\right)_C$ and t_{ss} due to Shabbir and Gupta (2007) and Singh and Solanki (2012) respectively at par with largest PRE $(401.74\%$ for Population I and 372.41% for Population II$)$ followed by the estimator t_{GK1} due to Grover and Kaur (2011) having second largest PRE (243.39% for Population I and 275.98% for Population II) while the regression estimator t_{lr} (and the estimators t_g, t_{RPe}, t_{Ai}, i=2,3,4,5,6, t_2, t_{sse}, t_{2e} and $t_{yz(2)}$) has the third largest PRE $(241.99\%$ for population I and 274.13% for Population II). The performances of the ratio (t_R) and product (t_P) are very poor even their performances are poorer than the conventional unbiased estimator $\bar{y}$ which does not utilize information on auxiliary attribute.

Tables 15 and 16 demonstrate that the proposed classes of estimators T_α^γ and $\dot{T}_\alpha^\gamma$ are more efficient than the regression estimator t_{lr} in both the Populations I and II for the selected values of the scalars (α,γ,d) closed in Table 16. It is also observed from Table 16 that one can identify the values of scalars (α,γ,d) in obtaining the estimators from the proposed classes of estimators T_α^γ and $\dot{T}_\alpha^\gamma$ better than the estimators $t_{(sg)_C}, t_{ss}$ and t_{lr} due to Shabbir and Gupta (2007), Singh and Solanki (2012) and the regression estimator t_{lr}.

For example for $(\alpha,\gamma,d) = (1,1,-0.99)$, the class estimators T_α^γ has PRE 803750.00% for Population I and PRE 470.90% for Population II. Also the suggested class of estimators $\dot{T}_\alpha^\gamma$ has the PRE 1273.21% in Population I and PRE 367.00% in Population II for the selected values of scalars $(\alpha,\gamma,d) = (1,1,-1)$. The substantial gain in efficiency is observed by using the proposed classes of estimators over $t_{(sg)_C}, t_{ss}$ and t_{lr}.

Table 16 also depicts that for other selected values of scalars (α,γ,d) the proposed classes of estimators T_α^γ and $\dot{T}_\alpha^\gamma$ are more efficient than the estimators $t_{(sg)_C}, t_{ss}$ and t_{lr} with considerable gain in efficiency. Thus our recommendation is in the favour of proposed classes of estimators T_α^γ and $\dot{T}_\alpha^\gamma$ for their use in practice.

5. Conclusions

This paper addressed the problem of estimating the PM $\bar{Y}$ of the SV y when the population proportion P of an auxiliary character ϕ is known. We have developed two classes of estimators for PM $\bar{Y}$ which includes a large number of known and unknown estimators as reported in Subsection 1.3 and Sections 2 and 3. The biases and mean squared errors expressions of the proposed classes of estimators have been derived to the first degree of approximation in single phase sampling. It is to be mentioned that the biases and mean squared errors of the estimators belonging to the suggested classes of estimators can be obtained from that of the envisaged classes of estimators just by inserting the suitable values of the scalars involved in the proposed classes of estimators. Asymptotic optimum estimator (AOE) in each proposed class of estimator is obtained along with minimum mean squared error formula. The realistic conditions derived under which the suggested classes of estimators are better than other existing estimators. Numerical illustrations along with appropriate recommendations are given to throw light on the merits of the suggested study.

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