Thailand Statistician
July 2023; 21(3): 660-674
http://statassoc.or.th
Contributed paper

Parameter Estimation for Inverted Topp-Leone Distribution Based on Different Ranked Set Sampling Schemes

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Received: 8 June 2021
Revised: 5 October 2021
Accepted: 7 October 2021

Abstract

Recently, an inverted Topp-Leone (IVT) distribution is introduced, which was useful for modeling lifetime phenomena. Some reliability measures of this distribution such as reliability, the maximum likelihood estimation and confidence intervals for the shape parameter are considered. In this paper, parameter estimation is discussed based on ranked set sampling (RSS) and neoteric ranked set sampling (NRSS) as a case of one stage ranked set sampling, double neoteric ranked set sampling (DNRSS) as a case of two stages ranked set sampling. Simulation studies are used to assess the two approaches from bias and efficiency aspects. The estimators are also compared with their analogs in simple random sampling. Moreover, it was shown that NRSS, RSS are more efficient than simple random sample (SRS) and they have small bias. The estimator based DNRSS, NRSS, and RSS are more efficient than the estimators based on SRS technique.

Keywords: Maximum likelihood estimation, neoteric ranked set sampling, inverted Topp-Leone distribution.

1. Introduction

Topp and Leone (1955) proposed the mathematical formulation of the family of J-shaped probability distributions. They also obtained its first four moments and showed its suitability to model failure data. Many contributions continued with studies of many authors, such as Nadarajah and Kotz (2003), Zghoul (2010, 2011). The inverted Topp-Leone (IVT) distribution is recently proposed by Muhammed (2019). This distribution has only one shape parameter, the probability density function (PDF) and, the cumulative distribution function (CDF) of the IVT distribution are, respectively, given by

$$F(x;\beta)=1-(x)^{-2\beta}(2x-1)^{\beta} \tag{1}$$

and

$$f(x;\beta)=2\beta(x-1)(2x-1)^{(\beta-1)}x^{(-2\beta-1)},$$

where $\beta>0,\ 1<x<\infty$ and β is a scale parameter.

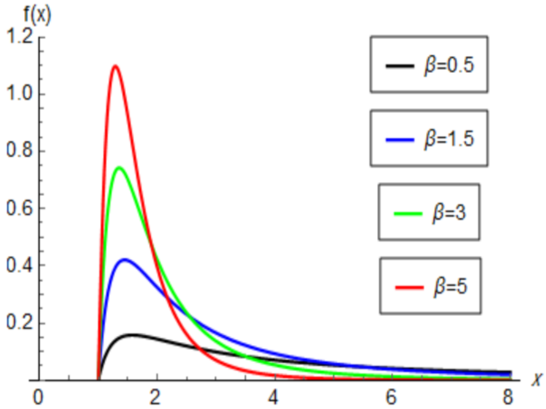

Figure 1 The PDF of The IVT Distribution

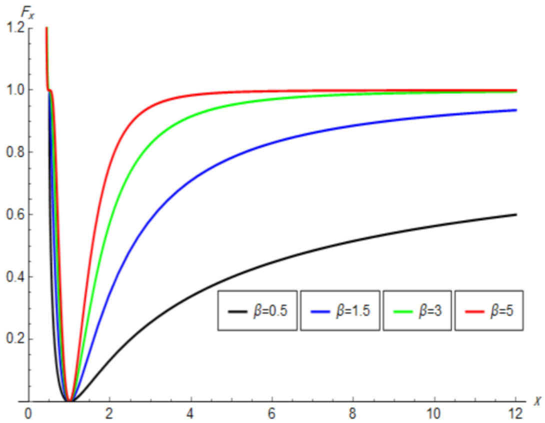

Figure 2 The CDF of The IVT Distribution

McIntyre (1952) proposed a way to estimate mean pasture yields with greater efficiency than simple random sample (SRS). The first theoretical result about RSS was obtained by Takahasi and Wakimoto (1968). They proved that, the RSS mean is an unbiased estimator of the population mean and the variance of the mean of a simple random sample is larger than the variance of the mean of a ranked set sample of the same size. The errors may creep in while ordering the randomly selected units of a set because of dependence upon the ranker's judgment. This aspect of the error of RSS was first considered theoretically by Dell and Clutter (1972) who have shown that, regardless of ranking errors, the RSS estimator of a population mean unbiased and at least as efficient as the SRS estimator with the same number of quantifications.

To reduce loss of efficiency in RSS due to errors in ranking and an improvement upon the efficiency of the estimator of the population mean are proposed extreme ranked set sampling by Samawi et al. (1996) and median ranked set sampling (MRSS) as a modification of the RSS by Muttlak (1997) as one stage of ranked set sampling Muttlak, (2003) introduced percentile ranked set sampling (PRSS). A recently developed extension of RSS, Neoteric ranked set sampling (NRSS) differs from the original RSS scheme by Zamanzade and Al-Omari (2016).

The two stage RSS designs double ranked set sampling (DRSS), as developed by Al-Saleh and Al-Kadiri, (2000). They used Takahasi definition and showed that the DRSS is more efficient in estimating the population mean than RSS and SRS. Thus, a gain in efficiency is obtained using DRSS without increasing the set size. Hence, DRSS is a more representative sample, and a k-stage RSS design uses $n^{(k+1)}$ sample units from the target population to produce a sample of size n after k stages of ranking developed by Al-Saleh and Al-Omari (2002).

Besides these studies, several authors have considered the estimation of the parameters of well-known distributions using RSS or its modifications. For example, estimation of unknown parameters of exponential, extreme-value, logistic, Weibull and Pareto distributions was studied by Lam et al. (1994), maximum likelihood estimators of the parameters of a modified Weibull distribution using extreme ranked set sampling was introduced by Al-Omari and Al-Hadrami (2011), Omar and Ibrahim (2013) estimated the shape and scale parameters of the Pareto distribution based on extreme RSS, Sabry and Shaaban (2020) derived the likelihood function for NRSS and double neoteric ranked set sampling (DRSS) for inverse Weibull distribution and Shaaban and Yahya (2020) is concerned with the estimation problem using maximum likelihood (ML) method of estimation for the unknown parameters of exponentiated Gumbel (EG) distribution based on different ranked set sampling schemes. For more about applications of RSS see Chen et al. (2005), Strzalkowska-Kominiak and Mahdizadeh (2014), Mahdizadeh (2015) and Mahdizadeh and Strzalkowska-Kominiak (2017). The main purpose of this paper is to compare the estimators of the IVT distribution that are obtained based on the different ranked set sampling schemes.

The paper is organized as follows: In Section 2, some sampling techniques are introduced. MLEs are derived for the shape parameter of IVT distribution using different sampling schemes and presented in Section 3. The numerical study for comparing the performance of DNRSS, NRSS, DRSS schemes with unknown parameters based on RSS and SRS techniques are presented in Section 4. Finally, the conclusion is in Section 5.

2. Some Ranked Set Sampling Techniques

This section is devoted to some various sampling procedures for selection of units in the sample will be considered such as RSS, NRSS and DNRSS schemes.

2.1. Ranked set sampling

Ranked set sampling proposed by McIntyre (1952) is a methodology which can improve the efficiency of techniques such as estimation and confidence intervals without increasing the number of substantial observations. It is designed to minimize the number of measured observations required to achieve the desired precision in making inferences. It uses additional information from the population to provide more structure to the data collection process and decreases the likelihood of an unrepresentative sample. Several studies have proved the higher efficiency of RSS, relative to SRS, for the estimation of a large number of population parameters. The RSS scheme can be described as follows**:**

Step 1 Randomly select m^2 sample units from the population.

Step 2 Allocate the m^2 selected units as randomly as possible into m sets, each of size m.

Step 3 Choose a sample for actual quantification by including the smallest ranked unit in the first set, the second smallest ranked unit in the second set, the process is continuous in this way until the largest ranked unit is selected from the last set.

Step 4 Repeat steps 1 through 4 for r cycles to obtain a sample of size mr.

cycle 1			cycle 2			cycle 3		
$\boxed{X_{(11)1}}$	$X_{(12)1}$	$X_{(13)1}$	$\boxed{X_{(11)2}}$	$X_{(12)2}$	$X_{(13)2}$	$\boxed{X_{(11)3}}$	$X_{(12)3}$	$X_{(13)3}$
$X_{(21)1}$	$\boxed{X_{(22)1}}$	$X_{(23)1}$	$X_{(21)2}$	$\boxed{X_{(22)2}}$	$X_{(23)2}$	$X_{(21)3}$	$\boxed{X_{(22)3}}$	$X_{(23)3}$
$X_{(31)1}$	$X_{(32)1}$	$\boxed{X_{(33)1}}$	$X_{(31)2}$	$X_{(32)2}$	$\boxed{X_{(33)2}}$	$X_{(31)3}$	$X_{(32)3}$	$\boxed{X_{(33)3}}$

Figure 3 RSS design

Let $\left\{X_{(ii)s}, i=1,2,...,m; s=1,2,...,r\right\}$ be a ranked set sample where m is the set size and r is the number of cycles. Then the likelihood function corresponding to RSS scheme of $X_{(ii)s}$ is given by

$$L_{RSS}(\theta \mid x)=\prod_{j=1}^{r}\prod_{i=1}^{m} C_i\, f(x_{(ii)j};\theta)[F(x_{(ii)j};\theta)]^{(i-1)}[1-F(x_{(ii)j};\theta)]^{(m-i)}, \tag{1}$$

where $c_i=\dfrac{m!}{(i-1)!(m-i)!}$.

2.2. Neoteric ranked set sampling

A recently developed extension of RSS is introduced NRSS by Zamanzade and Al-Omari (2016), NRSS has been shown to be effective, producing more efficient estimators for the population mean and variance compared to RSS and SRS. The following steps describe the NRSS sampling design:

Step 1 Select a simple random sample of size m^2 units from the target finite population.

Step 2 Ranked the m^2 selected units in an increasing magnitude based on a visual inspection or any other cost-free method with respect to a variable of interest.

Step 3 If m is an odd, then select the $\left[g+(i-1)m\right]^{\text{th}}$ ranked unit. If m is an even, then select the $\left[u+(i-1)m\right]^{\text{th}}$ ranked unit, if i is an even and $\left[(u+1)+(i-1)m\right]^{\text{th}}$ if i is an odd where $\left(u=\dfrac{m}{2}, g=\dfrac{m+1}{2}, \text{and } i=1,2,..,m\right)$.

Step 4 Repeat steps 1 through 3 r cycles if needed to obtain a NRSS of size $n=rm.$ The NRSS scheme can be described as follows, the observation after order:

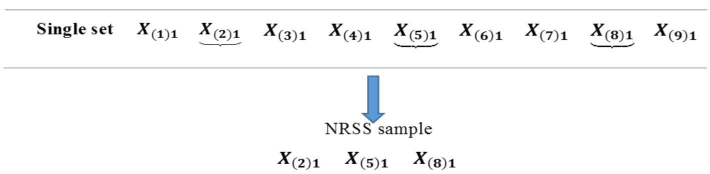

Figure 4 NRSS design in case of odd sample size

Using NRSS method, we have to choose the units with the rank 2, 5, 8 for actual quantification, then the measured NRSS units are $\left\{\boxed{X_{(2)1}}, \boxed{X_{(5)1}}, \boxed{X_{(8)1}}\right\}$ for one cycle.

Let $\left\{X_{(i)j},\ i=1,2,..,m,\ j=1,2,...,r\right\}$ be a neoteric ranked set sample where m is the set size and r is the number of cycles. Then the likelihood function corresponding to NRSS scheme that proposed by Sabry and shaaban (2020), is given by

$$L(\theta \mid x_{k(i)j}) = \frac{w!}{\prod_{i=1}^{m+1}(k(i)-k(i-1)-1)!}\prod_{i=1}^{m} f(x_{(k(i))j};\theta)\cdot\prod_{i=1}^{m+1}[F(x_{(k(i))j};\theta)-F(x_{(k(i-1))j};\theta)]^{k(i)-k(i-1)-1}, \tag{2}$$

where m is the set size, r is the number of cycles, $w=m^2!$, and $k(i)$ is chosen as

$$k(i)=\begin{cases} g+(i-1)m, & m \text{ odd} \\ u+(i-1)m, & m \text{ even, } i \text{ even} \\ (u+1)+(i-1)m, & m \text{ even, } i \text{ odd,} \end{cases}$$

where $k(0)=0,\ k(m+1)=w+1$ and $x_{(k(0))}=-\infty,\ x_{(k(m+1))}=\infty.$

2.3. Double ranked set sampling

Al-Saleh and Al-Kadiri (2000) proposed DRSS procedure for estimating the population mean as two stage ranked set sampling. The following steps are employed to obtain a DRSS of size m.

Step 1 Select m^3 elements from the target population and divide these elements randomly into n sets (of size m^2).

Step 2 Select a sample of size m in each set using RSS method.

Step 3 Apply the RSS procedure again to elements selected in step 2 to obtain a DRSS.

Step 4 The cycle may be repeated m times.

$$\begin{bmatrix} x_{(11)}^{(1)} & \cdots & x_{(1m)}^{(1)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(1)} & \cdots & x_{(mm)}^{(1)} \end{bmatrix}, \begin{bmatrix} x_{(11)}^{(2)} & \cdots & x_{(1m)}^{(2)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(2)} & \cdots & x_{(mm)}^{(2)} \end{bmatrix}, \begin{bmatrix} x_{(11)}^{(3)} & \cdots & x_{(1m)}^{(3)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(3)} & \cdots & x_{(mm)}^{(3)} \end{bmatrix} and \begin{bmatrix} x_{(11)}^{(m)} & \cdots & x_{(1m)}^{(m)} \\ \vdots & \ddots & \vdots \\ x_{(m1)}^{(m)} & \cdots & x_{(mm)}^{(m)} \end{bmatrix}$$

Figure 5 DRSS design in case of even sample size

So, we have four judgment ranked sets of size m each:

$X_{1,j} = \min\left\{\left\{x_{(11)}^{j}, x_{(22)}^{j},..., x_{(mm)}^{j}\right\}, j=1,2,...,r\right\}$, and

$X_{m,k} = \max\left\{\left\{x_{(11)}^{k}, x_{(22)}^{k},..., x_{(mm)}^{k}\right\}, k=r+1,r+2,...,m\right\}$

The likelihood function for DRSS that proposed by Sabry et al. (2019) is given as:

Case I: m even $(m=2r)$,

$$L(\theta)=\left[\prod_{j=1}^{r} m f_{(1:m)}(x_{(1,j)})[1-F_{(1:m)}(x_{(1,j)})]^{m-1}\right]\left[\prod_{k=r+1}^{m} m f_{(m:m)}(x_{(m,k)})[F_{(m:m)}(x_{(m,k)})]^{m-1}\right] \tag{3}$$

Case II: m odd $(m = 2r+1)$,

$$L(\theta)=\left[\prod_{j=1}^{r} m f_{(1:m)}(x_{(1,j)})[1-F_{(1:m)}(x_{(1,j)})]^{m-1}\right]\left[\prod_{k=r+2}^{m} m f_{(m:m)}(x_{(m,k)})[F_{(m:m)}(x_{(m,k)})]^{m-1}\right]$$
$$\times\left[\frac{(2r+1)!}{(r)!(r)!} f_{r+1:m}(x_{(r+1),(r+1)})\Big(F_{r+1:m}(x_{(r+1),(r+1)})\Big(1-F_{r+1:m}(x_{(r+1),(r+1)})\Big)\Big)^{r}\right]. \quad (4)$$

2.4. Double Neoteric ranked set sampling

DNRSS is defined by Taconeli and Cabral (2018) which is defined to be a two-stage design in which the first stage is defined as RSS scheme, while the NRSS procedure should be applied in the second stage. To draw a DRSS sample of size n, the following steps must be implemented:

Step 1 Identify m^3 elements from the target population and divide them, randomly, into m blocks with m sets of size m.

Step 2 Apply the RSS method to each block to obtain m RSS samples of size n.

Step 3 Employ the NRSS procedure to the m^2 elements selected in step 2 to obtain a DNRSS sample of size m. Only these sample units must be measured for the variable of interest.

Step 4 Steps 1-3 can be repeated r times to draw a sample of size mr.

In Figure 6, we show how to select a sample of size $m = 3$ and $r = 1$, then we have to select $m^3 = 27$ units as

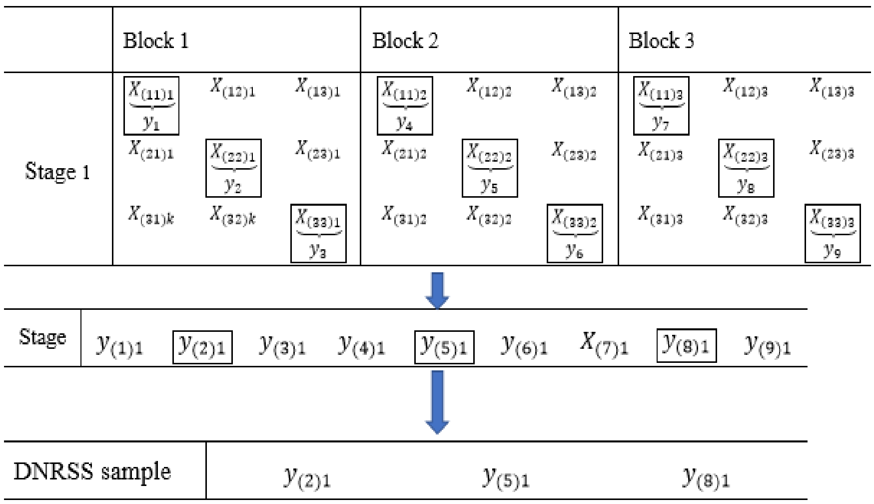

Figure 6 DNRSS design in case of odd sample size and one cycle

Let $\left\{X_{(i)j},\ i=1,2,..,m,\ j=1,2,...,r\right\}$ be a DNRSS where m is the set size and r is the number of cycles. Then the Likelihood function corresponding to DNRSS scheme that proposed by Sabry and shaaban (2020), is given by

$$L(\theta \mid x_{k(i)j}) = \frac{w!}{\prod_{i=1}^{m+1} (k(i)-k(i-1)-1)!} \prod_{i=1}^{m} f_m(y_{(k(i))j};\theta) \times \prod_{i=1}^{m+1} [F_m(y_{(k(i))j};\theta) - F_m(y_{(k(i-1))j};\theta)]^{k(i)-k(i-1)-1}. \tag{5}$$

3. Estimation of the IVT Distribution Parameters

In this Section MLE for the unknown parameter of IVT distribution is obtained based on RSS, NRSS, DRSS and DNRSS, the estimators are also compared with its analogs in simple random sampling.

3.1. Estimation based on SRS

Muhammed (2019) introduced the maximum likelihood estimator (MLE) for IVT distribution parameters based on SRS. In this subsection, MLE based on SRS will be reviewed. Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from IVT(β), then the likelihood function can be written as follows

$$L_{SRS}(x;\theta) = 2^n \beta^n \prod_{i=1}^{n} (x_i - 1) x_i^{-2\beta-1} (2x_i - 1)^{\beta-1},$$

and the log likelihood function is then derived as

$$l_{SRS}(\beta) \propto n \log \beta + \sum_{i=1}^{n} \log(x_i - 1) - (2\beta+1) \sum_{i=1}^{n} \log x_i + (\beta - 1) \sum_{i=1}^{n} \log(2x_i - 1),$$

The first derivative of the log-likelihood function denoted by l_{SRS} with respect to β is as follows

$$\frac{\partial l_{SRS}}{\partial \beta} = \frac{n}{\beta} - 2\sum_{i=1}^{n} \log x_i + \sum_{i=1}^{n} \log(2x_i - 1). \tag{6}$$

From (7), Muhammed (2019) showed that MLE of β can be obtained as follows

$$\beta = \frac{n}{\sum_{i=1}^{n} \log\left[\frac{x_i^2}{(2x_i - 1)}\right]}.$$

3.2. Estimation based on RSS

In this subsection, MLE of IVT distribution will be derived. Suppose $\left\{X_{(11)j}, X_{(22)j}, \ldots, X_{(mm)j};\ j = 1,2,\ldots,r\right\}$ denotes the ranked set sample of size $n = mr$ from IVT(β), where m is the set size and r is the number of cycles. By substituting (1) and (2) into (3), then the likelihood function based on RSS data is given by

$$L_{RSS}(x;\beta) = \prod_{j=1}^{r} \prod_{i=1}^{m} c_i \left(2\beta(x_{(ii)j} - 1) x_{(ii)j}^{-2\beta-1} \left(2x_{(ii)j} - 1\right)^{\beta-1}\right)$$
$$\times \left(1 - \left(x_{(ii)j}\right)^{-2\beta} \left(2x_{(ii)j} - 1\right)^{\beta}\right)^{i-1} \left(\left(x_{(ii)j}\right)^{-2\beta} \left(2x_{(ii)j} - 1\right)^{\beta}\right)^{n-i}$$

and the first derivative of l_{RSS} with respect to β is given by

$$\frac{\partial l_{RSS}}{\partial\beta}=\frac{rm}{\beta}-2\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(ii)j}+\sum_{j=1}^{r}\sum_{i=1}^{m}\log(2x_{(ii)j}-1)$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\frac{(2x_{(ii)j}-1)^{\beta}\left[2\log x_{(ii)j}-\log(2x_{(ii)j}-1)\right]}{(x_{(ii)j})^{2\beta}(2x_{(ii)j}-1)^{\beta}}+\sum_{j=1}^{r}\sum_{i=1}^{m}(n-i)\frac{\left[2\log x_{(ii)j}-\log(2x_{(ii)j}-1)\right]}{(x_{(ii)j})^{-2\beta}(2x_{(ii)j}-1)^{\beta}}.$$
(7)

It is clear that, is not easy to obtain a closed form of the non linear (9), so an iterative technique can be used to obtain MLEs of β as will be seen in Section 4.

3.3. Estimation based on NRSS

In this subsection, we will derive MLE for IVT(β) based on NRSS technique by substituting (1) and (2) in (4). Let $\left\{X_{(i)j},\ i=1,2,..,m,\ j=1,2,...,r\right\}$ be a neoteric ranked set sample where m is the set size and r is the number of cycles, then the likelihood function corresponding to NRSS scheme is given by

$$L_{NRSS}(x;\beta)\propto\prod_{j=1}^{r}\left(\prod_{i=1}^{m}\left(2\beta(x_{(k(i)j}-1)x_{(k(i)j}^{-2\beta-1}\left(2x_{(k(i)j}-1\right)^{\beta-1}\right)\right)$$
$$\times\prod_{i=1}^{m+1}\left[\left(1-x_{(k(i)j}^{-2\beta}\left(2x_{(k(i)j}-1\right)^{\beta}\right)-\left(1-x_{(k(i-1)j}^{-2\beta}\left(2x_{(k(i-1)j}-1\right)^{\beta}\right)\right]^{k(i)-k(i-1)-1}.$$

The associated log-likelihood function denoted by l_{NRSS} is as follows

$$l_{NRSS}(\beta)\propto mr\log\beta+\sum_{j=1}^{r}\sum_{i=1}^{m}\log\left(x_{(k(i))j}-1\right)-\left(2\beta+1\right)\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(k(i))j}+\left(\beta-1\right)\sum_{j=1}^{r}\sum_{i=1}^{m}\log\left(2x_{(k(i))j}-1\right)$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1)\log\left[x_{(k(i-1)j}^{-2\beta}\left(2x_{(k(i-1)j}-1\right)^{\beta}-x_{(k(i)j}^{-2\beta}\left(2x_{(k(i)j}-1\right)^{\beta}\right]$$

and the first derivative of the l_{NRSS} is given by

$$\frac{\partial l_{NRSS}}{\partial\beta}=\frac{rm}{\beta}-2\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(k(i))j}+\sum_{j=1}^{r}\sum_{i=1}^{m}\log(2x_{(k(i))j}-1)$$
$$+\sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1)$$
$$\frac{\left(2x_{(k(i-1)j}-1\right)^{\beta}\left[2\log\left(x_{(k(i-1)j}\right)-\log\left(2x_{(k(i-1)j}-1\right)\right]x_{(k(i)j}^{-2\beta}+x_{(k(i-1)j}^{-2\beta}\left(2x_{(k(i-1)j}-1\right)^{\beta}\left[\log\left(2x_{(k(i)j}-1\right)-2\log\left(x_{(k(i)j}\right)\right]}{x_{(k(i-1)j}^{-2\beta}\left(2x_{(k(i-1)j}-1\right)^{\beta}-x_{(k(i)j}^{-2\beta}\left(2x_{(k(i)j}-1\right)^{\beta}}.$$
(8)

MLE of IVT distribution parameters based on NRSS can be obtained by solving (10) using iterative technique.

3.4. Estimation based on DRSS

According to equation 5 the likelihood function for DRSS design is derived as follows.

Case I: m even $(m=2r)$

$$L_{DRSS}(x;\beta)=\left[\prod_{j=1}^{r} m^2\left(2\beta\left(x_{(1)j}-1\right)x_{(1)j}^{-2\beta-1}\left(2x_{(1)j}-1\right)^{\beta-1}\right)\left(x_{(1)j}^{-2\beta}\left(2x_{(1)j}-1\right)^{\beta}\right)^{m(m-1)}\right]$$
$$\times\left[\prod_{k=r+1}^{m} m\left(2m\beta\left(x_{(m)j}-1\right)x_{(m)j}^{-2\beta-1}\left(2x_{(m)j}-1\right)^{\beta-1}\right)\left(1-x_{(m)j}^{-2\beta}\left(2x_{(m)j}-1\right)^{\beta}\right)^{m(m-1)}\right].$$

Then, the associated log-likelihood function is obtained as

$$l_{DRSS}\propto m\log\beta+\sum_{j=1}^{r}\log(x_{(1)j}-1)-(2\beta+1)\sum_{j=1}^{r}\log(x_{(1)j})+(\beta-1)\sum_{j=1}^{r}\log(2x_{(1)j}-1)$$
$$+(m^2-1)\sum_{j=1}^{r}\log\left(x_{(1)j}^{-2\beta}\left(2x_{(1)j}-1\right)^{\beta}\right)+\sum_{k=r+1}^{m}\log(x_{(m)j}-1)-(2\beta+1)\sum_{k=r+1}^{m}\log\left(x_{(m)j}\right)$$
$$+(\beta-1)\sum_{k=r+1}^{m}\log(2x_{(m)j}-1)+(m^2-1)\sum_{k=r+1}^{m}\log\left(1-x_{(m)j}^{-2\beta}\left(2x_{(m)j}-1\right)^{\beta}\right)$$

and the first derivative is given by

$$\frac{\partial l_{DRSS}}{\partial\beta}=\frac{m}{\beta}-2\sum_{j=1}^{r}\log(x_{(1)j})+\sum_{j=1}^{r}\log(2x_{(1)j}-1)+(m^2-1)\sum_{j=1}^{r}\frac{2\log(x_{(1)j})-\log(2x_{(1)j}-1)}{\left(x_{(1)j}^{-2\beta}\left(2x_{(1)j}-1\right)^{\beta}\right)}$$
$$+2\sum_{k=r+1}^{m}\log\left(x_{(m)j}\right)+\sum_{k=r+1}^{m}\log(2x_{(m)j}-1)+(m^2-1)\sum_{k=r+1}^{m}\frac{\left(2x_{(m)j}-1\right)^{\beta}\left[\log(2x_{(m)j}-1)-2\log(x_{(m)j})\right]}{\left(\left(2x_{(m)j}-1\right)^{\beta}-x_{(m)j}^{2\beta}\right)}.$$

Case II: m odd $(m=2r+1)$

According to Equation (6) the log-likelihood function of the IVT distribution for odd set size is given by

$$l_{DRSS}\propto m\log\beta+\sum_{j=1}^{r}\log(x_{(1)j}-1)-(2\beta+1)\sum_{j=1}^{r}\log(x_{(1)j})+(\beta-1)\sum_{j=1}^{r}\log(2x_{(1)j}-1)$$
$$+(m^2-1)\sum_{j=1}^{r}\log\left(x_{(1)j}^{-2\beta}\left(2x_{(1)j}-1\right)^{\beta}\right)+\sum_{k=r+2}^{m}\log(x_{(m)j}-1)-(2\beta+1)\sum_{k=r+2}^{m}\log\left(x_{(m)j}\right)$$
$$+(\beta-1)\sum_{k=r+2}^{m}\log(2x_{(m)j}-1)+(m^2-1)\sum_{k=r+2}^{m}\log\left(1-x_{(m)j}^{-2\beta}\left(2x_{(m)j}-1\right)^{\beta}\right)$$
$$+\log(x_{(r+1),(r+1)}-1)-(2\beta+1)\log(x_{(r+1),(r+1)})+(\beta-1)\log(2x_{(r+1),(r+1)}-1)$$
$$+r\log\left(1-\left(x_{(r+1),(r+1)}^{-2\beta}\left(2x_{(r+1),(r+1)}-1\right)^{\beta}\right)\right)+r\log\left(x_{(r+1),(r+1)}^{-2\beta}\left(2x_{(r+1),(r+1)}-1\right)^{\beta}\right)$$
$$+r\log F_{r+1:m}\left(x_{(r+1),(r+1)}\right)+r\log\left(1-F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)$$

and the first derivative is given by

$$\frac{\partial l_{DRSS}}{\partial\beta}=\frac{m}{\beta}-2\sum_{j=1}^{r}\log(x_{(1)j})+\sum_{j=1}^{r}\log(2x_{(1)j}-1)+(m^2-1)\sum_{j=1}^{r}\frac{2\log(x_{(1)j})-\log(2x_{(1)j}-1)}{\left(x_{(1)j}^{-2\beta}\left(2x_{(1)j}-1\right)^{\beta}\right)}$$

$$-2\sum_{k=r+2}^{m}\log\left(x_{(m)j}\right)+\sum_{k=r+2}^{m}\log(2x_{(m)j}-1)+(m^2-1)\sum_{k=r+2}^{m}\frac{\left(2x_{(m)j}-1\right)^{\beta}\left[\log(2x_{(m)j}-1)-2\log(x_{(m)j})\right]}{\left(\left(2x_{(m)j}-1\right)^{\beta}-x_{(m)j}^{2\beta}\right)}$$

$$-2\log(x_{(r+1),(r+1)})+\log(2x_{(r+1),(r+1)}-1)+r\frac{\left(2x_{(r+1),(r+1)}-1\right)^{\beta}\left[2\log(x_{(r+1),(r+1)})-\log(2x_{(r+1),(r+1)}-1)\right]}{\left(\left(x_{(r+1),(r+1)}^{2\beta}\left(2x_{(r+1),(r+1)}-1\right)^{\beta}\right)\right)}$$

$$+r\frac{\left[2\log(x_{(r+1),(r+1)})-\log(2x_{(r+1),(r+1)}-1)\right]}{\left(\left(x_{(r+1),(r+1)}^{-2\beta}\left(2x_{(r+1),(r+1)}-1\right)^{\beta}\right)\right)}+rF_{\beta}\left(\frac{1-2F_{r+1:m}\left(x_{(r+1),(r+1)}\right)}{F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\left(1-F_{r+1:m}\left(x_{(r+1),(r+1)}\right)\right)}\right),$$

where $F_{\beta}=\partial F_{r+1:m}\left(x_{(r+1),(r+1)}\right)/\partial\beta.$

3.5. Estimation based on DNRSS

By substitution in Equation (7) based on IVT distribution the Likelihood function for set sizes m and with r cycles based on DNRSS scheme is given by

$$L_{DNRSS}(x;\beta)=\prod_{j=1}^{r}\left(h\prod_{i=1}^{m}c\left(2\beta(x_{(i)j}-1)x_{(i)j}^{-2\beta-1}\left(2x_{(i)j}-1\right)^{\beta-1}\right)\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{i-1}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-i}\right)$$

$$\times\prod_{i=1}^{m+1}\left[\begin{array}{l}\sum_{t=i}^{m}\binom{m}{t}\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{t}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-t}\\-\sum_{t=i-1}^{m}\binom{m}{t}\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{t}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-t}\end{array}\right]^{k(i)-k(i-1)-1}.$$

The associated log-likelihood function is as follows

$$l_{DNRSS}(\beta)\propto mr\log\beta+\sum_{j=1}^{r}\sum_{i=1}^{m}\log\left(x_{(i)j}-1\right)-(2\beta+1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j}+(\beta-1)\sum_{j=1}^{r}\sum_{i=1}^{m}\log\left(2x_{(i)j}-1\right)$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\log\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)+\sum_{j=1}^{r}\sum_{i=1}^{m}(m-i)\log\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1)\log\left[\begin{array}{l}\sum_{t=i}^{m}\binom{m}{t}\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{t}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-t}\\-\sum_{t=i-1}^{m}\binom{m}{t}\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{t}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-t}\end{array}\right]$$

and the first derivative of the l_{DNRSS} is given by

$$\frac{\partial l_{DNRSS}}{\partial\beta}=\frac{rm}{\beta}-2\sum_{j=1}^{r}\sum_{i=1}^{m}\log x_{(i)j}+\sum_{j=1}^{r}\sum_{i=1}^{m}\log(2x_{(i)j}-1)+\sum_{j=1}^{r}\sum_{i=1}^{m}(i-1)\frac{(2x_{(i)j}-1)^{\beta}\left[2\log x_{(i)j}-\log(2x_{(i)j}-1)\right]}{(x_{(i)j})^{2\beta}(2x_{(i)j}-1)^{\beta}}$$

$$+\sum_{j=1}^{r}\sum_{i=1}^{m}(n-i)\frac{\left[2\log x_{(i)j}-\log(2x_{(i)j}-1)\right]}{(x_{(i)j})^{-2\beta}(2x_{(i)j}-1)^{\beta}}+\sum_{j=1}^{r}\sum_{i=1}^{m+1}(k(i)-k(i-1)-1)\frac{\partial Q}{\partial\beta},$$

where

$$Q=\log\sum_{t=i}^{m}\binom{m}{t}\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{t}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-t}$$
$$-\sum_{t=i-1}^{m}\binom{m}{t}\left(1-x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{t}\left(x_{(i)j}^{-2\beta}\left(2x_{(i)j}-1\right)^{\beta}\right)^{m-t}.$$

4. Simulation Study

A simulation study is complementary to this study with an evaluation of the performance of the proposed maximum likelihood estimator of the shape parameter of IVT distribution based on RSS, NRSS, DRSS, DNRSS. The simulation is applied for 10,000 replications and different sample sizes, $m=\{20,30,40,50,100\}$. The simulation is made for different parameters values IVT(β)={IVT(0.5), IVT(1.5), IVT(3), IVT(5)}.

Comparison between the proposed estimators for β using SRS, RSS, MRSS, and NRSS schemes are carried out using mean square error (MSE) and efficiencies criteria. The efficiency between all estimators with respect to the MLE based on SRS are calculated. The efficiency of the estimator is defined as

$$eff\left(\theta_1,\ \theta_2\right)=\frac{MSE\left(\theta_1\right)}{MSE\left(\theta_2\right)}\ \text{ if } eff\left(\theta_1,\ \theta_2\right)>1,\ \text{then } \theta_2 \text{ is better than } \theta_1.$$

The results of relative biases, MSE and relative efficiency for the different estimator is listed in Figures 7-10 and Tables 1-2, are represented to clarify the simulation results.

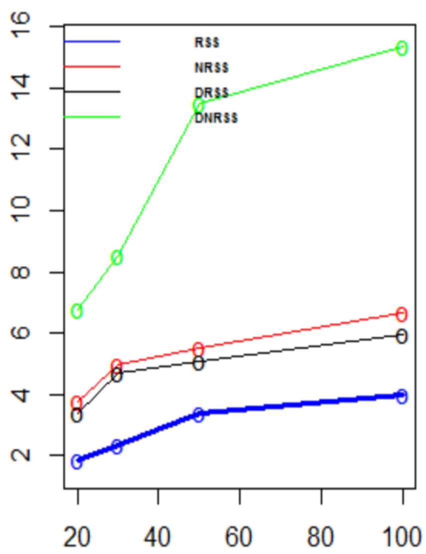

Figure 7 Relative efficiency of different RSS schemes for value of parameter 0.5

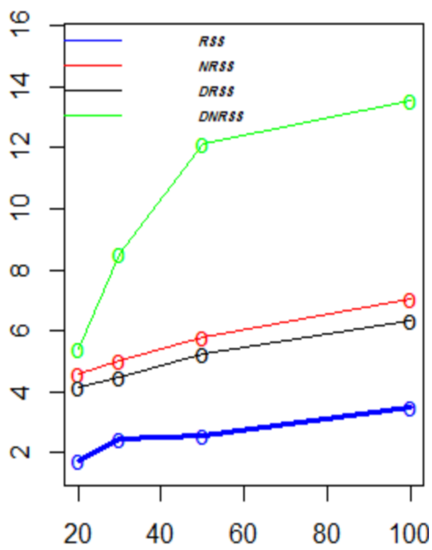

Figure 8 Relative efficiency of different RSS schemes for value of parameter 1.5

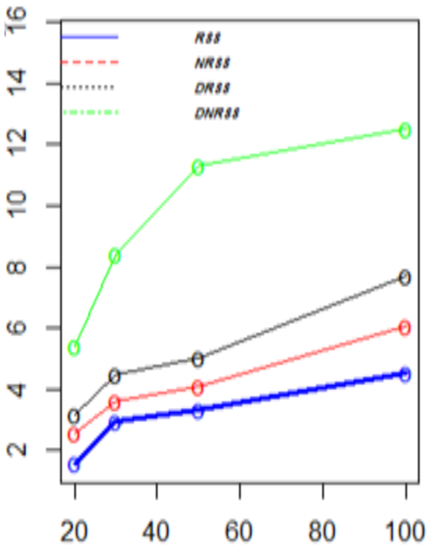

Figure 9 Relative efficiency of different RSS schemes for value of parameter 3

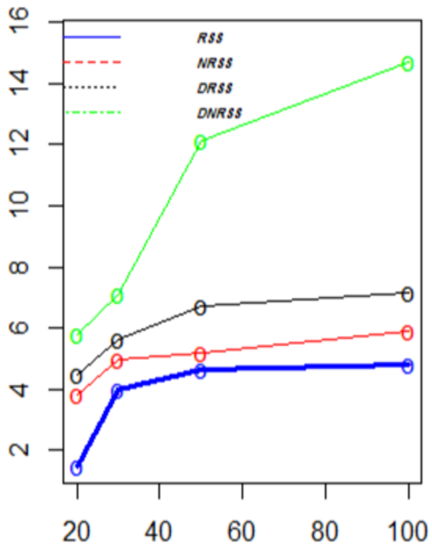

Figure 10 Relative efficiency of different RSS schemes for value of parameter 5

Simulation results are summarized in Figures 7-10 and Tables 1-2 and for some values of the estimation parameter.

1. It may be concluded that the DNRSS estimator is biased and more efficient than the SRS, RSS, NRSS estimator for all cases considered in this study. However, as demonstrated by Toconeli and Cabral (2018) it is better to use DNRSS with small sample size. Also, it is noted that the efficiency of the parameter estimation depends on the values of β as well as the sample size.

2. The relative efficiency from NRSS and RSS to the design with best performance than SRS for all cases.

3. The relative efficiency of the estimator based on NRSS have the largest efficiencies in cases for value of parameter $\beta = 0.5, 1.5,$ While increasing the value of parameter to $\beta = 3, 5$ the relative efficiency of the estimator based on RSS has become better than NRSS.

4. The biases are small for all cases.

5. MSE of the estimator based on SRS data are greater than MSE of the estimators based on RSS, NRSS, and DNRSS data.

6. Whenever, increasing the set size of the sample, MSE of estimator based on SRS, RSS, NRSS, and DNRSS decrease.

7. Approaching MSEs of the estimator based on SRS is nearly close from MSE of estimator based on RSS, NRSS as the sample size increases.

Table 1 MSE and Relative efficiency of the estimator for β based on RSS schemes compared to SRS estimator under perfect ranking

	m	SRS	RSS		NRSS		DRSS		DNRSS	
		MSE	MSE	RE	MSE	RE	MSE	RE	MSE	RE
IVT(0.5)	20	0.03342	0.01819	1.84	0.00888	3.76	0.00995	3.36	0.00494	6.76
	30	0.00935	0.00397	2.35	0.00187	4.99	0.00200	4.67	0.00110	8.52
	50	0.00117	0.00035	3.37	0.00021	5.54	0.00023	5.10	0.00009	13.49
	100	0.00031	0.00008	3.96	0.00005	6.64	0.00005	5.96	0.00002	15.37
IVT(1.5)	20	0.00772	0.00440	1.75	0.00168	4.60	0.00185	4.17	0.00142	5.43
	30	0.00082	0.00034	2.43	0.00016	5.04	0.00018	4.49	0.00010	8.54
	50	0.00055	0.00021	2.56	0.00009	5.81	0.00010	5.23	0.00005	12.15
	100	0.00033	0.00009	3.48	0.00005	7.04	0.00005	6.33	0.00002	13.56
IVT(3)	20	0.00310	0.00194	1.60	0.00120	2.58	0.00099	3.14	0.00057	5.40
	30	0.00035	0.00012	2.94	0.00010	3.59	0.00008	4.48	0.00004	8.40
	50	0.00025	0.00007	3.69	0.00006	4.07	0.00005	5.01	0.00002	11.29
	100	0.00014	0.00003	4.51	0.00002	6.05	0.00002	7.68	0.00001	12.49
IVT(5)	20	0.02232	0.01501	1.49	0.00588	3.79	0.00499	4.48	0.00386	5.79
	30	0.00441	0.00111	3.98	0.00088	4.98	0.00078	5.62	0.00062	7.10
	50	0.00229	0.00049	4.66	0.00044	5.20	0.00034	6.72	0.00019	12.12
	100	0.00064	0.00013	4.83	0.00011	5.91	0.00009	7.16	0.00004	14.69

5. Conclusion

Maximum likelihood estimator for the inverted Topp-Leone distribution is studied based on double neoteric ranked set sampling, ranked set sampling and neoteric ranked set sampling and simple random sample. These MLEs are not in closed forms, so numerical method is used. Results show that DNRSS estimator is biased and more efficient than the SRS, RSS, NRSS estimator for all cases considered in this study. Also, it was shown that NRSS, RSS are more efficient than SRS and they has small bias. Generally, the estimator based DNRSS, NRSS, and RSS are more efficient than the estimators based on SRS technique.

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Table 2 Relative bias of the estimator for β based on RSS schemes compared to SRS estimator under perfect ranking

	m	SRS	RSS	NRSS	DRSS	DNRSS
IVT(0.5)	20	–0.02312	–0.02136	–0.01391	–0.06219	–0.01505
	30	–0.01727	–0.02064	–0.00802	–0.05093	–0.00768
	50	–0.01342	–0.01097	–0.00651	–0.04218	–0.00503
	100	–0.00457	–0.00551	–0.00587	–0.00321	–0.00399
IVT(1.5)	20	–0.07095	0.07795	0.03053	0.00646	0.01443
	30	–0.05407	0.01539	0.01958	0.00547	0.00898
	50	–0.01473	0.01051	0.00496	0.34775	–0.00856
	100	–0.01266	0.00498	0.00325	0.21433	0.77446
IVT(3)	20	0.06032	0.07782	–0.10513	–0.03858	–0.00545
	30	0.03482	0.07510	–0.02413	–0.05456	–0.00846
	50	0.02383	0.01178	–0.00859	–0.08872	–0.00774
	100	0.01053	0.00969	–0.00985	–0.07412	–0.00421
IVT(5)	20	0.05411	0.07013	–0.10209	0.00479	0.00333
	30	0.04492	0.03605	–0.03567	0.00370	0.00234
	50	0.01467	0.02206	–0.02183	0.00321	0.00198
	100	0.00511	0.01688	0.00948	0.00215	0.00155

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