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Exact Moments of Generalized Akash Order Statistics

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Abstract

A generalized Akash distribution (GAD) has been proposed recently to analyze the lifetime data. For this distribution, exact explicit expressions for single moments, product moments, marginal moment generating functions and joint moment generating functions of order statistics are derived. By using these, the expected values, second moments, variances and covariances of order statistic from sample of sizes up to 10 for various values of the parameters are shown. In addition, best linear unbiased estimates of the parameters from location-scale GAD based on Type-II right-censored sample are obtained by using the derived moments.

Keywords: Best linear unbiased estimator, lifetime distribution, location and scale parameters, order statistics, type-II censoring.

1. Introduction

The generalized Akash distribution (GAD) has been introduced by Shanker et al. (2018) which is the recent lifetime distribution with two parameters which includes Akash and exponential distributions in particular cases. GAD plays important role in various areas of research such as, in the field of statistical theory (Almawajdeh and Al-Zou'bi, 2019; Alhyasat et al., 2020; Enogwe et al., 2021), medical science (Abushal, 2021), reliability theory and survival analysis (Al-Omari et al., 2020), and so on. Shanker discussed the various important statistical properties of the GAD including its moments, skewness, kurtosis, order statistics, hazard rate function, etc. Suppose $X_1, X_2, ..., X_n$ are the GAD random variables with parameters θ and α. The probability density function (pdf) and the cumulative distribution function (cdf) of GAD are given by, respectively,

$$f(x;\theta,\alpha)=\frac{\theta^3}{\theta^2+2\alpha}\left(1+\alpha x^2\right)e^{-\theta x} \quad ;x>0,\theta>0,\alpha>0, \tag{1}$$

$$F(x;\theta,\alpha)=1-\left[1+\frac{\alpha\theta x\left(\theta x+2\right)}{\theta^2+2\alpha}\right]e^{-\theta x} \quad ;x>0,\theta>0,\alpha>0. \tag{2}$$

The graphs of the pdf and cdf of GAD for specific values of θ and α are shown in Figures 1 and 2. The GAD is stochastically increasing with α for a fixed θ. Conversely, the GAD is stochastically decreasing with respect to θ for a fixed α (Shanker et al., 2018; Casella and Berger, 2002).

The order statistics are obtained by arranging them in size ordering, that is $X_{1:n}, X_{2:n}, ..., X_{n:n}$. The random variables $X_{1:n}$ and $X_{n:n}$ are minimum and maximum order statistics, respectively. The rth, $1\le r\le n$, order statistics is denoted by $X_{r:n}$. Order statistics play an important role in a

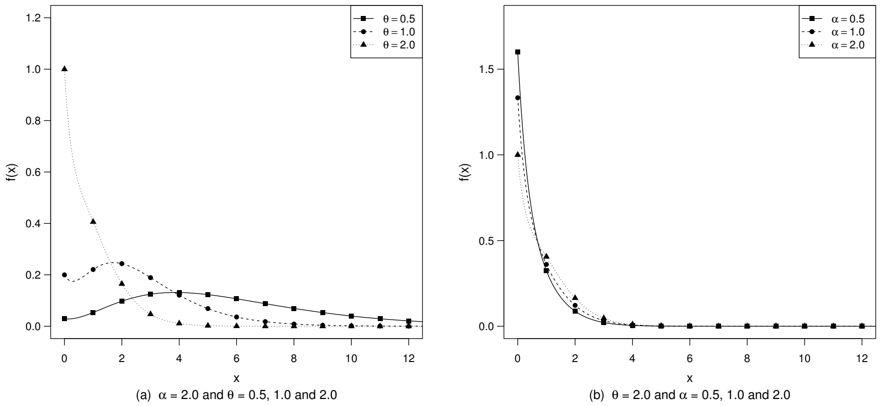

Figure 1 The probability density function of GAD for specific values of parameters θ and α

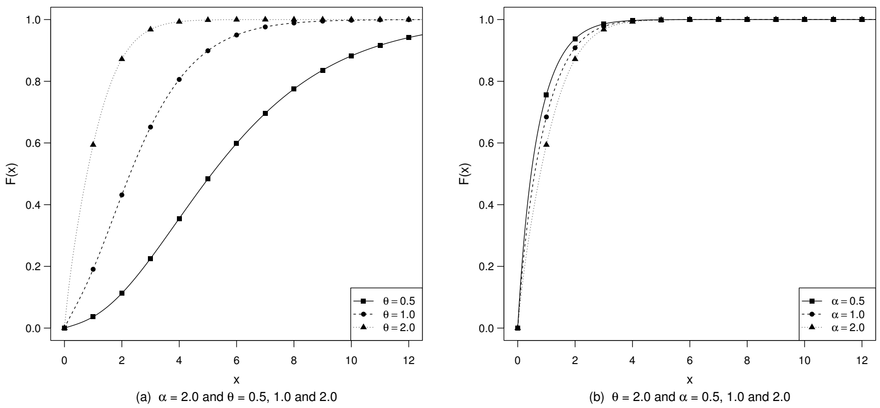

Figure 2 The cumulative distribution function of GAD for specific values of parameters θ and α

wide range of theoretical and practical problems such as characterization of probability distributions and goodness-of-fit test, entropy estimation, analysis of censored samples, reliability analysis, quality control and strength of materials. In recent year, many articles have carried out studies in moments of order statistics with several distributions; for example, Khan et al. (2015) investigated the exact moments of record values for computing the best linear unbiased estimators of the location and scale parameter of Burr distribution when shape parameter is known, Genc (2015) derived explicit expression of single and product moments of order statistics from the generalized *t* distribution, Sultan and Al-Thubyani (2016) considered single, double, triple and quadruple moments of order statistics from Lindley distribution, Sehgal et al. (2018) introduced extended normal distribution with single and product moments of order statistics and other statistical measures, Kumar and Goyal (2019) derived the explicit forms of single and double moments of power Lindley order statistics for computing the variance and best linear unbiased estimators of shape and scale parameters, etc. The potentiality of moments of order statistics can be seen in many areas such as quality control testing (Nadarajah and Pal, 2008), reliability (Dembinska and Goroncy, 2020), etc. For instance, the failure of products or materials can be predicted by using the times of a few early failures that are based on moments of order statistics. Thus, the expensive product have more reliable that are high reliability with long lifetime or expensive cost. In this paper, the explicit expression for the single and product moments of GAD order statistics are derived.

The rest of paper is organized as follows. In Section 2, we provide the useful technical lemmas for the derivation of the explicit expressions of order statistics. The closed form expression of single and product moments are given in Section 3. In Section 4, the location-scale GAD and BLUEs for μ and σ^2 are obtained using single and product moments in previous Section. The numerical study and illustrate example are shown by using R programming language in Sections 5 and 6. Finally, conclusion are presented in Section 7.

2. Technical lemmas

In this section, The proof of four technical lemmas are provided.

Lemma 1 Let $f(x)$ and $F(x)$ be given by Eqns. (1) and (2), respectively. For $a > 0, b > 0$ and $p > 0$, let

$$I_p\left(a,b\right) = \int_0^\infty x^p f\left(x\right)\left[F\left(x\right)\right]^a\left[1-F\left(x\right)\right]^b dx.$$

Applying the function with binomial expansion, then

$$\begin{aligned}I_p\left(a,b\right) &= \frac{\theta^3}{\theta^2+2\alpha}\sum_{l_1=0}^{a}\sum_{l_2=0}^{l_1+b}\sum_{l_3=0}^{l_2}\sum_{l_4=0}^{1}\binom{a}{l_1}\binom{l_1+b}{l_2}\binom{l_2}{l_3}\left(-1\right)^{l_1}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2}\left(\frac{2}{\theta}\right)^{l_3}\\ &\times\frac{\alpha^{l_4}\Gamma\left(p+2l_2-l_3+2l_4+1\right)}{\left[\theta\left(l_1+b+1\right)\right]^{p+2l_2-l_3+2l_4+1}},\end{aligned}$$

where $\Gamma\left(z\right)$ is a complete gamma function, $\Gamma\left(z\right)=\int_0^\infty t^{z-1}e^t dt$.

Proof: The $\left[F\left(x\right)\right]^a$ and $\left[1-F\left(x\right)\right]^b$ are expanded binomially and apply Eqns. (1) and (2), $I_p\left(a,b\right)$ are obtained as

$$\begin{aligned}I_p\left(a,b\right) &= \sum_{l_1=0}^{a}\binom{a}{l_1}\left(-1\right)^{l_1}\int_0^\infty x^p f\left(x\right)\left[1-F\left(x\right)\right]^{l_1+b}dx\\ &= \frac{\theta^3}{\theta^2+2\alpha}\sum_{l_1=0}^{a}\sum_{l_2=0}^{l_1+b}\binom{a}{l_1}\binom{l_1+b}{l_2}\left(-1\right)^{l_1}\frac{1}{\left(\theta^2+2\alpha\right)^{l_2}}\\ &\times\int_0^\infty x^p\left(1+\alpha x^2\right)\left[\alpha\theta x\left(\theta x+2\right)\right]^{l_2}e^{-\theta\left(l_1+b+1\right)x}dx\\ &= \frac{\theta^3}{\theta^2+2\alpha}\sum_{l_1=0}^{a}\sum_{l_2=0}^{l_1+b}\sum_{l_3=0}^{l_2}\binom{a}{l_1}\binom{l_1+b}{l_2}\binom{l_2}{l_3}\left(-1\right)^{l_1}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2}\left(\frac{2}{\theta}\right)^{l_3}\\ &\times\int_0^\infty x^{p+2l_2-l_3}\left(1+\alpha x^2\right)e^{-\theta\left(l_1+b+1\right)x}dx\\ &= \frac{\theta^3}{\theta^2+2\alpha}\sum_{l_1=0}^{a}\sum_{l_2=0}^{l_1+b}\sum_{l_3=0}^{l_2}\sum_{l_4=0}^{1}\binom{a}{l_1}\binom{l_1+b}{l_2}\binom{l_2}{l_3}\left(-1\right)^{l_1}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2}\left(\frac{2}{\theta}\right)^{l_3}\\ &\times\alpha^{l_4}\int_0^\infty x^{p+2l_2-l_3+l_4}e^{-\theta\left(l_1+b+1\right)x}dx\end{aligned}$$

$$
\begin{aligned}
I_p(a,b) &= \frac{\theta^3}{\theta^2+2\alpha}\sum_{l_1=0}^{a}\sum_{l_2=0}^{l_1+b}\sum_{l_3=0}^{l_2}\sum_{l_4=0}^{1}\binom{a}{l_1}\binom{l_1+b}{l_2}\binom{l_2}{l_3}(-1)^{l_1}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2}\left(\frac{2}{\theta}\right)^{l_3}\alpha^{l_4} \\
&\times \frac{\int_0^\infty z^{p+2l_2-l_3+l_4}e^{-z}dz}{\left[\theta(l_1+b+1)\right]^{p+2l_2-l_3+2l_4+1}} \\
&= \frac{\theta^3}{\theta^2+2\alpha}\sum_{l_1=0}^{a}\sum_{l_2=0}^{l_1+b}\sum_{l_3=0}^{l_2}\sum_{l_4=0}^{1}\binom{a}{l_1}\binom{l_1+b}{l_2}\binom{l_2}{l_3}(-1)^{l_1}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2}\left(\frac{2}{\theta}\right)^{l_3}\alpha^{l_4} \\
&\times \frac{\Gamma(p+2l_2-l_3+2l_4+1)}{\left[\theta(l_1+b+1)\right]^{p+2l_2-l_3+2l_4+1}},
\end{aligned}
$$

where $z=\theta(l_1+b+1)x$.

Lemma 2 Let $f(x)$ and $F(x)$ be given by Eqns. (1) and (2), respectively. For $a>0, b>0, p>0$ and $q>0$, let

$$I_{p,q}(a,b,c)=\int_0^\infty\int_x^\infty x^p y^q f(x)f(y)\left[F(x)\right]^a\left[F(y)-F(x)\right]^b\left[1-F(y)\right]^c dydx.$$

Applying the function with binomial expansion, then

$$
\begin{aligned}
I_{p,q}(a,b,c) &= \frac{\theta^6}{(\theta^2+2\alpha)^2}\sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\sum_{l_3=0}^{b+l_1-l_2}\sum_{l_4=0}^{c+l_2}\sum_{l_5=0}^{l_3}\sum_{l_6=0}^{l_4}\sum_{l_7=0}^{1}\sum_{l_8=0}^{1}\sum_{l_9=0}^{q+2l_4-l_6+2l_8}\binom{a}{l_1}\binom{b}{l_2} \\
&\times \binom{b+l_1-l_2}{l_3}\binom{c+l_2}{l_4}\binom{l_3}{l_5}\binom{l_4}{l_6}(-1)^{l_1+l_2}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4}\left(\frac{2}{\theta}\right)^{l_5+l_6} \\
&\times \frac{\alpha^{l_7+l_8}(q+2l_4-l_6+2l_8)!\,\Gamma(p+2l_3-l_5+2l_7+l_9+1)}{l_9!\left[\theta(c+l_2+1)\right]^{q+2l_4-l_6+2l_8-l_9+1}\left[\theta(b+c+l_1+2)\right]^{p+2l_3-l_5+2l_7+l_9+1}},
\end{aligned}
$$

where $\Gamma(z)$ is a complete gamma function, $\Gamma(z)=\int_0^\infty t^{z-1}e^t dt$.

Proof: The $\left[F(x)\right]^a$, $\left[F(y)-F(x)\right]^b$ and $\left[1-F(y)\right]^c$ are expanded binomially and apply Eqns. (1) and (2), the function $I_{p,q}(a,b,c)$ can be seen that

$$
\begin{aligned}
I_{p,q}(a,b,c) &= \sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\binom{a}{l_1}\binom{b}{l_2}(-1)^{l_1+l_2} \\
&\times \int_0^\infty\int_x^\infty x^p y^q f(x)f(y)\left[1-F(x)\right]^{b+l_1-l_2}\left[1-F(y)\right]^{c+l_2}dydx \\
&= \frac{\theta^6}{(\theta^2+2\alpha)^2}\sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\sum_{l_3=0}^{b+l_1-l_2}\sum_{l_4=0}^{c+l_2}\binom{a}{l_1}\binom{b}{l_2}\binom{b+l_1-l_2}{l_3}\binom{c+l_2}{l_4} \\
&\times \frac{(-1)^{l_1+l_2}}{(\theta^2+2\alpha)^{l_3+l_4}}\int_0^\infty\int_x^\infty x^p y^q e^{-\theta(b+l_1-l_2+1)x}e^{-\theta(c+l_2+1)y}\left[\alpha\theta x(\theta x+2)\right]^{l_3} \\
&\times \left[\alpha\theta y(\theta y+2)\right]^{l_4}\left(1+\alpha x^2\right)\left(1+\alpha y^2\right)dydx
\end{aligned}
$$

$$
\begin{aligned}
I_{p,q}\left(a,b,c\right) &= \frac{\theta^6}{(\theta^2+2\alpha)^2}\sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\sum_{l_3=0}^{b+l_1-l_2}\sum_{l_4=0}^{c+l_2}\sum_{l_5=0}^{l_3}\sum_{l_6=0}^{l_4}\binom{a}{l_1}\binom{b}{l_2}\binom{b+l_1-l_2}{l_3}\\
&\times\binom{c+l_2}{l_4}\binom{l_3}{l_5}\binom{l_4}{l_6}(-1)^{l_1+l_2}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4}\left(\frac{2}{\theta}\right)^{l_5+l_6}\\
&\times\int_0^\infty x^{p+2l_3-l_5}\left(1+\alpha x^2\right)e^{-\theta(b+l_1-l_2+1)x}\\
&\times\int_x^\infty y^{q+2l_4-l_6}\left(1+\alpha y^2\right)e^{-\theta(c+_2+1)y}dydx\\
&= \frac{\theta^6}{(\theta^2+2\alpha)^2}\sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\sum_{l_3=0}^{b+l_1-l_2}\sum_{l_4=0}^{c+l_2}\sum_{l_5=0}^{l_3}\sum_{l_6=0}^{l_4}\sum_{l_7=0}^{1}\sum_{l_8=0}^{1}\binom{a}{l_1}\binom{b}{l_2}\binom{b+l_1-l_2}{l_3}\\
&\times\binom{c+l_2}{l_4}\binom{l_3}{l_5}\binom{l_4}{l_6}(-1)^{l_1+l_2}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4}\left(\frac{2}{\theta}\right)^{l_5+l_6}\alpha^{l_7+l_8}\\
&\times\int_0^\infty x^{p+2l_3-l_5+2l_7}e^{-\theta(b+l_1-l_2+1)x}\int_x^\infty y^{q+2l_4-l_6+2l_8}e^{-\theta(c+l_2+1)y}dydx\\
&= \frac{\theta^6}{(\theta^2+2\alpha)^2}\sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\sum_{l_3=0}^{b+l_1-l_2}\sum_{l_4=0}^{c+l_2}\sum_{l_5=0}^{l_3}\sum_{l_6=0}^{l_4}\sum_{l_7=0}^{1}\sum_{l_8=0}^{1}\binom{a}{l_1}\binom{b}{l_2}\binom{b+l_1-l_2}{l_3}\\
&\times\binom{c+l_2}{l_4}\binom{l_3}{l_5}\binom{l_4}{l_6}(-1)^{l_1+l_2}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4}\left(\frac{2}{\theta}\right)^{l_5+l_6}\alpha^{l_7+l_8}\\
&\times\int_0^\infty x^{p+2l_3-l_5+2l_7}e^{-\theta(b+l_1-l_2+1)x}\frac{\int_{\theta(c+l_2+1)x}^\infty z^{(q+2l_4-l_6+2l_8+1)-1}e^{-z}dz}{\left[\theta\left(c+l_2+1\right)\right]^{q+2l_4-l_6+2l_8+1}}dx,
\end{aligned}
$$

where $z=\theta(c+l_2+1)y$. Apply incomplete gamma function to the integration of function z, $I_{p,q}\left(a,b,c\right)$ can be written as

$$
\begin{aligned}
I_{p,q}\left(a,b,c\right) &= \frac{\theta^6}{(\theta^2+2\alpha)^2}\sum_{l_1=0}^{a}\sum_{l_2=0}^{b}\sum_{l_3=0}^{b+l_1-l_2}\sum_{l_4=0}^{c+l_2}\sum_{l_5=0}^{l_3}\sum_{l_6=0}^{l_4}\sum_{l_7=0}^{1}\sum_{l_8=0}^{1}\binom{a}{l_1}\binom{b}{l_2}\binom{b+l_1-l_2}{l_3}\\
&\times\binom{c+l_2}{l_4}\binom{l_3}{l_5}\binom{l_4}{l_6}(-1)^{l_1+l_2}\left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4}\left(\frac{2}{\theta}\right)^{l_5+l_6}\alpha^{l_7+l_8}\\
&\times\int_0^\infty x^{p+2l_3-l_5+2l_7}e^{-\theta(b+l_1-l_2+1)x}\\
&\times\frac{\Gamma\left[q+2l_4-l_6+2l_8+1,\theta\left(c+l_2+1\right)x\right]}{\left[\theta\left(c+l_2+1\right)\right]^{q+2l_4-l_6+2l_8+1}}dx.
\end{aligned}
$$

Using the definition of the complementary incomplete gamma function,
$\Gamma\left(n,z\right)=(n-1)!e^{-z}\sum_{l=0}^{n-1}\frac{z^l}{l!}$, the function will be

$$
\begin{aligned}
I_{p,q}(a,b,c) &= \frac{\theta^6}{(\theta^2+2\alpha)^2} \sum_{l_1=0}^{a} \sum_{l_2=0}^{b} \sum_{l_3=0}^{b+l_1-l_2} \sum_{l_4=0}^{c+l_2} \sum_{l_5=0}^{l_3} \sum_{l_6=0}^{l_4} \sum_{l_7=0}^{1} \sum_{l_8=0}^{1} \binom{a}{l_1} \binom{b}{l_2} \binom{b+l_1-l_2}{l_3} \\
&\times \binom{c+l_2}{l_4} \binom{l_3}{l_5} \binom{l_4}{l_6} (-1)^{l_1+l_2} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4} \left(\frac{2}{\theta}\right)^{l_5+l_6} \alpha^{l_7+l_8} \\
&\times \frac{[\theta(c+l_2+1)]^{l_9} (q+2l_4-l_6+2l_8)!}{l_9! \, [\theta(c+l_2+1)]^{q+2l_4-l_6+2l_8+1}} \\
&\times \int_0^\infty x^{p+2l_3-l_5+2l_7+l_9} e^{-\theta(b+c+l_1+2)x} dx \\
&= \frac{\theta^6}{(\theta^2+2\alpha)^2} \sum_{l_1=0}^{a} \sum_{l_2=0}^{b} \sum_{l_3=0}^{b+l_1-l_2} \sum_{l_4=0}^{c+l_2} \sum_{l_5=0}^{l_3} \sum_{l_6=0}^{l_4} \sum_{l_7=0}^{1} \sum_{l_8=0}^{1} \binom{a}{l_1} \binom{b}{l_2} \binom{b+l_1-l_2}{l_3} \\
&\times \binom{c+l_2}{l_4} \binom{l_3}{l_5} \binom{l_4}{l_6} (-1)^{l_1+l_2} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4} \left(\frac{2}{\theta}\right)^{l_5+l_6} \frac{\alpha^{l_7+l_8}}{l_9!} \\
&\times \frac{(q+2l_4-l_6+2l_8)!}{[\theta(c+l_2+1)]^{q+2l_4-l_6+2l_8-l_9+1} \, [\theta(b+c+l_1+2)]^{p+2l_3-l_5+2l_7+l_9+1}} \\
&\times \int_0^\infty w^{(p+2l_3-l_5+2l_7+l_9+1)-1} e^{-w} dw \\
&= \frac{\theta^6}{(\theta^2+2\alpha)^2} \sum_{l_1=0}^{a} \sum_{l_2=0}^{b} \sum_{l_3=0}^{b+l_1-l_2} \sum_{l_4=0}^{c+l_2} \sum_{l_5=0}^{l_3} \sum_{l_6=0}^{l_4} \sum_{l_7=0}^{1} \sum_{l_8=0}^{1} \binom{a}{l_1} \binom{b}{l_2} \binom{b+l_1-l_2}{l_3} \\
&\times \binom{c+l_2}{l_4} \binom{l_3}{l_5} \binom{l_4}{l_6} (-1)^{l_1+l_2} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4} \left(\frac{2}{\theta}\right)^{l_5+l_6} \frac{\alpha^{l_7+l_8}}{l_9!} \\
&\times \frac{(q+2l_4-l_6+2l_8)! \, \Gamma(p+2l_3-l_5+2l_7+l_9+1)}{[\theta(c+l_2+1)]^{q+2l_4-l_6+2l_8-l_9+1} \, [\theta(b+c+l_1+2)]^{p+2l_3-l_5+2l_7+l_9+1}},
\end{aligned}
$$

where $w = \theta(b+c+l_1+2)x$.

Lemma 3 Let $f(x)$ and $F(x)$ be given by Eqns. (1) and (2), respectively. For $a > 0, b > 0$ and $t > 0$, let

$$
I_t(a,b) = \int_0^\infty e^{tx} f(x) [F(x)]^a [1-F(x)]^b \, dx.
$$

Applying the function with binomial expansion, then

$$
\begin{aligned}
I_t(a,b) &= \frac{\theta^3}{\theta^2+2\alpha} \sum_{l_1=0}^{a} \sum_{l_2=0}^{l_1+b} \sum_{l_3=0}^{l_2} \sum_{l_4=0}^{1} \binom{a}{l_1} \binom{l_1+b}{l_2} \binom{l_2}{l_3} (-1)^{l_1} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2} \left(\frac{2}{\theta}\right)^{l_3} \alpha^{l_4} \\
&\times \frac{\Gamma(2l_2-l_3+2l_4+1)}{[\theta(l_1+b+1)-t]^{2l_2-l_3+2l_4+1}},
\end{aligned}
$$

Proof: It is similar to the proof of Lemma 1.

Lemma 4 Let $f(x)$ and $F(x)$ be given by Eqns. (1) and (2), respectively. For $a > 0, b > 0, c > 0, t_1 > 0$ and $t_2 > 0$, let

$$I_{t_1,t_2}(a,b,c) = \int_0^\infty \int_x^\infty e^{t_1 x} e^{t_2 y} f(x) f(y) [F(x)]^a [F(y) - F(x)]^b [1 - F(y)]^c \, dy dx.$$

Applying the function with binomial expansion, then

$$\begin{aligned} I_{t_1,t_2}(a,b,c) &= \frac{\theta^6}{(\theta^2+2\alpha)^2} \sum_{l_1=0}^{a} \sum_{l_2=0}^{b} \sum_{l_3=0}^{b+l_1-l_2} \sum_{l_4=0}^{c+l_2} \sum_{l_5=0}^{l_3} \sum_{l_6=0}^{l_4} \sum_{l_7=0}^{1} \sum_{l_8=0}^{1} \sum_{l_9=0}^{2l_4-l_6+2l_8} \binom{a}{l_1} \binom{b}{l_2} \\ &\times \binom{b+l_1-l_2}{l_3} \binom{c+l_2}{l_4} \binom{l_3}{l_5} \binom{l_4}{l_6} (-1)^{l_1+l_2} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha} \right)^{l_3+l_4} \\ &\times \left(\frac{2}{\theta} \right)^{l_5+l_6} \frac{\alpha^{l_7+l_8} (2l_4 - l_6 + 2l_8)!}{l_9! [\theta(c+l_2+1) - t_2]^{2l_4-l_6+2l_8-l_9+1}} \\ &\times \frac{\Gamma(2l_3 - l_5 + 2l_7 + l_9 + 1)}{[\theta(b+c+l_1+2) - (t_1+t_2)]^{2l_3-l_5+2l_7+l_9+1}}. \end{aligned}$$

Proof: It is similar to the proof of Lemma 2.

3. Moments of Order Statistics

In this section, the explicit expressions for single and product moments, and marginal and joint moment generating function of order statistics for generalized Akash random sample are derived. Moments of order statistics play important role in reliability model to predict the failure items based in the times of few early failures. Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the order statistics corresponding to $X_1, X_2, \ldots, X_n$. The pdf of the rth order statistics is

$$f_{r:n}(x) = C_{r:n} f(x) F^{r-1}(x) [1 - F(x)]^{n-r}. \tag{3}$$

The joint pdf of the rth and sth order statistics,$1 \le r < s \le n$, is given by

$$f_{r,s:n}(x,y) = C_{r,s:n} f(x) f(y) F^{r-1}(x) [F(y) - F(x)]^{s-r-1} [1 - F(x)]^{n-s}, \tag{4}$$

where $C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$ and $C_{r,s:n} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$.

3.1. Single moments

The single moments of order statistics are very important to calculate the mean and draw the inferential techniques for the underlying distribution. The explicit expression for single moments of rth order statistics, $\mu_{r:n}^{(p)} = E\left[X_{r:n}^{(p)}\right]$, is established.

Theorem 1 For generalized Akash distribution and $1 \le r \le n$, the single moments of rth order statistic is given by

$$\begin{aligned} \mu_{r:n}^{(p)} &= \frac{\theta^3 C_{r:n}}{\theta^2+2\alpha} \sum_{l_1=0}^{r-1} \sum_{l_2=0}^{l_1+n-r} \sum_{l_3=0}^{l_2} \sum_{l_4=0}^{1} \binom{r-1}{l_1} \binom{l_1+n-r}{l_2} \binom{l_2}{l_3} \\ &\times (-1)^{l_1} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha} \right)^{l_2} \left(\frac{2}{\theta} \right)^{l_3} \frac{\alpha^{l_4} \Gamma(p + 2l_2 - l_3 + 2l_4 + 1)}{[\theta(l_1+n-r+1)]^{p+2l_2-l_3+2l_4+1}}. \end{aligned} \tag{5}$$

Proof: Using Eqn. (3) and Lemma 1, then

$$\begin{aligned} \mu_{r:n}^{(p)} &= C_{r:n} \int_0^\infty x^p f(x) [F(x)]^{r-1} [1 - F(x)]^{n-r} dx \\ &= C_{r:n} \times I_p(r-1, n-r). \end{aligned}$$

The validity of the single moments of order statistics in Equation (3) can be checked by using (Arnold et al., 2008)

$$\sum_{r=1}^{n} \mu_{r:n} = nE\left(X\right). \tag{6}$$

For the computability of the moments, the illustration of expected values, second moments and variances of order statistics for some value of parameters θ and α are shown in Table 1.

Table 1 Means and variances of rth order statistics from generalized Akash distribution for $n = 1, 2, ..., 7, \theta = 1$ and $\alpha = 0.5$

		r						
n		1	2	3	4	5	6	7
1	Mean	2.000000						
	Variance	3.000000						
2	Mean	1.078125	2.921875					
	Variance	1.009521	3.290771					
3	Mean	0.733539	1.767297	3.499164				
	Variance	0.510617	1.294893	3.288922				
4	Mean	0.552351	1.277103	2.257490	3.913055			
	Variance	0.305721	0.731358	1.377850	3.240723			
5	Mean	0.440989	0.997799	1.696060	2.631777	4.233375		
	Variance	0.201651	0.473970	0.824898	1.396258	3.188816		
6	Mean	0.365915	0.816357	1.360682	2.031439	2.931946	4.493661	
	Variance	0.141876	0.331445	0.561494	0.863346	1.392409	3.141605	
7	Mean	0.312064	0.689026	1.134686	1.662010	2.308510	3.181320	4.712384
	Variance	0.104604	0.243707	0.408923	0.606023	0.877211	1.380831	3.100188

Theorem 2 For generalized Akash distribution with $1 \leq r \leq n$ and $t > 0$, the marginal moment generating function of rth order statistics is

$$\begin{aligned} E\left(e^{tX_{r:n}}\right) &= \frac{\theta^3 C_{r:n}}{\theta^2 + 2\alpha} \sum_{l_1=0}^{r-1} \sum_{l_2=0}^{l_1+n-r} \sum_{l_3=0}^{l_2} \sum_{l_4=0}^{1} \binom{r-1}{l_1} \binom{l_1+n-r}{l_2} \binom{l_2}{l_3} (-1)^{l_1} \\ &\times \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_2} \left(\frac{2}{\theta}\right)^{l_3} \alpha^{l_4} \frac{\Gamma\left(2l_2 - l_3 + 2l_4 + 1\right)}{\left[\theta\left(l_1+n-r+1\right)-t\right]^{2l_2-l_3+2l_4+1}}. \end{aligned} \tag{7}$$

Proof: Using Eqn. (3) and Lemma 3, then

$$\begin{aligned} E\left(e^{tX_{r:n}}\right) &= C_{r:n} \int_0^\infty e^{tx} f\left(x\right) \left[F\left(x\right)\right]^{r-1} \left[1-F\left(x\right)\right]^{n-r} dx \\ &= C_{r:n} \times I_t\left(r-1, n-r\right). \end{aligned}$$

3.2. Product moments

The product moments of order statistics are very important to calculate the covariance and draw the inferential techniques for the underlying distribution. The explicit expression for product moments of rth and sth order statistics, $\mu_{r,s:n}^{(p,q)} = E\left[X_{r:n}^{(p)} X_{s:n}^{(q)}\right]$, is established.

Theorem 3 For generalized Akash distribution with $1 \leq r < s \leq n$, the joint moments of rth and sth order statistics is given by

$$\mu_{r,s:n}^{(p,q)} = \frac{\theta^6 C_{r,s:n}}{(\theta^2+2\alpha)^2} \sum_{l_1=0}^{r-1} \sum_{l_2=0}^{s-r-1} \sum_{l_3=0}^{s-r+l_1-l_2-1} \sum_{l_4=0}^{n-s+l_2} \sum_{l_5=0}^{l_3} \sum_{l_6=0}^{l_4} \sum_{l_7=0}^{1} \sum_{l_8=0}^{1} \sum_{l_9=0}^{q+2l_4-l_6+2l_8}$$
$$\times \binom{r-1}{l_1} \binom{s-r-1}{l_2} \binom{s-r+l_1-l_2-1}{l_3} \binom{n-s+l_2}{l_4} \binom{l_3}{l_5} \binom{l_4}{l_6}$$
$$\times (-1)^{l_1+l_2} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4} \left(\frac{2}{\theta}\right)^{l_5+l_6} \frac{\alpha^{l_7+l_8}(q+2l_4-l_6+2l_8)!}{l_9!\left[\theta(n-s+l_2+1)\right]^{q+2l_4-l_6+2l_8-l_9+1}}$$
$$\times \frac{\Gamma(p+2l_3-l_5+2l_7+l_9+1)}{\left[\theta(n-r+l_1+1)\right]^{p+2l_3-l_5+2l_7+l_9+1}}, \tag{8}$$

where $\Gamma(\cdot)$ is the complete gamma function.
Proof: Using Eqn. (4) and Lemma 2, then

$$\mu_{r,s:n}^{(p,q)} = C_{r,s:n} \int_0^\infty \int_x^\infty x^p y^q f(x) f(y) \left[F(x)\right]^{r-1} \left[F(y)-F(x)\right]^{s-r-1}$$
$$\times \left[1-F(y)\right]^{n-s} dydx$$
$$= C_{r,s:n} \times I_{p,q}(r-1, s-r-1, n-s).$$

The validity of the product moments of order statistics in Eqns. (4) can be checked by using (Arnold et al., 2008)

$$\sum_{r=1}^{n-1} \sum_{s=r+1}^{n} \mu_{r,s:n} = \binom{n}{2} \mu_{1,1}^2 = \binom{n}{2} \left[E(X)\right]^2. \tag{9}$$

The covariance can be obtained as

$$\sigma_{r,s:n} = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)} \mu_{s:n}^{(1)}, \qquad 1 \leq r < s \leq n. \tag{10}$$

Theorem 4 For generalized Akash distribution with $1 \leq r < s \leq n$ and $t_1, t_2 > 0$, the joint moment generating function of rth and sth order statistics is

$$E\left(e^{t_1 X_{r:n} + t_2 X_{s:n}}\right) = \frac{\theta^6 C_{r,s:n}}{(\theta^2+2\alpha)^2} \sum_{l_1=0}^{r-1} \sum_{l_2=0}^{s-r-1} \sum_{l_3=0}^{s-r+l_1-l_2-1} \sum_{l_4=0}^{n-s+l_2} \sum_{l_5=0}^{l_3} \sum_{l_6=0}^{l_4} \sum_{l_7=0}^{1} \sum_{l_8=0}^{1}$$
$$\times \sum_{l_9=0}^{2l_4-l_6+2l_8} \binom{r-1}{l_1} \binom{s-r-1}{l_2} \binom{s-r+l_1-l_2-1}{l_3}$$
$$\times \binom{n-s+l_2}{l_4} \binom{l_3}{l_5} \binom{l_4}{l_6} (-1)^{l_1+l_2} \left(\frac{\alpha\theta^2}{\theta^2+2\alpha}\right)^{l_3+l_4} \left(\frac{2}{\theta}\right)^{l_5+l_6}$$
$$\times \frac{\alpha^{l_7+l_8}(2l_4-l_6+2l_8)!}{l_9!\left[\theta(n-s+l_2)-t_2\right]^{2l_4-l_6+2l_8-l_9+1}}$$
$$\times \frac{\Gamma(2l_3-l_5+2l_7+l_9+1)}{\left[\theta(n-r+l_1+1)-(t_1+t_2)\right]^{2l_3-l_5+2l_7+l_9+1}}. \tag{11}$$

Proof: Using Eqn. (4) and Lemma 4, then

$$E\left(e^{t_1 X_{r:n} + t_2 X_{s:n}}\right) = C_{r,s:n} \int_0^\infty \int_x^\infty e^{t_1 x} e^{t_2 y} f(x) f(y) \left[F(x)\right]^{r-1}$$
$$\times \left[F(y)-F(x)\right]^{s-r-1} \left[1-F(y)\right]^{n-s} dydx$$
$$= C_{r,s:n} \times I_{t_1,t_2}(r-1, s-r-1, n-s).$$

The product moments and covariance of rth and sth order statistics for some value of parameters θ and α are shown in Table 2.

Table 2 Product moments and covariances of rth and sth order statistics from generalized Akash distribution for $n = 1, 2, ..., 5, \theta = 2$ and $\alpha = 0.5, 1$

α	n	s	r	$\mu_{r,s:n}$	$\sigma_{r,s:n}$	α	n	s	r	$\mu_{r,s:n}$	$\sigma_{r,s:n}$
0.5	2	2	1	0.490000	0.128702	1	2	2	1	0.694444	0.170730
	3	2	1	0.181860	0.053018		3	2	1	0.265939	0.070306
	3	3	1	0.349394	0.059618		3	3	1	0.506783	0.087208
	3	3	2	0.938746	0.188163		3	3	2	1.310611	0.238598
	4	2	1	0.095533	0.030014		4	2	1	0.143316	0.042469
	4	3	1	0.158288	0.033112		4	3	1	0.242919	0.054109
	4	4	1	0.274308	0.030106		4	4	1	0.388483	0.037268
	4	3	2	0.380805	0.079738		4	3	2	0.545238	0.091494
	4	4	2	0.687952	0.100613		4	4	2	0.996218	0.152186
	4	4	3	1.343114	0.220985		4	4	3	1.850493	0.270249
	5	2	1	0.058408	0.018904		5	2	1	0.088890	0.027956
	5	3	1	0.090842	0.020268		5	3	1	0.141633	0.033572
	5	4	1	0.137901	0.019589		5	4	1	0.202879	0.025956
	5	5	1	0.231857	0.019488		5	5	1	0.329453	0.026352
	5	3	2	0.212345	0.049443		5	3	2	0.312992	0.062303
	5	4	2	0.332471	0.059377		5	4	2	0.516135	0.105695
	5	5	2	0.538810	0.048609		5	5	2	0.755896	0.052738
	5	4	3	0.578465	0.090578		5	4	3	0.811271	0.083397
	5	5	3	1.009803	0.134052		5	5	3	1.457347	0.210368
	5	5	4	1.709101	0.240958		5	5	4	2.327949	0.286333

4. Location-scale Extension and Best Linear Unbiased Estimator

Let Z be random variable from GAD, and μ and σ be location parameter and scale parameter of transformation $X = \mu + \sigma Z$, respectively. The pdf of X is given by

$$f(x; \theta, \alpha, \mu, \sigma) = \frac{\theta^3}{\theta^2 + 2\alpha}\left[1 + \alpha\left(\frac{x-\mu}{\sigma}\right)^2\right] e^{-\theta\left(\frac{x-\mu}{\sigma}\right)}; \quad x > \mu, \quad \sigma, \theta, \alpha > 0. \tag{12}$$

The best linear unbiased estimators (BLUEs) of location and scale parameters based on Type-II right censored sample are derived in this section. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n-c:n}$, $c = 0, 1, \ldots, n-1$, be Type-II right censored samples from location-scale GAD with location parameter μ, scale parameter σ and pdf in Equation (12). Let $Z_{r:n} = \frac{X_{r:n}-\mu}{\sigma}$ denote the corresponding GAD order statistics and $\mathbf{Z} = (Z_{1:n}, Z_{2:n}, ..., Z_{n-c:n})^T$ be the vector of order statistics. The expectation and covariance are given by, respectively,

$$E(Z_{r:n}) = \mu_{r:n}^{(1)}, \qquad 1 \leq r \leq (n-c), \tag{13}$$

$$Cov(Z_{r:n}, Z_{s:n}) = \sigma_{r,s:n} = \mu_{r,s:n}^{(1,1)} - \mu_{r:n}^{(1)}\mu_{s:n}^{(1)}, \qquad 1 \leq r < s \leq (n-c). \tag{14}$$

Let the corresponding matrices be denoted as

$$\begin{aligned}
\mathbf{X} &= (X_{1:n}, X_{2:n}, ..., X_{n-c:n})^T, \\
\boldsymbol{\mu} &= (\mu_{1:n}, \mu_{2:n}, ..., \mu_{n-c:n})^T, \\
\mathbf{1} &= (1, 1, ..., 1)^T, \\
\boldsymbol{\Sigma} &= [\sigma_{r,s}], \qquad\qquad 1 \leq r, s \leq n-c.
\end{aligned}$$

The BLUEs of μ and σ can be obtained as (Arnold et al., 2008), respectively,

$$\hat{\mu} = \sum_{r=1}^{n-c} a_r X_{r:n} = \left[\frac{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \mathbf{1}^T \boldsymbol{\Sigma}^{-1} - \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}}{\left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right) \left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right) - \left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right)^2} \right] \mathbf{X}, \tag{15}$$

$$\hat{\sigma} = \sum_{r=1}^{n-c} b_r X_{r:n} = \left[\frac{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1} \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} - \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu} \mathbf{1}^T \boldsymbol{\Sigma}^{-1}}{\left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right) \left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right) - \left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right)^2} \right] \mathbf{X}. \tag{16}$$

The coefficients a_r and b_r in Eqns. (15) and (16) can be checked by using the conditions

$$\sum_{r=1}^{n-c} a_r = 1$$

and

$$\sum_{r=1}^{n-c} b_r = 0.$$

The corresponding variance and covariance are following

$$\operatorname{var}(\hat{\mu}) = \sigma^2 \left[\frac{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}}{\left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right) \left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right) - \left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right)^2} \right], \tag{17}$$

$$\operatorname{var}(\hat{\sigma}) = \sigma^2 \left[\frac{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right) \left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right) - \left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right)^2} \right] \tag{18}$$

and

$$\operatorname{cov}(\hat{\mu}, \hat{\sigma}) = -\sigma^2 \left[\frac{\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}}{\left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \boldsymbol{\mu}\right) \left(\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right) - \left(\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}\right)^2} \right]. \tag{19}$$

5. Simulation Study

In this section, the computation is done by using `R` programming language. The random sample follows location-scale GAD with $\mu = 0, \sigma = 1, \theta = 1$ and $\alpha = 0.25$. The coefficients of BLUEs, and variance and covariance for different values of sample size $(n = 6, 10)$, and various censoring cases $(c = 0, 1, 2, 3)$ are provided in Tables 3 and 4, respectively.

Table 3 Coefficients for the BLUE of μ and σ when $n = 6, 10$ and $c = 0, 1, 2, 3$

		$c = 0$		$c = 1$		$c = 2$		$c = 3$	
n	r	a_r	b_r	a_r	b_r	a_r	b_r	a_r	b_r
6	1	1.1200	-0.5431	1.0597	0.1152	1.1609	-0.5534	1.3963	-1.5792
	2	0.0338	0.1156	0.0067	0.4108	0.1494	-0.5320	-0.0906	0.5138
	3	-0.0177	0.1178	0.1366	-1.5660	-0.1765	0.5026	-0.3057	1.0654
	4	-0.0646	0.0461	-0.0831	0.2482	-0.1338	0.5829		
	5	-0.0578	0.1146	-0.1198	0.7918				
	6	-0.0136	0.1490						
10	1	1.1019	-0.5073	1.1031	-0.5230	1.1513	-0.8703	1.1898	-1.1177
	2	0.0097	-0.0662	0.0147	-0.1352	-0.0683	0.4636	-0.0367	0.2603
	3	-0.0965	0.4266	-0.1147	0.6775	0.0010	-0.1564	0.1982	-1.4212
	4	0.0363	-0.1796	0.0486	-0.3483	0.0574	-0.4120	-0.2941	1.8431
	5	0.0307	-0.1779	0.0403	-0.3102	-0.0700	0.4846	-0.1138	0.7661
	6	-0.0233	0.1332	-0.0287	0.2065	-0.0408	0.2942	0.0698	-0.4155
	7	-0.0200	0.1146	-0.0231	0.1577	-0.0112	0.0721	-0.0132	0.0849
	8	-0.0197	0.1105	-0.0221	0.1442	-0.0193	0.1241		
	9	-0.0166	0.1100	-0.0182	0.1309				
	10	-0.0026	0.0361						

From Table 4, the results show that the variances of the BLUE for μ and σ decrease as sample size increases while the variances of the BLUE for μ and σ tend to increase when the censoring level increases.

Table 4 Variance and covariance of BLUEs when $n = 6, 10$ and $c = 0, 1, 2, 3$

n	c	$\mathrm{var}(\hat{\mu})$	$\mathrm{var}(\hat{\sigma})$	$\mathrm{cov}(\hat{\mu}, \hat{\sigma})$
6	0	0.091863	0.175958	-0.047320
	1	0.090956	0.067941	-0.037424
	2	0.095412	0.262508	-0.066868
	3	0.106449	0.471992	-0.114953
10	0	0.028828	0.095727	-0.014680
	1	0.028837	0.097484	-0.014808
	2	0.029862	0.150730	-0.022197
	3	0.032549	0.261351	-0.039438

6. Real Data Analysis

The data set from Bhaumik et al. (2009) presents the concentration of vinyl chloride (in mg/L) from clean upgradient ground-water monitoring wells. Descriptive statistics of the actual data set are shown in Table 5. The random samples of size 10 are as follow 0.6, 5.1, 1.2, 2.0, 1.0, 0.4, 0.4, 3.2, 2.7 and 0.4. For the corresponding samples, the maximum likelihood estimations of θ and α based on GAD are 1 and 0.25, respectively. The graph of empirical cumulative distribution function (ecdf) is shown in Figure 3.

Table 5 Descriptive statistics of data set

Mean	1.8790
Median	1.1500
Standard Deviation	1.9526
Variance	3.8126
Minimum	0.1000
Maximum	8.0000

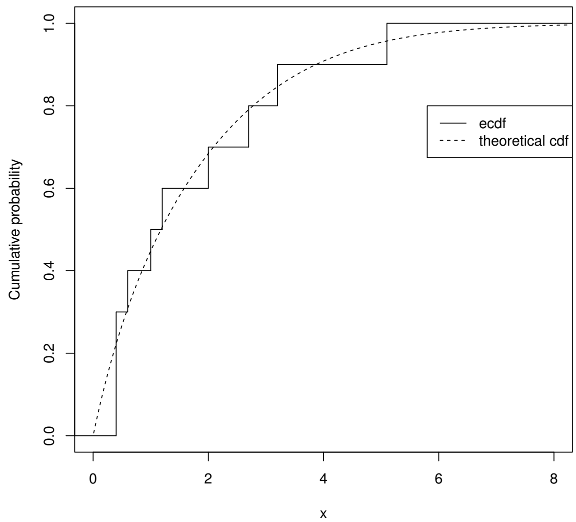

Figure 3 The ecdf of the sample based on GAD

To ensure that the random sample follows GAD, Kolmogorov-Smirnov statistic (K-S) is provided. The K-S statistics and its p-value are 0.22243 and 0.7056, respectively. It can conclude that the random sample comes from the population of GAD. By using Eqns. (15) and (16), and the BLUE coefficients in Table 3 when $c = 0$, the estimates of μ and σ will be

$$\hat{\mu} = \sum_{r=1}^{10} a_r X_{r:n} = 0.270897$$

and

$$\hat{\sigma} = \sum_{r=1}^{10} b_r X_{r:n} = 0.879042.$$

7. Conclusion

In this article, the single and product moments of GAD are derived in explicit forms. The single moment can be used to estimate the mean and variance of order statistic. The product moment is used to calculate the covariance of order statistics. The BLUE for location and scale parameter can be obtained by using both single and product moments. The variances of location and scale estimates are calculated to show the performance of the BLUEs for sample sizes up to 10 and several censoring cases. Then, we use these results to determine the BLUEs of parameters from location-scale GAD based on a Type-II censored sample and present an example to illustrate the results of the simulation.

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