Thailand Statistician
October 2023; 21(4): 759-766.
http://statassoc.or.th
Contributed paper

Distribution of Logit-type Link Function in a Generalized Quantile-based Asymmetric Distributional Setting

Md Rezaul Karim* **and Sejuti Haque**
Department of Statistics, Jahangirnagar University, Savar, Dhaka, Bangladesh
*Corresponding author; e-mail: rezaul@juniv.edu

Received: 31 March 2021
Revised: 2 October 2021
Accepted: 26 October 2021

Abstract

The article discusses the distribution of the logit-type link function for estimating quantile function in a generalized distributional setting. Besides this, the important properties including the quantile function of the proposed distribution are specified. In addition, the estimation of quantile function in a generalized quantile-based asymmetric family of the distributional framework via logit-type link function is proposed. The proposed method is illustrated in an actual data application concerning the daily proportion of SARS-CoV-2 infected people tested for COVID-19 infection.

Keywords: Logit-type link function, quantile estimation, quantile-based asymmetric family of distributions, SARS-CoV-2, COVID-19.

1. Introduction

The probability distribution is one of the fundamental statistical tools for statistical inferences. There are many classes of probability distributions are available in the literature [see, (Jones, 2015; Karim, 2019)]. Classical distributions such as normal distribution are very popular for estimating the convenient location (for example, mean, median, mode, etc.) of the response variable. But the mean provides a single characteristic of a distribution. It performs better result with excellent mathematical properties for the symmetric response variable. It is also not suitable when data comes from the skewed distribution [see, for example, Gijbels et al. (2019)]. When we focus on estimating parameters and asymptotic properties of the estimators, the exponential family is a very suitable class (Steland et al., 2019). The probability distribution function of an exponential family of the response variable Y can be written as

$$f_Y(y) = \exp\left(\frac{y\theta - b(\theta)}{a(\phi)} + c(y;\phi)\right), \tag{1}$$

where, $a(\cdot)$, $b(\cdot)$ and $c(\cdot,\cdot)$ are measurable functions [see for example, Fan et al. (1996)]. The parameter θ is called a canonical parameter and ϕ is a scale parameter. The mean and variance of Y are

$$E(Y) = b'(\theta) \qquad \text{and} \qquad \text{var}(Y) = \alpha(\phi)b'(\theta),$$

respectively. The function $g(b')^{-1}$, which links the mean to canonical parameter $(b')^{-1}(E(Y)) = \theta$ is called canonical link.

It is well known that the mean is highly influenced by extreme values. It is not usable when the quantile of distribution is the main interest [see for example, Koenker (2005)]. Therefore, Komunjer (2005) proposed a tick-exponential family for the (conditional) quantile estimation which is an analog to the linear exponential family (1) for the (conditional) mean estimation. The general form of tick exponential family for $y \in \mathbb{R}$ is given by

$$f_\alpha(y;\eta) = \alpha(1-\alpha)g\prime(y)\begin{cases} \exp\left[-(1-\alpha)(g(\eta)-g(y))\right] & \text{if} \quad y \leq \eta \\ \exp\left[\alpha(g(\eta)-g(y))\right] & \text{if} \quad y > \eta, \end{cases} \tag{2}$$

where η is the αth quantile of Y and g is a monotone link function. It is noted that the tick-exponential family (2) is only used for the whole real-line continuous variable. It is not useful for the semi-infinite supported response variable or boundary response variable. Besides this, asymmetric Laplace distribution is the only member of this family.

On the other hand, Koenker et al. (1978) proposed (conditional) quantile, which minimizes tick loss function. Unlike the mean function, the quantile function provides complete characteristics of the distribution. But this quantile proposed by Koenker et al. (1978) is actually nonparametric because it does not need the underlying parametric assumption. It is more robust to outliers than mean estimation. In conditional settings, it is only a tool used to find the effect of the covariate on different quantile levels of the response variable. A friendly discussion of conditional quantile estimation was presented by Koenker (2005). Many problems arise in nonparametric quantile due to the unknown underlying distribution—for example, crossing problem in quantile curves which leads to invalid inference, less efficiency, etc.

To solve the problems of nonparametric quantile, Gijbels et al. (2019) proposed a generalized quantile-based asymmetric family of distributions for estimating quantile function of any continuous variable Y which takes the form

$$f_\alpha^g(y;\eta,\phi) = \frac{2\alpha(1-\alpha)g'(y)}{\phi}\begin{cases} f\left((1-\alpha)\left(\frac{g(\eta)-g(y)}{\phi}\right)\right) & \text{if} \quad y \leq \eta \\ f\left(\alpha\left(\frac{g(y)-g(\eta)}{\phi}\right)\right) & \text{if} \quad y > \eta, \end{cases} \tag{3}$$

where, η is the location parameter (which is a αth quantile of Y), ϕ is the scale parameter and f is a unimodal symmetric density at zero.

One of the specialties of the family (3) is that the location parameter (η) is a specific quantile of this family. There are many members of this family available in the literature. For instance, asymmetric normal, asymmetric Laplace, asymmetric t, and at least three big families such as tick exponential family (2), asymmetric power family (see, Komunjer (2007)), quantile-based asymmetric family (see, Gijbels et al. (2019)) are are a subset of family (3). For any $\beta \in (0,1)$, the βth-quantile of $Y \sim f_\alpha^g(y;\eta,\phi)$ in (3) of the family equals

$$\{F_\alpha^g\}^{-1}(\beta;\eta,\phi) = \begin{cases} g^{-1}\left(g(\eta) + \frac{\phi}{1-\alpha}F^{-1}\left(\frac{\beta}{2\alpha}\right)\right) & \text{if} \quad \beta \leq \alpha \\ g^{-1}\left(g(\eta) + \frac{\phi}{\alpha}F^{-1}\left(\frac{1+\beta-2\alpha}{2(1-\alpha)}\right)\right) & \text{if} \quad \beta > \alpha, \end{cases} \tag{4}$$

with in particular $\{F_\alpha^g\}^{-1}(\alpha;\eta,\phi) = \eta$ and F^{-1} the quantile function associated to the reference symmetric density f. The family (3) and the quantile function (4) depend on two vital elements:

- the reference symmetric density f or its cumulative distribution function F and
- the monotone strictly increasing link function g.

When the link function is identity (i.e., $g(y) = y$) then family tends to quantile-based asymmetric family given in Gijbels et al. (2019). In this study, the main focus is to estimate the logit-type link function g.

Section 2 discusses the logit-type link function based on a general distribution function. The possible logit-type link functions for some known distribution are mentioned in this section. The distribution of the logit-type link function is derived here. We also discuss the distributional properties and parameter estimation of the proposed distribution in this section. Section 3 illustrates applying the proposed method for a real dataset. Finally, the concluding remarks are presented in Section 4.

2. Logit-type Link Function

Let the density of Y is a member of the generalized quantile-based family of distributions, and G is a distribution function of Y. Suppose g be a logit-type link function of Y depends on G such that

$$g(Y) = \text{logit}(G(Y)) = \ln\left(\frac{G(Y)}{1-G(Y)}\right). \tag{5}$$

If we know the distribution function of G, we can easily derive the logit-type link function by using (5). The link function of different probability distributions is presented in Table 1 in Appendix. The graphical presentation of the some link functions under some real-valued random variable and semi-infinite supported random variables are depicted in Figure 1. From Table 1, it is noticed that the link function of the logistic distribution is an "identity" function, i.e., $g(\eta) = \eta$ and Figure 1(a) also confirm this.

Since η is the αth quantile of Y and g is the monotone strictly increasing link function, then $g(\eta)$ is also the αth quantile of $Z = g(Y)$ [see for example, Koenker (2005)]. By introducing the αth quantile parameter $\mu \in \mathbb{R}$ and a scale parameter $\phi > 0$ in the density (7), we get

$$f_\alpha(z;\mu,\phi) = \frac{2\alpha(1-\alpha)}{\phi}\begin{cases} \dfrac{e^{-\alpha(\frac{z-\mu}{\phi})}}{\left(1+e^{-\alpha(\frac{z-\mu}{\phi})}\right)^2} & \text{if} \quad z > \mu \\ \dfrac{e^{-(1-\alpha)(\frac{\mu-z}{\phi})}}{\left(1+e^{-(1-\alpha)(\frac{\mu-z}{\phi})}\right)^2} & \text{if} \quad z \le \mu, \end{cases} \tag{6}$$

where $F_\alpha^{-1}(\alpha) = \mu$. The density given in (6) is denoted by ALD(μ, ϕ, α) and called quantile-based asymmetric logistic density (ALD) proposed in Gijbels et al. (2019). The graphical presentation of this distribution is displayed in Figure 2. From Figure 2, it is seen that the curves are unimodal and the densities are right-skewed (left-skewed) for the value of $\alpha < 0.50$ (for the value of $\alpha > 0.50$). The density is symmetric if and only if the index parameter $\alpha = 0.50$.

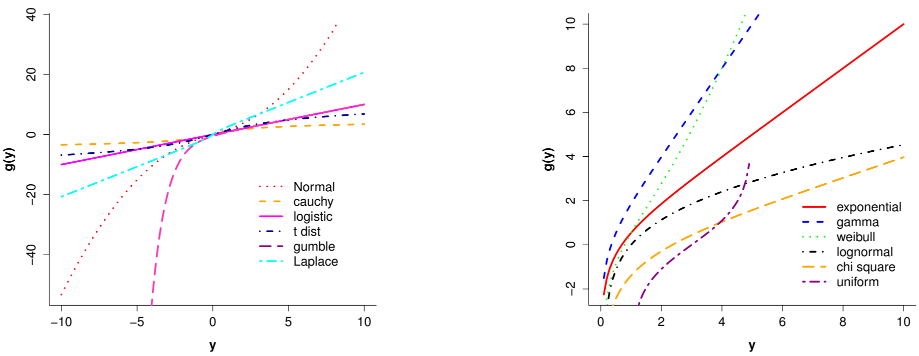

Figure 1 Link function curve for (a). the real-valued random variable; (b). the semi-infinite supported random variable

2.1. Distribution of logit-type link function

We now want to find the distribution of $Z = g(Y)$ when $Y \sim G$. We get

$$\begin{aligned}\Pr(Z \le z) &= \Pr\left(\ln\left(\frac{G(Y)}{1-G(Y)}\right) \le z\right)\\ &= \Pr\left(\frac{G(Y)}{1-G(Y)} \le e^{z}\right)\\ &= \Pr\left(G(Y) \le \frac{e^{z}}{1+e^{z}}\right) \qquad [\text{since}, \quad G(Y) \sim U(0,1)]\\ &= \frac{e^{z}}{1+e^{z}} = \frac{1}{1+e^{-z}} \qquad ; \quad -\infty < z < \infty.\end{aligned}$$

This is the cumulative distribution function of a standard logistic distribution. The probability density function of Z can be written as

$$f_Z(z) = \frac{d}{dz}\left[\frac{1}{1+e^{-z}}\right] = \frac{e^{-z}}{(1+e^{-z})^2}. \tag{7}$$

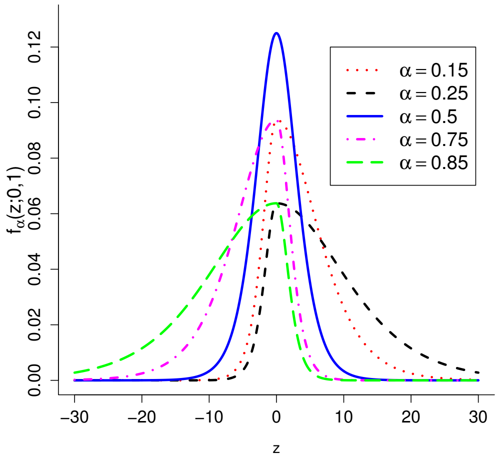

Figure 2 The density plots of a quantile-based asymmetric logistic distribution with $\alpha = (0.15, 0.25, 0.50, 0.75, 0.85)$th Quantile of $\mu = 0$ and $\phi = 1$

The mean and variance of $Y \sim \text{ALD}(\mu, \phi, \alpha)$ are following:

$$E(Z) = \mu + 2\phi\left[\frac{1-2\alpha}{\alpha(1-\alpha)}\right]\ln(2),$$

$$\text{var}(Z) = \frac{\phi^2}{\alpha^2(1-\alpha)^2}\left[(1-2\alpha)^2\left\{\frac{\pi^2}{3} - 4(\ln(2))^2\right\} + \frac{\pi^2\alpha(1-\alpha)}{3}\right].$$

We can easily find the cumulative distribution function and the quantile function of Z which respectively are

$$F_\alpha(z) = \begin{cases} \frac{2\alpha}{1+\exp\left\{-(1-\alpha)\left(\frac{z-\mu}{\phi}\right)\right\}}; & \text{if} \quad z < \mu \\ 2\alpha - 1 + \frac{2(1-\alpha)}{1+\exp\left\{-\alpha\left(\frac{z-\mu}{\phi}\right)\right\}}; & \text{if} \quad z \ge \mu, \end{cases}$$

and

$$F_\alpha^{-1}(\beta) = \begin{cases} \mu - \frac{\phi}{1-\alpha}\ln(\frac{2\alpha}{\beta} - 1); & \text{if} \quad \beta < \alpha \\ \mu - \frac{\phi}{\alpha}\ln(\frac{1-\beta}{\beta-2\alpha+1}); & \text{if} \quad \beta \geq \alpha. \end{cases}$$

Figure 3 depicts the cumulative distribution function (left panel) and the quantile function (right panel) when $Z \sim \text{ALD}(\mu, \phi, \alpha)$, for two values of α. Recall that $F_\alpha(\mu; \mu, \phi) = \alpha$ and $F_\alpha^{-1}(\alpha) = \mu$.

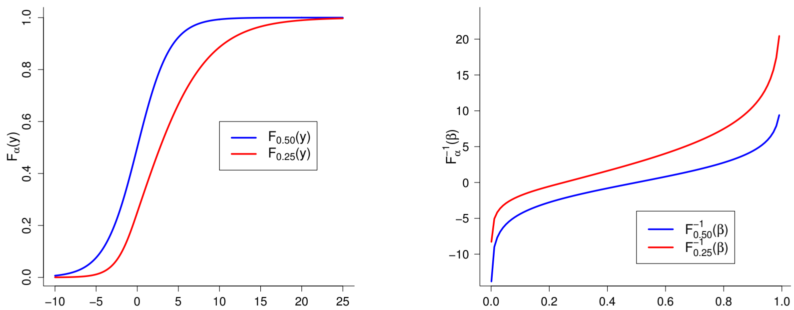

Figure 3 Cumulative distribution function (left) and quantile function (right) for $\mu = 0$, $\phi = 1$ and $\alpha = (0.25, 0.50)$

2.2. Maximum Likelihood Estimation

Based on an independent and identically distributed (i.i.d.) sample $Z_1, \ldots, Z_n$ from $Z \sim \text{ALD}(\mu, \phi, \alpha)$ the likelihood function of $\boldsymbol{\theta} = (\mu, \phi, \alpha)^T$ is defined by

$$L_n(\mu, \phi, \alpha) = \Big[\frac{2\alpha(1-\alpha)}{\phi}\Big]^n \prod_{i=1}^{n} \Big[\frac{e^{-(1-\alpha)(\frac{\mu - Z_i}{\phi})}}{\Big(1 + e^{-(1-\alpha)(\frac{\mu - Z_i}{\phi})}\Big)^2}\Big]^{\mathbb{I}(Z_i \leq \mu)} \times \Big[\frac{e^{-\alpha(\frac{Z_i - \mu}{\phi})}}{\Big(1 + e^{-\alpha(\frac{Z_i - \mu}{\phi})}\Big)^2}\Big]^{\mathbb{I}(Z_i > \mu)}.$$

And the log-likelihood function for $\boldsymbol{\theta} = (\mu, \phi, \alpha)^T$ can be written as

$$\begin{aligned} \ln[L_n(\mu, \phi, \alpha)] = n\ln[2\alpha(1-\alpha)] - n\ln(\phi) - (1-\alpha)\sum_{i=1}^{n}(\frac{\mu - Z_i}{\phi})\mathbb{I}(Z_i \leq \mu) \\ - 2\sum_{i=1}^{n}\ln\Big(1 + e^{-(1-\alpha)(\frac{\mu - Z_i}{\phi})}\Big)\mathbb{I}(Z_i \leq \mu) - \alpha\sum_{i=1}^{n}(\frac{Z_i - \mu}{\phi})\mathbb{I}(Z_i > \mu) \\ - 2\sum_{i=1}^{n}\ln\Big(1 + e^{-\alpha(\frac{Z_i - \mu}{\phi})}\Big)\mathbb{I}(Z_i > \mu). \end{aligned}$$

The MLE of μ, ϕ and α is obtained from the optimization problem $\max_{\boldsymbol{\theta} \in \boldsymbol{\Theta}} \ln[L_n(\mu, \phi, \alpha)]$. We now can easily estimate η by using the inverted link function $\eta = g^{-1}(\mu)$ and hence can easily estimate the quantile function by using (4).

Notably, this log-likelihood function is nonlinear and complex as well. It is also a nondifferentiable function at $y = \mu$. In this situation, we can estimate the parameter by using an algorithm offered by Gijbels et al. (2019). They also implemented this algorithm in the R package `QBAsyDist`.

3. Real Data Application

For illustrative purposes, we consider the daily proportion of severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) infected people who have tested for Coronavirus disease (COVID-19) infection from August 3, 2020 to February 12, 2021 in Bangladesh. The number of daily new SARS-CoV-2 infected cases and daily new tested peoples are reported by the Institute of Epidemiology Disease Control and Research (IEDCR), Dhaka, Bangladesh. The data are available on the website with web-link https://covid19.who.int/. Notice that the daily proportion of SARS-CoV-2 infected people (Y) is a bounded variable with support $[0, 1]$. It is observed that on average each day, 13.61% of peoples are infected who have tested for COVID-19 infection.

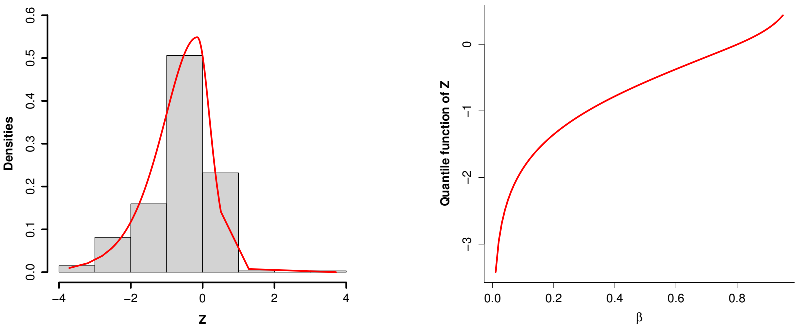

Figure 4 (a) Histogram and fitted density estimate (solid red line) of the uniform logit-type transformation of the proportion of daily SARS-CoV-2 infected people (left); (b) the estimated quantile function (right) the uniform logit-type transformation of the proportion of daily SARS-CoV-2 infected people

For the bounded random variable, we can not directly compute the quantile function of the distribution. Therefore, many authors including Bottai et al. (2010) and Columbu (2016) used the uniform logit-type link function in quantile estimation. That is the link function is $Z = \text{logit}(G(Y))$, where $G(y) = (y - a)/(b - a)$. We also consider this link function to estimate the quantile function of the proportion of daily SARS-CoV-2 infected cases among the people who have tested for COVID-19 infection on that day. In Section 2, we have shown that the distribution of Z is a quantile-based asymmetric logistic distribution given in (6).

For the uniform logit-type link function for this data set, we consider a as the minimum proportion of infected people minus k and b as the maximum infected people plus k, where k is very small number. In this case, we use $k = 0.01$. To add (subtract) a small value of k to b (a) to avoid the zero value of denomination (numerator) in the logit-type link function. The resulting link function is $z = g(y) = \ln\left(\frac{y-a}{b-y}\right)$ for $y \in (a, b)$. Using this link function $Z = g(Y)$, we estimate parameter $\boldsymbol{\theta} = (\mu, \phi, \alpha)^T$ of the distribution of Z via the method of maximum likelihood estimation. The maximum likelihood estimates of $\widehat{\boldsymbol{\theta}}$ are $(-0.1467, 0.1824, 0.7234)^T$. Using these maximum likelihood estimates, we draw estimated density and estimate quantile function. The histogram of Z with estimated density is presented in Figure 4 (a). From the histogram, it is clear that the variable Z is left-skewed which also confirm by getting $\widehat{\alpha} = 0.7234$ (¿0.5). Based on the estimated density, the estimated quantile function is also depicted in Figure 4 (b). We now can easily estimate the quantile function of Y using the link function. In this case, $Y = (a + be^Z)/(1 + e^Z)$ and for any $\beta \in (0, 1)$, the estimated βth-quantile of Y equals

$$\{\widehat{F}_{\alpha}^{g}\}^{-1}(\beta;\eta,\phi) = \begin{cases} \dfrac{\left(a+b\exp\left(\widehat{\mu}-\frac{\widehat{\phi}}{1-\widehat{\alpha}}\ln(\frac{2\widehat{\alpha}}{\beta}-1)\right)\right)}{1+\exp\left(\widehat{\mu}-\frac{\widehat{\phi}}{1-\widehat{\alpha}}\ln(\frac{2\widehat{\alpha}}{\beta}-1)\right)} & \text{if} \quad \beta \le \widehat{\alpha} \\ \dfrac{a+b\exp\left(\widehat{\mu}-\frac{\widehat{\phi}}{\widehat{\alpha}}\ln(\frac{1-\beta}{\beta-2\widehat{\alpha}+1})\right)}{1+\exp\left(\widehat{\mu}-\frac{\widehat{\phi}}{\widehat{\alpha}}\ln(\frac{1-\beta}{\beta-2\widehat{\alpha}+1})\right)} & \text{if} \quad \beta > \widehat{\alpha}. \end{cases}$$

The estimated quantile curve of Y and the QQ-plot are presented in Figure 5. The QQ-plot looks like a curve, but actually, it is very close to a straight line. Because the scale of both axes is tiny, therefore it seems like a curve. Otherwise, it is very close to the 45° line. This is confirmed by looking at the linear correlation coefficient of theoretical quantile and sample quantile which is 0.9863, indicating a nearly perfect relationship between theoretical quantile and sample quantile.

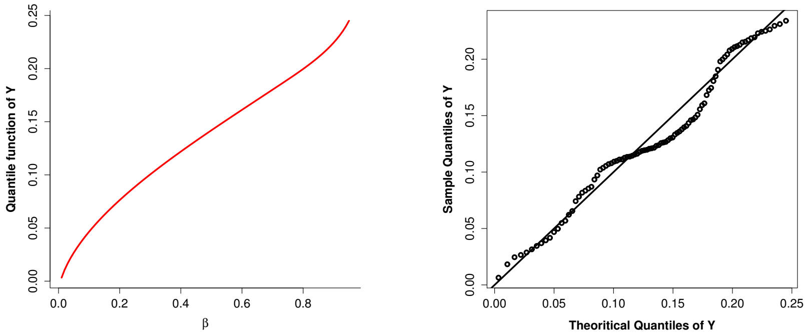

Figure 5 (a) Estimated quantile function of Y; (b) The QQ-plot (right)

4. Concluding Remarks

This article discusses the quantile estimation of the generalized quantile-based asymmetric family of distributions discussed in Gijbels et al. (2019). This quantile function depends on two fundamental elements: (i) the link function g and (ii) reference density f. We derive the distribution of the logit-type link function of the response variable Y. The properties with the parameter estimation of this distribution are discussed. A real data application regarding the proportion of daily SARS-Cov-2 infected people tested for COVID-19 infection is added to illustrate the proposed methods to estimate the quantile function. Note that the proportion of daily SARS-Cov-2 infected people tested for COVID-19 infection may depend on meteorological factors such as temperature, humidity, wind speed, etc. In that case, the parameter η and ϕ would be a function of these covariates. Therefore, further research would be to develop a method for estimating the conditional quantile function in the regression settings.

5. Declarations

- **Funding:** This research received no specific grant from any funding agency in the public, commercial, or not-for-profit sectors.
- **Conflicts of interest/Competing interests:** The authors have no conflicts of interest to declare.
- **Availability of data and material:** Yes
- **Code availability:** Yes
- **Authors' contributions:** The first author identified the research problem and supervised the whole work. He collected data and wrote a draft article. The second author analyzed data. All authors finalized the final version for the publications.
- **Ethics approval:** The Authors guarantee that the Contribution to the Work has not been previously published elsewhere.
- **Consent to participate:** Not applicable
- **Consent for publication:** We give our consent for the publication of identifiable details, which can include a photograph(s) and/or videos and/or case history and/or details within the text ("Material") to be published in the above Journal and Article.

Acknowledgements

We would like to thank the references for his comments and suggestions on the manuscript.

References

Jones MC. On families of distributions with shape parameters. Int Stat Rev. 2015; 83(1): 175–192.

Karim RM. Study of a family of asymmetric densities and flexible quantile regression. PhD [dissertation]. KU Leuven & Hasselt University; 2019.

Gijbels I, Karim R, Anneleen V. Quantile estimation in a generalized asymmetric distributional setting. In: Steland A, Rafajłowicz E, Okhrin O. Stochastic Models, Statistics and Their Application. SMSA 2019. Springer Proceedings in Mathematics & Statistics, vol 294; pp. 13-40.

Steland A and Rafajłowicz E, and Okhrin O. Stochastic Models, Statistics and Their Applications. SMSA 2019. Springer Proceedings in Mathematics & Statistics, vol. 294.

Fan, Jianqing and Gijbels I. Local Polynomial Modelling and Its Applications. London: Chapman & Hall/CRC; 1996.

Koenker R. Quantile regression. New York: Cambridge University Press; 2005.

Komunjer I. Quasi-maximum likelihood estimation for conditional quantiles. J Econom. 2005; 128(1): 137-164.

Koenker R, Bassett Jr G. Regression quantiles: Econometrica. 1978; 33-50.

Komunjer I. Asymmetric power distribution: Theory and applications to risk measurement. J Appl Econ. 2007; 22(1): 891-921.

Gijbels I, Karim R, Anneleen V. On Quantile-based Asymmetric Family of Distributions: Properties and Inference. Int Stat Rev. 2010; 87(3): 471-504.

Bottai M, Cai B and McKeown RE. Logistic quantile regression for bounded outcomes. Stat Medi. 2010; 29(2): 309-317.

Columbu S , Bottai M. Logistic Quantile Regression to Model Cognitive Impairment in Sardinian Cancer Patients. In: Battista DT, Moreno E, Racugno W. Topics on Methodological and Applied Statistical Inference. Springer. 2016; 65-73.

Appendix

Table 1 Link function for different distribution

Distribution name	Link function $g(y)$	Support
Normal	$\log\left(\frac{1+\text{erf}(y/\sqrt{2})}{1-\text{erf}(y/\sqrt{2})}\right)$	$y \in \mathbb{R}$
Exponential	$\log\left(\frac{1-\exp(-y)}{\exp(-y)}\right)$	$y \in [0, \infty)$
Gamma	$\log\left(\frac{\Gamma_y(\gamma)}{\gamma-\Gamma_y(\gamma)}\right)$	$y \in (0, \infty)$
Cauchy	$\log\left(\frac{0.5-\frac{\arctan(y)}{\pi}}{0.5-\frac{\arctan(y)}{\pi}}\right)$	$y \in (-\infty, \infty)$
Weibull	$\log\left(\frac{1-\exp(y^{\gamma})}{\exp(y^{\gamma})}\right)$	$y \in [0, \infty)$
Gumble	$\log\left(\frac{\exp[-\exp(-y)]}{1-\exp[-\exp(-y)]}\right)$	$y \in \mathbb{R}$
Lognormal	$\log\left(\frac{\phi(\ln(y)/\sigma\Big)}{1-\phi\Big(\ln(y)/\sigma)}\right)$	$y \in (0, \infty)$
Logistic	y	$y \in (-\infty, \infty)$
Laplace	$\begin{cases} \log\left(\frac{\frac{1}{2}\exp(y)}{1-\frac{1}{2}\exp(y)}\right) \text{ if } y \leq 0 \\ \log\left(\frac{1-\frac{1}{2}exp(y)}{\frac{1}{2}exp(y)}\right) \text{ if } y \geq 0 \end{cases}$	$y \in \mathbb{R}$
Chi-square	$\log\left(\frac{\frac{1}{\Gamma\frac{1}{2}}\gamma(\frac{1}{2}, \frac{y}{2})}{1-\frac{1}{\Gamma\frac{1}{2}}\gamma(\frac{1}{2}, \frac{y}{2})}\right)$	$y \in (-\infty, \infty)$ if $k = 1$ $y \in [-\infty, \infty)$ otherwise
Student-t	$\log\left(\frac{\frac{1}{2} + y\Gamma\left(\frac{1}{2}\right) \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -y^2\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}}{\frac{1}{2} - y\Gamma\left(\frac{1}{2}\right) \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; \frac{3}{2}; -y^2\right)}{\sqrt{\pi}\,\Gamma\left(\frac{1}{2}\right)}}\right)$	$y \in (-\infty, \infty)$
Uniform	$\log\big(y\ /\ (1-y)\big)$	$0 \leq y \leq 1$