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Bayesian Modelling of Exponentiated Weibull Generated Family for Interval-censored Data with rstan

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Abstract

The purpose of this paper is to fit the Exponentiated Weibull generated (EW-G) family to interval-censored survival data in the Bayesian environment, which is an extended version of the Weibull-G family and generated by Exponentiated Weibull random variable. Based on the EW-G family, three special models, namely Exponentiated Weibull Exponential, Exponentiated Weibull Weibull, and Exponentiated Weibull Rayleigh are applied for the analysis. The models are fitted in R using the probabilistic programming language Stan, which offers full Bayesian inference through Hamiltonian Monte Carlo algorithm and their extension No-U-turn sampler algorithm. Stan codes for the analysis are provided. We compared the model with the leave-one-out-cross validation information criterion, and we found that simpler models with fewer parameters give a better fitting to the data.

Keywords: Bayesian analysis, survival models, EW-generated family, interval censored data, LOO, R, Stan.

1. Introduction

In the past decades, numerous parametric models have been extensively used to model lifetime data. However, in many applied areas, such as survival analysis, insurance, and finance, there is a clear need of extended versions of these models. In recent years, researchers are introducing generated families that are generalized versions of classical distributions, because it is observed that generated families fit real data better than the parent distributions. Many authors have used generalized families to model the lifetime data under different censoring schemes. Sindhu et al. (2016) studied Bayesian estimation on the generalized logistic distribution under left type-II censoring. Mahmoud and Ghazal (2017) have discussed the estimations from the exponentiated Rayleigh distribution based on generalized Type-II hybrid censored data. Sindhu et al. (2017) have discussed the Bayesian analysis of the two-component mixture of lifetime distribution, with a particular case for Rayleigh distribution, under doubly censored samples. Almetwaly and Almongy (2018) have discussed the estimation of the generalized power Weibull distribution parameters using progressive censoring. AbuJarad and Khan (2018) have studied Bayesian survival analysis of Topp-Leone generalized family for right-censored data. Sindhu and Hussain (2020) studied the predictive inference and parameter estimation from the half-normal distribution for the left Censored data. AbuJarad et al. (2020) have used the Marshall-Olkin family to model the prognosis of women with breast cancer with the Bayesian approach.

In this paper, to model the interval-censored data, we have used the Exponentiated Weibull-generated (EW-G) family introduced by Hassan and Elgarhy (2016), which is an extended version

of Weibull-G family (Bourguignon et al., 2014) and generated by Exponentiated Weibull random variable. Exponentiated Weibull distribution is the generalization of the most popular Weibull distribution, which provides a more flexible model for analyzing lifetime data (Hassan and Elgarhy, 2016). Application of Exponentiated Weibull distribution in the field of survival and reliability were illustrated by Mudholkar et al. (1995). This paper illustrates the fitting of the Exponentiated Weibull generated (EW-G) family to the interval-censored data using the Bayesian approach. We have considered three models based on the EW-G family for analyzing real data. The first special model is Exponentiated Weibull Exponential (EWE) distribution proposed by Elgarhy et al. (2017), and the model was fitted to lifetime data. The second is Exponentiated Weibull Weibull (EWW) distribution analyzed by Hassan and Elgarhy (2016) and also discussed by Cordeiro et al. (2017), and finally, Exponentiated Weibull Rayleigh (EWR) discussed by Cordeiro et al. (2017) is considered. These three models have been used for fitting interval-censored survival data with the Bayesian approach.

For interval-censored data, the exact failure times are not known, but the failures are known to have occurred within an interval (c_1, c_2) , e.g., in clinical trials, patients are often seen at pre-scheduled visits, but the event of interest may have occurred in-between visits. These types of data are known as interval-censored data. Interval-censored data commonly arise in studies where there is a non-lethal point, such as recurrence of a disease or condition (Collett, 2015). If we set $c_1 = 0$ and $c_2 = \infty$ in the definition of an interval-censored observation, interval censoring becomes a natural generalization of the commonly encountered right censoring and less common left censoring (Bogaerts et al., 2017). Dealing with Interval censoring is very complicated as compared to right censoring and left censoring. The classical statistical methods are not capable of handling interval-censored data well. However, Bayesian methods can be applied to analyze interval-censored data effectively and efficiently.

The whole analysis is done by R (R Core Team, 2021) and Stan (Stan Development Team., 2017). Stan software is developed mainly for Bayesian analysis. Stan uses Hamiltonian Monte Carlo (HMC) algorithm (Duane et al., 1987) and its extension No-U-turn Sampler (NUT) algorithm (Hoffman and Gelman, 2014) for simulation. The Hamiltonian Monte Carlo is a more sophisticated and efficient Markov chain Monte Carlo (MCMC) algorithm (Duane et al., 1987), and it is a combination of MCMC and deterministic simulation methods. HMC and NUT sampler algorithms converge quickly, especially, for high dimensional models regardless of whether the priors are conjugate or not (Hoffman and Gelman, 2014). Stan codes have been developed for fitting models. The main objective of this paper is to fit three EW-G models to the same interval-censored data in the Bayesian environment through `rstan` (Stan Development Team, 2020) package of R and to compare models based on Leave One Out-Cross Validation information criterion (LOOIC) (Gelfand et al., 1992; Vehtari et al., 2015, 2017) which is a fully Bayesian criterion.

2. Exponentiated Weibull Generated Family

The cumulative distribution function (CDF) of the Exponentiated Weibull Generated (EW-G) family is given by

$$F(t) = \left[1 - \exp \left(-\alpha \left[\frac{G(t)}{1 - G(t)} \right]^\beta \right) \right]^{a-1}, \quad (1)$$

where $a, \beta > 0$ are the two shape parameters and $\alpha > 0$ is scale parameter. The eq(1) provides a wider family of continuous distributions. The probability density function (pdf) corresponding to Eqn. (1) is

$$f(t) = \frac{a\alpha\beta(G(t))^{\beta-1}g(t)}{(1 - G(t))^{\beta+1}} e^{-\alpha \left[\frac{G(t)}{1 - G(t)} \right]^\beta} \left[1 - \exp \left(-\alpha \left[\frac{G(t)}{1 - G(t)} \right]^\beta \right) \right]^{a-2}, \quad (2)$$

where $t > 0, a, \alpha, \beta > 0$.

2.1. Eponentiated Weibull exponential distribution

Based on the EW-G family, the exponentiated Weibull exponential (EWE) distribution is obtained by considering the baseline distribution G as the exponential distribution. Let the random variable T follows the following exponential distribution with scale parameter $\lambda > 0$,

$$g(t; \lambda) = \frac{1}{\lambda} e^{-\frac{t}{\lambda}}. \quad (3)$$

The CDF of the exponential distribution is given by

$$G(t; \lambda) = 1 - e^{-\frac{t}{\lambda}}. \quad (4)$$

Substituting Eqns. (3) and (4) into Eqn. (1) to get the CDF of exponentiated Weibull exponential distribution, EWE($a, \alpha, \beta, \lambda$) takes following form

$$F(t; \Psi) = \left[1 - \exp(-\alpha(e^{\frac{t}{\lambda}} - 1)^\beta) \right]^a, \quad (5)$$

where, $\Psi = (a, \alpha, \beta, \lambda)$ is the set of parameters, inserting Eqns. (3) and (4) into Eqn. (2), then the pdf of EWE takes the following form

$$f(t; \Psi) = \frac{a\alpha\beta}{\lambda} \times [e^{\frac{t}{\lambda}} - 1]^{\beta-1} \times \exp[-\{\alpha(e^{\frac{t}{\lambda}})^\beta - \frac{t}{\lambda}\}] \times [1 - \exp(-\alpha(e^{\frac{t}{\lambda}} - 1)^\beta)]^{a-1}. \quad (6)$$

Survival function of EWE is defined as

$$S(t; \Psi) = 1 - \left[1 - \exp(-\alpha(e^{\frac{t}{\lambda}} - 1)^\beta) \right]^a.$$

Hazard function is given by

$$h(t; \Psi) = \frac{f(t; \Psi)}{S(t; \Psi)}$$

$$h(t; \Psi) = \frac{\frac{a\alpha\beta}{\lambda} \times [e^{\frac{t}{\lambda}} - 1]^{\beta-1} \times \exp[-\{\alpha(e^{\frac{t}{\lambda}})^\beta - \frac{t}{\lambda}\}] \times [1 - \exp(-\alpha(e^{\frac{t}{\lambda}} - 1)^\beta)]^{a-1}}{1 - \left[1 - \exp(-\alpha(e^{\frac{t}{\lambda}} - 1)^\beta) \right]^a}.$$

The joint likelihood function of EWE for interval censored data in the interval (t_1, t_2) , right censored data in the interval (t_1, ∞) , and left censored data in the interval $(0, t_2)$ can be defined as

$$L(t; \Psi) = \begin{cases} \prod_{i=1}^n \{S(t_{1i}) - S(t_{2i})\}, & \delta_i = 0 \\ \prod_{i=1}^n S(t_{1i}), & \delta_i = 1 \\ \prod_{i=1}^n \{1 - S(t_{2i})\}, & \delta_i = 2. \end{cases}$$

$$L(t; \Psi) = \begin{cases} \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha(e^{\frac{t_{2i}}{\lambda}} - 1)^\beta) \right]^a - \left[1 - \exp(-\alpha(e^{\frac{t_{1i}}{\lambda}} - 1)^\beta) \right]^a \right\}, & \delta_i = 0 \\ \prod_{i=1}^n \left\{ 1 - \left[1 - \exp(-\alpha(e^{\frac{t_{1i}}{\lambda}} - 1)^\beta) \right]^a \right\}, & \delta_i = 1 \\ \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha(e^{\frac{t_{2i}}{\lambda}} - 1)^\beta) \right]^a \right\}, & \delta_i = 2. \end{cases}$$

Here t_1 and t_2 are lower limit and upper limit of the survival time, and δ_i is a variable which takes values 0,1,2 and is defined as

$$\delta_i = \begin{cases} 0, & \text{Interval censored} \\ 1, & \text{Right censored} \\ 2, & \text{Left censored.} \end{cases}$$

2.2. Exponentiated Weibull Weibull Distribution

The exponentiated Weibull Weibull (EWW) distribution is obtained assuming G as the weibull distribution in the EW-G family. Let the random variable T follows the following Weibull distribution with scale parameter $\lambda > 0$ and shape parameter $k > 0$,

$$g(t; k, \lambda) = \frac{k}{\lambda} \left(\frac{t}{\lambda}\right) \exp\left(-\left(\frac{t}{\lambda}\right)^k\right). \quad (7)$$

The CDF of Weibull distribution is given by

$$G(t; k, \lambda) = 1 - \exp\left(-\left(\frac{t}{\lambda}\right)^k\right). \quad (8)$$

Substituting Eqns. (7) and (8) into Eqn. (1) to get the CDF of exponentiated Weibull Weibull distribution, EWW($a, \alpha, \beta, k, \lambda$) takes the following form

$$F(t; \Psi) = \left[1 - \exp(-\alpha[e^{(\frac{t}{\lambda})^k} - 1]^\beta)\right]^a, \quad (9)$$

where, $\Psi = (a, \alpha, \beta, k, \lambda)$ is the set of parameters, inserting Eqns. (7) and (8) into Eqn. (2), we obtained the pdf of EWW, and it takes the following form

$$\begin{aligned} f(t; \Psi) &= \frac{a\alpha\beta k}{\lambda} \times \left(\frac{t}{\lambda}\right)^{k-1} \times [e^{(\frac{t}{\lambda})^k} - 1]^{\beta-1} \times \exp\left(\left(\frac{t}{\lambda}\right)^k - \alpha\left(e^{(\frac{t}{\lambda})^k} - 1\right)^\beta\right) \\ &\times \left[1 - \exp\left(-\alpha\left[e^{(\frac{t}{\lambda})^k} - 1\right]^\beta\right)\right]^{a-1}. \end{aligned} \quad (10)$$

Survival function of EWW is defined as

$$S(t; \Psi) = 1 - \left[1 - \exp(-\alpha[e^{(\frac{t}{\lambda})^k} - 1]^\beta)\right]^a.$$

Hazard function is given by

$$\begin{aligned} h(t; \Psi) &= \frac{a\alpha\beta k}{\lambda} \times \left(\frac{t}{\lambda}\right)^{k-1} \times [e^{(\frac{t}{\lambda})^k} - 1]^{\beta-1} \times \exp\left(\left(\frac{t}{\lambda}\right)^k - \alpha\left(e^{(\frac{t}{\lambda})^k} - 1\right)^\beta\right) \\ &\times \left[1 - \exp\left(-\alpha\left(e^{(\frac{t}{\lambda})^k} - 1\right)^\beta\right)\right]^{a-1} \times \frac{1}{1 - \left[1 - \exp(-\alpha[e^{(\frac{t}{\lambda})^k} - 1]^\beta)\right]^a}. \end{aligned}$$

The joint likelihood function is given by

$$L(t; \Psi) = \begin{cases} \prod_{i=1}^n \{S(t_{1i}) - S(t_{2i})\}, & \delta_i = 0 \\ \prod_{i=1}^n S(t_{1i}), & \delta_i = 1 \\ \prod_{i=1}^n \{1 - S(t_{2i})\}, & \delta_i = 2. \end{cases}$$

$$L(t; \Psi) = \begin{cases} \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[e^{(\frac{t_{2i}}{\lambda})^k} - 1]^\beta)\right]^a - \left[1 - \exp(-\alpha[e^{(\frac{t_{1i}}{\lambda})^k} - 1]^\beta)\right]^a \right\}, & \delta_i = 0 \\ \prod_{i=1}^n \left\{ 1 - \left[1 - \exp(-\alpha[e^{(\frac{t_{1i}}{\lambda})^k} - 1]^\beta)\right]^a \right\}, & \delta_i = 1 \\ \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[e^{(\frac{t_{2i}}{\lambda})^k} - 1]^\beta)\right]^a \right\}, & \delta_i = 2. \end{cases}$$

Here t_1 and t_2 are lower limit and upper limit of the survival time, and δ_i is already define in Subsection 2.1.

2.3. Exponentiated Weibull Rayleigh distribution

Taking G as the Rayleigh distribution, the exponentiated Weibull Rayleigh (EWR) model is obtained from the EW-G family. Let the random variable T follows the following Rayleigh distribution with scale parameter $\lambda > 0$,

$$g(t; \lambda) = \frac{t}{\lambda^2} \exp\left(-\frac{t^2}{2\lambda^2}\right). \quad (11)$$

The CDF of Rayleigh distribution is given by

$$G(t; \lambda) = 1 - \exp\left(-\frac{t^2}{2\lambda^2}\right). \quad (12)$$

Substituting Eqns. (11) and (12) into Eqn. (1) to get the CDF of exponentiated Weibull Rayleigh distribution, EWR(a, α , β , λ) takes the following form

$$F(t; \Psi) = \left[1 - \exp(-\alpha[\exp(\frac{t^2}{2\lambda^2}) - 1]^\beta)\right]^a, \quad (13)$$

where, $\Psi = (a, \alpha, \beta, \lambda)$ is the set of parameters, inserting Eqns. (11) and (12) into Eqn. (2), we obtained the pdf of EWR, and it takes the following form

$$\begin{aligned} f(t; \Psi) &= \frac{a\alpha\beta t}{\lambda} \times \left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^{\beta-1} \times \left[1 - \exp\left(-\alpha\left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^\beta\right)\right]^{a-1} \\ &\times \exp\left(\frac{t^2}{2\lambda^2} - \alpha\left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^\beta\right). \end{aligned} \quad (14)$$

Survival function of EWR is defined as

$$S(t; \Psi) = 1 - \left[1 - \exp(-\alpha[\exp(\frac{t^2}{2\lambda^2}) - 1]^\beta)\right]^a.$$

Hazard function is given by

$$\begin{aligned} h(t; \Psi) &= \frac{a\alpha\beta t}{\lambda} \times \left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^{\beta-1} \times \left[1 - \exp\left(-\alpha\left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^\beta\right)\right]^{a-1} \\ &\times \exp\left(\frac{t^2}{2\lambda^2} - \alpha\left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^\beta\right) \times \frac{1}{1 - \left[1 - \exp\left(-\alpha\left[\exp(\frac{t^2}{2\lambda^2}) - 1\right]^\beta\right)\right]^a}. \end{aligned}$$

The joint likelihood function is given by

$$L(t; \Psi) = \begin{cases} \prod_{i=1}^n \{S(t_{1i}) - S(t_{2i})\}, & \delta_i = 0 \\ \prod_{i=1}^n S(t_{1i}), & \delta_i = 1 \\ \prod_{i=1}^n \{1 - S(t_{2i})\}, & \delta_i = 2. \end{cases}$$

$$L(t; \Psi) = \begin{cases} \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[\exp(\frac{t_{2i}^2}{2\lambda^2}) - 1]^\beta)\right]^a - \left[1 - \exp(-\alpha[\exp(\frac{t_{1i}^2}{2\lambda^2}) - 1]^\beta)\right]^a \right\}, & \delta_i = 0 \\ \prod_{i=1}^n \left\{ 1 - \left[1 - \exp(-\alpha[\exp(\frac{t_{1i}^2}{2\lambda^2}) - 1]^\beta)\right]^a \right\}, & \delta_i = 1 \\ \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[\exp(\frac{t_{2i}^2}{2\lambda^2}) - 1]^\beta)\right]^a \right\}, & \delta_i = 2. \end{cases}$$

3. Bayesian Regression Analysis of EW-Generated Family

3.1. Prior specification

To build the posterior distribution, we must first specify the prior distribution for the regression coefficients, shape parameters, and scale parameters of the models.

3.1.1 Gaussian prior

We have chosen regularizing prior for regression coefficient that is Gaussian prior with mean 0 and standard deviation 1. This prior prevent a model from getting too excited by the data that slows the rate of over-excitement of model and reduces the overfitting of data to the model (McElreath, 2018). Here the question arises, why do we choose Gaussian prior with mean zero, because this prior places as much probability below zero as it does above zero, which gives more conservative estimates as compared to perfectly flat prior (McElreath, 2018).

3.1.2 Half-Cauchy prior

For the shape and scale parameter of the model, we have chosen half-Cauchy prior because Gelman et al. (2006) recommended the half-Cauchy prior as a better choice for a scale parameter. The half-Cauchy distribution with scale 25 is nearly flat prior but not wholly. The prior distribution that is not entirely flat provides enough information for the numerical approximation algorithm to continue to explore the target density, the posterior distribution (Akhtar and Khan, 2014; Khan and Khan, 2018).

3.2. Model specification

To construct the Bayesian regression model, we need to introduce covariates in the scale parameter for the all model through a log link function

$$\lambda = \exp(Xb),$$

where b is regression coefficients of the covariates.

3.2.1 Posterior distribution of EWE

Using Bayes theorem, joint posterior density can be written as

$$P(\Psi|X, t) \propto L(t|\Psi, X) \times P(a) \times P(\alpha) \times P(\beta) \times P(b).$$

We have defined Posterior density separately for the above three censoring types.

First for Interval censoring ($\delta_i=0$),

$$\begin{aligned} P(\Psi|X, t) &\propto \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha(e^{\frac{t_{2i}}{\exp(x_i b)}} - 1)^\beta) \right]^a - \left[1 - \exp(-\alpha(e^{\frac{t_{1i}}{\exp(x_i b)}} - 1)^\beta) \right]^a \right\} \\ &\times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right). \end{aligned}$$

Right censoring ($\delta_i=1$),

$$\begin{aligned} P(\Psi|X, t) &\propto \prod_{i=1}^n \left\{ 1 - \left[1 - \exp(-\alpha(e^{\frac{t_{1i}}{\exp(x_i b)}} - 1)^\beta) \right]^a \right\} \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \\ &\times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right). \end{aligned}$$

Left censoring ($\delta_i=2$),

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha(e^{\frac{t_{2i}}{\exp(x_i b)}} - 1)^\beta) \right]^a \right\} \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \\ \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

3.2.2 Posterior density of EWW

Using Bayes theorem, joint posterior density can be written as

$$P(\Psi|X, t) \propto L(t|\Psi, X) \times P(a) \times P(\alpha) \times P(\beta) \times P(k) \times P(b).$$

We have defined Posterior density separately for the above three censoring types.

First for Interval censoring ($\delta_i=0$),

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[e^{\frac{t_{2i}}{\exp(x_i b)}} - 1]^\beta) \right]^a - \left[1 - \exp(-\alpha[e^{\frac{t_{1i}}{\exp(x_i b)}} - 1]^\beta) \right]^a \right\} \\ \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \\ \times \frac{2 \times 25}{\pi(k^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

Right censoring ($\delta_i=1$)

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ 1 - \left[1 - \exp(-\alpha[e^{\frac{t_{1i}}{\exp(x_i b)}} - 1]^\beta) \right]^a \right\} \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \\ \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \frac{2 \times 25}{\pi(k^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

Left censoring ($\delta_i=2$),

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[e^{\frac{t_{2i}}{\exp(x_i b)}} - 1]^\beta) \right]^a \right\} \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \\ \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \frac{2 \times 25}{\pi(k^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

3.2.3 Posterior density of EWR

Using Bayes theorem, joint posterior density can be written as

$$P(\Psi|X, t) \propto L(t|\Psi, X) \times P(a) \times P(\alpha) \times P(\beta) \times P(b).$$

We have defined Posterior density separately for the above three censoring types.

First for Interval censoring ($\delta_i=0$),

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[\exp(\frac{t_{2i}^2}{2e^{2x_i b}}) - 1]^\beta) \right]^a - \left[1 - \exp(-\alpha[\exp(\frac{t_{1i}^2}{2e^{2x_i b}}) - 1]^\beta) \right]^a \right\}$$

$$\times \frac{2 \times 25}{\pi(a^2 + 25^2)} \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

Right censoring ($\delta_i=1$),

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ 1 - \left[1 - \exp(-\alpha[\exp(\frac{t_{1i}^2}{2e^{2x_i}}) - 1]^\beta) \right]^a \right\} \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \\ \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

Left censoring ($\delta_i=2$),

$$P(\Psi|X, t) \propto \prod_{i=1}^n \left\{ \left[1 - \exp(-\alpha[\exp(\frac{t_{2i}^2}{2e^{2x_{i,b}}}) - 1]^\beta) \right]^a \right\} \times \frac{2 \times 25}{\pi(a^2 + 25^2)} \\ \times \frac{2 \times 25}{\pi(\alpha^2 + 25^2)} \times \frac{2 \times 25}{\pi(\beta^2 + 25^2)} \times \prod_{j=0}^J \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}b_j^2\right).$$

3.3. Implementation using Stan

3.3.1 Stan code for EWE

```
modelEWE="functions{
  real EWE_lpdf(real t, real a, real beta, real alpha, real lambda){
    real log_f;
    log_f=log((a*beta*alpha/lambda)*(exp(t/lambda)-1)^(beta-1)*
    exp(-(alpha*(exp(t/lambda)-1)^beta-t/lambda)))*
    (1-exp(-(alpha)*(exp(t/lambda)-1)^beta))^(a-1));
    return log_f;
  }
  real EWE_lccdf(real t, real a, real beta, real alpha, real lambda){
    real log_ccf;
    log_ccf=log(1-(1-exp((-alpha)*(exp(t/lambda)-1)^beta))^a);
    return log_ccf;
  }
  real EWE_lcdf(real t, real a, real beta, real alpha, real lambda){
    real log_cf;
    log_cf=log((1-exp((-alpha)*(exp(t/lambda)-1)^beta))^a);
    return log_cf;
  }
}
data {
  int N;
  vector<lower=0>[N] y1;
  vector<lower=0>[N] y2;
  vector<lower=0,upper=2>[N] event;
  int M;
  matrix[N, M] x;
}
parameters {
  vector[M] b;
  real<lower=0> a;
  real<lower=0> beta;
  real<lower=0>alpha;
}
transformed parameters {
  vector[N] linpred;
  vector<lower=0>[N] lambda;
  linpred=x*b;
  for(i in 1:N){
```

```

lambda[i]=exp(linpred[i]);
}}
model {
target += cauchy_lpdf(a | 0,25) - 1 * cauchy_lccdf(0|0,25);
target += cauchy_lpdf(beta | 0,25) - 1 * cauchy_lccdf(0|0,25);
target += cauchy_lpdf(alpha| 0,25) - 1 * cauchy_lccdf(0|0,25);
target += normal_lpdf(b|0,1);
for (n in 1:N) {
if (event[n] == 0) {
target +=log_diff_exp( EWE_lccdf(y1[n]|a, beta, alpha,lambda[n]),
EWE_lccdf(y2[n]|a,beta,alpha,lambda[n]));
}
else if (event[n] == 1) {
target += EWE_lccdf(y1[n] | a,beta,alpha,lambda[n]);
}
else if (event[n] == 2) {
target += loglm_exp(EWE_lccdf(y2[n] | a,beta,alpha,lambda[n]));
}}
}
generated quantities{
real log_lik[N];
for (n in 1:N){
if (event[n] == 0) {
log_lik[n]=log_diff_exp( EWE_lccdf(y1[n]|a,beta,alpha,lambda[n]),
EWE_lccdf(y2[n]|a,beta,alpha,lambda[n]));
}
else if (event[n] == 1) {
log_lik[n]= EWE_lccdf(y1[n] | a,beta,alpha,lambda[n]);
}
else if (event[n] == 2) {
log_lik[n]= loglm_exp(EWE_lccdf(y2[n] | a,beta,alpha,lambda[n]));
}}
}
}

```

3.3.2 Stan code for EWW

```

modelEWW="functions{
real EWW_lpdf(real t, real a, real beta, real k, real alpha,
real lambda){
real log_f;
log_f=log(alpha*a*beta*k*(1/lambda)*(t/lambda)^(k-1)*
(exp(t/lambda)^(k-1)^(beta-1)*exp((t/lambda)^k-alpha*
(exp(t/lambda)^(k-1)^(beta)*(1-exp(-alpha*(exp(t/lambda)^(k-1)^(beta))^(a-1));
return log_f;
}
real EWW_lcdf(real t, real a, real beta, real k, real alpha,
real lambda){
real log_cf;
log_cf=log((1-exp(-alpha*(exp(t/lambda)^(k-1)^(beta))^(a);
return log_cf;
}
real EWW_lccdf(real t, real a, real beta, real k, real alpha,
real lambda){
real log_ccf;
log_ccf=log(1-(1-exp(-alpha*(exp(t/lambda)^(k-1)^(beta))^(a);
return log_ccf;
}}
data {
int N;
vector<lower=0>[N] y1;
vector<lower=0>[N] y2;
vector<lower=0,upper=2>[N]event;
int M;

```

```

matrix[N, M] x;
}
parameters {
vector[M] b;
real<lower=0> a;
real<lower=0> beta;
real<lower=0>k;
real<lower=0>alpha;
}
transformed parameters {
vector[N] linpred;
vector<lower=0>[N] lambda;
linpred=x*b;
for(i in 1:N){
lambda[i]=exp(linpred[i]);
} }
model {
target += cauchy_lpdf(a | 0,25) - 1 * cauchy_lccdf(0|0,25);
target += cauchy_lpdf(beta | 0,25) - 1 * cauchy_lccdf(0|0,25);
target += cauchy_lpdf(k | 0,25) - 1 * cauchy_lccdf(0|0,25);
target += cauchy_lpdf(alpha | 0,25) - 1 * cauchy_lccdf(0|0,25);
target += normal_lpdf(b|0,1);
for (n in 1:N) {
if (event[n] == 0) {
target +=log_diff_exp( EWW_lccdf(y1[n]|a,beta,k,alpha,lambda[n]),
EWW_lccdf(y2[n]|a,beta,k,alpha,lambda[n]));
}
else if (event[n] == 1) {
target += EWW_lccdf(y1[n]|a,beta,k,alpha,lambda[n]);
}
else if (event[n] == 2) {
target += loglm_exp(EWW_lccdf(y2[n]|a,beta,k,alpha,lambda[n]));
}
}
}
generated quantities{
real log_lik[N];
for (n in 1:N){
if (event[n] == 0) {
log_lik[n]=log_diff_exp( EWW_lccdf(y1[n]|a,beta,k,alpha,lambda[n]),
EWW_lccdf(y2[n]|a,beta,k,alpha,lambda[n]));
}
else if (event[n] == 1) {
log_lik[n]= EWW_lccdf(y1[n] | a,beta,k,alpha,lambda[n]);
}
else if (event[n] == 2) {
log_lik[n]= loglm_exp(EWW_lccdf(y2[n] | a,beta,k,alpha,lambda[n]));
}
}
}
}

```

3.3.3 Stan code for EWR

```

modelEWR="functions{
real EWR_lpdf(real t, real a, real beta, real alpha, real lambda){
real log_f;
log_f=log(a*alpha*beta*(t/lambda)*(exp(t^2/(2*lambda^2))-1)^(
(beta-1)*(1-exp(-alpha*(exp(t^2/(2*lambda^2))-1)^beta))^(a-1)*
exp((t^2/(2*lambda^2))-alpha*(exp(t^2/(2*lambda^2))-1)^beta));
return log_f;
}
real EWR_lcdf(real t, real a, real beta, real alpha, real lambda){
real log_cf;
log_cf=log((1-exp(-alpha*(exp(t^2/(2*lambda^2))-1)^beta))^a);
return log_cf;
}
}

```

```

real EWR_lccdf(real t, real a, real beta, real alpha, real lambda){
  real log_ccf;
  log_ccf=log(1-(1-exp(-alpha*(exp(t^2/(2*lambda^2))-1)^beta))^a);
  return log_ccf;
}
data {
  int N;
  vector<lower=0>[N] y1;
  vector<lower=0>[N] y2;
  vector<lower=0,upper=2>[N]event;
  int M;
  matrix[N, M] x;
}
parameters {
  vector[M] b;
  real<lower=0> a;
  real<lower=0> beta;
  real<lower=0>alpha;
}
transformed parameters {
  vector[N] linpred;
  vector<lower=0.3>[N] lambda;
  linpred=x*b;
  for(i in 1:N){
    lambda[i]=exp(linpred[i]);
  }
}
model {
  target += cauchy_lpdf(a | 0,25) - 1 * cauchy_lccdf(0|0,25);
  target += cauchy_lpdf(beta | 0,25) - 1 * cauchy_lccdf(0|0,25);
  target += cauchy_lpdf(alpha| 0,25) - 1 * cauchy_lccdf(0|0,25);
  target += normal_lpdf(b|0,1);
  for (n in 1:N) {
    if (event[n] == 0) {
      target +=log_diff_exp( EWR_lccdf(y1[n]|a, beta, alpha,lambda[n]),
        EWR_lccdf(y2[n]|a,beta,alpha,lambda[n]));
    }
    else if (event[n] == 1) {
      target += EWR_lccdf(y1[n] | a,beta,alpha,lambda[n]);
    }
    else if (event[n] == 2) {
      target += log1m_exp(EWR_lccdf(y2[n] | a,beta,alpha,lambda[n]));
    }
  }
  generated quantities{
    real log_lik[N];
    for (n in 1:N){
      if (event[n] == 0) {
        log_lik[n]=log_diff_exp( EWR_lccdf(y1[n]|a,beta,alpha,lambda[n]),
          EWR_lccdf(y2[n]|a,beta,alpha,lambda[n]));
      }
      else if (event[n] == 1) {
        log_lik[n]= EWR_lccdf(y1[n] | a,beta,alpha,lambda[n]);
      }
      else if (event[n] == 2) {
        log_lik[n]= log1m_exp(EWR_lccdf(y2[n] | a,beta,alpha,lambda[n]));
      }
    }
  }
}

```

3.4. Recurrence of an ulcer data

Let us introduce the recurrence of an ulcer data set taken from Collett (2015), so that the whole analysis roam around that data, where the same data were discussed by Whitehead (1989). Patients who have been cured of ulcers, carcinomas, or other recurrent conditions are usually provided with medication to maintain their recovery. These patients are subsequently examined at regular intervals

(0-6 months or 6-12 months) to detect whether a recurrence has occurred. Regular visits to a clinic were arranged for the patients, and endoscopies were performed 6 months and 12 months after the randomization of two treatments. Naturally, some patients may experience symptoms of a recurrence and be subsequently diagnosed as having had a recurrence at a time other than one of the scheduled screening times.

The data frame contains each patient's age in years and the duration of verified disease (1=less than five years, 2=greater than or equal to five years). In addition, two treatment groups (A or B) have randomly given the patients to suppress the recurrence of an ulcer. There are two variables associated with ulcer detection in the data sets, namely the time of the last visit, in months, and the endoscopy result (1=no ulcer detected, 2=ulcer detected). Using the above two variables, we extracted lower and upper time points. The time between the two points is survival time and, the event is the recurrence of an ulcer.

3.5. Creation of data frame in R

```
ID=seq(1,43)
Age=c(48,73,54,58,56,49,71,41,23,37,38,76,38,27,47,54,38,27,58,
75,25,58,63,41,47,58,74,75,72,59,52,75,76,34,36,59,44,28,62,23,
49,61,33)

Duration=c(2,1,1,2,1,2,rep(1,5),2,2,rep(1,4),2,2,rep(1,7),2,2,
rep(1,5),2,1,1,1,2,rep(1,4),2)

#Time of last visits
TimeLV=c(7,rep(12,8),5,rep(12,3),6,6,6,10,7,rep(12,7),3,2,6,
rep(12,5),6,rep(12,9))

# 1= no ulcer detected, 2= ulcer detected
Result=c(2,rep(1,8),2,1,1,1,2,2,1,2,2,rep(1,7),2,2,1,1,2,1,2,
1,1,1,1,2,rep(1,6))

Lower=c(6,rep(12,8),0,12,12,12,0,0,6,6,6,rep(12,7),0,0,6,12,6,
12,6,12,6,12,12,6,rep(12,6))

Upper=c(7,rep(Inf,8),5,rep(Inf,3),6,6,Inf,10,7,rep(Inf,7),3,2,
Inf,Inf,12,Inf,12,rep(Inf,4),12,rep(Inf,6))

#teartment A=0,B=1
treat=c(rep(1,4),0,0,1,0,rep(1,4),0,0,1,0,1,1,0,1,0,0,1,0,1,
rep(0,4),rep(1,3),0,0,1,1,0,1,1,0,1,0,1)

# 0=Interval ,1=Right, 2=left
Status=c(0,rep(1,8),2,rep(1,3),2,2,1,0,0,rep(1,7),2,2,1,1,0,
1,0,rep(1,4),0,rep(1,6))

Ulcer=data.frame(ID,Lower,Upper,Age,Duration,treat,Status)
Ulcer$Dur=as.numeric(Ulcer$Duration==2)
X=cbind(1,Age,Ulcer$Dur,treat)
N=nrow(X)
M=ncol(X)
data=list(y1=Lower,y2=Upper,x=X,event=Status,N=N,M=M)
```

3.6. Fitting and analysis with Stan

To run the stan code in R, we need of `rstan` package, and then we will fit all three models with stan. The results are discussed in the Output Summary. First, we start with the EWE model.

3.6.1 Fitting of exponentiated Weibull exponential model

```
fitEWE=stan(model_code = modelEWE,data=data,iter=5000,chains = 2,
control=list(list(adapt_delta=0.99),list(max_treedepth=20)))
```

Output Summary

Using print command output of the fitted model is obtained and are given in the Table 1. Relevant graphs are also shown in the Figure 1 and Figure 2.

```
print (fitEWE)
```

Table 1 Posterior summary of the Model EWE using function `stan()`

parameters	mean	se_mean	sd	2.5% (Lower)	97.5% (Upper)	n_eff	Rhat
b[1]	0.74	0.02	0.91	-0.99	2.60	1546	1.00
b[2]	0.05	0.00	0.05	-0.01	0.19	394	1.00
b[3]	0.14	0.01	0.69	-1.18	1.55	2864	1.00
b[4]	-0.08	0.02	0.69	-1.39	1.36	1984	1.00
alpha	2.71	0.12	1.78	0.30	6.41	226	1.01
beta	0.24	0.03	0.56	0.01	2.08	275	1.00
a	41.27	2.89	123.92	0.16	246.39	1842	1.00

According to Gelman et al. (2013), acceptable limit of effective sample size as $n_{eff} > 100$ and potential scale reduction factor $\hat{R} < 1.1$. It is evident from Table 1 that the Rhat is close to 1 and the effective sample size is enough to get convergence; traceplot, Figure 1a, shows also convergence of MCMC algorithm to the target posterior distribution. It is clear from Table 1 and caterpillar plot, Figure 1b, that the intercept b1, regression coefficients b2, b3 and b4 are not statistically significant as they do not belong to the 95% credible intervals. The covariates age, Durations, and treatment have no effects on the recurrence time. A negative sign in the posterior estimate of treatment tells that treatment A is superior to treatment B, but it has no significant effect on the recurrence time.

Graphical Analysis

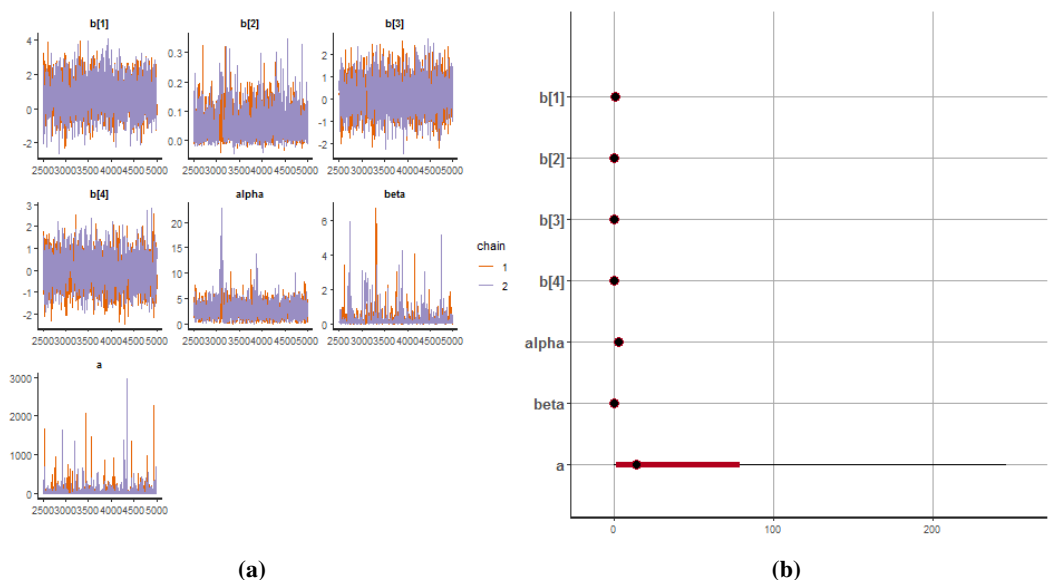


Figure 1 (a) Traceplot for each parameter of EWE, which tells mixing of two chains is good that means Markov chains converge to target distribution, (b) Caterpillar plot for each parameter of EWE

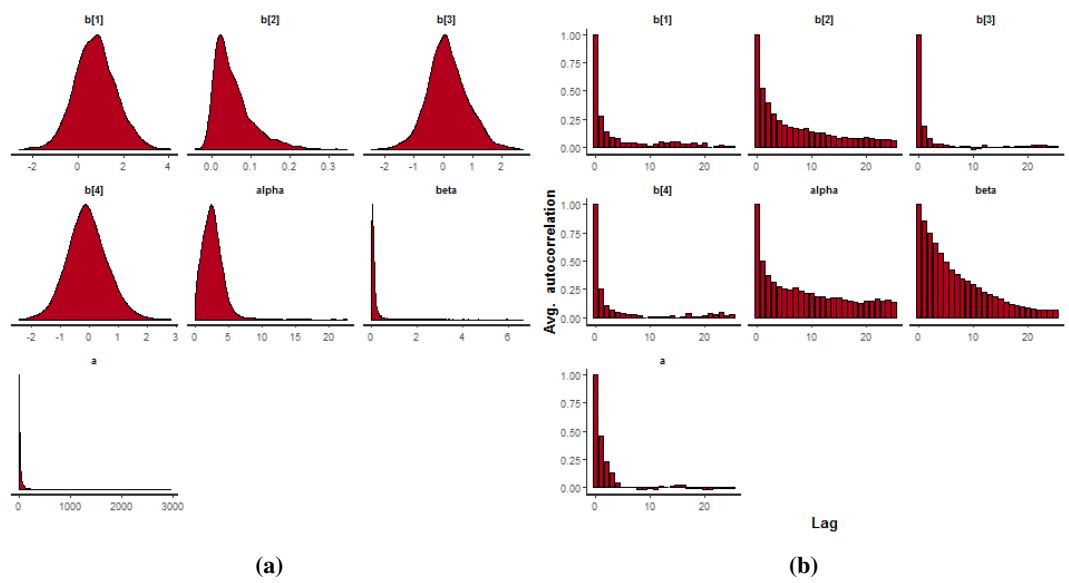


Figure 2 (a) Posterior density plot for each parameter of EWE, (b) Autocorrelation plot for each parameter of EWE, and autocorrelation declining towards zero

3.6.2 Fitting of Exponentiated Weibull Weibull

```
fitEWW=stan(model_code = modelEWW,data=data,iter=5000,chains = 2,
control=list(list(adapt_delta=0.99),list(max_treedepth=20)) )
```

Output Summary

```
print (fitEWW)
```

Table 2 Posterior summary of the Model EWW using function stan ()

parameters	mean	se_mean	sd	2.5% (Lower)	97.5% (Upper)	n_eff	Rhat
b[1]	0.80	0.02	0.99	-1.18	2.68	1578	1.00
b[2]	0.06	0.00	0.07	-0.01	0.26	412	1.00
b[3]	0.13	0.02	0.68	-1.21	1.58	1651	1.00
b[4]	-0.12	0.02	0.70	-1.52	1.35	1456	1.00
alpha	7.71	2.16	29.18	0.34	43.51	182	1.01
beta	0.76	0.11	1.56	0.00	5.73	185	1.02
a	33.67	2.27	103.11	0.09	233.89	2061	1.00
k	6.91	0.64	21.94	0.00	42.70	1191	1.00

It is evident from Table 2 that the Rhat is close to 1 and the effective sample size is enough to get convergence; traceplot, Figure 3a, shows also convergence of MCMC algorithm to the target posterior distribution. It is clear from Table 2 and caterpillar plot, Figure 3b, that the intercept b1, regression coefficients b2, b3 and b4 are not statistically significant as they do not belong to the 95% credible intervals. The covariates age, Durations, and treatment have no effects on the recurrence time.

Graphical Analysis

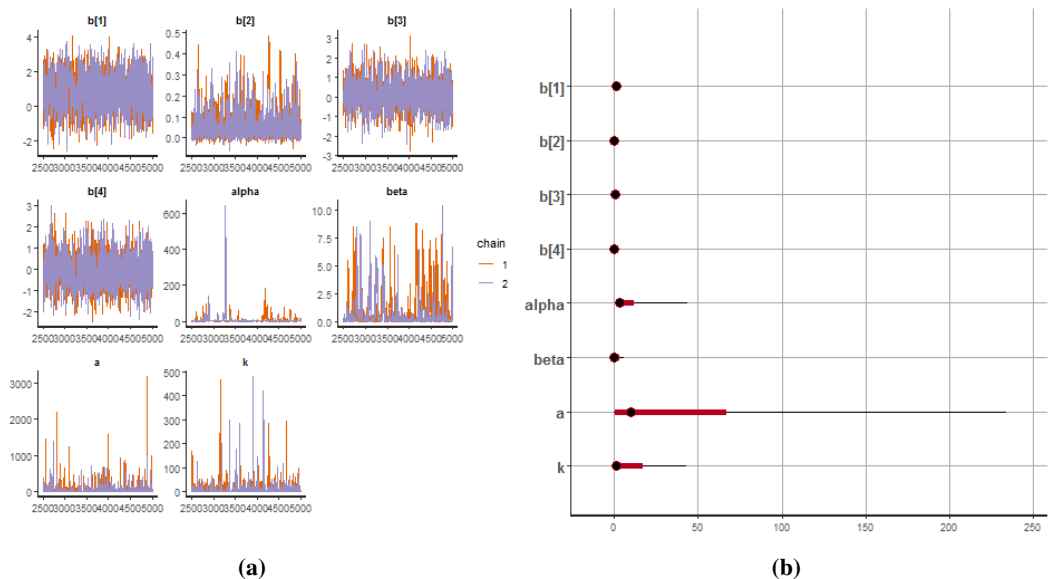


Figure 3 (a) Traceplot for each parameter of EWW, which tells mixing of two chains is good that means Markov chains converge to target distribution, (b) Caterpillar plot for each parameter of EWW

3.6.3 Fitting of Exponentiated Weibull Reyleigh

```
fitEWR=stan(model_code = modelEWR,data=dataUEWE,iter=5000,chains=2,
control=list(list(adapt_delta=0.99),list(max_treedepth=20)))
```

Output Summary

```
print(fitEWR)
```

Table 3 Posterior summary of the Model EWR using function `stan()`

parameters	mean	se_mean	sd	2.5% (Lower)	97.5% (Upper)	n_eff	Rhat
b[1]	1.13	0.03	0.89	-0.72	2.86	752	1.00
b[2]	0.07	0.00	0.05	0.00	0.19	424	1.00
b[3]	0.16	0.02	0.73	-1.26	1.69	1199	1.00
b[4]	0.13	0.01	0.69	-1.17	1.60	2233	1.00
alpha	4.06	0.65	9.68	0.24	14.46	219	1.01
beta	0.32	0.05	0.65	0.02	2.48	159	1.00
a	36.42	5.33	141.19	0.09	213.25	701	1.00

It is evident from Table 3 that the Rhat is close to 1 and the effective sample size is enough to get convergence; traceplot, Figure 5a, shows also convergence of MCMC algorithm to the target posterior distribution. It is clear from Table 3 and caterpillar plot, Figure 5b, that the intercept b1, regression coefficients b2, b3 and b4 are not statistically significant as they do not belong to the 95% credible intervals. The covariates age, Durations, and treatment have no effects on the recurrence time.

Graphical analysis

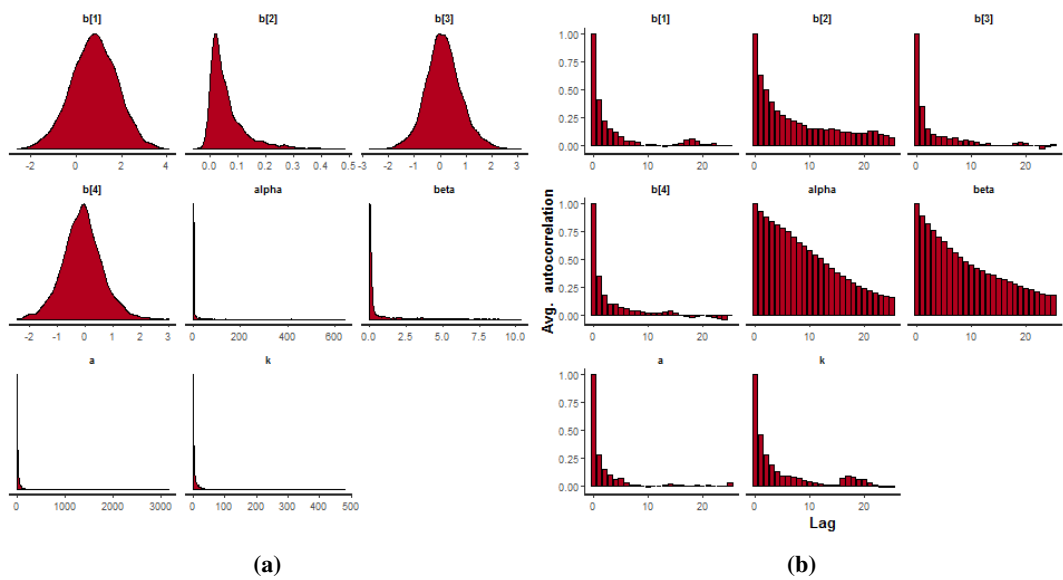


Figure 4 (a) Posterior density plot for each parameter of EWW, (b) Autocorrelation plot for each parameter of EWW, and autocorrelation declining towards zero

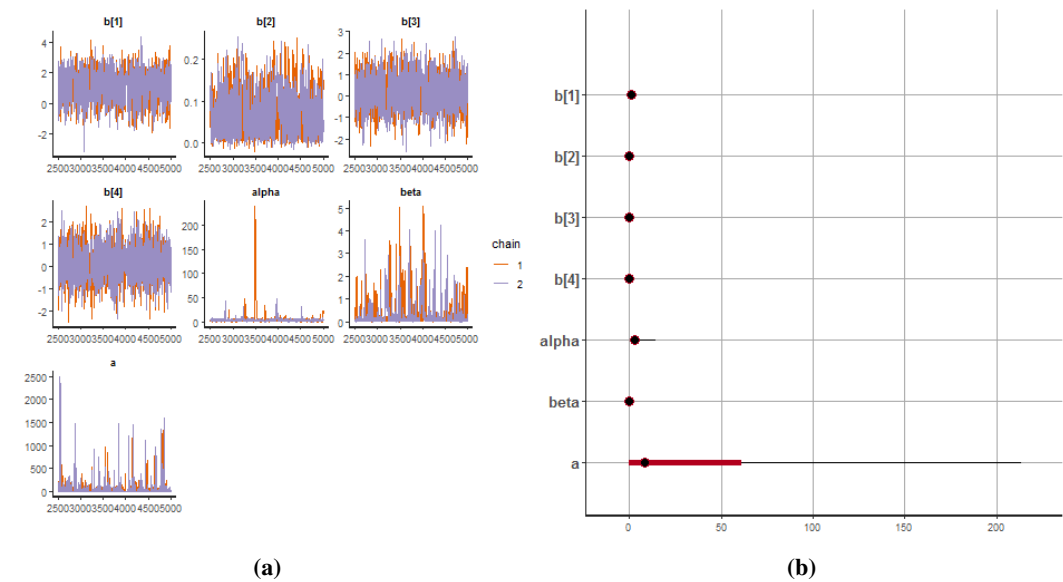


Figure 5 (a) Traceplot for each parameter of EWR, which tells mixing of two chains is good that means Markov chains converge to target distribution, (b) Caterpillar plot for each parameter of EWR

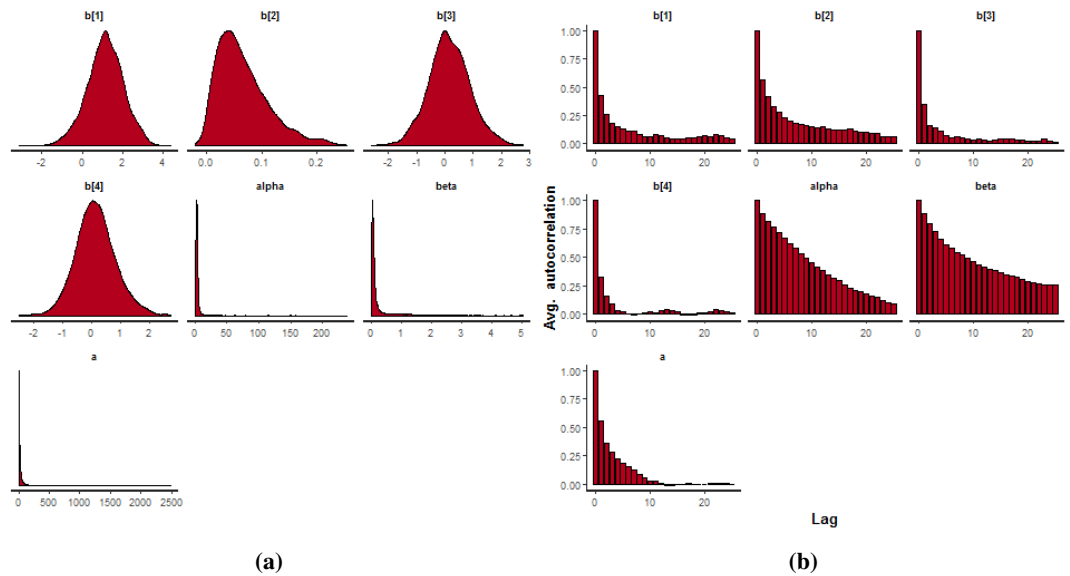


Figure 6 (a) Posterior density plot for each parameter of EWR, (b) Autocorrelation plot for each parameter of EWR, and autocorrelation declining towards zero

4. Model Comparisons

Models are compared and selected on the basis of leave-one-out cross-validation information criterion (LOOIC) and the best model has the least LOOIC.

Table 4 LOOIC value of the three models

Models	EWE	EWV	EWR
LOOIC	85.1	86.4	86

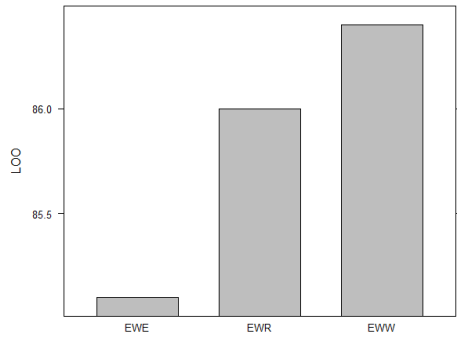


Figure 7 Barplot of looic of the three models

It can be seen from Table 4 and Figure 7, that EWE has the lowest LOOIC value as compared to the EWW and EWR models. It means EWE is the best model for the recurrence of ulcer data and could be chosen as the adequate model for the given data.

5. Discussion and Conclusions

This paper presents the Bayesian Analysis of three Exponentiated Weibull Generated family of distribution for interval-censored data that is the recurrence of an ulcer data, taken from Collett (2015). Three special models of EWG family, namely EWE, EWW, and EWR have been used to analyze the survival data using the softwares R and Stan under Bayesian paradigm. These models are an extension of Weibull distribution as we know that Weibull distribution is a very potent and essential distribution in survival analysis and reliability. Here, all three models are suitable for fitting the data. As the model EWE has the least LOOIC value (see Table 4 and Figure 7), so EWE is the best model for analyzing the recurrence of ulcer data. The EWE distribution is best not only for the given data, but we have also checked it for another interval-censored data, that is, the occurrence of breast retraction data given in Collett (2015). We have used the same generalized family of distribution, same Bayesian setup for analysis of this data, and again, we have found that EWE is the best model compared to the other two models. However, we have not reported the whole analysis to save the space. We conclude that the model that is less complex and has fewer parameters gives better fitting or performance to the data. There is no doubt that the model with more parameters should fit the data better than a model with less parameters but this does not happens always. Sometimes simple models give better performance than complex models.

References

- AbuJarad MH, Khan AA. Bayesian survival analysis of Topp-Leone generalized family with Stan. *Int J Stat Appl*. 2018; 8(5): 274-290.
- AbuJarad MH, Khan AA, Khaleel MA, AbuJarad ES, AbuJarad AH, Oguntunde PE. Bayesian reliability analysis of Marshall and Olkin model. *Ann Data Sci*. 2020; 7(3): 461-489.
- Akhtar MT, Khan AA. Bayesian analysis of generalized log-Burr family with R. *SpringerPlus*. 2014; 3(1): 1-10.
- Almetwaly EM, Almongy HM. Estimation of the generalized power Weibull distribution parameters using progressive censoring schemes. *Int J Probab Stat*. 2018; 7(2): 51-61.
- Bogaerts K, Komárek A, Lesaffre E. Survival analysis with interval-censored data: A practical approach with examples in R, SAS, and BUGS. *Chapman and Hall/CRC*; 2017.
- Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. *J Data Sci*. 2014; 12(1): 53-68.
- Collett D. Modelling survival data in medical research. *CRC press*; 2015.
- Cordeiro GM, Afify AZ, Yousof HM, Pescim RR, Aryal GR. The exponentiated Weibull-H family of distributions: Theory and Applications. *Mediterr J Math*. 2017; 14(4): 1-22.
- Duane S, Kennedy AD, Pendleton BJ, Roweth D. Hybrid monte carlo. *Phys Lett. B*. 1987; 195(2): 216-222.
- Elgarhy M, Shakil M, Kibria G. Exponentiated Weibull-exponential distribution with applications. *Appl Appl Math*. 2017; 12(2): 710-725.
- Gelfand AE, Dey DK, Chang H. Model determination using predictive distributions with implementation via sampling-based methods. *Stanford Univ CA Dept of Statistics*; 1992.
- Gelman A. Prior distributions for variance parameters in hierarchical models (comment on article by Browne and Draper). *Bayesian Anal*. 2006; 1(3): 515-534.
- Gelman A, Carlin JB, Stern HS, Rubin DB. Bayesian data analysis. *Chapman and Hall/CRC*; 2013.
- Hassan AS, Elgarhy M. A new family of exponentiated Weibull-generated distributions. *Int. J. Math. Appl*. 2016; 4(1):135-148.

- Hoffman MD, Gelman A. The No-U-Turn sampler: adaptively setting path lengths in Hamiltonian Monte Carlo *J Mach Learn Res.* 2014; 15(1): 1593-1623.
- Khan N, Khan AA. Bayesian Analysis of Topp-Leone Generalized Exponential Distribution. *Austrian J Stat.* 2018; 47(4): 1-15.
- Mahmoud MAW, Ghazal MGM. Estimations from the exponentiated rayleigh distribution based on generalized Type-II hybrid censored data. *J Egyptian Math Soc.* 2017; 25(1): 71-78.
- McElreath R. Statistical rethinking: A Bayesian course with examples in R and Stan. Chapman and Hall/CRC; 2018.
- Mudholkar GS, Srivastava DK, Freimer M. The exponentiated Weibull family: A reanalysis of the bus-motor-failure data. *Technometrics.* 1995; 37(4): 436-445.
- R Core Team. R: A language and environment for statistical computing. R Foundation for Statistical Computing, Vienna, Austria. 2021. Available from: <https://www.R-project.org/>.
- Sindhu TN, Aslam M, Hussain Z. Bayesian estimation on the generalized logistic distribution under left type-II censoring. *Thail Stat.* 2016; 14(2): 181-195.
- Sindhu TN, Feroze N, Aslam M. A class of improved informative priors for bayesian analysis of two-component mixture of failure time distributions from doubly censored data. *J Stat Manage Syst.* 2017; 20(5): 871-900.
- Sindhu TN, Hussain Z. Predictive inference and parameter estimation from the half-normal distribution for the left censored data. *Ann Data Sci.* 2020: 1-15.
- Stan Development Team. RStan: the R interface to Stan. R package version 2.21.2. 2020. Available from: <http://mc-stan.org/>.
- Stan Development Team. Stan Modeling Language Users Guide and Reference Manual Version 2.17.0. 2017.
- Vehtari A, Gelman A, Gabry J. Efficient implementation of leave-one-out cross-validation and WAIC for evaluating fitted Bayesian models. *arXiv preprint arXiv:1507.04544.* 2015.
- Vehtari A, Gelman A, Gabry J. Practical Bayesian model evaluation using leave-one-out cross-validation and WAIC *Stat. Comput.* 2017; 27(5): 1413-1432.
- Whitehead J. The analysis of relapse clinical trials, with application to a comparison of two ulcer treatments. *Stat Med.* 1989; 8(12): 1439-1454.