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Adaptive Test for Periodic ARFIMA Models

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Abstract

The goal of this article is to construct an adaptive test for periodic long memory models, using the main technical tool of Le Cam (1986)'s Local Asymptotic Normality (LAN) property constructed in Amimour and Belaide (2020), and a Correlogram-Based LAN Result. We consider the problem of testing a given ARFIMA model in which the density of the generating white noise is specified against a periodic ARFIMA model. The perspectives of this work are, first to establish a locally asymptotically most stringent parametric tests, when the density of the innovations and the long-memory parameter are unspecified. Second to define and investigate so-called residual rank tests.

Keywords: Local asymptotic normality property, long memory process, locally asymptotically optimal test; residual autocorrelations, periodically correlated models.

1. Introduction

Short rang dependent (SRD) time series are characterized by the fast decrease in their autocorrelations, since these lasts constitute the principal source of information in time series. The term long range dependence (LRD) is introduced as an alternative of SRD, that can model the time series, which exhibit strong correlations, and take into account the correlations between more distant observations, their autocorrelations decrease slowly than the SRD, for more details see [Samorodnitsky (2006), Pipiras and Taqqu (2017)]. The fractional ARIMA (ARFIMA) is the most simple and popularized LRD model, that is widely studied and used since the pioneer Hosking (1981)'s article. Through the ARFIMA models, in international economics Gil-Alana and Toro (2002) studied the Purchasing Power Parity Hypothesis, modeling the real exchange rate of five industrialized countries against the U.S. dollar, in order to capture the low-frequency dynamics relevant to the examination of long-term parity. In medicine, Al Zahrani et al. (2020) applied the ARFIMA model, to the monthly diabetic patient data records, through pattern identification, estimation, diagnosis verification and prediction. Confirming the existence of long memory behavior in the data via Hurst test. Jibrin et al. (2015) used the model to study and forecast crude oil prices in West Texas Intermediate and Brent, identifying long-memory features in the series.

Currently, it is well known, that many meteorology, climate, economic and financial time series, encountered in practice exhibit a periodical autocorrelation structure, as mentioned in Seymour (2001) periodicity occurs naturally in many environmental time series - hourly tide levels, daily stream flow, monthly average temperature, and carbon dioxide (C02) exchange of growing plants - the author focused on the study of the two last series, she described the methods for detecting and modeling periodic SRD time series, and used the average squared coherence for detecting the periodicity. However, the periodic structure can be detected in many LRD time series that can be described by the seasonal LRD time series models, such that the ARFIMA process with seasonality (SARFIMA), the asymptotic properties of these models are studied extensively in Bisognin and Lopes (2009). The SARFIMA models are a particular case of the ARFIMA models, the stationary condition is that the autocorrelations do not depend on the season. Although in practice for most cases, the seasonally adjusted data still show seasonal variations in the autocovariance function. A flexible class of periodic ARFIMA models, analogue to ARFIMA models and LRD stationary series. PARFIMA models are periodic and periodically correlated (PC) time series. Indeed, these models have not received enough attention, despite its importance in modeling time series which present both a long memory phenomenon and a periodic structure, for example the modelisation of inflation rate see Franses and Ooms (1997) and the administrative planifications see Hui and Li (1995), the asymptotic properties of PARFIMA models have been studied recently see (Amimour and Belaide (2020a), Amimour and Belaide (2020b)), and the estimation of the periodic long memory parameter is established in (Amimour et al. (2022)).

It is well known from the local asymptotic normality (LAN) property, that a sequence of statistical experiments converges weakly to a Gaussian position experiment. Moreover, in the latter experiment the optimal solutions in a local and asymptotical sense exist for any classical inference problem, in particular the locally asymptotically optimal (LAO) tests, which are widely used for SRD model, such that the LAO tests for autorregressive against diagonal bilinear time series models derived by Benghabrit and Hallin (1996b), they verified the LAN condition for the model and constructed the LAO test using the correlograme-based LAN result, and using the Rank-Based residual autocorrelation in Benghabrit and Hallin (1996a). Since the periodic models are only to be considered if the periodic behavior of the autocovariance function is revealed, it is therefore, interesting to establish procedures to detect this periodic behavior. For the periodic SRD models, Benterzi and Hallin (1996) derived the LAO tests for the null hypothesis of traditional AR dependence with unspecified AR coefficients and unspecified innovation densities, against an alternative of periodically correlated AR dependence, using the parametric and nonparametric rank-based versions. Benterzi and Merzougui (2010) considered the problem of testing the periodicity in Autoregressive Conditional Heteroskedastic (ARCH) process, establishing the LAO test, when the density of the generating white noise is unspecified and symmetric satisfying some mild conditions, for the null hypothesis of classical ARCH process against an alternative of PCARCH dependence using the main technical tool of Le Cam (1986). On the other hand and in the case of non-periodic LRD models, particularly the regression model with stationary non-Gaussian ARFIMA(p,d,q) long-memory errors considered by Hallin et al. (1999), they proved that the log-likelihood ratios exhibits the typical LAN behavior, then they deal with the problem of testing the null hypothesis under which the parameters of interest belong to the intersection of some open subset of $\mathbb{R}^{p+q+1}$ against the alternative, which the parameters do not belong to this intersection.

Our objective is to establish a LAO parametric test, based on local asymptotic normality results for Periodically time-varying ARFIMA (PtvARFIMA) models constructed in Amimour and Belaide (2020). Since the ARFIMA model has a very important parameter, which give the long memory feature to the model, it seems very important to test the periodicity of this parameter. The present article is mainly devoted to test a pure stationary ARFIMA model, whose innovations are independent, of the same specified density, which is not necessarily Gaussian while satisfies some mild conditions, and the specified long memory parameter, against a contiguous local alternative hypothesis consisting of a locally periodic of the specified period and unspecified parameter. The asymptotic linearity property is established for the central sequence, which allows for controlling (under arbitrary innovation densities) the asymptotic consequences of alignment, that is, the effects of substituting estimated residuals for the exact ones, and the quadratic statistic of the test is expressed in terms of the function of the residual f-autocorrelations. This problem can be encountered, for example, during the validation step. Indeed, if the residuals calculated from a stationary ARFIMA model transformed from a seasonal process still reveal the seasonal behavior, it is natural, in this situation, to introduce an ARFIMA model whose long memory parameter is a periodic function in time, this model can take into account the periodicity of the autocorrelation function.

Outline of the article. We first present, in Section 2, the notation and main technical assumptions to be considered in the paper, then briefly, we recalls the LAN property of PARFIMA models and stat the definition of the statistical tool such that f-autocorrelation coefficients, in order to address the problem of the Correlogram-Based LAN, these results are then investigated in Section 3, where we show the locally asymptotically optimal test for the periodic model in case when the error density is specified.

2. LAN Property for PtvARFIMA Models

2.1. The simple and alternative hypotheses

Let, $H_f^{(n)}(\mathbf{d})$, where $\mathbf{d}$ is the p-dimensional vector $(d_1, ..., d_p)'$, be the sequence of the simple hypothesis under which the observed process $(X_t^{(n)}, t \in \mathbb{Z})$ satisfies the stochastic linear difference equation called stationary ARFIMA$(0, d, 0)$ and given by:

$$(1 - B)^d X_t^{(n)} = \varepsilon_t, \tag{1}$$

where, d is the so-called long memory parameter whose values lie in $(0, \frac{1}{2})$, and $(\varepsilon_t, t \in \mathbb{Z})$ is a zero mean white noise with finite variance σ^2, admits a specified density f which is not necessarily Gaussian, and σ^2 is not specified, B denotes the lag operator such that $X_{t-j} = B^j X_t$. According to Hosking (1981) the process (1) is invertible and causal when $d \in (0, \frac{1}{2})$.

Let, $H_f^{(n)}\left(d^{(n)}\right)$, where $d^{(n)}$ is the p-dimensional vector $(d_1^{(n)}, ..., d_p^{(n)})'$, be a sequence of local alternative hypotheses which are contiguous to $H_f^{(n)}(\mathbf{d})$, under which the process $(X_t^{(n)}, t \in \mathbb{Z})$ satisfies the stochastic linear difference equation called PARFIMA$(0, d_i, 0)$ and given by:

$$(1 - B)^{d_i^{(n)}} X_{i+pm} = \varepsilon_{i+pm}(d_i^{(n)}), \tag{2}$$

where for all $t \in \mathbb{Z}$, there exists $i = \{1, \ldots, p\}$, $m \in \mathbb{Z}$, such that $t = i + pm$ and $p > 1$ represents the period $\in \mathbb{N}$ and $d_i^{(n)} = d + n^{-1/2}\delta_i^{(n)}$,

$$d^{(n)} = d + n^{-1/2}\delta^{(n)},$$

with $\delta^{(n)} = (\delta_1^{(n)}, \ldots, \delta_p^{(n)})'$, $(\varepsilon_t, t \in \mathbb{Z})$ is a zero mean white noise with finite variance σ^2, which is supposed to be constant in time and admits the density f previously defined, and $\delta_i^{(n)}$ is the periodic local perturbation of the parameter d. X_{i+pm} represents X_t during the i-th season of the year. According to Amimour (2020) the process (2) is invertible and causal when $d_i \in (0, \frac{1}{2})$.

2.2. Main assumptions and comments

In the whole paper, the following assumptions are required:

Assumption 1 $f(x) > 0, x \in \mathbb{R}, \int_{-\infty}^{\infty} x f(x) dx = 0, \int_{-\infty}^{\infty} x^4 f(x) dx < \infty.$

Assumption 2 f is absolutely continuous on finite intervals (see Hajek et al. (1999) p 14), i.e. there exists an integrable function f' on all bounded intervals, such that, $a < b \in \mathbb{R}$, we have, $f(b) - f(a) = \int_a^b f'(x) dx.$

Assumption 3 Letting $\Phi_f(x) = -f^{'}/f$, assume that the Fisher information associated with f is finite, and $0 < \int\limits_a^b |\Phi_f(x)|^{2+\delta} f(x)dx < \infty,$ for some $\delta > 0$. Implies that $0 < I(f) = E(\Phi_f^2(\varepsilon_t)) < \infty$.

Assumption 4 f is strongly unimodal,(see Hajek et al. (1999), p 14) i.e., the function $-\log(f(x))$ is convexe on all intervals $(a\,;b)$, or Φ_f is monotone increasing on $\mathbb{R}$.

Assumption 5 The score function Φ_f satisfies (almost everywhere) the Lipschitzian condition (global) (see, e.g. Hallin and Puri (1988)):

$$|\Phi_f(x) - \Phi_f(y)| < A_f|x - y|,$$

where A_f is a constant independent only of f.

Denote by $X^{(n)} = (X_1^{(n)}, \ldots, X_n^{(n)})$ a realization of a finite size n and suppose for simplicity of notation, that n is a multiple of p, i.e., $n = n'p, n' \in \mathbb{N}^*$ and let the log-likelihood ratio for $H_f^{(n)}\left(d^{(n)}\right)$ with respect to $H_f^{(n)}(\mathbf{d})$ given by

$$\Lambda_{f,d(n)/\mathbf{d}}^{(n)}(X^{(n)}) = \log \frac{I_{d^{(n)}/f}(X^{(n)})}{I_{\mathbf{d}/f}(X^{(n)})}, \tag{3}$$

where, $I_{d^{(n)}/f}(X^{(n)})$ represents the conditional likelihood function under $H_f^{(n)}\left(d^{(n)}\right)$, and $I_{d/f}(X^{(n)})$ the conditional likelihood function under $H_f^{(n)}(\mathbf{d})$.

Denote also by $\Delta_f^{(n)}(\mathbf{d}) = (\Delta_{1,f}^{(n)}, \ldots, \Delta_{p,f}^{(n)})'$ the central sequence:

$$\Delta_{i,f}^{(n)} = \frac{1}{\sqrt{n}}\sum_{k=1}^{q}\frac{1}{k}\sum_{\left[m=\frac{k-i+1}{p}\right]}^{n^{'}-1}(Z_{i+mp-k}\Phi_f(Z_{i+mp})), \tag{4}$$

where, Z_{i+pm} is the subsequent quantities of the calculated residuals under $H_f^{(n)}(\mathbf{d})$, and $q \in \mathbb{N}$ is the truncation parameter. Noting that all convergences are taken under $H_f^{(n)}(\mathbf{d})$.

Lemma 1 Amimour and Belaide (2020). Suppose that Assumptions 1 - 4 hold. Then

(i) The log-likelihood ratio admits, under $H_f^{(n)}(\mathbf{d})$, as $n \to \infty$, the asymptotic representation

$$\Lambda_{f,d^{(n)}/\mathbf{d}}^{(n)}(X^{(n)}) = \left(\delta^{(n)}\right)^{'}\Delta_f^{(n)}(\mathbf{d}) - \frac{1}{2}\left(\delta^{(n)}\right)^{'}\Gamma_f(\mathbf{d})\delta^{(n)} + O_{\mathbb{P}_\mathbf{d}}(1). \tag{5}$$

(ii) Still under $H_f^{(n)}(\mathbf{d})$, as $n \to \infty$; $\Delta_f^{(n)}(\mathbf{d})$ is asympotically normal,with mean 0 and covariance matrix $\Gamma_f(\mathbf{d})$, where

$$\Gamma_f(\mathbf{d}) = \sigma^2 I(f)\begin{pmatrix} \sum\limits_{k=1}^{\infty}\frac{1}{k^2} & & 0 \\ & \ddots & \vdots \\ 0 & \cdots & \sum\limits_{k=1}^{\infty}\frac{1}{k^2} \end{pmatrix}, \tag{6}$$

Proof: See Lemma 3.1 in Amimour and Belaide (2020) or (Amimour (2020); p 70).

The following corollary establish the asymptotic distributions of the central sequence and the logarithm of the likelihood ratio under the alternative hypothesis $H_f^{(n)}(d^{(n)})$.

Corollary 1 Under the same assumptions and notations of the Lemma (1) we have

$$\Lambda_{f,d^{(n)}/\mathbf{d}}(X^{(n)}) - \delta^{(n)'}\Delta_f^{(n)}(\mathbf{d}) + \frac{1}{2}\delta^{(n)'}\Gamma_f(\mathbf{d})\delta^{(n)} = O_{P_{\mathbf{d}}}(1),\ under\ H_f^{(n)}(d^{(n)}). \quad (7)$$

$$\Delta_f^{(n)}(\mathbf{d}) \underset{n\to\infty}{\to} N\left(\Gamma_f(\mathbf{d})\delta^{(n)}, \Gamma_f(\mathbf{d})\right),\ under\ H_f^{(n)}(d^{(n)}). \quad (8)$$

$$\Lambda_{f,d^{(n)}/\mathbf{d}}(X^{(n)}) \underset{n\to\infty}{\to} N\left(\frac{1}{2}\delta^{(n)'}\Gamma_f(\mathbf{d})\delta^{(n)}, \delta^{(n)'}\Gamma_f(\mathbf{d})\delta^{(n)}\right),\ under\ H_f^{(n)}(d^{(n)}). \quad (9)$$

Now, we will focus on the development of the locally asymptotically maximin test, corresponding to the specified simple hypothesis (the long memory parameter is specified), against the unspecified local alternative hypothesis.

2.3. Equivalent forms of the residual autocorrelation function

Let the residual autocorrelation function of the i-em period, $i = 1, 2, ..., p$ and of order $k = 1, 2, ..., n-1$ associated with the density of the white noise process:

$$r_{i,k,n} = \frac{1}{\sigma\sqrt{I(f)}\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)} \sum_{m=\left\lfloor\frac{k-i+1}{p}\right\rfloor}^{n'-1} \Phi(Z_{i+pm})Z_{i+pm-k}.$$

Replacing σ and $I(f_1) = \sigma^2 I(f)$ by these estimators

$$S^{(n)2} = \frac{1}{n}\sum_{i=1}^{p}\sum_{m=0}^{n'-1} Z_{i+pm}^2,$$

and

$$\hat{I}(f_1) = \frac{1}{n}\sum_{i=1}^{p}\sum_{m=0}^{n'-1} \Phi_1^2\left(\frac{Z_{i+pm}^{(n)}}{S^{(n)}}\right),$$

respectively, where $\Phi(x) = \frac{\Phi_1(\frac{x}{\sigma})}{\sigma}$, we obtain

$$\hat{r}_{i,k,n} = \frac{1}{\sqrt{\hat{I}(f_1)}\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)} \sum_{m=\left\lfloor\frac{k-i+1}{p}\right\rfloor}^{n'-1} \Phi_1\left(\frac{Z_{i+pm}^{(n)}}{S^{(n)}}\right)\frac{Z_{i+pm-k}}{S^{(n)}}.$$

Proposition 1 Assuming that Assumptions 1 and 5 hold, then we have the following assertions:

- The random variables $\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}} r_{i,k,n}$ and $\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}} \hat{r}_{i,k,n}$ are asymptotically equivalents.
- $\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}} r_{i,k,n}$ (resp. $\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}} \hat{r}_{i,k,n}$) is exactly (resp, asymptotically) standardized under $H_f^{(n)}(\mathbf{d})$.

Proof: Putting

$$T_i^{(n)}(Z^{(n)}, \alpha) = \frac{1}{\sigma I(f)\left(n' - \left\lfloor\frac{k-i+1}{p}\right\rfloor\right)} \sum_{m=\left\lfloor\frac{k-i+1}{p}\right\rfloor}^{n'-1} \Phi_f(\alpha Z_{i+pm})Z_{i+pm-k}, \text{for } i = 1, ..., p,$$

we have

$$\begin{aligned}\left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}r_{i,k,n} &= (I(f))^{-\frac{1}{2}}\left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}\\ &\times\frac{1}{\sigma}T_i^{(n)}(Z^{(n)},\frac{1}{\sigma}).\\ \left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}\hat{r}_{i,k,n} &= (I(f))^{-\frac{1}{2}}\left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}\\ &\times\frac{1}{S^{(n)}}T_i^{(n)}(Z^{(n)},\frac{1}{S^{(n)}}).\end{aligned}$$

Since $S^{(n)}$ is the estimate of σ, it is sufficient to prove that

$$\begin{aligned}\Delta_i^{(n)}(Z^{(n)},\alpha) &= \left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}\\ &\times\left[T_i^{(n)}(Z^{(n)},\frac{1}{\sigma})-T_i^{(n)}(Z^{(n)},\frac{1}{S^{(n)}})\right]\\ &= O_{p_d}(1),\text{for } i=1,...,p,\end{aligned}$$

according to Hallin and Puri (1988), $T_i^{(n)}(Z^{(n)},\alpha)$, is differentiable with respect to α at point $\alpha=\frac{1}{\sigma}$.

$$\begin{aligned}\Delta_i^{(n)}(Z^{(n)},\alpha) &= \left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}\\ &\times\left[\left(\frac{1}{S^{(n)}}-\frac{1}{\sigma}\right)\partial_\alpha T_i^{(n)}\left(Z^{(n)},\alpha^*(Z^{(n)})\right)\right],\end{aligned}$$

where $\alpha^*(Z^{(n)})\in\left(\min\left(\frac{1}{S^{(n)}},\frac{1}{\sigma}\right);\max\left(\frac{1}{S^{(n)}},\frac{1}{\sigma}\right)\right)$,
so we can show that the quantity

$$\partial_\alpha T_i^{(n)}\left(Z^{(n)},\alpha^*(Z^{(n)})\right)-\partial_\alpha T_i^{(n)}(Z^{(n)},\frac{1}{\sigma}),$$

converges to 0 in quadratic mean (see Bentarzi (1995), p 175).

From the asymptotic equivalent and it is clear that $\left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}r_{i,k,n}$ is centred with variance 1, so $\left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)^{\frac{1}{2}}r_{i,k,n}$ is exactly standardized under $H_f^{(n)}(\mathbf{d})$.

2.4. LAN property based on f-residual autocorrelation

The correlogram-Based LAN result, involving a central sequence which is measurable with respect to the residual autocorrelation, this version of correlogram-Based LAN allows for a better intuitive interpretation of the central sequence and easier diagnosis of its adaptivity/non adaptivity features. Noting

$$\hat{\Delta}_{i,f}^{(n)}=\left[\sigma\sqrt{I(f)}\left(n^{'}-\left\lfloor\frac{k-i+1}{p}\right\rfloor\right)\right]\frac{1}{\sqrt{n}}\sum_{k=1}^{q}\frac{1}{k}\hat{r}_{i,k,n}.$$

The following proposition provides an alternative formulation of LAN based on central sequence

Proposition 2 Under Assumptions 1 - 4 we have:

$$\hat{\Delta}_f^{(n)}(\mathbf{d})\underset{n\to\infty}{\to}N\left(0,\Gamma_f(\mathbf{d})\right),\text{ under } H_f^{(n)}(\mathbf{d}).$$

$$\hat{\Delta}_f^{(n)}(\mathbf{d})\underset{n\to\infty}{\to}N\left(\Gamma_f(\mathbf{d})\delta^{(n)},\Gamma_f(\mathbf{d})\right),\text{ under } H_f^{(n)}(d^{(n)}).$$

Proof: See Corollary 1 and Lemma 1.

So, $\hat{\Delta}_{i,f}^{(n)}$ relies on a generalized concepte of residual autocorrelations and higher order residual moments Akharif and Hallin (2003).

Proposition 3 The central sequence $\Delta_f^{(n)}(\mathbf{d})$ (and so $\hat{\Delta}_f^{(n)}(\mathbf{d})$) have under $H_f^{(n)}(\mathbf{d})$, (and so under $H_f^{(n)}(d^{(n)})$) the following asymptotic linearity property

$$\Delta_f^{(n)}(d^{(n)}) - \Delta_f^{(n)}(\mathbf{d}) = -\Gamma_f(\mathbf{d})\delta^{(n)} + O_{P_{\mathbf{d}}}(1).$$

Proof: Let $V^{(n)}$ be a sequence of vectors, $p-$dimensional, such that $\sup_n V^{(n)'}V^{(n)} < \infty$. Noting $\Lambda_{f,d+n^{-\frac{1}{2}}V^{(n)}/d+n^{-\frac{1}{2}}\delta^{(n)}}(X^{(n)})$ the logarithm of likelihood ratio for $H_f^{(n)}(d+n^{-\frac{1}{2}}V^{(n)})$ with respect to $H_f^{(n)}(d+n^{-\frac{1}{2}}\delta^{(n)})$, so we can see that

$$\begin{aligned}\Lambda_{f,d+n^{-\frac{1}{2}}\left(V^{(n)}+\delta^{(n)}\right)/d+n^{-\frac{1}{2}}\delta^{(n)}}(X^{(n)}) &= \Lambda_{f,d+n^{-\frac{1}{2}}\left(V^{(n)}+\delta^{(n)}\right)/\mathbf{d}}(X^{(n)}) \\ &\quad -\Lambda_{f,d+n^{-\frac{1}{2}}\delta^{(n)}/\mathbf{d}}(X^{(n)}).\end{aligned}$$

Using the LAQ decomposition under $H_f^{(n)}(d+n^{-\frac{1}{2}}\delta^{(n)})$, and so under $H_f^{(n)}(\mathbf{d})$ by contiguity we obtain

$$\begin{aligned}\Lambda_{f,d+n^{-\frac{1}{2}}\left(V^{(n)}+\delta^{(n)}\right)/d+n^{-\frac{1}{2}}\delta^{(n)}}(X^{(n)}) &= \Lambda_{f,d+n^{-\frac{1}{2}}V^{(n)}+n^{-\frac{1}{2}}\delta^{(n)}/d+n^{-\frac{1}{2}}\delta^{(n)}}(X^{(n)}) \\ &= V^{(n)'}\Delta_f^{(n)}(d+n^{-\frac{1}{2}}\delta^{(n)}) \\ &\quad -\frac{1}{2}V^{(n)'}\Gamma_f(d+n^{-\frac{1}{2}}\delta^{(n)})V^{(n)} + O_{P_{\mathbf{d}}}(1),\\ \text{and } \Lambda_{f,d+n^{-\frac{1}{2}}\left(V^{(n)}+\delta^{(n)}\right)/d+n^{-\frac{1}{2}}\delta^{(n)}}(X^{(n)}) &= \Lambda_{f,d+n^{-\frac{1}{2}}\left(V^{(n)}+\delta^{(n)}\right)/\mathbf{d}}(X^{(n)}) \\ &\quad -\Lambda_{f,d+n^{-\frac{1}{2}}\delta^{(n)}/d}(X^{(n)}) \\ &= V^{(n)'}\Delta_f^{(n)}(\mathbf{d}) - V^{(n)'}\Gamma_f(\mathbf{d})\delta^{(n)} \\ &\quad -\frac{1}{2}V^{(n)'}\Gamma_f(\mathbf{d})V^{(n)} + O_{P_{\mathbf{d}}}(1).\end{aligned}$$

Hence we get under $H_f^{(n)}(\mathbf{d})$

$$V^{(n)'}\left(\Delta_f^{(n)}(d+n^{-\frac{1}{2}}\delta^{(n)}) - \Delta_f^{(n)}(\mathbf{d})\right) = -V^{(n)'}\Gamma_f(d)\delta^{(n)} + O_{P_{\mathbf{d}}}(1).$$

3. Locally Asymptotically Optimal Test

Definition 1 The envelopes of the power function denoted by $\beta\left(\alpha, H_f^{(n)}, K_f^{(n)}\right)$, for the sequences of the simple hypotheses $H_f^{(n)}$ against the alternatives $K_f^{(n)}$, is defined as follow:

$$\beta\left(\alpha, H_f^{(n)}, K_f^{(n)}\right) = \sup_{\Phi\in\Psi_\alpha}\inf_{d\in H^{(n)}} E\Phi\ \alpha \in]0,1[,$$

where $\Psi_\alpha = \left\{\Phi / \underset{d\in H^{(n)}}{E}\Phi \le \alpha\right\}$.

Let the sequences of alternative hypotheses

$$K_f^{(n)}(c) = \underset{\delta^{(n)}\in\mathbb{R}^p}{\cup}\left\{H_f^{(n)}(d+n^{\frac{1}{2}}\delta^{(n)});\delta^{(n)}\in\mathbb{R}^p/\delta^{(n)'}\left(\Gamma_f(\mathbf{d})\right)^{-1}\delta^{(n)} \ge c\right\}, c>0,$$

and considering the quadratic form

$$Q_f^{(n)}(\mathbf{d}) = \hat{\Delta}_f^{(n)}(\mathbf{d})' \left(\Gamma_f(\mathbf{d})\right)^{-1} \hat{\Delta}_f^{(n)}(\mathbf{d}).$$

The previous definitions and notations allow us to give the essential result concerning the problem of testing a specified stationary model, given by 1, against an unspecified local p-periodic model given by 2. That is, a testing problem clenched with the Gaussian experiment position, i.e testing the simple hypothesis

$$H_f^{(n)} : \delta^{(n)} = 0,$$

versus the alternative one

$$K_f^{(n)} : \delta^{(n)} \neq 0.$$

The following proposition describes the optimal test to test this hypothesis.

Proposition 4 Under Assumptions 1 - 5 we have:

(i) The quadratic form $Q_f^{(n)}(\mathbf{d})$ follows asymptotically the central chi-square with p degrees of freedom $\chi^2(p)$ under $H_f^{(n)}(\mathbf{d})$.

(ii) The quadratic form $Q_f^{(n)}(\mathbf{d})$ follows asymptotically the non central chi-square with p degrees of freedom and noncentrality parameter v, $\chi^2(p, v)$, where $v = \delta^{(n)'} \Gamma_f(\mathbf{d}) \delta^{(n)}$ under $H_f^{(n)}\left(d^{(n)}\right)$.

The sequence of tests $\Phi^{(n)} = 1$ if and only if $Q_f^{(n)}(\mathbf{d}) > \chi^2_{1-\alpha}(p)$.

(1) has asymptotic level α (under $H_f^{(n)}$ $(\mathbf{d})$).

(2) has asymptotic power:

$$1 - F\left(\chi^2_{1-\alpha}; p, \delta^{(n)'} \Gamma_f(\mathbf{d}) \delta^{(n)}\right), \text{ under } H_f^{(n)}\left(d^{(n)}\right),$$

where $F\left(\chi^2_{1-\alpha}; p, v\right)$ denotes the non central chi-square distribution function with p degrees of freedom and non centrality parameter v.

(3) is locally asymptotically optimal for testing a specified time-invariant ARFIMA$(0, d, 0)$ model, given by (1), against, a unspecified local p-periodic ARFIMA model (2).

Proof: The asymptotic law of the quadratic form $Q_f^{(n)}(\mathbf{d})$ of the central sequence $\hat{\Delta}_f^{(n)}(\mathbf{d})$ follows immediately from the asymptotic normality of $\hat{\Delta}_f^{(n)}(\mathbf{d})$, under $H_f^{(n)}(\mathbf{d})$, and under $H_f^{(n)}\left(d^{(n)}\right)$ see proposition (2).

Example Testing an ARFIMA $(0, d, 0)$ against a local 2-periodic ARFIMA $(0, d^{(n)}, 0)$, $p = 2, i = 1, 2$. The LAO test for testing $H_f^{(n)}$ against the local alternative hypothesis of type $K_f^{(n)}$, consists in rejecting the null hypothesis if:

$$\frac{6}{n\pi^2}\left[\left(n' - \lfloor \tfrac{k}{2} \rfloor\right) \sum_{k=1}^{q} \tfrac{1}{k} \hat{r}_{1,k,n}\right]^2 + \frac{6}{n\pi^2}\left[\left(n' - \lfloor \tfrac{k-1}{2} \rfloor\right) \sum_{k=1}^{q} \tfrac{1}{k} \hat{r}_{2,k,n}\right]^2 > \chi^2_{1-\alpha}(2).$$

The power function of the test, is given by

$$1 - F\left(\chi^2_{1-\alpha}; 2; \frac{\pi^2}{6} \sigma^2 I(f) \left(\delta_1^2 + \delta_2^2\right)\right).$$

4. Conclusion

In this work, we solved the problem for testing periodicity, in the long memory models driven by a periodically time varying long memory parameter, that is, the problem of detecting periodicity in the ARFIMA model against the alternative local p-periodic ARFIMA model, using the LAN property of PARFIMA model, which is the key result for virtually all problems in asymptotic inference connected with this type of model, and the residual autocorrelation structure. The test constructed here is valid and optimal when the innovation density f, which intervenes in the statistic of the test through the score function is specified. So, this note opens the way for constructing a most stringent test in future consideration, when the error density is unknown and considered as a nuisance parameter and the long-memory parameter is unspecified. Also for establishing the locally asymptotically optimal Rank-Based tests, or generalizing the test constructed here by considering the short term component of PARFIMA model.

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