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Accelerated Life Testing and Its Application in the Field of Warranty Policy under Type-I Censoring Scheme

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Abstract

In this paper, we describe how to propose and analyze the accelerated life testing plans to develop the excellence and reliability of the item for consumption. We also focus on finding the expected cost rate and the expected total cost for age replacement under pro rate rebate warranty policy. The problem is studied using constant stress, under the assumption that the lifetimes of the units follow the Fréchet failure model for predicting the cost of age replacement under the warranty policy. Asymptotic variance and covariance matrix of the estimators are obtained by using the Fisher information matrix. Confidence intervals for parameters and respective errors are also obtained. A simulation study is performed to illustrate the statistical properties.

Keywords: Accelerated life test, constant stress, Fréchet distribution, warranty policy, age-replacement, Type-I censoring, Fisher information matrix, simulation.

1. Introduction

Due to the exercise of high expertise in manufacturing items, the reliability of items has become very high, thus it is very tough nowadays to get a satisfactory number of failure times in classical life investigation experiments and reliability experimentations; the majority of items have an exponential life span and it may be thousands of hours. Our intent is to find an appropriate way to generate early failures, because, under normal use conditions for products, producing failures in the lifetime experiments with a restricted testing era may construct very few failures and it is not enough to study the information and make a very good model for them. Thus, we use the most excellent way for decreasing the life spans of the goods: Accelerated life tests (ALT). In ALT, units or goods are exposed to tough circumstances and high-stress levels (humidity, temperature, pressure, voltage, etc.) according to the situations of manufacturing. There are many methods and models of ALT; goods are exposed to stress according to the reason of the experiment and the type of goods. Scientists apply different types and methods of acceleration, e.g., constant ALT, step stress ALT, and progressive stress ALT.

Failure data obtained from ALTs can be divided into two categories: complete or censored. Complete data consist of the exact failure time of test units, which means that the failure time of each sample unit is observed or known. In many cases, when life data are analyzed. All units in the sample may not fail. This type of data is called censored or incomplete data and this scheme is called censoring. There are several types of censoring, including left, right, interval, time, and failure censoring. Here, we are only focusing on Type-I censoring scheme. Type I censoring, this censoring occurs when a test end after a fixed amount of time. It is also called time censoring.

In this paper, we design an accelerated life test plan using a Type-I censoring scheme when the lifetime of test units follows the Fréchet failure model and provides its application in the field of warranty policy.

Now, we present brief literature on accelerated life testing (ALTg) and warranty models related to our study in this paper. El-Dessouky (2015) described ALTg and age-replacement policy under warranty with the exponentiated Pareto distribution. El-Dessouky (2017) also described maintenance service policy using step stress partially ALTg for the extension of the Exponential distribution using Type-II censored samples. Zhao and Xie (2017) presented a framework to predict the warranty cost and risk under one-dimensional. In the present time, the two-dimensional (Jack et al. 2009, Gupta et al. 2014, Huang et al. 2017) and extended warranty (Jung et al. 2015, Ye and Murthy, 2016) have taken an important place in warranty policy analysis. However, it is very tough and challenging to design warranty policies and predict warranty costs for new products that have not been come into the market, because the failure rate of such types of products is not available. A literature review is presented by Murthy and Djamaludin (2002) on new product warranty by considering marketing, logistics, etc., and a joint optimization model that involved reliability, warranty, and price for new products are presented by Huang et al. (2007). Xie and Ye (2016) proposed an aggregate discounted warranty cost forecast under the new item.

ALTs could be conducted to adumbrate the reliability of new products. In an ALT, the ecology stresses animated to advance the occurrences of failures. By celebratory abortion times (either exact or censored) from an ALT, the accelerated life (AL) model ambit estimated, and analysis planners can use these estimators to accomplish statistical inferences of the lifetime administration of analysis products, which helps to adumbrate the number of warranty claims. In Yang (2010), optimal 3-level accommodation ALT affairs were discussed to minimize the asymptotic about-face of best likelihood appraisal of the assurance cost and proposed a technique for item population for predicting the warranty cost and its confidence interval. For an overview of accelerated believability tests, one can refer to Meeker and Escobar (1993). Borgia et al. (2013) presented a case study for the household's gadgets; he proposed accelerated life tests for new product qualification. Currently, Alam et al. (2020) presented a study on age replacement policy under pro rebate warranty policy for Burr Type-X distribution using Type-II censoring scheme. Alam et al. (2019a, 2019b) handled with step stress partially ALT with multiple and adaptive Type-II progressively hybrid censoring schemes. Alam et al. (2021) handled constant-stress ALT under a progressive Type-II censoring scheme and also presents its application in the area of maintenance strategy. Chaudhary and Tomer (2018) presented an estimation of stress–strength reliability for the Maxwell model under a progressive Type-II censoring. Chaudhary et al. (2017) considered the estimation of $R = P(Y < X)$ under Maxwell failure model.

The research is motivated by two factors. The first aims to establish ALT under Type-I censoring scheme for the Fréchet failure model. The second step is to provide the application of ALT in the field of warranty policy. The originality of this work stems from the fact that no earlier research has been conducted in this area using proposed censoring strategies for the Fréchet distribution.

The rest of the paper is organized as follows: Section 2 provides the introduction of Fréchet distribution and test procedure also. The likelihood function, Fisher Information matrix, the inverse of Fisher Information matrix is developed in Section 3. A simulation study is carried out in Section 4. Estimation of shape parameter and reliability function is presented in Section 5. The age-replacement policy under pro-rate rebate warranty for Fréchet distribution is presented in Section 6. The real-world implementation of the proposed problem is discussed in Section 7. Finally, a conclusion is made in Section 8.

2. Model Description and Test Method

ALT is generally performed by one of the two approaches; (i) Accelerated failure time, which means ALT is performed for the item or component by experiencing usual circumstances but more intensively than normal. This approach is excellent for items or components that are exercised on a continuous-time basis. (ii) Accelerated stress means ALT is performed by with items or components at higher stress than usual. For designing ALT plans, the following points are needed

- (i) The stress application testing process.
- (ii) The stress levels for each stress type selected and the type of stress to be applied in the investigation.
- (iii) Relationship between life and stress (life-stress relationship) which is expressed by the mathematical model.
- (iv) At last, the proportion of experiment units to be allocated to each level of stress.

Some authors dealt with constant stress, such as Abdel-Ghaly et al. (1998) tackled with an accelerated life test plan for the Pareto failure model and estimate reliability function and parameters of the model. Attia et al. (2011, 2013) handled with Accelerated life test plan for Birnbaum-Saunders and generalized logistic distributions using different censoring plans with constant stress. In the following section:

- (i) Stress S_j has k -levels.
- (ii) Assuming that S_u is normal use situation and fulfilling $S_u < S_1 < S_2 < S_3 < \dots < S_k$.
- (iii) There are m_j units put on the investigation at each stress level.
- (iv) The test ends when r_j units attain among these m_j units.

This current study is dealt with Type-I censoring and constant stress with the assumption that the lifetime of the units follows the Fréchet failure model.

Let the random variable (r.v.) X follows Fréchet distribution, then its probability density function and cumulative density function are given as:

The probability density function (pdf) of the model is given as

$$f(x, \lambda, \mu) = \lambda \mu x^{-(\mu+1)} e^{-\lambda x^{-\mu}}, \quad x > 0, \lambda > 0, \mu > 0, \quad (1)$$

where λ and μ are scale and shape parameters, respectively.

The cumulative distribution function (cdf) of the model is given as

$$F(x, \lambda, \mu) = e^{-\lambda x^{-\mu}}, \quad x > 0, \lambda > 0, \mu > 0. \quad (2)$$

The reliability function of the model is given as

$$R(x, \lambda, \mu) = 1 - e^{-\lambda x^{-\mu}}. \quad (3)$$

The hazard function of the model is given as

$$h(x, \lambda, \mu) = \frac{\lambda \mu x^{-(\mu+1)} e^{-\lambda x^\mu}}{1 - e^{-\lambda x^\mu}}. \quad (4)$$

This failure model is also known as inverse Weibull distribution (IWD). The hazard function is always unimodal while the probability density function depends on the shape parameter as it is unimodal or decreasing. This distribution is a max stable distribution and the negative of a r.v. having a Fréchet distribution is a min stable distribution. Chi-Chao (1997) provided details in his research if the product consists of various components and each component has a similar failure time distribution. The product falls when the powerless component fails. Then the Weibull failure is a suitable model of that failure mode. Similarly, the Fréchet failure model will be suitable for modeling when electrical or mechanical parts lie in the life testing experimentation. The pdf, cdf, reliability, and hazard curves are shown in Figures 1-4, respectively.

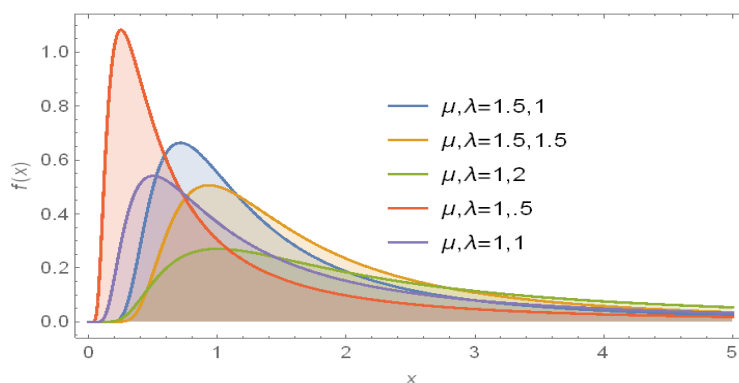


Figure 1 Probability density curve of Fréchet distribution

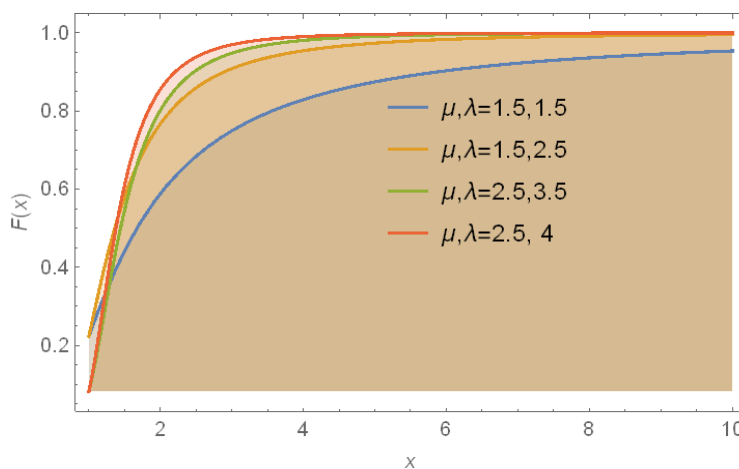


Figure 2 Cumulative distribution curve of Fréchet distribution

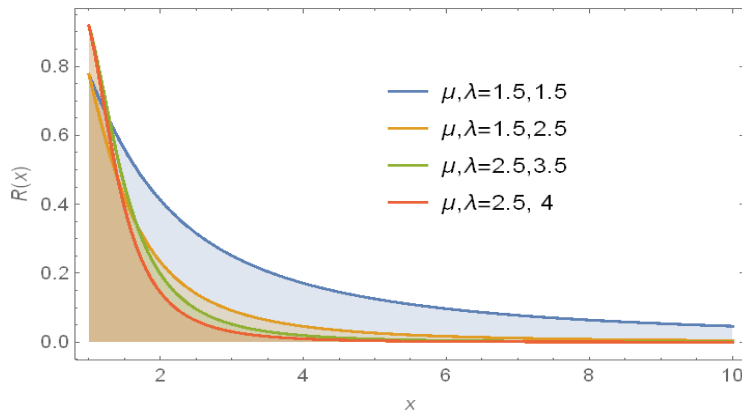


Figure 3 Reliability curve of Fréchet distribution x

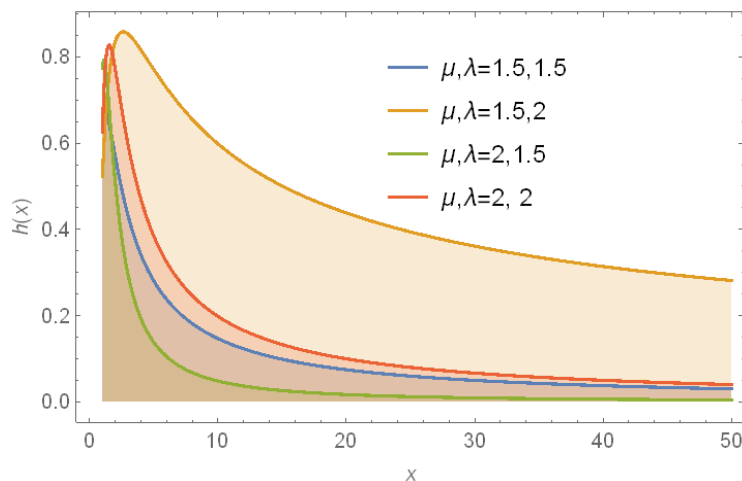


Figure 4 Hazard curve of Fréchet distribution

Several authors have studied different aspects of inferential procedures for the Fréchet failure model. The maximum likelihood (ML) estimators and the ordinary least-square estimators for this model are discussed by Calabria and Pulcini (1989) and Erto (1989), respectively. In the occurrence of cure fraction, the ML estimators of this model are presented by Ramos et al. (2019).

This study is based on constant stress and Type-I censoring scheme. We have considered the Stress $S_j, j = 1, 2, \dots, k$ which affects the shape parameter of the used distribution, μ_j through the following (5) called the power rule model.

$$\mu_j = US_j^{-q}; \quad U > 0, q > 0, \quad j = 1, 2, \dots, k, \quad (5)$$

where U and q are the proportionality constant; and power of the applied stress respectively.

3. Estimation Procedure

In this section, the ML estimation method is used. The reliability practitioner used this method because it is very robust and provides estimates of the parameters with excellent properties. At the

stress level S_j , the authors constructed the likelihood function of an observation (time to failure) and at each stress level S_j , m_j units were put on the test.

Therefore, $N = \sum_{j=1}^k m_j$ is the total number of units in the test. When a Type-I (time) censoring scheme is adopted at each stress level, the experiment ends once the censoring time “ x_0 ” is reached. It is assumed that $r_j (\leq m_j)$ units are observed at the j^{th} stress level before the test is terminated and $(m_j - r_j)$ units still carry on till the end of the test. In this situation, the likelihood function of the experiment for the Fréchet failure model is taking the following form

$$L(\lambda, U, \mu) = \prod_{j=1}^k \frac{m_j}{(m_j - r_j)!} \left[\prod_{i=1}^{r_j} f(x_{ij}; \mu_j, U, \lambda) \right] [1 - F(x_0)]^{m_j - r_j}, \quad (6)$$

where is the time of cessation of the test.

The log-likelihood function is obtained by taking the logarithm of the above (6) and given as

$$\begin{aligned} \ln L = Q + \sum_{j=1}^k r_j (\ln \lambda + \ln U - q \ln S_j) + \sum_{i=1}^{r_j} \sum_{j=1}^k (1 - US_j^{-q}) \ln x_{ij} + \sum_{i=1}^{r_j} \sum_{j=1}^k (-\lambda x_{ij} - US_j^{-q}) \\ + \sum_{j=1}^k (m_j - r_j) \ln(1 - e^{-\lambda x_0 - US_j^{-q}}), \end{aligned} \quad (7)$$

where Q is the constant.

The ML estimates of λ, U and q can be estimated from the following three equations.

$$\begin{aligned} \frac{\partial \ln L}{\partial \lambda} &= \sum_{j=1}^k \frac{r_j}{\lambda} + \sum_{i=1}^{r_j} \sum_{j=1}^k x_{ij} + \sum_{j=1}^k (m_j - r_j) \frac{x_0 e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} = 0, \\ \frac{\partial \ln L}{\partial q} &= -\sum_{j=1}^k r_j \ln S_j + \sum_{i=1}^{r_j} \sum_{j=1}^k \ln x_{ij} U q S_j^{-q-1} + \sum_{i=1}^{r_j} \sum_{j=1}^k U q V_j^{-q-1} - \sum_{j=1}^k (m_j - r_j) \frac{e^{-\lambda x_0 - US_j^{-q}} U S_j^{-q-1}}{1 - e^{-\lambda x_0 - US_j^{-q}}} = 0, \\ \frac{\partial \ln L}{\partial U} &= \sum_{j=1}^k \frac{r_j}{U} - \sum_{i=1}^{r_j} \sum_{j=1}^k \ln x_{ij} S_j^{-q} - \sum_{i=1}^{r_j} \sum_{j=1}^k S_j^{-q} + \sum_{j=1}^k (m_j - r_j) \frac{e^{-\lambda x_0 - US_j^{-q}} S_j^{-q}}{1 - e^{-\lambda x_0 - US_j^{-q}}} = 0. \end{aligned}$$

It looks impossible to solve the above three non-linear equations manually. Therefore, an iterative technique (Newton-Raphson) can be used to get ML estimates of parameters.

In mathematical statistics, the Fisher information or information matrix is a technique of evaluating the amount of information that an observable r.v. carries about an unknown parameter of a distribution. Simply, the information matrix is the variance of the score or the expected value of the observed information. The information matrix for the Fréchet failure model under Type-I censoring is given as

$$F = \begin{bmatrix} \frac{-\partial^2 \ln L}{\partial \lambda^2} & \frac{-\partial^2 \ln L}{\partial \lambda \partial U} & \frac{-\partial^2 \ln L}{\partial \lambda \partial q} \\ \frac{-\partial^2 \ln L}{\partial U \partial \lambda} & \frac{-\partial^2 \ln L}{\partial U^2} & \frac{-\partial^2 \ln L}{\partial U \partial q} \\ \frac{-\partial^2 \ln L}{\partial q \partial \lambda} & \frac{-\partial^2 \ln L}{\partial q \partial U} & \frac{-\partial^2 \ln L}{\partial q^2} \end{bmatrix}. \quad (8)$$

The elements of the matrix F are obtained by second partial derivatives of log-likelihood function with respect to parameters λ, q and U . Consequently, the elements are expressed by the following equations

$$\begin{aligned}
 -\frac{\partial^2 \ln L}{\partial \lambda^2} &= \sum_{j=1}^k \frac{r_j}{\lambda^2} + \sum_{j=1}^k (m_j - r_j) x_0 \frac{e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \left[x_0 + \frac{x_0 e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \right], \\
 -\frac{\partial^2 \ln L}{\partial U^2} &= \sum_{j=1}^k \frac{r_j}{U^2} - \sum_{j=1}^k (m_j - r_j) U q S_j^{-q-1} \frac{e^{-\lambda x_0 - US_j^{-q}} S_j^{-q}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \left[1 - \frac{e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \right], \\
 \frac{\partial^2 \ln L}{\partial q^2} &= -\sum_{i=1}^{r_j} \sum_{j=1}^k \ln x_{ij} U q S_j^{-q-1} (q^{-1} - \ln S_j) - \sum_{i=1}^{r_j} \sum_{j=1}^k U q S_j^{-q-1} (q^{-1} - \ln S_j) \\
 &\quad + \sum_{j=1}^k (m_j - r_j) \frac{e^{-\lambda x_0 - US_j^{-q}} U S_j^{-p}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \left[-\ln S_j + q U S_j^{-q-1} - \frac{e^{-\lambda x_0 - US_j^{-q}} U S_j^{-q} \ln S_j}{1 - e^{-\lambda x_0 - US_j^{-q}}} \right], \\
 -\frac{\partial^2 \ln L}{\partial \lambda \partial U} &= -\frac{\partial^2 \ln L}{\partial U \partial \lambda}, \\
 -\frac{\partial^2 \ln L}{\partial \lambda \partial U} &= -\sum_{j=1}^k (m_j - r_j) \frac{x_0 e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \left[S_j^{-q} - \frac{e^{-\lambda x_0 - US_j^{-q}} S_j^{-q}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \right], \\
 -\frac{\partial^2 \ln L}{\partial \alpha \partial p} &= -\frac{\partial^2 \ln L}{\partial p \partial \alpha}, \\
 -\frac{\partial^2 \ln L}{\partial \lambda \partial q} &= -\sum_{j=1}^k (m_j - r_j) \frac{x_0 e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \left[U S_j^{-q} + U S_j^{-q} \ln S_j \frac{e^{-\lambda x_0 - US_j^{-q}}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \right], \\
 -\frac{\partial^2 \ln L}{\partial U \partial q} &= -\frac{\partial^2 \ln L}{\partial q \partial U}, \\
 -\frac{\partial^2 \ln L}{\partial U \partial q} &= \sum_{i=1}^{r_j} \sum_{j=1}^k \ln x_{ij} S_j^{-q} \ln S_j + \sum_{i=1}^{r_j} \sum_{j=1}^k S_j^{-q} \ln S_j \\
 &\quad - \sum_{j=1}^k (m_j - r_j) \frac{e^{-\lambda x_0 - US_j^{-q}} S_j^{-p}}{1 - e^{-\lambda x_0 - US_j^{-q}}} \left[-U S_j^{-q} \ln S_j - \ln S_j - \frac{e^{-\lambda x_0 - US_j^{-q}} S_j^{-q} U \ln S_j}{1 - e^{-\lambda x_0 - US_j^{-q}}} \right].
 \end{aligned}$$

Now, the variance-covariance matrix is the inverse of the Fisher Information matrix and given as

$$\Sigma = F^{-1}, \quad (9)$$

$$\Sigma = \begin{bmatrix} -\frac{\partial^2 \ln L}{\partial \lambda^2} & -\frac{\partial^2 \ln L}{\partial \lambda \partial U} & -\frac{\partial^2 \ln L}{\partial \lambda \partial q} \\ -\frac{\partial^2 \ln L}{\partial U \partial \lambda} & -\frac{\partial^2 \ln L}{\partial U^2} & -\frac{\partial^2 \ln L}{\partial U \partial q} \\ -\frac{\partial^2 \ln L}{\partial q \partial \lambda} & -\frac{\partial^2 \ln L}{\partial q \partial U} & -\frac{\partial^2 \ln L}{\partial q^2} \end{bmatrix}^{-1} = \begin{bmatrix} AVar(\hat{\lambda}) & ACov(\hat{\lambda}\hat{U}) & ACov(\hat{\lambda}\hat{q}) \\ ACov(\hat{U}\hat{\lambda}) & AVar(\hat{U}) & ACov(\hat{U}\hat{q}) \\ ACov(\hat{q}\hat{\lambda}) & ACov(\hat{q}\hat{U}) & AVar(\hat{q}) \end{bmatrix}, \quad (10)$$

$AVar$ and $ACov$ are asymptotic variance and asymptotic covariance, respectively.

The $100(1-\gamma)\%$ approximated two-sided limits of confidence for parameters λ, U and q are given as

$$\hat{\lambda} \pm Z_{\gamma/2} \sqrt{I_{22}^{-1}(\hat{\lambda}, \hat{U}, \hat{q})}, \quad \hat{U} \pm Z_{\gamma/2} \sqrt{I_{22}^{-1}(\hat{\lambda}, \hat{U}, \hat{q})} \quad \text{and} \quad \hat{q} \pm Z_{\gamma/2} \sqrt{I_{33}^{-1}(\hat{\lambda}, \hat{U}, \hat{q})},$$

$Z_{\gamma/2}$ is the $100(1-\gamma/2)\%$ percentile of a standard normal variate.

4. Simulation Results

In this section, we apply the Monte-Carlo simulation technique to examine the performances of the MLEs through their absolute relative bias (RAB) and mean square error (MSE). Using the invariance property of MLEs, we can estimate the MLEs of shape parameter μ_j through the following equation,

$$\mu_j = US_j^{-q}; \quad U > 0, q > 0, \quad j = 1, 2, \dots, k.$$

The detailed steps are given below:

1. First, 1,000 random samples of sizes 40, 80, 120, and 160 are generated from Fréchet distribution by inverse CDF method. Different initial values are selected for all sets of parameters.

2. The stress has three levels, and the values of are $(S_1 = 1, S_2 = 1.5, S_3 = 2)$,
 $m_j = \frac{n}{3}$ & $r_j = 60\% n_j$.

3. The parameters of the model are estimated under Type-I censoring for all sample sizes.

4. The Newton-Raphson method is applied for solving all non-linear equations.

5. The estimates of the shape parameter μ_j are calculated from (5).

6. The RABs and MSEs are tabulated for all sets of (μ_0, U_0, q_0) .

7. We determine the MLEs of the shape parameter μ_u at the usual stress level $S_u = 0.5$ by the invariance property of MLEs.

8. The reliability function at the similar usual stress for various values λ, U, q and t_0 is calculated, $\hat{R}_u(t) = 1 - e^{-\lambda_0 t^{-\mu_0}}$.

9. At mission time ($t_0 = 1.2, 1.5, 1.9$), the MLEs of survival function are predicted in the same usual circumstances for every parameter set.

Table 1 The estimates, relative bias and MSE of the parameters $(\mu, S, q, \lambda_1, \lambda_2, \lambda_3)$

under Type-I censoring

<i>n</i>	Parameters	$(\lambda_0 = 0.40, S_0 = 1.7, q_0 = 1)$			$(\lambda_0 = 1.2, S_0 = 1.7, q_0 = 1)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
40	λ	2.239	0.095	0.088	1.330	0.080	0.086
	S	1.811	0.070	0.069	1.743	0.062	0.073
	q	2.130	0.080	0.072	1.325	0.095	0.081
	μ_1	1.556	0.065	0.057	2.321	0.061	0.077
	μ_2	1.443	0.062	0.051	1.965	0.053	0.050
	μ_3	1.778	0.055	0.062	2.201	0.061	0.069

Table 1 (Continued)

n	Parameters	$(\lambda_0 = 0.40, S_0 = 1.7, q_0 = 1)$			$(\lambda_0 = 1.2, S_0 = 1.7, q_0 = 1)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
80	λ	1.830	0.086	0.076	1.076	0.073	0.076
	S	1.543	0.067	0.055	2.046	0.052	0.061
	q	1.791	0.077	0.061	2.298	0.088	0.097
	μ_1	2.062	0.054	0.040	2.189	0.064	0.051
	μ_2	2.388	0.054	0.031	1.960	0.050	0.042
	μ_3	1.197	0.049	0.055	1.486	0.036	0.057
120	λ	2.139	0.044	0.042	2.345	0.069	0.061
	S	2.349	0.054	0.044	2.492	0.050	0.051
	q	2.046	0.065	0.054	2.052	0.072	0.086
	μ_1	1.556	0.046	0.035	2.528	0.042	0.047
	μ_2	2.290	0.048	0.040	1.404	0.039	0.037
	μ_3	1.196	0.041	0.022	1.967	0.011	0.050
160	λ	2.252	0.018	0.029	2.763	0.064	0.050
	S	2.612	0.023	0.013	2.160	0.042	0.023
	q	1.313	0.113	0.010	1.885	0.045	0.031
	μ_1	1.950	0.027	0.014	1.443	0.062	0.069
	μ_2	2.322	0.031	0.023	2.111	0.018	0.016
	μ_3	1.089	0.032	0.023	1.025	0.033	0.025

Table 2 The estimates, relative bias and MSE of the parameters $(\mu, S, q, \lambda_1, \lambda_2, \lambda_3)$ under Type-I censoring

n	Parameters	$(\lambda_0 = 0.40, S_0 = 1.3, q_0 = 1)$			$(\lambda_0 = 1.2, S_0 = 1.3, q_0 = 1.9)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
40	λ	2.346	0.076	0.089	2.987	0.099	0.087
	S	1.055	0.223	0.087	1.171	0.210	0.087
	q	1.327	0.090	0.088	2.621	0.082	0.071
	μ_1	2.401	0.093	0.062	1.523	0.081	0.071
	μ_2	1.970	0.212	0.081	2.210	0.077	0.068
	μ_3	1.783	0.099	0.077	2.809	0.093	0.096
80	λ	1.832	0.062	0.077	1.667	0.071	0.075
	S	1.014	0.092	0.072	1.158	0.144	0.070
	q	1.737	0.073	0.070	1.136	0.052	0.095
	μ_1	1.064	0.081	0.058	1.283	0.074	0.061
	μ_2	2.986	0.188	0.065	1.170	0.069	0.061
	μ_3	1.789	0.068	0.065	2.986	0.088	0.079

Table 2 (Continued)

<i>n</i>	Parameters	$(\lambda_0 = 0.40, S_0 = 1.3, q_0 = 1)$			$(\lambda_0 = 1.2, S_0 = 1.3, q_0 = 1.9)$		
		Estimator	RABs	MSEs	Estimator	RABs	MSEs
120	λ	1.639	0.054	0.045	2.998	0.063	0.062
	S	1.919	0.089	0.063	2.791	0.081	0.056
	q	1.778	0.052	0.050	1.990	0.050	0.066
	μ_1	2.061	0.086	0.061	2.930	0.066	0.055
	μ_2	1.398	0.070	0.047	1.132	0.054	0.016
	μ_3	1.494	0.059	0.076	2.998	0.077	0.098
160	λ	1.052	0.030	0.037	1.997	0.049	0.056
	S	2.168	0.060	0.023	1.439	0.055	0.041
	q	2.016	0.043	0.061	1.967	0.042	0.055
	μ_1	1.192	0.050	0.019	1.209	0.055	0.066
	μ_2	1.921	0.059	0.033	1.432	0.032	0.014
	μ_3	2.595	0.042	0.016	2.955	0.030	0.084

5. Estimation of the Survival Function and Shape Parameter at Normal Stress

In the following table, we estimate the survival function at the usual stress level $S_u = 0.5$ for various values of parameters λ, U, q and t_0 , also find the shape parameter for the same stress level.

Table 3 Estimated reliability function and shape parameter at normal stress

λ_0	U_0	q_0	μ_0	t_0	$R_u(t_0)$
0.40	1.7	1	4.121	1.2	0.312
				1.5	0.309
				1.9	0.242
1.2	1.7	1	3.804	1.2	0.436
				1.5	0.418
				1.9	0.303
0.40	1.3	1	3.477	1.2	0.610
				1.5	0.417
				1.9	0.388
1.2	1.3	1.9	3.526	1.2	0.195
				1.5	0.184
				1.9	0.107

6. The Age-Replacement Policy under Pro-rate Rebate Warranty for Fréchet Distribution

Under this warranty policy, the following assumptions are made

(i) A non-repairable product is replaced at a certain time τ or upon failure, which takes place earliest.

(ii) When the product is unsuccessful at the time $t \leq \tau$, a failure replacement is carried out with a purchasing cost P_c and downtime cost D_c , where $D_c, P_c > 0$.

(iii) The client is reimbursed by a quantity of sales price P_c if the item be unsuccessful over the warranty period (w_p). So, the rebate function in the pro-rata warranty is

$$R(t) = \begin{cases} P_c \left(1 - \frac{t}{w_p}\right) & 0 \leq t \leq w_p \\ 0 & t > w_p. \end{cases} \quad (11)$$

Mamer (1982) handled with a price tag investigation of pro-rata with a without charge replacement warranty approaches; he studied the long-run typical and total expenses of things with warranty. Matis et al. (2008) tackled a pro-rata study for joint warranty problems; he used several repair selections in his investigation. Huang et al. (2008) presented a study on estimating the predictable warranty cost for the approach where the item usage is intermittent and of heterogeneous usage intensity by the item existence cycle when sales occur regularly.

Key Assumptions: The main assumptions in this policy are

1. Product is replaced at the point of failure (corrective replacement), or age τ (preventive replacement), which arrives earliest.
2. The pro-rata rebate warranty approach is applied to the sale of products.
3. There is no salvage charge for the preventive replaced item.
4. The warranty time is less than age replacement, i.e. $w_p < \tau$.

When the item's life arrives τ , then the preventive replacement is carried out with cost P_c only because it is a planned preventive safeguarding action. The total cost incurred in a renewal cycle for this strategy is

$$C(d) = \begin{cases} D_c + P_c - R(t), & 0 \leq t \leq w_p \\ D_c + P_c, & w_p < t < \tau \\ P_c, & t \geq \tau. \end{cases} \quad (12)$$

The expected total cost (Chien 2010, Chien et al. 2014) under this policy is given by

$$E(C_p(t)) = D_c F(\tau) + D_c \frac{\int_0^{w_p} \bar{F}(u) du}{w_p}. \quad (13)$$

The expected cost rate is

$$E(CR(t)) = \frac{E(C_p(t))}{\int_0^{\tau} \bar{F}(u) du}, \quad (14)$$

where $\int_0^{\tau} \bar{F}(u) du$ is the expected cycle time, which is represented by $E(T(\tau))$.

Under the Fréchet distribution, the cdf is

$$F(u) = e^{-\lambda u^{-\mu}}, \quad u, \mu, \lambda > 0.$$

$$\text{So, } \int_0^{w_p} \bar{F}(u) du = w_p - \int_0^{w_p} e^{-\lambda u^{-\mu}} du = w_p (1 - e^{-\lambda w_p^{-\mu}}).$$

$$\text{And, } \int_0^{\tau} \bar{F}(u) du = \tau - \int_0^{\tau} e^{-\lambda u^{-\mu}} du = \tau (1 - e^{-\lambda \tau^{-\mu}}).$$

We can get the expected total cost and expected cost rate for the non-repairable product using the above values in (13) and (14). For example, if the failure replacement is carried out with a downtime cost $D_c = 70$ and purchasing cost $P_c = 1100$. The expected total cost, expected cycle time and expected cost rate are estimated for age-replacement under warranty policy for several values of warranty periods w_p and the parameters of Fréchet distribution λ and μ at normal use.

Table 4 The expected total cost, the expected cycle time and the expected cost rate

μ	λ	w_p	τ	Expected Total Cost $E(C_p(\tau))$	Expected Cycle Time $E(T(\tau))$	Expected Cost Rate $CR(\tau)$
0.5	4	8	9.2	1,032.45	7.34	476.76
0.5	4	8	7.5	998.45	6.87	509.66
0.8	4	8	9.2	992.65	6.03	480.87
0.8	4	8	7.5	887.14	4.44	556.35
1.3	4	8	9.2	808.18	4.98	490.98
1.3	5	8	9.2	945.76	5.72	485.98
1.3	5	8	7.5	970.97	4.12	515.76
1.3	7	8	9.2	991.64	8.20	484.90
1.3	7	8.5	7.8	1,011.92	7.55	508.97
1.3	7	8.5	9.2	1,014.83	8.42	483.71
1.3	7	8.5	9.8	1,012.58	9.11	345.39

7. Real World Implementation of Proposed Work

In the recent era, every manufacturer producing objects want to increase their sales by providing some incentives to customers in the form of warranty and guaranty. It is a written statement presented by manufacturers to the customers, promising them to repair or replace the product they purchased, if necessary within a specified period of time. In addition, it is also a means of advertising the quality of the product and thereby boosting sales. Other times, a legislative process, in an effort to protect the public interest, requires that manufacturers provide a warranty. A rebate warranty is one of the most common types of warranty policies.

The manufacturers shall only provide incentives when they have the faith in the products that their product has the ability to serve at least in the mentioned warranty time period. Therefore, it is essential for manufacturers to test the reliability and performance of the products before letting them serve in the market. This can be done by using accelerated life testing on products, where products are put at higher stresses than normal to induce failure and predict their life under normal use conditions. Accelerated life testing also helps manufacturers to predict the various costs associated with the product under the warranty policy. The main aim of undertaking accelerated life testing is the longtime of today’s products, the small-time between design and release, and the challenge of testing products that are continuously used under normal conditions. Hence accelerated life testing is used to provide quick information about the life distribution of products.

8. Conclusion

From Tables 1 and 2, increases in the sample size lead to a decrease in the mean squares and absolute relative bias. So, the asymptotically normally distributed and consistent estimators are provided by MLEs.

From Table 3, an inverse relationship is developed between mission time and the reliability function. The following conclusion is made on the basis of Table 4. The expected total cost and expected time cycle are inversely related to the parameter's value, while the expected cost rate is directly related to the value of the parameter. The expected total cost and expected cycle time are directly related to the value of the parameter, while the expected cost rate is inversely related to the value of the parameter. Increases in the warranty resulted in decreases in the expected total cost and expected cost rate and also does not affect the expected life cycle. Finally, an inverse relationship is developed between the age of replacement and the expected cost rate, while direct relationships are developed between the expected total cost and expected time cycle.

In the future, this work can be extended with different censoring schemes for other failure models. The application of ALT can be done with other warranty policies under the Bayesian approach in the extended work.

ALTg procedure has a limitation that such testing could be very expensive and thus, should be carried out with less iteration. To remove such type of limitation, it is necessary first to recognize the failure approach with qualitative analysis, after this, go on to quantitative analysis only to those essential failure approaches. Expenditure of testing can be reduced in this way.

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