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Gamma Zero-Truncated Poisson Distribution with the Minimum Compounded Function

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Abstract

The gamma zero-truncated Poisson (GZTP) distribution is introduced in this work as a novel lifetime distribution created by compounding the gamma and zero-truncated Poisson distributions with the minimum function. The proposed distribution's features are examined, including proofs of its probability density function and cumulative distribution function and formulas for its survival function, hazard function, moment, mean, variance, and quantile. The shape of the GZTP distribution's hazard function is flexible and can be increasing, decreasing, or unimodal. The estimation process utilizes maximum likelihood. Asymptotic properties of maximum likelihood estimators are studied, and simulations are used to test how well parameter estimation works.

Keywords: Compounding, gamma distribution, Zero-truncated Poisson distributions.

1. Introduction

The modeling of lifetimes is an important statistical work in many fields. The new lifetime distributions have been proposed in a lot of literature. One way to make a lifetime distribution is to combine a lifetime model with a discrete distribution. The most common idea of a compound model is that it has a lifetime of N (discrete random variables) components and a non-negative continuous random variable, X_i. The minimum of positive continuous random variables can be denoted by $Y = \min\{X_1, X_2, \ldots, X_N\}$. The model is obtained under the concept of a series system with identical components. In a series system, if any part fails, the whole system fails. This means that the distribution of Y can be used to model the time to the first failure of a system with N protected parts or the time it takes for a person to get sick again after treatment.

Several authors have proposed new distributions for the minimum of X_i. Adamidis and Loukas (1998) proposed an exponential-geometric (EG) distribution by compounding geometric distribution and exponential distribution for modeling the time to the first failure of the devices and the time interval in days between explosions in coalmines. Adamidis et al. (2005) investigated the extended exponential geometric (EEG) distribution. The different estimation procedures for the unknown parameters of the EEG distribution presented by Louzada et al. (2016). The Weibull-geometric (WG) distribution with the minimum compounded function was proposed by Barreto-Souza et al. (2011) and can be applied to the data on the fatigue life for 67 specimens of Alloy T7987. Zakerzadeh and Mahmoudi (2013) introduced a Lindley-geometric (LG) distribution, which is a strong competitor to other distributions used in fitting the data on the waiting times before service of 100 bank customers. Tahmasbi and Rezaei (2008) introduced an exponential-logarithmic (EL) distribution. Ciumara and Preda (2009) proposed a Weibull-logarithmic distribution that generalizes the EL distributions. In addition, several new compounds of Poisson distribution and some lifetime models have been introduced in their closed forms. Kus (2007) proposed an exponential-Poisson (EP) distribution and fitted the model to the successive earthquake data. Barreto-Souza and Silva (2015) and Louzada et al. (2020), respectively, discuss a likelihood ratio test to discriminate between the EP and gamma distributions and various frequentist estimation techniques for the parameters of the EP distribution. In addition, Xu et al. (2016) examined Bayes estimations of the parameter of the EP distribution under some symmetrical and unsymmetrical loss functions. Hemmati et al. (2011) and Lu and Shi (2012) proposed a Weibull-Poisson (WP) and discussed various of its statistical properties along with its reliability features. Alkarni and Oraby (2012) defined the class of Poisson with some lifetime distributions, presented the density, survival, and hazard functions, and gave some of their properties. In their works, they also present some Rayleigh-Poisson and Pareto-Poisson distribution properties. Gui et al. (2014) developed the Lindley-Poisson (LP) distribution and used it to model the time between earthquakes and the length of time guinea pigs lived after being injected with varying amounts of tubercle bacilli.

One of the most common ways to model lifetime data is by using the gamma distribution. But, the gamma distribution gave a monotone hazard function, which is different from the hazard functions of many physical phenomena. In many situations, the hazard function goes through three phases: it first goes up, then stays almost the same, and then goes down. This hazard function, which we shall refer to as upside-down bathtub-shaped, can be discovered through reliability and biological research. Consequently, life-cycle models that exhibit a hazard function with an upside-down bathtub shape are very helpful in survival analysis. In this article, the gamma and zero- truncated Poisson distributions are compounded to create a new lifetime distribution by using the minimum function, which the hazard function can perform in an upside-down bathtub shape.

The remainder of the paper is structured as follows: After defining the density function of the GZTP distribution, probability density function plots of the GZTP distribution are shown. Second, the GZTP distribution's properties are introduced. This section derives its moment-generating function, quantile, survival, and hazard rate functions. The generation of random numbers for the GZTP distribution is also discussed. Thirdly, estimation of the parameters using the maximum likelihood method, inference for large samples, and simulation analysis are provided. Fourth, an application to real datasets is provided. Finally, there is a discussion and some conclusions.

2. Gamma Zero-Truncated Poisson Distribution

Let $X_1, X_2, \ldots, X_N$ be N independent and identically distributed random variables from gamma distribution with the following probability density function (pdf):

$$f(x;\alpha,\beta) = \frac{\beta^{\alpha} x^{\alpha-1} e^{-\beta x}}{\Gamma(\alpha)}, \;\; x > 0,$$

where $\alpha > 0$ is a shape parameter, $\beta > 0$ is a rate parameter, and $\Gamma(\alpha)$ is a gamma function of α, and N is itself a random variable with a zero-truncated Poisson distribution and independence of X_i's. The probability mass function of N is the following:

$$P(N=n)=\frac{e^{-\lambda}\lambda^{n}}{n!\left(1-e^{-\lambda}\right)},\ n=1,2,\ldots \text{ and } \lambda>0.$$

Assuming that random variables X and N are independent, and $Y=\min\{X_1,X_2,\ldots,X_n\}$. Then, $g(y|n)=n\left[1-F(y)\right]^{n-1}f(y)$, where $f(y)$ is a pdf and $F(y)$ is the cumulative distribution function (cdf) of Y. The joint distribution between Y and N are obtained as follow:

$$g(y,n)=g(y\mid n)\,p(N=n)=n\left[1-F(y)\right]^{n-1}f(y)\left(\frac{e^{-\lambda}\lambda^{n}}{n!\left(1-e^{-\lambda}\right)}\right)$$

$$=\frac{\lambda e^{-\lambda}f(y)}{\left(1-e^{-\lambda}\right)}\frac{\left[\lambda\left(1-F(y)\right)\right]^{n-1}}{(n-1)!},$$

and the marginal distribution for Y is

$$g(y;\alpha,\beta,\lambda)=\sum_{n=1}^{\infty}g(y,n)=\sum_{n=1}^{\infty}\frac{\lambda e^{-\lambda}f(y)}{\left(1-e^{-\lambda}\right)}\frac{\left[\lambda\left(1-F(y)\right)\right]^{n-1}}{(n-1)!}=\frac{\lambda e^{-\lambda}f(y)}{\left(1-e^{-\lambda}\right)}e^{\lambda(1-F(y))},$$

where $F(y)=1-\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}$ and $\Gamma(\alpha,\beta y)=\int_{\beta y}^{\infty}t^{\alpha-1}e^{-t}dt$ is the upper incomplete gamma function. The pdf of the compound gamma zero-truncated Poisson distribution can be written as

$$g(y;\mathbf{\theta})=\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}\right)e^{\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}, \tag{1}$$

where $\mathbf{\theta}=(\lambda,\alpha,\beta)$. .

In the sequel, the distribution of Y will be referred to as the GZTP, and the plots of its pdf are displayed in Figure 1 for selected parameter values. When $0<\alpha\le 1$, the shape of the density is strictly decreasing as shown in Figure 1(a), whereas when $\alpha>1$, the density becomes unimodal and the curves show that the GZTP has a positively skewed distribution as shown in Figure 1(b). For $\alpha=1$, the GZTP distribution reduces to the density of the EP distribution introduced by Kus (2007).

Theorem 1 *Considering the GZTP distribution with the pdf of Equation (1), the distribution is reduced to a two-parameter gamma distribution as* $\lambda\to 0$.

Proof: The proof is shown in Appendix.

Theorem 2 *The density function of GZTP distribution is strictly decreasing if* $0<\alpha\le 1$.

Proof: The first derivative of the GZTP distribution is

$$g'(y;\boldsymbol{\theta}) = \left[\frac{\lambda \beta^{\alpha} y^{\alpha-2} e^{\lambda\left(\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}\right) - \beta y - \lambda}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)} \right] \left[\alpha - 1 - \beta y - \frac{\lambda(\beta y)^{\alpha} e^{-\beta y}}{\Gamma(\alpha)} \right].$$

The sign of $g'(y;\boldsymbol{\theta})$ depends on the sign of the second bracket. Let $h = \beta y$, the second bracket will be $\alpha - 1 - \left(h + \lambda h^{\alpha} e^{-h} / \Gamma(\alpha)\right)$. If $0 < \alpha \le 1$, then $g'(y;\boldsymbol{\theta}) < 0$. Hence, $g(y;\boldsymbol{\theta})$ is a decreasing function.

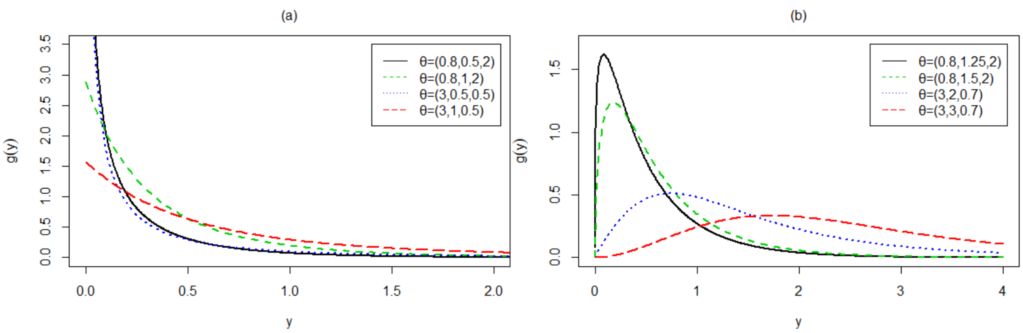

Figure 1 Probability density functions of the GZTP distribution with (a) $0 < \alpha \le 1$ and (b) $\alpha > 1$

3. Properties of the GZTP

3.1. Distribution function and moments

The cdf of the GZTP distribution is given by

$$G(y;\boldsymbol{\theta}) = \left(1 - e^{-\lambda + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right) \Big/ \left(1-e^{-\lambda}\right). \tag{2}$$

The proof of the cdf is given in Appendix. The r^{th} quantile for this distribution is defined as the value y_r such that

$$\Gamma(\alpha,\beta y_r) = \frac{\Gamma(\alpha)}{\lambda} \ln\left(\frac{1 - r\left(1-e^{-\lambda}\right)}{e^{-\lambda}} \right).$$

Proof: Since the r^{th} quantile denoted by y_r and $y_r = G^{-1}(r)$. This implies that $G(y_r) = r$. As $Y \sim GZTP$, then

$$G(y_r;\boldsymbol{\theta}) = \left(1 - e^{-\lambda + \frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right) \Big/ \left(1-e^{-\lambda}\right) = r.$$

Hence, it can be analytically solved for y_r to obtain

$$\Gamma(\alpha,\beta y_r) = \frac{\Gamma(\alpha)}{\lambda} \ln\left(\frac{1 - r\left(1-e^{-\lambda}\right)}{e^{-\lambda}} \right).$$

The moment-generating function is defined by

$$M_Y(t) = \frac{\lambda \beta^{\alpha} e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^{\infty} y^{\alpha-1} e^{ty-\beta y+\lambda \frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} dy.$$

The numerical values of the k^{th} moment can be obtained by using numerical integration. By using direct integration, we can calculate the raw moments of Y from Equation (1). The k raw moments are given by

$$E(Y^k) = \int_0^{\infty} y^k g(y) dy = \frac{\lambda \beta^{\alpha} e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^{\infty} y^{\alpha-1+k} e^{\lambda \frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y} dy.$$

Therefore, the mean and variance of the GZTP distribution are given, respectively, as follows:

$$E(Y) = \frac{\lambda \beta^{\alpha} e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^{\infty} y^{\alpha} e^{\lambda \frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y} dy,$$

and

$$Var(Y) = \frac{\lambda \beta^{\alpha} e^{-\lambda}}{\Gamma(\alpha)\left(1-e^{-\lambda}\right)} \int_0^{\infty} y^{\alpha+1} e^{\lambda \frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y} dy - \left[E(Y)\right]^2.$$

3.2. The survival and hazard functions

From Equations (1) and (2), the survival function and hazard function of the GZTP distribution are given by

$$S(y;\boldsymbol{\theta}) = 1 - G(y;\boldsymbol{\theta}) = 1 - \frac{\left(1-e^{-\lambda+\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)} = \frac{1-e^{-\lambda}-1+e^{-\lambda+\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\left(1-e^{-\lambda}\right)}$$

$$= \frac{e^{-\lambda+\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}} - e^{-\lambda}}{\left(1-e^{-\lambda}\right)} = -\frac{e^{-\lambda}\left(1-e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}{\left(1-e^{-\lambda}\right)},$$

and

$$H(y;\boldsymbol{\theta}) = \frac{g(y;\boldsymbol{\theta})}{s(y;\boldsymbol{\theta})} = \frac{\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma(\alpha)}\right) e^{\lambda\left(\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}\right)}}{-e^{-\lambda}\left(1-e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)\Big/\left(1-e^{-\lambda}\right)} = -\frac{\lambda \beta^{\alpha} y^{\alpha-1} e^{-\beta y+\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\Gamma(\alpha)\left(1-e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}\right)}.$$

Let $\eta(y) = -\frac{g'(y;\boldsymbol{\theta})}{g(y;\boldsymbol{\theta})}$, then $\eta(y) = \beta\left(1 + \frac{\lambda e^{-\beta y}(\beta y)^{\alpha-1}}{\Gamma(\alpha)}\right) - \frac{(\alpha-1)}{y}$,

and $\eta'(y) = \frac{1}{\Gamma(\alpha) y^2} \times \left[(\alpha-1)\Gamma(\alpha) - \lambda(\beta y)^{\alpha}(\beta y - \alpha + 1)e^{-\beta y}\right].$

For $0 < \alpha \le 1$, $\eta'(y) < 0$ for all $y's$. Then, the hazard function is a decreasing function, which follows Glaser (1980). Figure 2 illustrates some of the possible shapes of the hazard functions for selected values of $\boldsymbol{\theta}$.

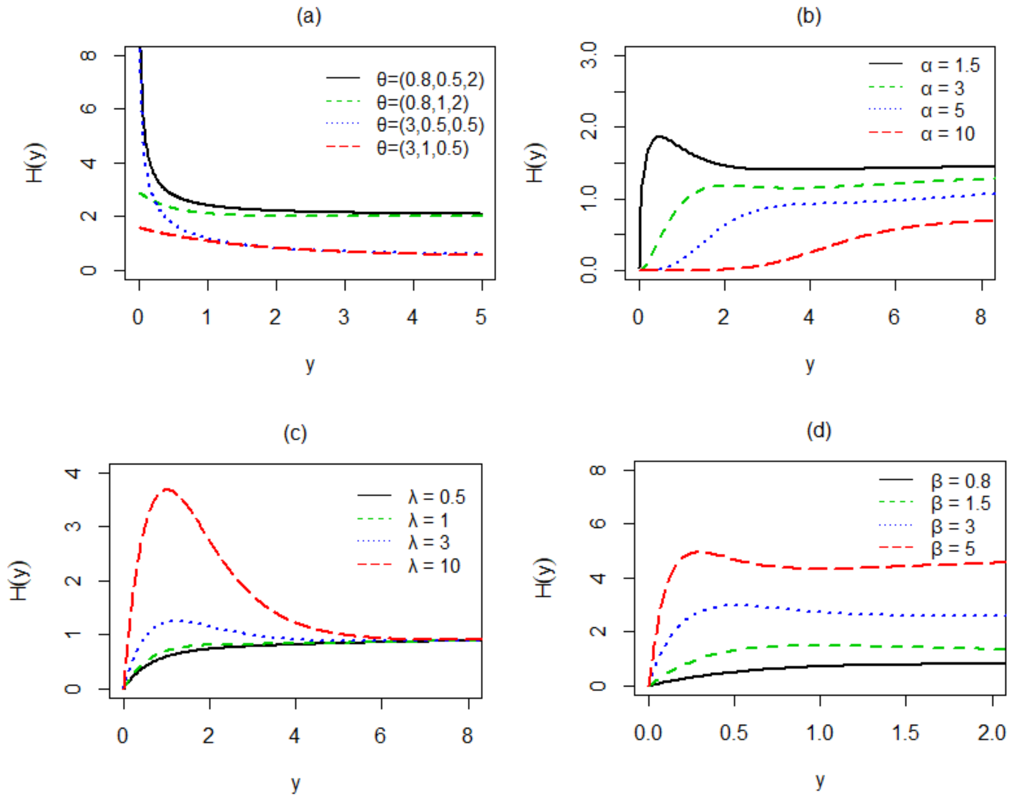

Figure 2 Hazard functions of the GZTP distribution (a) $0<\alpha\leq 1$, (b) $\lambda=2,\ \beta=1.5$, (c) $\alpha=2,\ \beta=1$, (d) $\lambda=2,\ \alpha=2$

3.3. Random number generation

The rejection sampling algorithm is used to generate random samples from the target distribution or Equation (1) by using random samples from a convenient distribution, which is called a proposal distribution. Here, the continuous uniform distribution, $p(y;\boldsymbol{\theta})$, is selected as a proposal distribution. The algorithm is shown as follows:

Step 1: find a constant c such that $cp(y)\geq g(y;\boldsymbol{\theta})$;

Step 2: obtain a sample y from the proposal;

Step 3: obtain a sample u from $U(0,1)$;

Step 4: check whether $cp(y)u\leq g(y;\boldsymbol{\theta})$. If this holds, accept y as a sample drawn from g. Otherwise, y will be rejected.

4. Estimation of the Parameters

4.1. Maximum likelihood estimators

Let $Y_1,Y_2,\ldots,Y_n$ be random samples with observed values $y_1,y_2,\ldots,y_n$ from the GZTP distribution with parameters $\boldsymbol{\theta}$. The likelihood function based on the observed random sample size of n, $Z_{obs}=(y_1,y_2,\ldots,y_n)$ is given by

$$L(\boldsymbol{\theta}; z_{obs}) = \left(\frac{\lambda e^{-\lambda}}{1-e^{-\lambda}}\right)^n \left(\frac{\beta^{n\alpha}}{(\Gamma(\alpha))^n}\right)\left(\prod_{i=1}^{n} y_i\right)^{\alpha-1} e^{-\beta\left(\sum_{i=1}^{n} y_i\right)+\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)}.$$

The corresponding log-likelihood function is

$$l(\boldsymbol{\theta}; z_{obs}) = n\left(\log\lambda - \lambda - \log\left(1-e^{-\lambda}\right)\right) + n\alpha\log\beta - n\log\Gamma(\alpha) + (\alpha-1)\sum_{i=1}^{n}\log y_i$$
$$-\beta\left(\sum_{i=1}^{n} y_i\right) + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i),$$

and subsequently the associated gradients are found to be

$$\frac{\partial l(\boldsymbol{\theta}; z_{obs})}{\partial\lambda} = n\left(\frac{1}{\lambda} - 1 - \frac{e^{-\lambda}}{1-e^{-\lambda}}\right) + \frac{\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)}{\Gamma(\alpha)}, \tag{3}$$

$$\frac{\partial l(\boldsymbol{\theta}; z_{obs})}{\partial\alpha} = n\log\beta - n\psi_0(\alpha) + \sum_{i=1}^{n}\log y_i + \lambda\sum_{i=1}^{n}\frac{\left(G_{2,3}^{3,0}\left(\beta y_i \middle| \begin{matrix}1,1\\ 0,0,\alpha\end{matrix}\right) + \Gamma(\alpha,\beta y_i)\times \left(\log(\beta y_i) - \psi_0(\alpha)\right)\right)}{\Gamma(\alpha)}, \tag{4}$$

$$\frac{\partial l(\boldsymbol{\theta}; z_{obs})}{\partial\beta} = \frac{n\alpha}{\beta} - \sum_{i=1}^{n} y_i - \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i}, \tag{5}$$

where $\psi_0(\alpha)$ is a digamma function that define as the 1st derivative of the logarithm of gamma function and $G_{p,q}^{m,n}\left(\beta y_i \middle| \begin{matrix}a_1,\ldots a_p\\ b_1,\ldots b_q\end{matrix}\right)$ is Meijer G-function. Equation (5) can be solved analytically for λ as follows:

$$\frac{\partial l(\boldsymbol{\theta}; z_{obs})}{\partial\beta} \stackrel{set}{=} 0,$$

$$\frac{n\alpha}{\beta} + \sum_{i=1}^{n} y_i - \frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i} \stackrel{set}{=} 0,$$

$$\lambda = \frac{\Gamma(\alpha)}{\beta^{\alpha-1}\sum_{i=1}^{n} y_i^{\alpha} e^{-\beta y_i}}\left(\frac{n\alpha}{\beta} + \sum_{i=1}^{n} y_i\right).$$

Therefore, the maximum likelihood estimator (MLE) of λ is $\hat{\lambda} = \frac{\Gamma(\hat{\alpha})}{\hat{\beta}^{\hat{\alpha}-1}\sum_{i=1}^{n} y_i^{\hat{\alpha}} e^{-\hat{\beta} y_i}}\left(\frac{n\hat{\alpha}}{\hat{\beta}} + \sum_{i=1}^{n} y_i\right),$

conditional upon the value of $\hat{\alpha}$ and $\hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are maximum likelihood estimates. For α and β, there are no closed forms, but the estimates can be calculated by numerical methods such as the Newton–Raphson method or probabilistic methods such as simulated annealing. The following theorems express the conditions that must be met in order for the MLEs to exist.

Theorem 3

(a) Let $l_1(\lambda;\alpha,\beta,z_{obs}) = \dfrac{\partial l(\boldsymbol{\theta};z_{obs})}{\partial\lambda}$, *$\alpha$ and β are known. Then $\hat{\lambda}$ is the uniquely exists root of* $l_1(\lambda;\alpha,\beta,z_{obs}) = 0$ if $\dfrac{\sum_{i=1}^{n}\Gamma(\alpha,\beta y_i)}{\Gamma(\alpha)} > \dfrac{n}{2}.$

(b) Let $l_3(\beta;\lambda,\alpha,z_{obs}) = \dfrac{\partial l(\boldsymbol{\theta};z_{obs})}{\partial\beta}$, *$\lambda$ and α are known. Then, there exists at least one solution of* $l_3(\beta;\lambda,\alpha,z_{obs}) = 0.$

Proof: The proofs of Theorem 3 are given in the Appendix.

4.2. Asymptotic variance-covariance matrix of the MLEs

The MLE of $\boldsymbol{\theta}$ is approximately multivariate normal with mean $\boldsymbol{\theta}$ and a variance-covariance matrix that is the inverse of expected information matrix $J(\boldsymbol{\theta}) = E[I(\boldsymbol{\theta})]$, where $I(\boldsymbol{\theta})$ is the observed information matrix with elements $I_{ij} = -\partial l^2 / \partial\boldsymbol{\theta}_i \partial\boldsymbol{\theta}_j$, $i, j = 1,2,3.$ By differentiating Equations (3)-(5), the elements of the symmetric and second order observed information matrix, $I(\boldsymbol{\theta})$, are found as follows:

$$I_{11} = \frac{n\left(e^{2\lambda} - (\lambda^2+2)e^{\lambda} + 1\right)}{\lambda^2\left(e^{\lambda}-1\right)^2},$$

$$I_{22} = n\psi^{(1)}(\alpha) - \lambda\sum_{i=1}^{n}\left[\frac{1}{\Gamma(\alpha)}\left(2G_{3,4}^{4,0}\left(\beta y_i \middle| \begin{matrix}1,1,1\\0,0,0,\alpha\end{matrix}\right) + 2\left(\log(\beta y_i) - \psi^{(0)}(\alpha)\right)G_{2,3}^{3,0}\left(\beta y_i \middle| \begin{matrix}1,1\\0,0,\alpha\end{matrix}\right)\right.\right.$$
$$\left.\left. + \Gamma(\alpha,\beta y_i)\left(\begin{matrix}-2\psi^{(0)}(\alpha)\log(\beta y_i) + \psi^{(0)}(\alpha)^2\\ -\psi^{(1)}(\alpha) + \log^2(\beta y_i)\end{matrix}\right)\right)\right],$$

$$I_{33} = \frac{n\alpha}{\beta^2} + \frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha}\left((\alpha-1-\beta y_i)\beta^{\alpha-2}e^{-\beta y_i}\right),$$

$$I_{12} = I_{21} = -\sum_{i=1}^{n}\frac{G_{2,3}^{3,0}\left(\beta y_i \middle| \begin{matrix}1,1\\0,0,\alpha\end{matrix}\right) + \Gamma(\alpha,\beta y_i)\left(\log(\beta y_i) - \psi^{(0)}(\alpha)\right)}{\Gamma(\alpha)},$$

$$I_{13} = I_{31} = \frac{\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha}e^{-\beta y_i}, \text{ and}$$

$$I_{23} = I_{32} = -\frac{n}{\beta} + \lambda\sum_{i=1}^{n} e^{-\beta y_i} \times \left[\frac{y_i^{\alpha}\beta^{\alpha-1}\left(-\psi^{(0)}(\alpha) + \log(\beta) + \log(y_i)\right)}{\Gamma(\alpha)}\right].$$

4.3. Simulation study

The samples were generated by using the rejection sampling method, where $\lambda = 0.5, 3, 7,$ $\alpha = 0.25, 0.5, 1, 2,$ and $\beta = 0.05, 1, 3.$ These values of parameters are selected such that all different shapes of distributions are represented. When all parameters are assumed unknown, the MLEs of λ, α and β are numerically calculated by the simulated-annealing method via the function maxLik in R program. The maxLik package (Henningsen and Toomet 2011) is used to calculate the MLEs, and the simulated annealing method is chosen because it gives stable solutions. The number of replications is chosen to be 3,000 as it provided stable results. Tables 1–3 give the averages of the MLEs, $AV(\hat{\boldsymbol{\theta}})$, and the corresponding standard errors, $SE(\hat{\boldsymbol{\theta}})$. The values of $AV(\hat{\boldsymbol{\theta}})$ and $SE(\hat{\boldsymbol{\theta}})$ suggest that the MLEs performed consistently. The standard errors of MLEs decrease as the sample size increases, and the bias of the MLEs is reduced for a large sample size, i.e., $n = 1{,}000$. For example, when $n = 50, 100, 1000$ for $\boldsymbol{\theta} = (7, 2, 3)$, the averages of MLEs are (7.9780, 2.1839, 5.1052), (7.4028, 2.1696, 4.9561), and (6.8333, 2.0391, 3.4088), respectively. The estimates tend to be close to their actual values as sample sizes increase; however, it is noticed that, with the same sample size, $SE(\hat{\lambda})$ is larger than $SE(\hat{\alpha})$ and $SE(\hat{\beta})$ in most situations.

For the variances, the estimates of $Var(\hat{\lambda}), Var(\hat{\alpha}), Var(\hat{\beta}), Cov(\hat{\lambda},\hat{\alpha}), Cov(\hat{\lambda},\hat{\beta})$, and $Cov(\hat{\alpha},\hat{\beta})$ can be directly calculated using the derived formulas in the previous section, i.e., $I_{11}, I_{12}, I_{13}, I_{22}$, and I_{23}. Then, I_{11}, I_{12}, I_{13}, I_{22}, and I_{23} are estimated by replacing all unknown parameters with the corresponding MLEs. Taking the inverse of the negative Hessian matrix results in the variance-covariance matrix. In addition, Monte-Carlo simulations (MCs) can also be used to estimate such variances. For example, the estimate of $Var(\hat{\lambda})$ is $\sum_{i=1}^{N}\left(\hat{\lambda}-\bar{\hat{\lambda}}\right)^2 \Big/ (N-1)$ and the estimate of $Cov(\hat{\lambda},\hat{\alpha})$ is

$$\frac{\sum_{i=1}^{N}\left(\hat{\lambda}-\bar{\hat{\lambda}}\right)\left(\hat{\alpha}-\bar{\hat{\alpha}}\right)}{\sqrt{\sum_{i=1}^{N}\left(\hat{\lambda}-\bar{\hat{\lambda}}\right)^2 \sum_{i=1}^{N}\left(\hat{\alpha}-\bar{\hat{\alpha}}\right)^2}},$$

where N is the number of iterations. Tables 4-6 summarize variance estimates obtained using the analytic method and MCs.

It is observed that the variance of $\hat{\lambda}$ increases as λ increases given that α and β are fixed. For instance, comparing cases with $\lambda = 0.5$ and $\lambda = 7$ at $\alpha = 0.25$, $\beta = 3$ and $n = 50$, $\widehat{Var}(\hat{\lambda})$ from MCs increases from 1.2809 to 7.4205, respectively, and the variance from the analytical method rises from 3.9718 to 11.2405. Also, when λ and β are fixed, $\widehat{Var}(\hat{\alpha})$ increases as α increases. For example, for cases with $\alpha = 0.25$ and $\alpha = 2$ at $\lambda = 3$, $\beta = 1$ and $n = 100$, $\widehat{Var}(\hat{\alpha})$ from MCs increases from 0.0008 to 0.0666, and by analytic method, $\widehat{Var}(\hat{\alpha})$ rises from 0.0013 to 0.1010. Likewise, when λ and α are fixed, the variance of $\hat{\beta}$ increases as β increases. In particular, comparing cases with $\beta = 0.05$ and $\beta = 3$, at $\lambda = 0.5$, $\alpha = 0.25$ and $n = 100$, $\widehat{Var}(\hat{\beta})$ increases from 0.0004 to 1.2414 in MCs and from 0.0011 to 1.9535 in the analytic method.

For large values of n, it is observed that the variance estimates derived from the MC simulations are relatively close to the analytic estimates. For example, at $\lambda = 7$, $\alpha = 0.25$ and $\beta = 3$, the $\widehat{Var}(\hat{\alpha})'s$ from MCs for $n = 50, 100$ and $1{,}000$ are $0.0029, 0.0014$, and 0.0002, while $\widehat{Var}(\hat{\alpha})'s$ from the analytic method are $0.0040, 0.0019$ and 0.0002, respectively. It is noted that the approximation becomes quite accurate as n increases. In the same way, Tables 7-9 show that the covariances of the MLEs found in the Hessian matrix are very close to the covariances found in simulations when n is large, i.e., $n = 1,000$.

5. Applications

In this section, two real datasets are used to illustrate the use of the proposed GZTP distribution. The remission time of bladder cancer patients and March precipitation are considered. Because the probability density function of these datasets is unimodal, the GZTP, WP, and gamma distributions are employed to model the data. For these distributions, the MLEs are used to estimate the parameters, and the p-values of the Kolmogorov-Smirnov (K-S) test are compared. Those comparative pdfs are given, respectively:

WP: $f_1\left(y;\boldsymbol{\theta}_1\right) = \dfrac{\alpha\beta\lambda y^{\alpha-1}}{1-e^{-\lambda}} e^{-\lambda-\beta y^{\alpha}+\lambda e^{-\beta y^{\alpha}}}$, $y > 0$, $\boldsymbol{\theta}_1 = \left(\lambda,\alpha,\beta\right)$

Gamma: $f_2(y;\boldsymbol{\theta}_2) = \dfrac{\beta^{\alpha} y^{\alpha-1} e^{-\beta y}}{\Gamma\left(\alpha\right)}$, $y > 0$, $\boldsymbol{\theta}_2 = \left(\alpha,\beta\right)$.

5.1. Remission time of bladder cancer patients

According to Lee and Wang (2003), the dataset consists of the number of months that 128 bladder cancer patients spent in remission. Table 10 shows the MLEs and Kolmogorov-Smirnov (K-S) statistics for the GZTP, WP, and gamma models, along with their p-values. The results show that all distributions can be used to model the data at a significance level of 0.05. However, the K-S test statistic has the largest p-value under the GZTP distribution, so this means the GZTP distribution is the most suitable for the data.

5.2. March precipitation

A dataset has 30 measurements of how much rain fell in March in Minneapolis/St. Paul. Each measurement is in inches. Lu and Shi (2012) have discussed this data. It can be observed from Table 10 that the GZTP distribution fits the model as well as any other comparative distribution.

6. Conclusions and Discussion

The GZTP distribution is newly constructed by compounding the gamma and zero-truncated Poisson distributions. The plots of the probability density function and hazard function were presented to show the flexibility of this distribution. The maximum likelihood estimators were studied, and it was found that some MLEs have no closed form. The formula of asymptotic variance-covariance matrix of the MLEs was also explicitly derived.

Simulations were performed to demonstrate the behavior of MLEs in the GZTP distribution. The result showed that as sample sizes increase, both standard errors and biasness diminish. When standard error is taken into account, the parameters of the zero-truncated Poisson distribution are more difficult to estimate than those of the gamma distribution. When the variance and covariance are taken into account, estimates from Monte Carlo simulations are close to those from the analytical method when the sample size is large.

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Appendix

The proof of Theorem 1

If λ approaches to 0, then

$$\lim_{\lambda\to 0} g(y;\boldsymbol{\theta}) = \lim_{\lambda\to 0}\frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}\right)e^{\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)} = \lim_{\lambda\to 0}\frac{\lambda\beta^{\alpha}y^{\alpha-1}e^{-\lambda-\beta y+\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}$$

$$=\frac{\beta^{\alpha}y^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda}{\left(1-e^{-\lambda}\right)}\right)\left(\lim_{\lambda\to 0} e^{-\lambda-\beta y+\lambda\left(\frac{\Gamma(\alpha,By)}{\Gamma(\alpha)}\right)}\right)=\frac{\beta^{\alpha}y^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda}{\left(1-e^{-\lambda}\right)}\right)\left(e^{-\beta y}\right)$$

$$=\frac{\beta^{\alpha}y^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\lambda e^{\lambda}}{\left(e^{\lambda}-1\right)}\right)\left(e^{-\beta y}\right)=\frac{\beta^{\alpha}y^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{\frac{d\lambda}{d\lambda}}{\frac{d}{d\lambda}\left(e^{\lambda}-1\right)}\right)\left(\lim_{\lambda\to 0} e^{\lambda}\right)\left(e^{-\beta y}\right)$$

$$=\frac{\beta^{\alpha}y^{\alpha-1}}{\Gamma(\alpha)}\left(\lim_{\lambda\to 0}\frac{1}{e^{\lambda}}\right)(1)\left(e^{-\beta y}\right)=\frac{\beta^{\alpha}y^{\alpha-1}}{\Gamma(\alpha)}(1)(1)\left(e^{-\beta y}\right)=\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}.$$

Therefore, the GZTP distribution reduces to the two- parameter gamma distribution.

The proof of cdf

The cumulative distribution function of the GZTP distribution is given by

$$G(y;\boldsymbol{\theta}) = \int_0^y g(y;\boldsymbol{\theta})dy = \int_0^y \frac{\lambda e^{-\lambda}}{\left(1-e^{-\lambda}\right)}\left(\frac{\beta^{\alpha}y^{\alpha-1}e^{-\beta y}}{\Gamma(\alpha)}\right)e^{\lambda\left(\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}\right)}dy$$

$$=\frac{\lambda e^{-\lambda}\beta^{\alpha}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\int_0^y y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy.$$

Now, find the value of $\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy$ by letting $\Gamma(\alpha,\beta y)$ be u, then $\dfrac{du}{dy}=-\beta^{\alpha}y^{\alpha-1}e^{-\beta y}$.

Therefore, $\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy = \int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}\times\left(-\dfrac{y^{1-\alpha}e^{\beta y}}{\beta^{\alpha}}du\right) = -\dfrac{1}{\beta^{\alpha}}\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du.$

Consider $\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du$. Let $\dfrac{\lambda u}{\Gamma(\alpha)}$ be v, then $\dfrac{dv}{du}=\dfrac{\lambda}{\Gamma(\alpha)}$ and $du=\dfrac{\Gamma(\alpha)}{\lambda}dv$, and

$$\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du = \frac{\Gamma(\alpha)}{\lambda}\int e^{v}dv = \frac{\Gamma(\alpha)e^{v}}{\lambda} = \frac{\Gamma(\alpha)e^{\frac{\lambda u}{\Gamma(\alpha)}}}{\lambda}.$$

Therefore, $\int y^{\alpha-1}e^{\lambda\frac{\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\beta y}dy = -\dfrac{1}{\beta^{\alpha}}\int e^{\frac{\lambda u}{\Gamma(\alpha)}}du = -\dfrac{\Gamma(\alpha)e^{\frac{\lambda u}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}} = -\dfrac{\Gamma(\alpha)e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}$, and then

$$G(y;\boldsymbol{\theta}) = \left[\frac{\lambda e^{-\lambda}\beta^{\alpha}}{\left(1-e^{-\lambda}\right)\Gamma(\alpha)}\left(-\frac{\Gamma(\alpha)e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}}}{\lambda\beta^{\alpha}}\right)+C\right]_0^y = \left[-\frac{e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}}{\left(1-e^{-\lambda}\right)}+C\right]_0^y$$

$$= -\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda} - e^{\frac{\lambda\Gamma(\alpha,0)}{\Gamma(\alpha)}-\lambda}\right] = -\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda} - e^{\frac{\lambda\Gamma(\alpha)}{\Gamma(\alpha)}-\lambda}\right]$$

$$= -\frac{1}{\left(1-e^{-\lambda}\right)}\left[e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda} - e^{\lambda-\lambda}\right] = -\frac{\left(e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda} - 1\right)}{\left(1-e^{-\lambda}\right)} = \frac{\left(1-e^{\frac{\lambda\Gamma(\alpha,\beta y)}{\Gamma(\alpha)}-\lambda}\right)}{\left(1-e^{-\lambda}\right)}.$$

The proof of Theorem 3

(a) Since $l_1\left(\lambda;\alpha,\beta,z_{obs}\right) = n\left(\frac{1}{\lambda}-1-\frac{e^{-\lambda}}{1-e^{-\lambda}}\right)+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)}$,

$\lim_{\lambda\to 0} l_1\left(\lambda;\alpha,\beta,z_{obs}\right) = -\frac{n}{2}+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)}$ and $\lim_{\lambda\to\infty} l_1\left(\lambda;\alpha,\beta,z_{obs}\right) = -n+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)}$.

It can be shown that $\lim_{\lambda\to 0} l_1\left(\lambda;\alpha,\beta,z_{obs}\right) = -\frac{n}{2}+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)} > 0$ as $\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)} > \frac{n}{2}$.

Since $\frac{\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)} < 1$ for all y_i, then $\lim_{\lambda\to\infty} l_1\left(\lambda;\alpha,\beta,z_{obs}\right) = -n+\frac{\sum_{i=1}^{n}\Gamma\left(\alpha,\beta y_i\right)}{\Gamma(\alpha)} < 0$. Therefore, at least one solution of $l_1\left(\lambda;\alpha,\beta,z_{obs}\right)=0$ exists. For the proof of uniqueness of solution, it is needed to show function l_1 is strictly decreasing in λ. The first derivative of l_1 is considered and given by

$$l_1'\left(\lambda;\alpha,\beta,z_{obs}\right) = -\frac{n(-e^{\lambda}(\lambda^2+2)+e^{2\lambda}+1)}{\left(e^{\lambda}-1\right)^2\lambda^2} = -\frac{ne^{\lambda}(e^{-\lambda}+e^{\lambda}-(\lambda^2+2))}{e^{\lambda}}.$$

If $e^{-\lambda}+e^{\lambda}-(\lambda^2+2) > 0$, then $l_1'\left(\lambda;\alpha,\beta,z_{obs}\right) < 0$ and l_1 is strictly decreasing in λ.

Consider $e^{\lambda} = 1+\lambda+\frac{1}{2}\lambda^2+\frac{1}{3!}\lambda^3+\ldots$ and $e^{-\lambda} = 1-\lambda+\frac{1}{2}\lambda^2-\frac{1}{3!}\lambda^3+\ldots$, then

$e^{-\lambda}+e^{\lambda} = 2+\lambda^2+\frac{2}{4!}\lambda^4+\ldots > \lambda^2+2$, or $e^{-\lambda}+e^{\lambda}-(\lambda^2+2) > 0$.

Therefore, $l_1'\left(\lambda;\alpha,\beta,z_{obs}\right) < 0$ for $\lambda > 0$. This completes the proof.

(b) Since $l_3\left(\beta;\lambda,\alpha,z_{obs}\right) = \frac{n\alpha}{\beta}-\sum_{i=1}^{n} y_i-\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha}e^{-\beta y_i}$,

$\lim_{\beta\to 0} l_3\left(\beta;\lambda,\alpha,z_{obs}\right) = \lim_{\beta\to 0}\frac{n\alpha}{\beta}-\lim_{\beta\to 0}\sum_{i=1}^{n} y_i-\lim_{\beta\to 0}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha}e^{-\beta y_i} = \infty$, and

$\lim_{\beta\to\infty} l_3\left(\beta;\lambda,\alpha,z_{obs}\right) = \lim_{\beta\to\infty}\frac{n\alpha}{\beta}-\lim_{\beta\to\infty}\sum_{i=1}^{n} y_i-\lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha}e^{-\beta y_i} = 0-\sum_{i=1}^{n} y_i-\lim_{\beta\to\infty}\frac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum_{i=1}^{n} y_i^{\alpha}e^{-\beta y_i}$

Consider $\lim\limits_{\beta\to\infty}\dfrac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum\limits_{i=1}^{n}y_i^{\alpha}e^{-\beta y_i}=\dfrac{\lambda}{\Gamma(\alpha)}\lim\limits_{\beta\to\infty}\sum\limits_{i=1}^{n}\beta^{\alpha-1}y_i^{\alpha}e^{-\beta y_i}$

$$=\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(\lim_{\beta\to\infty}\beta^{\alpha-1}y_i^{\alpha}e^{-\beta y_i}\right)=\frac{\lambda}{\Gamma(\alpha)}\sum_{i=1}^{n}\left(y_i\lim_{\beta\to\infty}\frac{(\beta y_i)^{\alpha-1}}{e^{\beta y_i}}\right),$$

and

$$\lim_{\beta\to\infty}\frac{(\beta y_i)^{\alpha-1}}{e^{\beta y_i}}=\lim_{\beta\to\infty}\left(\frac{\beta y_i}{e^{\beta y_i/\alpha-1}}\right)^{\alpha-1}=\lim_{\beta\to\infty}(\alpha-1)^{\alpha-1}\left(\frac{c}{e^{c}}\right)^{\alpha-1},c=\beta y_i/\alpha-1$$

$$=(\alpha-1)^{\alpha-1}\left(\lim_{\beta\to\infty}\frac{c}{e^{c}}\right)^{\alpha-1}=(\alpha-1)^{\alpha-1}\left(\lim_{\beta\to\infty}\frac{1}{e^{c}}\right)^{\alpha-1}=0$$

Then, $\lim\limits_{\beta\to\infty}l_3(\beta;\lambda,\alpha,z_{obs})=\lim\limits_{\beta\to\infty}\dfrac{n\alpha}{\beta}-\lim\limits_{\beta\to\infty}\sum\limits_{i=1}^{n}y_i-\lim\limits_{\beta\to\infty}\dfrac{\lambda\beta^{\alpha-1}}{\Gamma(\alpha)}\sum\limits_{i=1}^{n}y_i^{\alpha}e^{-\beta y_i}=0-\sum\limits_{i=1}^{n}y_i-0=-\sum\limits_{i=1}^{n}y_i<0$

Therefore, at least one solution of $l_3(\beta;\lambda,\alpha,z_{obs})=0$ exists.

Table 1 The averages of MLEs and standard errors of $\hat{\boldsymbol{\theta}}$ from 3,000 samples with $n = 50$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	$AV(\hat{\boldsymbol{\theta}})$	$SE(\hat{\lambda})$	$SE(\hat{\alpha})$	$SE(\hat{\beta})$
50	(0.5, 0.25, 0.05)	(0.7740, 0.3419, 0.1089)	0.0228	0.0016	0.0010
	(0.5, 0.25, 1)	(1.8920, 0.4072, 1.0290)	0.0369	0.0017	0.0168
	(0.5, 0.25, 3)	(1.9664, 0.4360, 3.0202)	0.0361	0.0017	0.0474
	(0.5, 0.5, 0.05)	(0.1536, 0.5788, 0.1259)	0.0094	0.0035	0.0009
	(0.5, 0.5, 1)	(1.0143, 0.5492, 1.0175)	0.0350	0.0060	0.0132
	(0.5, 0.5, 3)	(1.1301, 0.5737, 3.0636)	0.0441	0.0031	0.0377
	(0.5, 1, 0.05)	(0.0666, 1.4296, 0.1808)	0.0070	0.0062	0.0007
	(0.5, 1, 1)	(1.1179, 1.0645, 0.9856)	0.0537	0.0064	0.0112
	(0.5, 1, 3)	(1.3310, 1.0695, 2.8908)	0.0743	0.0066	0.0328
	(0.5, 2, 0.05)	(1.7073, 3.2283, 0.3143)	0.1231	0.0432	0.0049
	(0.5, 2, 1)	(1.4524, 2.1125 0.9549)	0.0406	0.0075	0.0059
	(0.5, 2, 3)	(1.6255, 2.1118, 2.8408)	0.0498	0.0078	0.0189
	(3, 0.25, 0.05)	(3.0282, 0.3361, 0.1329)	0.0330	0.0011	0.0031
	(3, 0.25, 1)	(3.9823, 0.3959, 1.6154)	0.0381	0.0013	0.0423
	(3, 0.25, 3)	(4.5719, 0.4297, 3.6947)	0.0626	0.0014	0.0759
	(3, 0.5, 0.05)	(1.2933, 0.5037, 0.1572)	0.0336	0.0027	0.0018
	(3, 0.5, 1)	(2.6305, 0.5260, 1.5961)	0.0437	0.0035	0.0334
	(3, 0.5, 3)	(3.3085, 0.5455, 4.0771)	0.0768	0.0024	0.0790
	(3, 1, 0.05)	(0.2925, 1.1489, 0.1955)	0.0253	0.0071	0.0013
	(3, 1, 1)	(2.5052, 0.9942, 1.3947)	0.0752	0.0055	0.0219
	(3, 1, 3)	(2.8688, 0.9983, 4.0033)	0.0931	0.0053	0.0672
	(3, 2, 0.05)	(1.2485, 3.7371, 0.3098)	0.1052	0.0345	0.0041
	(3, 2, 1)	(2.3025, 2.0699, 1.3427)	0.0693	0.0128	0.0180
	(3, 2, 3)	(2.7822, 2.0512, 3.7918)	0.0970	0.0126	0.0549
	(7, 0.25, 0.05)	(5.0467, 0.3695, 0.7512)	0.0530	0.0012	0.0310
	(7, 0.25, 1)	(6.9508, 0.4514, 5.5962)	0.0888	0.0019	0.1276
	(7, 0.25, 3)	(8.4013, 0.4879, 9.2526)	0.0995	0.0020	0.1533
	(7, 0.5, 0.05)	(3.9643, 0.5126, 0.3498)	0.0910	0.0025	0.0094
	(7, 0.5, 1)	(5.1392, 0.5569, 4.0841)	0.1315	0.0025	0.1107
	(7, 0.5, 3)	(8.6819, 0.5842, 5.0566)	0.1532	0.0023	0.1299
	(7, 1, 0.05)	(4.2653, 1.0818, 0.2413)	0.1026	0.0064	0.0038
	(7, 1, 1)	(4.2506, 1.0432, 3.1315)	0.1601	0.0060	0.0696
	(7, 1, 3)	(7.3358, 1.0310, 5.5968)	0.2252	0.0063	0.1729
	(7, 2, 0.05)	(4.0828, 2.9785, 0.2220)	0.1573	0.0249	0.0050
	(7, 2, 1)	(3.8889, 2.3360, 2.4612)	0.1548	0.0170	0.0455
	(7, 2, 3)	(6.9780, 2.1839, 5.1052)	0.2380	0.0171	0.1312

Table 2 The averages of MLEs and standard errors of $\hat{\boldsymbol{\theta}}$ from 3,000 samples with $n=100$

n	$\boldsymbol{\theta}=(\lambda,\alpha,\beta)$	$AV(\hat{\boldsymbol{\theta}})$	$SE(\hat{\lambda})$	$SE(\hat{\alpha})$	$SE(\hat{\beta})$
100	(0.5, 0.25, 0.05)	(0.6503, 0.3356, 0.1082)	0.0161	0.0011	0.0007
	(0.5, 0.25, 1)	(1.9562, 0.4042, 0.9342)	0.0277	0.0012	0.0114
	(0.5, 0.25, 3)	(2.0653, 0.4355, 2.8513)	0.0331	0.0013	0.0355
	(0.5, 0.5, 0.05)	(0.0748, 0.5660, 0.1236)	0.0044	0.0025	0.0006
	(0.5, 0.5, 1)	(0.9284, 0.5508, 1.0028)	0.0298	0.0023	0.0089
	(0.5, 0.5, 3)	(1.0714, 0.5670, 2.9489)	0.0345	0.0023	0.0271
	(0.5, 1, 0.05)	(0.0243, 1.3968, 0.1781)	0.0009	0.0077	0.0009
	(0.5, 1, 1)	(1.0183, 1.0434, 0.9648)	0.0485	0.0048	0.0085
	(0.5, 1, 3)	(1.1229, 1.0501, 2.8765)	0.0596	0.0057	0.0260
	(0.5, 2, 0.05)	(0.7135, 3.2249, 0.3359)	0.0863	0.0335	0.0036
	(0.5, 2, 1)	(1.2695, 2.0442, 0.9367)	0.0626	0.0091	0.0083
	(0.5, 2, 3)	(1.6502, 2.0593, 2.7370)	0.0864	0.0096	0.0286
	(3, 0.25, 0.05)	(3.2172, 0.3373, 0.1075)	0.0251	0.0008	0.0016
	(3, 0.25, 1)	(4.3747, 0.3941, 1.1819)	0.0314	0.0009	0.0256
	(3, 0.25, 3)	(4.4825, 0.4278, 3.4289)	0.0485	0.0015	0.0978
	(3, 0.5, 0.05)	(1.5568, 0.4936, 0.1372)	0.1929	0.0150	0.0070
	(3, 0.5, 1)	(2.9546, 0.5271, 1.3430)	0.0416	0.0016	0.0230
	(3, 0.5, 3)	(3.3096, 0.5489, 3.6419)	0.0505	0.0016	0.0622
	(3, 1, 0.05)	(0.1898, 1.1017, 0.1920)	0.0249	0.0049	0.0010
	(3, 1, 1)	(2.5910, 0.9841, 1.2950)	0.0619	0.0037	0.0183
	(3, 1, 3)	(2.6937, 0.9896, 3.8590)	0.0646	0.0039	0.0536
	(3, 2, 0.05)	(0.8058, 3.6090, 0.3096)	0.0869	0.0242	0.0032
	(3, 2, 1)	(2.8093, 2.0122, 1.1734)	0.0732	0.0039	0.0570
	(3, 2, 3)	(3.0028, 2.0120, 3.4572)	0.0866	0.0086	0.0426
	(7, 0.25, 0.05)	(5.4649, 0.3661, 0.4725)	0.0507	0.0013	0.0192
	(7, 0.25, 1)	(6.5506, 0.4469, 5.5509)	0.0593	0.0014	0.1241
	(7, 0.25, 3)	(7.8929, 0.4849, 9.5024)	0.0711	0.0014	0.1265
	(7, 0.5, 0.05)	(4.1455, 0.5194, 0.2557)	0.0896	0.0018	0.0069
	(7, 0.5, 1)	(4.8483, 0.5524, 3.6499)	0.0910	0.0017	0.0828
	(7, 0.5, 3)	(6.5427, 0.5802, 6.6263)	0.1184	0.0019	0.1571
	(7, 1, 0.05)	(2.2317, 1.0855, 0.2415)	0.1026	0.0065	0.0037
	(7, 1, 1)	(4.1421, 1.0517, 3.1891)	0.1543	0.0065	0.0695
	(7, 1, 3)	(7.1496, 1.0116, 5.0158)	0.1465	0.0032	0.1163
	(7, 2, 0.05)	(4.0264, 2.9367, 0.2188)	0.1545	0.0240	0.0048
	(7, 2, 1)	(4.2215, 2.3144, 2.3648)	0.1611	0.0173	0.0467
	(7, 2, 3)	(7.4028, 2.1696, 4.9561)	0.2388	0.0168	0.1333

Table 3 The averages of MLEs and standard errors of $\hat{\boldsymbol{\theta}}$ from 3,000 samples with $n = 1{,}000$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	$AV(\hat{\boldsymbol{\theta}})$	$SE(\hat{\lambda})$	$SE(\hat{\alpha})$	$SE(\hat{\beta})$
1,000	(0.5, 0.25, 0.05)	(0.5732, 0.3259, 0.1062)	0.0088	0.0005	0.0003
	(0.5, 0.25, 1)	(1.9473, 0.4041, 0.8584)	0.0120	0.0005	0.0044
	(0.5, 0.25, 3)	(2.1669, 0.4386, 2.5179)	0.0118	0.0005	0.0125
	(0.5, 0.5, 0.05)	(0.0170, 0.5482, 0.1214)	0.0006	0.0009	0.0002
	(0.5, 0.5, 1)	(0.8081, 0.5403, 0.9743)	0.0131	0.0011	0.0036
	(0.5, 0.5, 3)	(0.9409, 0.5608, 2.8985)	0.0128	0.0010	0.0111
	(0.5, 1, 0.05)	(0.0075, 1.3425, 0.1710)	0.0002	0.0027	0.0003
	(0.5, 1, 1)	(0.5189, 0.9991, 0.9951)	0.0160	0.0024	0.0033
	(0.5, 1, 3)	(0.5471, 1.0026, 2.9828)	0.0168	0.0018	0.0081
	(0.5, 2, 0.05)	(0.0039, 3.0754, 0.3404)	0.0001	0.0106	0.0008
	(0.5, 2, 1)	(0.5951, 2.0036, 0.9893)	0.0297	0.0044	0.0041
	(0.5, 2, 3)	(0.5971, 2.0084, 2.9734)	0.0223	0.0033	0.0089
	(3, 0.25, 0.05)	(3.3539, 0.3372, 0.0903)	0.0124	0.0004	0.0005
	(3, 0.25, 1)	(4.8809, 0.3921, 0.7578)	0.0166	0.0004	0.0070
	(3, 0.25, 3)	(3.3429, 0.4338, 2.5060)	0.0599	0.0006	0.0173
	(3, 0.5, 0.05)	(1.1796, 0.4921, 0.1484)	0.0134	0.0010	0.0006
	(3, 0.5, 1)	(3.4121, 0.5320, 0.9663)	0.0265	0.0006	0.0096
	(3, 0.5, 3)	(3.5870, 0.5507, 2.8279)	0.0240	0.0006	0.0262
	(3, 1, 0.05)	(0.3405, 1.0618, 0.1822)	0.0483	0.0017	0.0012
	(3, 1, 1)	(3.0369, 0.9946, 1.0316)	0.0337	0.0011	0.0083
	(3, 1, 3)	(3.0159, 0.9933, 3.0920)	0.0308	0.0010	0.0235
	(3, 2, 0.05)	(4.0241, 3.2407, 0.2164)	0.1829	0.0153	0.0047
	(3, 2, 1)	(3.0607, 1.9917, 1.0210)	0.0368	0.0032	0.0075
	(3, 2, 3)	(2.9388, 1.9939, 3.1180)	0.0323	0.0028	0.0216
	(7, 0.25, 0.05)	(6.4174, 0.3611, 0.2239)	0.0227	0.0004	0.0031
	(7, 0.25, 1)	(6.7969, 0.4452, 4.4480)	0.0335	0.0005	0.0609
	(7, 0.25, 3)	(7.7530, 0.4865, 9.4380)	0.0333	0.0006	0.0889
	(7, 0.5, 0.05)	(6.9969, 0.5185, 0.0646)	0.0521	0.0006	0.0016
	(7, 0.5, 1)	(5.2841, 0.5530, 2.4849)	0.0513	0.0006	0.0434
	(7, 0.5, 3)	(5.8831, 0.5763, 6.2887)	0.0392	0.0006	0.0799
	(7, 1, 0.05)	(7.7062, 1.0288, 0.0539)	0.0596	0.0017	0.0016
	(7, 1, 1)	(6.1109, 1.0176, 1.5166)	0.1127	0.0022	0.0439
	(7, 1, 3)	(6.8551, 1.0084, 3.6359)	0.1049	0.0019	0.0919
	(7, 2, 0.05)	(9.4414, 2.4936, 0.0781)	0.0407	0.0050	0.0003
	(7, 2, 1)	(6.8195, 2.0363, 1.1450)	0.0945	0.0066	0.0240
	(7, 2, 3)	(6.8333, 2.0391, 3.4088)	0.0930	0.0063	0.0645

Table 4 Estimated variances of MLEs for $n = 50$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	Monte-Carlo simulations			Analytic method		
		$Var(\hat{\lambda})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$	$Var(\hat{\lambda})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$
50	(0.5, 0.25, 0.05)	0.5148	0.0025	0.0010	1.5843	0.0062	0.0027
	(0.5, 0.25, 1)	1.3467	0.0028	0.2793	3.8851	0.0059	0.4865
	(0.5, 0.25, 3)	1.2809	0.0029	2.2036	3.9718	0.0063	4.7511
	(0.5, 0.5, 0.05)	0.0837	0.0114	0.0007	2.0434	0.0256	0.0020
	(0.5, 0.5, 1)	1.1883	0.0354	0.1680	3.1241	0.0181	0.2181
	(0.5, 0.5, 3)	1.9358	0.0094	1.4144	2.0658	0.0170	1.7158
	(0.5, 1, 0.05)	0.1294	0.0989	0.0013	6.0531	0.1950	0.0043
	(0.5, 1, 1)	2.6950	0.0385	0.1162	7.4961	0.0862	0.1850
	(0.5, 1, 3)	5.0574	0.0394	0.9844	6.2039	0.0829	1.6006
	(0.5, 2, 0.05)	10.1080	1.2450	0.0159	12.6787	1.0056	0.0114
	(0.5, 2, 1)	4.5356	0.1553	0.0951	8.6845	0.2852	0.1505
	(0.5, 2, 3)	6.6810	0.1629	0.9634	10.9382	0.3287	1.3033
	(3, 0.25, 0.05)	1.0438	0.0012	0.0091	6.0534	0.0026	0.0223
	(3, 0.25, 1)	1.3888	0.0015	1.7175	9.3077	0.0029	7.7007
	(3, 0.25, 3)	3.4624	0.0018	5.0874	9.5963	0.0032	5.8813
	(3, 0.5, 0.05)	1.0979	0.0070	0.0032	3.0873	0.0119	0.0067
	(3, 0.5, 1)	1.8580	0.0120	1.0857	7.9966	0.0082	1.6357
	(3, 0.5, 3)	5.3327	0.0052	5.6391	7.4157	0.0077	14.7862
	(3, 1, 0.05)	0.5971	0.0472	0.0017	4.4781	0.1219	0.0046
	(3, 1, 1)	5.2591	0.0277	0.4475	9.5822	0.0439	0.7465
	(3, 1, 3)	7.9690	0.0255	4.1552	13.2599	0.0564	9.2401
	(3, 2, 0.05)	8.1678	0.8753	0.0124	11.4873	0.8723	0.0127
	(3, 2, 1)	4.5226	0.1546	0.3058	7.2964	0.1977	0.3590
	(3, 2, 3)	8.5590	0.1439	2.7418	8.7134	0.1971	4.2162
	(7, 0.25, 0.05)	2.5686	0.0014	0.8756	16.9288	0.0021	2.0530
	(7, 0.25, 1)	5.9948	0.0027	12.3860	7.6041	0.0028	18.8064
	(7, 0.25, 3)	7.4205	0.0029	17.5966	11.2459	0.0040	21.8596
	(7, 0.5, 0.05)	7.1020	0.0055	0.0766	16.0608	0.0074	0.0847
	(7, 0.5, 1)	13.0415	0.0046	9.2447	15.9675	0.0073	22.7584
	(7, 0.5, 3)	19.7570	0.0043	14.1992	27.8015	0.0068	28.6425
	(7, 1, 0.05)	8.9690	0.0348	0.0125	9.6951	0.0608	0.0132
	(7, 1, 1)	20.9617	0.0299	3.9588	12.1720	0.0450	3.7378
	(7, 1, 3)	31.8948	0.0250	18.8088	27.3020	0.0602	25.0770
	(7, 2, 0.05)	18.1048	0.4534	0.0183	21.1529	0.5630	0.0073
	(7, 2, 1)	19.6920	0.2375	1.7050	14.5430	0.2756	1.4748
	(7, 2, 3)	36.4370	0.1872	11.0638	24.3434	0.2780	20.3718

Table 5 Estimated variances of MLEs for $n = 100$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	Monte-Carlo simulations			Analytic method		
		$Var(\hat{\lambda})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$	$Var(\hat{\lambda})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$
100	(0.5, 0.25, 0.05)	0.2591	0.0012	0.0004	0.7401	0.0031	0.0011
	(0.5, 0.25, 1)	0.7636	0.0014	0.1289	1.6064	0.0027	0.2001
	(0.5, 0.25, 3)	1.0804	0.0016	1.2414	1.9674	0.0031	1.9535
	(0.5, 0.5, 0.05)	0.0168	0.0055	0.0003	1.1341	0.0139	0.0009
	(0.5, 0.5, 1)	0.8619	0.0052	0.0770	2.4689	0.0122	0.1453
	(0.5, 0.5, 3)	1.1622	0.0051	0.7177	1.7142	0.0091	0.9841
	(0.5, 1, 0.05)	0.0007	0.0485	0.0006	1.9577	0.0755	0.0016
	(0.5, 1, 1)	2.1622	0.0214	0.0671	3.8082	0.0597	0.1068
	(0.5, 1, 3)	3.2651	0.0300	0.6193	3.0352	0.0395	0.7425
	(0.5, 2, 0.05)	4.5836	0.6907	0.0082	4.7394	0.4891	0.0063
	(0.5, 2, 1)	3.5698	0.0746	0.0634	3.7908	0.1339	0.0640
	(0.5, 2, 3)	6.7487	0.0836	0.7426	4.2481	0.1218	0.8205
	(3, 0.25, 0.05)	0.6069	0.0006	0.0024	5.0483	0.0013	0.0106
	(3, 0.25, 1)	0.9296	0.0008	0.6169	9.8013	0.0013	2.0758
	(3, 0.25, 3)	0.8218	0.0008	3.3405	5.4918	0.0015	8.5921
	(3, 0.5, 0.05)	0.5581	0.0034	0.0007	1.1545	0.0045	0.0029
	(3, 0.5, 1)	1.7149	0.0026	0.5260	4.1991	0.0036	0.6759
	(3, 0.5, 3)	2.2422	0.0024	3.3976	6.6116	0.0036	9.4159
	(3, 1, 0.05)	0.5710	0.0226	0.0010	1.9538	0.0522	0.0018
	(3, 1, 1)	3.6297	0.0130	0.3185	5.2027	0.0177	0.3912
	(3, 1, 3)	4.9457	0.0139	2.9974	5.0933	0.0178	3.4364
	(3, 2, 0.05)	5.2210	0.4037	0.0072	3.9637	0.3791	0.0053
	(3, 2, 1)	4.8152	0.0666	0.1931	7.0270	0.1010	0.2707
	(3, 2, 3)	6.9059	0.0679	1.6732	7.3455	0.0920	2.3630
	(7, 0.25, 0.05)	1.1526	0.0008	0.1655	6.9089	0.0010	0.4321
	(7, 0.25, 1)	2.1204	0.0012	9.2776	4.2917	0.0014	13.6663
	(7, 0.25, 3)	3.3119	0.0014	10.4939	7.8267	0.0019	15.5661
	(7, 0.5, 0.05)	6.3296	0.0026	0.0372	14.5278	0.0033	0.0473
	(7, 0.5, 1)	6.4718	0.0024	5.3612	8.8099	0.0029	9.7279
	(7, 0.5, 3)	7.9378	0.0022	13.9765	9.6126	0.0028	19.9803
	(7, 1, 0.05)	8.9059	0.0358	0.0118	39.7043	0.0759	0.0178
	(7, 1, 1)	19.0367	0.0335	3.8634	13.9862	0.0430	4.0673
	(7, 1, 3)	23.6772	0.0115	14.9294	15.7604	0.0146	20.8320
	(7, 2, 0.05)	17.7533	0.4270	0.0172	18.4690	0.5155	0.0073
	(7, 2, 1)	21.1447	0.2452	1.7752	19.3761	0.3020	1.5975
	(7, 2, 3)	36.6068	0.1821	11.4014	57.5472	0.3381	15.7911

Table 6 Estimated variances of MLEs for $n = 1{,}000$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	Monte-Carlo simulations			Analytic method		
		$Var(\hat{\lambda})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$	$Var(\hat{\lambda})$	$Var(\hat{\alpha})$	$Var(\hat{\beta})$
1,000	(0.5, 0.25, 0.05)	0.0384	0.0001	0.0000	0.0673	0.0003	0.0001
	(0.5, 0.25, 1)	0.0856	0.0002	0.0113	0.1260	0.0003	0.0169
	(0.5, 0.25, 3)	0.0904	0.0002	0.1011	0.1489	0.0003	0.1673
	(0.5, 0.5, 0.05)	0.0002	0.0005	0.0000	0.0991	0.0012	0.0001
	(0.5, 0.5, 1)	0.0964	0.0007	0.0074	0.1163	0.0009	0.0089
	(0.5, 0.5, 3)	0.0976	0.0006	0.0742	0.1202	0.0009	0.0849
	(0.5, 1, 0.05)	0.0000	0.0045	0.0001	0.7758	0.0195	0.0004
	(0.5, 1, 1)	0.1273	0.0030	0.0056	0.1740	0.0038	0.0065
	(0.5, 1, 3)	0.2748	0.0032	0.0646	0.1983	0.0042	0.0612
	(0.5, 2, 0.05)	0.0000	0.0621	0.0004	0.1873	0.0370	0.0004
	(0.5, 2, 1)	0.4400	0.0099	0.0086	0.2433	0.0127	0.0056
	(0.5, 2, 3)	0.4763	0.0106	0.0757	0.2440	0.0126	0.0487
	(3, 0.25, 0.05)	0.0783	0.0001	0.0001	0.2946	0.0001	0.0007
	(3, 0.25, 1)	0.1477	0.0001	0.0261	0.8861	0.0001	0.1197
	(3, 0.25, 3)	1.8345	0.0002	0.1537	0.7022	0.0002	0.7337
	(3, 0.5, 0.05)	0.0894	0.0005	0.0002	0.1173	0.0006	0.0003
	(3, 0.5, 1)	0.3899	0.0002	0.0507	0.8282	0.0003	0.0906
	(3, 0.5, 3)	0.3265	0.0002	0.3894	1.3872	0.0003	1.2635
	(3, 1, 0.05)	1.8817	0.0023	0.0012	0.3339	0.0055	0.0002
	(3, 1, 1)	1.0363	0.0011	0.0630	1.2085	0.0013	0.0823
	(3, 1, 3)	0.9611	0.0011	0.5603	1.0361	0.0013	0.6053
	(3, 2, 0.05)	1.1011	0.1406	0.0135	0.7770	0.0238	0.0003
	(3, 2, 1)	1.2201	0.0093	0.0507	1.8760	0.0122	0.0672
	(3, 2, 3)	0.9669	0.0074	0.4328	1.0888	0.0091	0.4621
	(7, 0.25, 0.05)	0.2573	0.0001	0.0047	2.3858	0.0001	0.0208
	(7, 0.25, 1)	0.5568	0.0001	1.8371	0.7094	0.0002	1.8145
	(7, 0.25, 3)	0.5587	0.0002	3.9725	0.9507	0.0002	4.4882
	(7, 0.5, 0.05)	1.3938	0.0002	0.0014	3.6506	0.0002	0.0034
	(7, 0.5, 1)	1.3175	0.0002	0.9428	3.0960	0.0003	1.9842
	(7, 0.5, 3)	0.7830	0.0002	3.2487	0.8338	0.0003	3.1858
	(7, 1, 0.05)	1.3853	0.0012	0.0010	1.9336	0.0012	0.0002
	(7, 1, 1)	5.7144	0.0021	0.8653	2.4750	0.0015	0.2991
	(7, 1, 3)	4.5041	0.0015	3.4543	3.5295	0.0017	2.1812
	(7, 2, 0.05)	0.7022	0.0108	0.0000	1.9520	0.0092	0.0001
	(7, 2, 1)	4.1513	0.0201	0.2679	9.1907	0.0291	0.2156
	(7, 2, 3)	4.0194	0.0183	1.9366	3.2583	0.0196	1.1301

Table 7 Estimated covariances of MLEs for $n = 50$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	Monte-Carlo simulations			Analytic method		
		$Cov(\hat{\lambda}, \hat{\alpha})$	$Cov(\hat{\lambda}, \hat{\beta})$	$Cov(\hat{\alpha}, \hat{\beta})$	$Cov(\hat{\lambda}, \hat{\alpha})$	$Cov(\hat{\lambda}, \hat{\beta})$	$Cov(\hat{\alpha}, \hat{\beta})$
50	(0.5, 0.25, 0.05)	0.0167	-0.0074	0.0004	0.0730	-0.0434	-0.0009
	(0.5, 0.25, 1)	0.0260	-0.3685	0.0021	0.0785	-1.0115	-0.0118
	(0.5, 0.25, 3)	0.0260	-1.0383	0.0086	0.0777	-3.1869	-0.0365
	(0.5, 0.5, 0.05)	0.0082	0.0002	0.0019	0.1792	-0.0414	-0.0015
	(0.5, 0.5, 1)	0.0527	-0.2516	0.0140	0.1504	-0.5952	-0.0148
	(0.5, 0.5, 3)	0.0614	-0.9937	0.0112	0.1311	-1.3203	-0.0298
	(0.5, 1, 0.05)	0.0098	-0.0013	0.0100	0.8501	-0.1269	-0.0091
	(0.5, 1, 1)	0.0826	-0.3587	0.0215	0.3485	-0.8593	-0.0212
	(0.5, 1, 3)	0.0827	-1.4204	0.0647	0.3461	-2.2946	-0.0506
	(0.5, 2, 0.05)	-0.9600	-0.3150	0.1106	0.6624	-0.2406	0.0279
	(0.5, 2, 1)	-0.0020	-0.4329	0.0721	0.4511	-0.8379	0.0253
	(0.5, 2, 3)	-0.0705	-1.7181	0.2408	0.5200	-2.6111	0.0695
	(3, 0.25, 0.05)	0.0120	-0.0469	0.0009	0.0533	-0.2703	-0.0015
	(3, 0.25, 1)	0.0025	-0.8461	0.0256	0.0433	-5.6072	-0.0099
	(3, 0.25, 3)	0.0187	-1.9375	0.0392	0.0186	-14.7122	0.0385
	(3, 0.5, 0.05)	0.0450	-0.0311	0.0004	0.1129	-0.1019	-0.0020
	(3, 0.5, 1)	0.0334	-0.9693	0.0176	0.0930	-2.7954	-0.0186
	(3, 0.5, 3)	0.0524	-3.4378	0.0144	0.0779	-8.0506	-0.0447
	(3, 1, 0.05)	0.0403	-0.0081	0.0062	0.5758	-0.1046	-0.0063
	(3, 1, 1)	0.0861	-1.0252	0.0251	0.1366	-2.0500	-0.0006
	(3, 1, 3)	0.0762	-3.8528	0.0638	0.1965	-9.0941	-0.0445
	(3, 2, 0.05)	-0.6992	-0.2432	0.0830	1.2138	-0.2739	0.0061
	(3, 2, 1)	-0.0730	-0.7981	0.1382	0.0475	-1.2044	0.0994
	(3, 2, 3)	-0.1983	-3.3283	0.3961	-0.0612	-4.5892	0.3633
	(7, 0.25, 0.05)	-0.0023	-0.7181	0.0180	0.0261	-4.0791	0.0011
	(7, 0.25, 1)	0.0447	-3.0238	0.0984	0.0274	-7.8428	0.0608
	(7, 0.25, 3)	0.0821	-2.7226	0.1110	0.0673	-14.8206	0.1457
	(7, 0.5, 0.05)	0.0511	-0.4967	-0.0001	0.0693	-0.6612	-0.0039
	(7, 0.5, 1)	0.0580	-6.8978	0.0365	-0.0044	-16.1750	0.0794
	(7, 0.5, 3)	0.0640	-11.6383	0.0566	-0.0588	-29.9746	0.2744
	(7, 1, 0.05)	-0.0115	-0.2802	0.0069	0.1803	-0.2204	-0.0006
	(7, 1, 1)	0.0577	-6.7938	0.0600	0.0603	-4.3486	0.0315
	(7, 1, 3)	-0.0510	-19.2249	0.1667	0.0082	-33.4684	-0.4394
	(7, 2, 0.05)	-1.2498	-0.5239	0.0668	0.0360	-0.2732	0.0117
	(7, 2, 1)	-0.7951	-4.4318	0.4517	-0.4409	-3.0468	0.3499
	(7, 2, 3)	-1.2627	-15.5863	1.0759	-1.1463	-16.0017	1.6004

Table 8 Estimated covariances of MLEs for $n = 100$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	Monte-Carlo simulations			Analytic method		
		$Cov(\hat{\lambda}, \hat{\alpha})$	$Cov(\hat{\lambda}, \hat{\beta})$	$Cov(\hat{\alpha}, \hat{\beta})$	$Cov(\hat{\lambda}, \hat{\alpha})$	$Cov(\hat{\lambda}, \hat{\beta})$	$Cov(\hat{\alpha}, \hat{\beta})$
100	(0.5, 0.25, 0.05)	0.0084	-0.0037	0.0002	0.0365	-0.0188	-0.0004
	(0.5, 0.25, 1)	0.0167	-0.2194	-0.0010	0.0391	-0.4453	-0.0064
	(0.5, 0.25, 3)	0.0187	-0.8036	-0.0008	0.0417	-1.5448	-0.0198
	(0.5, 0.5, 0.05)	0.0014	0.0002	0.0009	0.1027	-0.0213	-0.0009
	(0.5, 0.5, 1)	0.0377	-0.1710	-0.0006	0.1233	-0.4897	-0.0180
	(0.5, 0.5, 3)	0.0358	-0.6267	-0.0010	0.0805	-0.9822	-0.0257
	(0.5, 1, 0.05)	0.0003	0.0001	0.0049	0.2875	-0.0404	-0.0018
	(0.5, 1, 1)	0.0758	-0.2760	0.0055	0.3186	-0.5176	-0.0315
	(0.5, 1, 3)	0.0707	-1.0591	0.0198	0.1737	-1.1253	-0.0260
	(0.5, 2, 0.05)	-0.4530	-0.1468	0.0606	0.5685	-0.1266	0.0086
	(0.5, 2, 1)	0.0025	-0.3729	0.0317	0.2101	-0.3593	0.0134
	(0.5, 2, 3)	-0.1178	-1.7554	0.1370	0.1295	-1.4932	0.0669
	(3, 0.25, 0.05)	0.0075	-0.0207	0.0003	0.0332	-0.1929	-0.0010
	(3, 0.25, 1)	0.0012	-0.4622	0.0101	0.0274	-3.6721	-0.0040
	(3, 0.25, 3)	-0.0004	-0.9413	0.0281	0.0106	-7.2806	0.0175
	(3, 0.5, 0.05)	0.0323	-0.0076	0.0001	0.0458	-0.0457	-0.0007
	(3, 0.5, 1)	0.0239	-0.7209	0.0007	0.0401	-1.3175	-0.0084
	(3, 0.5, 3)	0.0232	-1.9506	0.0051	0.0389	-6.5461	-0.0293
	(3, 1, 0.05)	0.0113	-0.0116	0.0031	0.2265	-0.0383	-0.0017
	(3, 1, 1)	0.0421	-0.8400	0.0086	0.0440	-1.1681	0.0071
	(3, 1, 3)	0.0428	-2.9136	0.0224	0.0441	-3.2982	0.3356
	(3, 2, 0.05)	-0.3975	-0.1575	0.0408	0.5790	-0.1047	0.0035
	(3, 2, 1)	-0.1374	-0.7827	0.0637	-0.3028	-1.1604	0.0978
	(3, 2, 3)	-0.1541	-2.5909	0.1930	-0.2575	-3.4547	0.2455
	(7, 0.25, 0.05)	-0.0036	-0.2745	0.0068	0.0103	-1.2154	0.0015
	(7, 0.25, 1)	0.0082	-1.9151	0.0680	0.0087	-5.6556	0.0405
	(7, 0.25, 3)	0.0384	-1.8403	0.0514	0.0607	-8.3301	0.0289
	(7, 0.5, 0.05)	0.0182	-0.3725	0.0002	0.0262	-0.5455	-0.0022
	(7, 0.5, 1)	0.0343	-4.1091	0.0092	0.0143	-9.2530	0.0017
	(7, 0.5, 3)	0.0195	-7.5173	0.0508	-0.0017	-12.5261	0.0718
	(7, 1, 0.05)	0.0265	-0.2651	0.0064	0.1609	-0.4968	-0.0050
	(7, 1, 1)	0.0240	-6.4625	0.0838	0.0244	-4.5572	0.0466
	(7, 1, 3)	-0.0709	-14.8243	0.1200	-0.1172	-12.9119	0.1621
	(7, 2, 0.05)	-1.0704	-0.5007	0.0606	-0.2290	-0.2290	0.0202
	(7, 2, 1)	-0.8718	-4.7650	0.4776	-0.5568	-3.3656	0.3783
	(7, 2, 3)	-1.0943	-16.0831	1.0025	-3.1229	-25.3552	1.7878

Table 9 Estimated covariances of MLEs for $n = 1,000$

n	$\boldsymbol{\theta} = (\lambda, \alpha, \beta)$	Monte-Carlo simulations			Analytic method		
		$Cov(\hat{\lambda}, \hat{\alpha})$	$Cov(\hat{\lambda}, \hat{\beta})$	$Cov(\hat{\alpha}, \hat{\beta})$	$Cov(\hat{\lambda}, \hat{\alpha})$	$Cov(\hat{\lambda}, \hat{\beta})$	$Cov(\hat{\alpha}, \hat{\beta})$
1,000	(0.5, 0.25, 0.05)	0.0016	-0.0006	0.0000	0.0035	-0.0016	0.0000
	(0.5, 0.25, 1)	0.0021	-0.0234	-0.0002	0.0039	-0.0381	-0.0007
	(0.5, 0.25, 3)	0.0020	-0.0759	-0.0004	0.0041	-0.1341	-0.0020
	(0.5, 0.5, 0.05)	0.0000	0.0000	0.0001	0.0088	-0.0018	-0.0001
	(0.5, 0.5, 1)	0.0064	-0.0194	-0.0007	0.0079	-0.0241	-0.0009
	(0.5, 0.5, 3)	0.0056	-0.0637	-0.0017	0.0078	-0.0776	-0.0027
	(0.5, 1, 0.05)	0.0000	0.0000	0.0005	0.1132	-0.0155	-0.0019
	(0.5, 1, 1)	0.0143	-0.0180	-0.0004	0.0200	-0.0249	-0.0013
	(0.5, 1, 3)	0.0169	-0.1008	-0.0025	0.0228	-0.0832	-0.0048
	(0.5, 2, 0.05)	0.0000	0.0000	0.0046	0.0336	-0.0061	0.0015
	(0.5, 2, 1)	0.0142	-0.0523	0.0016	0.0352	-0.0271	-0.0003
	(0.5, 2, 3)	0.0150	-0.1592	0.0052	0.0343	-0.0789	-0.0005
	(3, 0.25, 0.05)	0.0010	-0.0023	0.0000	0.0031	-0.0129	-0.0001
	(3, 0.25, 1)	0.0002	-0.0475	0.0006	0.0017	-0.3044	0.0001
	(3, 0.25, 3)	-0.0073	-0.0808	0.0007	0.0025	-0.6656	-0.0002
	(3, 0.5, 0.05)	0.0043	-0.0027	0.0000	0.0058	-0.0043	-0.0001
	(3, 0.5, 1)	0.0031	-0.1253	-0.0003	0.0056	-0.2523	-0.0011
	(3, 0.5, 3)	0.0016	-0.3187	0.0008	0.0082	-1.2608	-0.0052
	(3, 1, 0.05)	0.0287	-0.0447	-0.0004	0.0238	-0.0050	-0.0002
	(3, 1, 1)	-0.0019	-0.2356	0.0015	-0.0031	-0.2928	0.0017
	(3, 1, 3)	-0.0018	-0.6776	0.0046	-0.0023	-0.7279	0.0048
	(3, 2, 0.05)	-1.4915	-0.5162	0.0408	-0.0066	-0.0082	0.0009
	(3, 2, 1)	-0.0670	-0.2347	0.0164	-0.1065	-0.3273	0.0214
	(3, 2, 3)	-0.0449	-0.6157	0.0386	-0.0564	-0.6707	0.0450
	(7, 0.25, 0.05)	-0.0010	-0.0274	0.0004	0.0019	-0.2083	0.0001
	(7, 0.25, 1)	-0.0022	-0.7947	0.0115	-0.0005	-0.9451	0.0072
	(7, 0.25, 3)	0.0020	-1.0225	0.0122	0.0056	-1.5546	0.0059
	(7, 0.5, 0.05)	-0.0017	-0.0393	0.0001	-0.0059	-0.0936	0.0001
	(7, 0.5, 1)	-0.0033	-0.9917	0.0056	-0.0051	-2.2976	0.0062
	(7, 0.5, 3)	-0.0012	-1.4329	0.0097	0.0000	-1.4271	0.0080
	(7, 1, 0.05)	-0.0124	-0.0324	0.0004	-0.0208	-0.0163	0.0003
	(7, 1, 1)	-0.0705	-2.0288	0.0270	-0.0315	-0.7068	0.0110
	(7, 1, 3)	-0.0535	-3.4799	0.0496	-0.0456	-2.4165	0.0404
	(7, 2, 0.05)	-0.0096	-0.0039	0.0006	-0.0564	-0.0123	0.0008
	(7, 2, 1)	-0.2439	-0.9459	0.0644	-0.4560	-1.2852	0.0722
	(7, 2, 3)	-0.2196	-2.4574	0.1641	-0.2043	-1.7280	0.1349

Table 10 Maximum likelihood estimates and goodness-of-fit testing for two datasets

Data	Distribution	Estimate	K-S	p-value
1 $(n=128)$	GZTP	$\hat{\boldsymbol{\theta}}=(3.9201,\ 1.4169,\ 0.0623)$	0.03598	0.9964
	WP	$\hat{\boldsymbol{\theta}}_1=(4.0130,\ 1.2744,\ 0.0171)$	0.04551	0.9536
	Gamma	$\hat{\boldsymbol{\theta}}_2=(1.1726,\ 0.1252)$	0.07330	0.4974
2 $(n=30)$	GZTP	$\hat{\boldsymbol{\theta}}=(0.3811,\ 3.1587,\ 1.7838)$	0.05708	0.9999736
	WP	$\hat{\boldsymbol{\theta}}_1=(2.1745,\ 2.1041,\ 0.1358)$	0.05709	0.9999734
	Gamma	$\hat{\boldsymbol{\theta}}_2=(2.9582,\ 1.7661)$	0.05601	0.9999834