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A New Estimator for the Gaussian Linear Regression Model with Multicollinearity

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Abstract

In this study, a new two-parameter biased estimator for estimating the parameter of a multicollinearity linear regression model is developed. We compared the performance of the proposed estimator with some existing estimators in terms of the mean squared error matrix. The theoretical comparisons and simulation results revealed that the proposed two-parameter estimator dominate other estimators under certain conditions using the mean squared error. The simulation result shows that the ordinary least squares estimator has the lowest performance. Among the one-parameter estimator, the ridge regression estimator dominates the Liu estimator. However, the proposed method dominates both the one-parameter and two-parameter estimators. We also discuss the estimations of the biasing parameters. A famous real-life dataset is analyzed to support both theoretical and simulation results.

Keywords: Liu estimator, mean squared error, proposed two-parameter estimator, real-life dataset application, ridge estimator, simulation.

1. Introduction

We usually assume that the regressors or the explanatory variables of the multiple linear regression model are independent. In practice, strong or near to perfect linear relationships exist among the explanatory variables (regressors), which causes the problem of multicollinearity. In the presence of multicollinearity, the ordinary least squares (OLS) estimator becomes inefficient and leads to wrong inference with wrong model parameters signs (Hoerl and Kennard, 1970). Several solutions exist in literature to solve the multicollinearity problem. One of the widely used solutions to this problem is the ridge regression estimator proposed by Hoerl and Kennard (1970). However, the ridge regression estimator depends on the ridge or shrinkage parameter k for the one-parameter method. Estimation of this parameter is a crucial aspect of the ridge regression estimator. Different authors have given some one-parameter estimators for various m k odels. To mention a few, Hoerl and Kennard (1970), Mayer and Willke (1973), Swindel (1976), Khalaf and Shukur (2005), Kibria and Banik (2016), Kibria and Lukman (2020), Suhail et al. (2021), Amin et al. (2023), Aladeitan et al. (2021), Ugwuowo et al. (2021), Arashi et al. (2021) and others. Since the ridge regression estimator is a non-linear function of the shrinkage parameter k, Liu (1993) proposed a new estimator, which combines the advantages of the ORR estimator and the Stein (1956) estimator and it is a linear function of biasing parameter d. Several researchers consider Liu estimator for different models, among them Liu (1993, 2003), Li and Yang (2012), Mansson et al. (2015), Akdeniz and Kaçiranlar (1995), Ozkale and Kaçiranlar (2007), Sakallıoglu and Kaçıranlar (2008), Qasim et al. (2020), Lukman et al. (2020) are notable. To take the advantage of ridge regression and Liu (1993) estimators, several researchers consider two-parameter estimators, to mention a few, Ozkale and Kaçiranlar (2007), Sakallıoglu and Kaçıranlar (2008), Yang and Chang (2010), Roozbeh (2018), Lukman et al. (2019a, 2019b), Dawoud and Kibria (2020a, 2020b), and Qasim et al. (2022) among others.

The main objective of this paper is to propose a new kind of two-parameter estimator for the regression parameters and then to compare the proposed two-parameter estimator performance with the OLS, the ordinary ridge regression (ORR), the Liu estimators, the two-parameter (TP) estimator of Ozkale and Kaçiranlar (2007) and the new two-parameter (NTP) estimator of Yang and Chang (2010). The organization of the article is as follows: The model and the parameter estimation techniques are given in Section 2. The theoretical comparisons between the proposed two-parameter (PTP) estimator and some existing estimators are given and the biasing parameters are also given in Section 3. A simulation study is performed in Section 4 while real-life data is analyzed in Section 5. This article ends up with some concluding remarks in Section 6.

2. The Model and Estimation of the Parameter

Consider the following linear regression model:

$$y = X\beta + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2 I), \tag{1}$$

where y is an $n \times 1$ vector of the response variable, X is a known $n \times p$ full rank design matrix, β is an $p \times 1$ vector of unknown model parameters, and ε is known as an $n \times 1$ disturbances vector having zero mean and variance-covariance matrix, $Cov(\varepsilon) = \sigma^2 I_n$, I_n is called as an identity matrix with the order $n \times n$.

To define different estimators, canonical form of the model (1) is given by

$$y = Z\alpha + \varepsilon, \tag{2}$$

where $Z = XE$, $\alpha = E'\beta$, and E is an orthogonal matrix such that $Z'Z = E'X'XE = W = diag(w_1, w_2, \ldots, w_p)$. Then, the OLS estimator of α is given as

$$\hat{\alpha} = W^{-1} Z'y, \tag{3}$$

and the matrix mean squared error (MMSE) of the OLS estimator is given by

$$MMSE(\hat{\alpha}) = \sigma^2 W^{-1}. \tag{4}$$

The ORR of α (Hoerl and Kennard 1970) is given by

$$\hat{\alpha}_k = AW\hat{\alpha}, \tag{5}$$

where $A = [W + kI_p]^{-1}$, $k > 0$ is the biasing parameter and then the MMSE of $\hat{\alpha}_k$ would be

$$MMSE(\hat{\alpha}_k) = \sigma^2 AWA' + (AW - I_p)\alpha\alpha'(AW - I_p)'. \tag{6}$$

Hoerl et al. (1975) had defined k for $\hat{\alpha}_k$ as

$$\hat{k}_{HM} = \frac{p\,\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2}. \tag{7}$$

The Liu estimator of α (Liu 1993) is given as

$$\hat{\alpha}_d = F\,\hat{\alpha}, \tag{8}$$

where $F = [W + I_p]^{-1}[W + d\,I_p]$, $0 < d < 1$ is the biasing parameter and d is defined for $\hat{\alpha}_d$ as

$$\hat{d}_{opt} = 1 - \hat{\sigma}^2\left[\frac{\sum_{i=1}^{p}(1/(w_i(w_i+1)))}{\sum_{i=1}^{p}(\hat{\alpha}_i^2/(w_i+1)^2)}\right], \tag{9}$$

$$MMSE(\hat{\alpha}_d) = \sigma^2\,F\,W^{-1}F' + (1-d)^2(W+I_p)^{-1}\alpha\alpha'(W+I_p)^{-1}. \tag{10}$$

If $\hat{d}_{opt}$ is negative, Ozkale and Kaciranlar (2007) had given the alternative $\hat{d}_{alt}$ as:

$$\hat{d}_{alt} = \min\left[\frac{\hat{\alpha}_i^2}{(\hat{\sigma}^2/w_i) + \hat{\alpha}_i^2}\right]_{i=1}^{p}. \tag{11}$$

The TP estimator of α (Ozkale and Kaciranlar 2007) is

$$\hat{\alpha}_{TP} = R\,\hat{\alpha}, \tag{12}$$

where $R = (W + k\,\mathrm{I}_p)^{-1}(W + k\,d\,\mathrm{I}_p)$ and the biasing parameters k and d of the TP estimator are defined by

$$\hat{k}_{TP} = \frac{\hat{\sigma}^2}{\hat{\alpha}_i^2 - d\,((\hat{\sigma}^2/w_i) + \hat{\alpha}_i^2)}, \tag{13}$$

$$\hat{d}_{TP} = \frac{\sum_{i=1}^{p}\frac{(\hat{k}\hat{\alpha}_i^2 - \hat{\sigma}^2)}{(w_i+k)^2}}{\sum_{i=1}^{p}\frac{\hat{k}\,(\hat{\sigma}^2 + \hat{\alpha}_i^2 w_i)}{w_i\,(w_i+k)^2}}, \tag{14}$$

$$MMSE(\hat{\alpha}_{TP}) = \sigma^2\,RW^{-1}R' + [R - I_p]\,\alpha\,\alpha'[R - I_p]'. \tag{15}$$

If $\hat{d}_{opt}$ is negative, we can use $\hat{d}_{alt}$ that was considered by Ozkale and Kaciranlar (2007). The NTP estimator of α (Yang and Chang, 2010) is given as

$$\hat{\alpha}_{NTP} = FAW\hat{\alpha}, \tag{16}$$

and the biasing parameters k and d of the NTP estimator are defined by

$$\hat{k}_{NTP} = \frac{\hat{\sigma}^2(w_i+d) - (1-d)\,w_i\,\hat{\alpha}_i^2}{(w_i+1)\,\hat{\alpha}_i^2}, \tag{17}$$

$$\hat{d}_{NTP} = \frac{\sum_{i=1}^{p}\left[((k+1)\,w_i + k)\,w_i\,\hat{\alpha}_i^2 - w_i^2\hat{\sigma}^2\right]/\left[(w_i+1)^2\,(w_i+k)^2\right]}{\sum_{i=1}^{p}\left[(\hat{\sigma}^2 + w_i\,\hat{\alpha}_i^2)\,w_i\right]/\left[(w_i+1)^2\,(w_i+k)^2\right]}, \tag{18}$$

$$MMSE(\hat{\alpha}_{NTP}) = \sigma^2\,FAWA'\,F' + [FAW - I_p]\,\alpha\,\alpha'[FAW - I_p]'. \tag{19}$$

The PTP estimator is found by minimizing the objective function below

$$(y-Z\alpha)'(y-Z\alpha)+[(\sqrt{k(1+d)}\,\alpha+\frac{k}{\sqrt{k(1+d)}}\,\hat{\alpha})'(\sqrt{k(1+d)}\,\alpha+\frac{k}{\sqrt{k(1+d)}}\,\hat{\alpha})-c] \quad (20)$$

with respect to α, it will yield the normal equations

$$(W+k(1+d)\mathrm{I}_p)\alpha = Z'y-k\,\hat{\alpha}, \quad (21)$$

where $k \geq 0$, $0<d<1$ and c is a constant. The solution to (21) gives the PTP estimator as follows

$$\hat{\alpha}_{PTP} = (W+k(1+d)\mathrm{I}_p)^{-1}(W-k\,\mathrm{I}_p)\hat{\alpha} = CB\,\hat{\alpha}, \quad (22)$$

where $C=(W+k(1+d)\mathrm{I}_p)^{-1}$ and $B=(W-k\,\mathrm{I}_p)$.

Moreover, the new PTP estimator is also obtained by augmenting $\frac{-k}{\sqrt{k(1+d)}}\,\hat{\alpha}=\sqrt{k(1+d)}\;\alpha+\varepsilon^*$, where ε^* is $p\times 1$ a random vector with mean zero and dispersion matrix $\sigma^2 I$, to Equation (2) and then using the OLS estimate. The MMSE of the new PTP estimator of α is given by

$$MMSE(\hat{\alpha}_{PTP}) = \sigma^2 CBW^{-1}B'C' + [CB-I_p]\,\alpha\alpha'[CB-I_p]'. \quad (23)$$

Some lemmas are used for theoretical comparisons among estimators in the next section.

Lemma 2.1 (Farebrother 1976) *Let G be an $n\times n$ positive definite matrix, that is $G>0$ and α be some vector; then, $G-\alpha\alpha'>0$ if and only if $\alpha' G^{-1}\alpha<1$.*

Lemma 2.2 (Trenkler and Toutenburg 1990) *Let $\alpha_i = C_i y$, $i=1,2$ be two linear estimators of α. Suppose that $D=Cov(\hat{\alpha}_1)-Cov(\hat{\alpha}_2)>0$, where $Cov(\hat{\alpha}_i)\, i=1,2$ be the covariance matrix of $\hat{\alpha}_i$ and $b_i = Bias(\hat{\alpha}_i)=(C_iX-I)\alpha$, $i=1,2$. Consequently,*

$$\Delta(\hat{\alpha}_1-\hat{\alpha}_2) = MSEM(\hat{\alpha}_1)-MSEM(\hat{\alpha}_2) = \sigma^2 D + b_1b_1' - b_2b_2' > 0 \quad (24)$$

if and only if $b_2'[\sigma^2 D + b_1'b_1]^{-1}b_2 < 1$ where $MSEM(\hat{\alpha}_i) = Cov\,(\hat{\alpha}_i)+b_ib_i'$.

3. Comparison Among the Estimators

3.1. Comparison between $\hat{\alpha}$ and $\hat{\alpha}_{PTP}$

Theorem 3.1 *$MMSE\,(\hat{\alpha}) - MMSE\,(\hat{\alpha}_{PTP})>0$ if and only if*

$$\alpha'[CB-I_p]'\,[\sigma^2\,(W^{-1}-CBW^{-1}B'C')][CB-I_p]\alpha<1. \quad (25)$$

Proof: Consider $Cov\,(\hat{\alpha})-Cov\,(\hat{\alpha}_{PTP}) = \sigma^2\,(W^{-1}-CBW^{-1}B'C')$

$$= \sigma^2\, diag\left\{\frac{1}{w_i}-\frac{(w_i-k)^2}{w_i(w_i+k(1+d))^2}\right\}_{i=1}^{p} \quad (26)$$

where $W^{-1}-CBW^{-1}B'C'$ will be positive definite if and only if $(w_i+k(1+d))^2-(w_i-k)^2>0$ or $(w_i+k(1+d))-(w_i-k)>0$. Clearly, for $k>0$ and $0<d<1$, $(w_i+k(1+d))-(w_i-k)$ $=k(2+d)>0$. By Lemma 2.2, the proof is completed.

3.2. Comparison between $\hat{\alpha}_k$ and $\hat{\alpha}_{PTP}$

Theorem 3.2 *MMSE* $(\hat{\alpha}_k) - MMSE\,(\hat{\alpha}_{PTP}) > 0$ *if and only if*

$$\alpha'[CB - I_p]'\,[V_1 + (AW - I_p)\alpha\,\alpha'(AW - I_p)'][CB - I_p]\alpha < 1, \tag{27}$$

where $V_1 = \sigma^2\,(AWA' - CBW^{-1}B'C')$.

Proof:

$$\begin{aligned} V_1 &= \sigma^2\,(AWA' - CBW^{-1}B'C') \\ &= \sigma^2\, diag\left\{\frac{w_i}{(w_i + k)^2} - \frac{(w_i - k)^2}{w_i(w_i + k(1+d))^2}\right\}_{i=1}^{p} \end{aligned} \tag{28}$$

where $AWA' - CBW^{-1}B'C'$ will be positive definite if and only if $w_i^2\,(w_i + k(1+d))^2 - (w_i + k)^2(w_i - k)^2 > 0$ or $w_i\,(w_i + k(1+d)) - (w_i + k)(w_i - k) > 0.$ Clearly, for $k > 0$ and $0 < d < 1$, $w_i\,(w_i + k(1+d)) - (w_i + k)(w_i - k) = w_i\,k(1+d) + k^2 > 0.$ By Lemma 2.2, the proof is completed.

3.3. Comparison between $\hat{\alpha}_d$ and $\hat{\alpha}_{PTP}$

Theorem 3.3 *MMSE* $(\hat{\alpha}_d) - MMSE\,(\hat{\alpha}_{PTP}) > 0$ *if and only if*

$$\alpha'[CB - I_p]'\,[V_2 + (F - I_p)\alpha\,\alpha'(F - I_p)'][CB - I_p]\alpha < 1 \tag{29}$$

where $V_2 = \sigma^2\,(FW^{-1}F - CBW^{-1}B'C')$

Proof:

$$\begin{aligned} V_2 &= \sigma^2\,(FW^{-1}F - CBW^{-1}B'C') \\ &= \sigma^2\, diag\left\{\frac{(w_i + d)^2}{w_i(w_i + 1)^2} - \frac{(w_i - k)^2}{w_i(w_i + k\,(1+d))^2}\right\}_{i=1}^{p} \end{aligned} \tag{30}$$

where $FW^{-1}F - CBW^{-1}B'C'$ will be positive definite if and only if $(w_i + d)^2\,(w_i + k(1+d))^2 - (w_i + 1)^2(w_i - k)^2 > 0$ or $(w_i + d)(w_i + k(1+d)) - (w_i + 1)(w_i - k) > 0.$ Clearly, for $k > 0$ and $0 < d < 1$, $(w_i + d)(w_i + k(1+d)) - (w_i + 1)(w_i - k) = w_i\,(2k + kd + d - 1)$ $+k\,(d\,(1+d) + 1) > 0.$ By Lemma 2.2, the proof is completed.

3.4. Comparison between $\hat{\alpha}_{TP}$ and $\hat{\alpha}_{PTP}$

Theorem 3.4 *MMSE* $(\hat{\alpha}_{TP}) - MMSE\,(\hat{\alpha}_{PTP}) > 0$ *if and only if*

$$\alpha'[CB - I_p]'\,[V_3 + (R - I_p)\alpha\,\alpha'(R - I_p)'][CB - I_p]\alpha < 1, \tag{31}$$

where $V_3 = \sigma^2\,(RW^{-1}R' - CBW^{-1}B'C')$.

Proof:

$$V_3 = \sigma^2\,(RW^{-1}R' - CBW^{-1}B'C') = \sigma^2\, diag\left\{\frac{(w_i + kd)^2}{w_i(w_i + k)^2} - \frac{(w_i - k)^2}{w_i(w_i + k\,(1+d))^2}\right\}_{i=1}^{p} \tag{32}$$

where $RW^{-1}R' - CBW^{-1}B'C'$ will be positive definite if and only if $(w_i + kd)^2\,(w_i + k(1+d))^2$ $-(w_i + k)^2(w_i - k)^2 > 0$ or $(w_i + kd)(w_i + k(1+d)) - (w_i + k)(w_i - k) > 0.$ Clearly, for $k > 0$ and

$0 < d < 1,\ (w_i + kd)(w_i + k(1+d)) - (w_i + k)(w_i - k) = w_i\, k(1+2d) + k^2\,(d(1+d)+1) > 0.$ By Lemma 2.2, the proof is completed.

3.5. Comparison between $\hat{\alpha}_{NTP}$ and $\hat{\alpha}_{PTP}$

Theorem 3.5 *$MMSE(\hat{\alpha}_{NTP}) - MMSE(\hat{\alpha}_{PTP}) > 0$ if and only if*

$$\alpha'[CB - I_p]'\,[V_4 + (FAW - I_p)\alpha\alpha'(FAW - I_p)'][CB - I_p]\alpha < 1. \quad (33)$$

where $V_4 = \sigma^2\,(FAWA'F' - CBW^{-1}B'C')$.

Proof:

$$V_4 = \sigma^2\,(FAWA'F' - CBW^{-1}B'C') = \sigma^2\, diag\left\{\frac{w_i\,(w_i+d)^2}{(w_i+1)^2(w_i+k)^2} - \frac{(w_i-k)^2}{w_i(w_i+k\,(1+d))^2}\right\}_{i=1}^{p}, \quad (34)$$

where $FAWA'F' - CBW^{-1}B'C'$ will be positive definite if and only if $w_i^2(w_i+d)^2\,(w_i+k\,(1+d))^2$ $-(w_i+1)^2(w_i+k)^2(w_i-k)^2 > 0$ or $w_i\,(w_i+d)(w_i+k\,(1+d)) - (w_i+1)(w_i+k)(w_i-k) > 0$. Clearly, for $k > 0$ and $0 < d < 1$, $w_i(w_i+d)(w_i+k(1+d)) - (w_i+1)(w_i^2 - k^2)$ $= w_i^2(k(1+d)+d-1) + w_i(kd\,(1+d)+k^2) + k^2 > 0$. By Lemma 2.2, the proof is completed.

3.6. Selection of the parameters k and d

Different estimators of the biasing parameters k and d are obtained and proposed in different studies. These include Hoerl and Kennard (1970), Liu (1993), Hoerl et al. (1975), Kibria (2003), Khalaf and Shukur (2005), Mansson et al. (2015), Kibria and Banik (2016), Lukman and Ayinde (2017), Lukman et al. (2017), Lukman and Olatunji (2018) and Qasim et al. (2020), among others.

Here, the optimal values of k and d for the new PTP estimator will be found. First, by minimizing the following equation m, we get the optimal value of k for a fixed d as

$$MMSE(\hat{\alpha}_{PTP}) = E\,((\hat{\alpha}_{PTP} - \alpha)'(\hat{\alpha}_{PTP} - \alpha)),$$

$$m = tr\,(MMSE(\hat{\alpha}_{PTP})),$$

$$m = \sigma^2 \sum_{i=1}^{p} \frac{(w_i - k)^2}{w_i\,(w_i + k\,(1+d))^2} + k^2(2+d)^2 \sum_{i=1}^{p} \frac{\alpha_i^2}{(w_i + k\,(1+d))^2}. \quad (35)$$

Differentiating m with respect to k and setting $(\partial m/\partial k) = 0$, we find

$$k = \frac{w_i\,\sigma^2}{w_i\,\alpha_i^2\,(d+2) + \sigma^2}. \quad (36)$$

Therefore, the estimated optimal value of k using the unbiased estimators of the unknown parameter σ^2 and α_i^2 is as follows:

$$\hat{k} = \frac{w_i\,\hat{\sigma}^2}{w_i\,\hat{\alpha}_i^2\,(d+2) + \hat{\sigma}^2}, \quad (37)$$

and

$$\hat{k}_{\min(PTP)} = \min\left\{\frac{w_i\,\hat{\sigma}^2}{w_i\,\hat{\alpha}_i^2\,(d+2) + \hat{\sigma}^2}\right\}_{i=1}^{p}. \quad (38)$$

Also, the optimal value of d can be found by differentiating m with respect to d for a fixed k and setting $(\partial m/\partial d) = 0$, we get

$$d = \frac{w_i\,\sigma^2 - k\,(2\,w_i\,\alpha_i^2 + \sigma^2)}{w_i\,\alpha_i^2 k}. \tag{39}$$

Then, the estimated optimal d with the unbiased estimators of their parameters is

$$d = \frac{w_i\,\hat{\sigma}^2 - \hat{k}\,(2\,w_i\,\hat{\alpha}_i^2 + \hat{\sigma}^2)}{w_i\,\hat{\alpha}_i^2 \hat{k}}, \tag{40}$$

and

$$\hat{d}_{\min(PTP)} = \min\left\{\frac{w_i\,\hat{\sigma}^2 - \hat{k}\,(2\,w_i\,\hat{\alpha}_i^2 + \hat{\sigma}^2)}{w_i\,\hat{\alpha}_i^2 \hat{k}}\right\}_{i=1}^{p}. \tag{41}$$

The PTP estimator in this study is obtained by employing the following biasing parameters k and d:

1. Propose 1:

$$\hat{k} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2} \text{ and } \hat{d} = \min\left(\frac{\hat{\alpha}_i^2}{\left(\hat{\sigma}^2 / w_i\right) + \hat{\alpha}_i^2}\right). \tag{42}$$

2. Propose 2:

$$\hat{k} = \sqrt{\frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2}} \text{ and } \hat{d} = \sqrt{\min\left(\frac{\hat{\alpha}_i^2}{\left(\hat{\sigma}^2 / w_i\right) + \hat{\alpha}_i^2}\right)}. \tag{43}$$

3. Propose 3:

$$\hat{k} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2} \text{ and } \hat{d} = \sqrt{\max\left(\frac{\hat{\alpha}_i^2}{\left(\hat{\sigma}^2 / w_i\right) + \hat{\alpha}_i^2}\right)}. \tag{44}$$

4. Propose 4:

$$\hat{k} = \sqrt{\frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2}} \text{ and } \hat{d} = \max\left(\frac{\hat{\alpha}_i^2}{\left(\hat{\sigma}^2 / w_i\right) + \hat{\alpha}_i^2}\right). \tag{45}$$

5. Propose 5:

$$\hat{k}_{\min(PTP)} = \min\left\{\frac{w_i\,\hat{\sigma}^2}{w_i\,\hat{\alpha}_i^2\,(d+2) + \hat{\sigma}^2}\right\}_{i=1}^{p} \text{ and } \hat{d} = \min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right). \tag{46}$$

6. Propose 6:

$$\hat{k}_{\min(PTP)} = \sqrt{\min\left\{\frac{w_i\,\hat{\sigma}^2}{w_i\,\hat{\alpha}_i^2\,(d+2) + \hat{\sigma}^2}\right\}_{i=1}^{p}} \text{ and } \hat{d} = \sqrt{\min\left(\frac{\hat{\sigma}^2}{\hat{\alpha}_i^2}\right)}. \tag{47}$$

4. Simulation Study

A Monte Carlo simulation is conducted for the purpose of judging the performances of the estimators at different specification. The simulation study was conducted using the RStudio programming language. The following factors affect the performance of the estimators: the sample size n, the level of multicollinearity ρ, the error variance σ^2, the number of explanatory variables p and the biasing parameters k and d. The biasing parameters k and d are defined in Equations (42) to (47). The sample sizes in this study is considered as 50 and 100. The explanatory is generated in Equation (48) in accordance with the following authors: Gibbons (1981) and Kibria (2003), among others

$$X_{ji} = (1-\rho^2)^{1/2} z_{ji} + \rho z_{j(p+1)} \;, j = 1, 2, ..., n, \; i = 1, 2, ..., p, \tag{48}$$

where z_{ji} are independent standard normal pseudo random numbers, ρ^2 is the correlation between any two independent variables. The values of ρ were taken as 0.70, 0.80, 0.90, 0.95 and 0.99, respectively. The response variable is generated for $p = 4$ and 8, respectively and defined as follows:

$$Y_j = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + e_j, \tag{49}$$

$$Y_j = \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6 + \beta_7 X_7 + \beta_8 X_8 + e_j, \tag{50}$$

where e_j has a normal distribution with mean equals 0 and variance equals σ^2. β is taken such that $\beta'\beta = 1$ (Newhouse and Oman 1971). Values of σ are as 5 and 10. The estimators' performances are judged through the mean squared error (MSE) in Equation (51). The smaller the mean squared error the better the estimator.

$$MSE\left(\hat{\beta}\right) = \frac{1}{2000} \sum_{j=1}^{2000} \left(\hat{\beta}_{ij} - \beta_i\right)' \left(\hat{\beta}_{ij} - \beta_i\right), \tag{51}$$

where $\hat{\beta}_{ij}$ is the estimate of the i^{th} parameter in j^{th} replication, β_i are the true parameter values and the experiment is replicated 2000 times. The MSE results are presented for different values of n, ρ, σ in Table 1 for $p = 4$ and Table 2 for $p = 8$, respectively. For a graphical representation, we also plotted MSE vs ρ in Figure 1 and MSE vs n in Figure 2.

From both Figures 1 and 2 and Tables 1 and 2, the following observation can be made about the results based on some of the determining factors:

1. The OLS estimator exhibited the least performance at all levels of multicollinearity.

2. The PTP estimator performs better than all the estimators with one parameter (ORR and the Liu) and two parameters (TP and NTP) in terms of the MSE. However, its performance is a function of the biasing parameters. The proposed estimator with the biasing parameters $\hat{k} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p} \hat{\alpha}_i^2}$ and $\hat{d} = \sqrt{\max\left(\frac{\hat{\alpha}_i^2}{\left(\hat{\sigma}^2 / w_i\right) + \hat{\alpha}_i^2}\right)}$ performed the best in the simulation study in most cases.

3. The ORR estimator outperforms the Liu estimator.

4. The MSE decreases as the sample size increases at a particular level of multicollinearity.

5. Increasing the value of σ increases the mean square errors of each of the estimators keeping other variables constant.

6. The mean squared error of all the estimators increases for a given level of multicollinearity and σ as the number of explanatory variables increases.

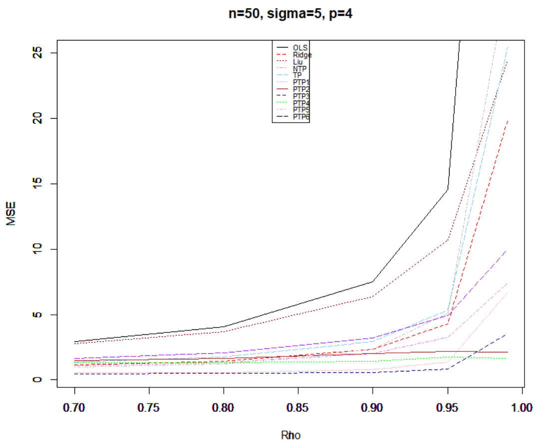

Figure 1 Estimated MSE vs ρ for fixed n, σ and p

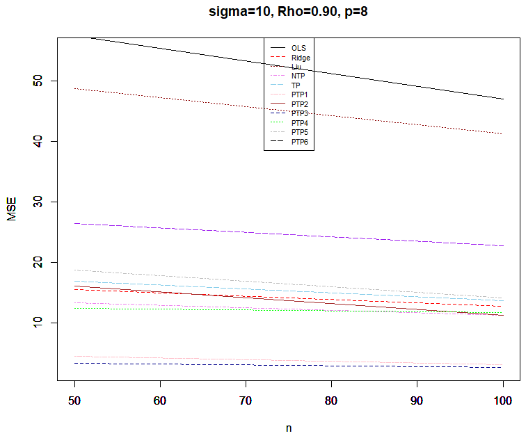

Figure 2 Estimated MSE vs n for fixed ρ, σ and p

5. Application

This dataset was originally adopted by Eledum and Alkhaklifa (2012) and later adopted by Eledum and Zahri (2013) and Lukman et al. (2015) among others. The dependent variable (y) is the product value in the manufacturing sector and the independent variables are defined as follows: the values of the imported intermediate commodities (X_1), imported capital commodities (X_2) and the value of imported raw materials (X_3). The period of the data span from 1960 to 1990. The correlation coefficient between the independent variables are high and as follows: $r_{X_1X_2} = 0.995$, $r_{X_1X_3} = 0.992$, $r_{X_2X_3} = 0.991$. The computed values of the variance inflation factors are 128.264, 103.428, and 70.871, respectively. According to Lukman et al. (2019a), there is multicollinearity when the variance inflation factor exceeds ten (10). The condition number is computed to be 1176 which also indicate the presence of severe multicollinearity. The regression analysis results are presented in Table 3, which agrees with the simulation study to some extent. The PTP estimator with the biasing parameters $\hat{k} = \frac{p\hat{\sigma}^2}{\sum_{i=1}^{p}\hat{\alpha}_i^2}$ and $\hat{d} = \sqrt{\max\left(\frac{\hat{\alpha}_i^2}{\left(\hat{\sigma}^2 / w_i\right) + \hat{\alpha}_i^2}\right)}$ outperforms other.

Table 1 Estimated MSE of the estimators when $p = 4$

σ	n	ρ	Estimators										
			OLS	Ridge	Liu	NTP	TP	PTP1	PTP2	PTP3	PTP4	PTP5	PTP6
5	50	0.7	2.939	1.154	2.767	1.099	1.357	0.561	1.450	0.429	1.294	0.967	1.660
		0.8	4.035	1.425	3.700	1.323	1.716	0.570	1.649	0.522	1.305	1.224	2.065
		0.9	7.499	2.347	6.365	2.027	2.908	0.776	2.010	0.535	1.431	2.338	3.174
		0.95	14.56	4.267	10.71	3.226	5.369	1.384	2.199	0.827	1.725	5.084	4.899
		0.99	71.66	19.86	24.51	7.404	25.45	6.703	2.133	3.493	1.660	30.82	10.01
	100	0.7	1.457	0.672	1.417	0.657	0.772	0.319	0.938	0.281	0.866	0.558	0.961
		0.8	1.904	0.783	1.832	0.757	0.924	0.350	1.099	0.299	0.994	0.597	1.170
		0.9	3.292	1.164	3.074	1.094	1.426	0.361	1.474	0.358	1.281	0.908	1.765
		0.95	6.090	1.952	5.362	1.735	2.44	0.546	1.907	0.470	1.579	1.799	2.770
		0.99	28.42	8.283	16.91	5.101	10.55	2.537	2.070	1.412	1.534	11.24	7.424
10	50	0.7	11.75	3.915	11.05	3.714	4.687	1.554	5.408	1.164	4.791	2.600	6.068
		0.8	16.14	5.073	14.78	4.699	6.189	1.870	6.188	1.291	5.350	3.753	7.649
		0.9	29.99	8.828	25.43	7.613	11.00	3.100	7.663	1.897	6.309	8.260	12.01
		0.95	58.24	16.54	42.81	12.49	20.91	5.737	8.515	3.239	6.674	19.15	18.88
		0.99	286.6	78.96	97.72	29.35	101.0	27.18	8.670	14.03	5.271	121.8	39.49
	100	0.7	5.829	2.120	5.664	2.069	2.525	0.899	3.476	0.743	3.182	1.163	3.392
		0.8	7.614	2.592	7.324	2.505	3.146	0.954	4.078	0.756	3.665	1.439	4.172
		0.9	13.16	4.135	12.28	3.882	5.127	1.352	5.518	0.914	4.770	2.702	6.422
		0.95	24.36	7.298	21.42	6.479	9.165	2.328	7.243	1.416	5.978	6.149	10.34
		0.99	113.6	32.62	67.59	20.08	41.65	10.49	8.143	5.702	6.040	43.62	28.96

Table 2 Estimated MSE of the estimators when $p=8$

σ	n	ρ	Estimators										
			OLS	Ridge	Liu	NTP	TP	PTP1	PTP2	PTP3	PTP4	PTP5	PTP6
5	50	0.7	6.414	2.000	5.958	1.876	2.156	1.636	2.910	1.464	2.440	1.834	3.606
		0.8	8.370	2.483	7.595	2.278	2.687	1.706	3.310	1.478	2.663	2.509	4.448
		0.9	14.36	4.033	12.18	3.474	4.386	2.092	4.140	2.612	2.702	4.983	6.792
		0.95	26.40	7.182	19.74	5.506	7.829	3.955	4.855	3.009	3.550	10.67	10.69
		0.99	122.3	32.38	43.70	12.42	35.38	8.988	4.450	4.316	3.211	62.21	26.48
	100	0.7	4.133	1.334	3.945	1.278	1.425	1.319	2.251	1.238	1.919	0.976	2.508
		0.8	6.017	1.824	5.621	1.711	1.951	1.400	2.817	1.254	2.319	1.557	3.417
		0.9	11.75	3.335	10.30	2.945	3.577	1.742	3.926	1.404	3.027	3.799	5.871
		0.95	105.6	28.12	38.97	10.71	30.71	6.013	5.007	4.034	3.678	41.54	22.12
		0.99	115.4	30.73	43.27	11.99	33.13	7.574	4.667	4.610	4.189	59.15	26.32
10	50	0.7	25.65	7.317	23.82	6.855	7.915	2.229	11.19	1.866	9.340	6.149	13.78
		0.8	33.48	9.297	30.37	8.524	10.09	2.750	12.78	2.572	10.39	8.907	17.13
		0.9	57.47	15.52	48.73	13.37	16.92	4.481	16.11	3.366	12.46	18.78	26.49
		0.95	130.2	37.37	95.34	22.12	55.67	6.565	16.78	5.576	13.34	46.78	63.91
		0.99	489.5	128.9	174.7	49.45	140.8	36.21	17.77	17.29	15.62	247.8	105.4
	100	0.7	16.53	4.740	15.77	4.539	5.065	2.199	8.616	1.733	7.284	3.032	9.451
		0.8	24.06	6.705	22.47	6.290	7.180	2.654	10.82	1.921	8.858	5.296	13.03
		0.9	47.02	12.75	41.21	11.25	13.68	3.148	11.23	2.615	11.70	14.10	22.79
		0.95	93.12	24.92	72.38	19.61	26.80	6.191	13.96	3.053	13.64	35.50	38.94
		0.99	461.8	122.3	173.3	47.84	132.2	30.55	15.62	15.48	14.72	235.4	104.7

Table 3 Parameters estimation and their MSE of the manufacturing data

Coefficients	Estimators										
	OLS	Ridge	Liu	NTP	TP	PTP1	PTP2	PTP3	PTP4	PTP5	PTP6
$\hat{\beta}_0$	208.9	178.6	191.9	164.1	178.7	148.4	162.81	129.6	147.3	198.3	186.2
$\hat{\beta}_1$	0.613	0.863	0.753	0.982	0.862	1.113	0.993	1.267	1.122	0.701	0.800
$\hat{\beta}_2$	1.256	1.157	1.201	1.110	1.157	1.058	1.106	0.997	1.055	1.221	1.182
$\hat{\beta}_3$	–1.221	–1.264	–1.245	–2.547	–1.263	–1.306	–1.286	–1.332	–2.543	–1.236	–1.253
MSE	1851	1354	1562	1143	1354	934.8	1125	714.1	920.9	1667	1471

6. Some Concluding Remarks

The ordinary least square estimator becomes inefficient in the presence of multicollinearity for the linear regression models (LRM). Alternative estimators such as the ridge regression, Liu estimator, the two-parameter estimator by Ozkale and Kaciranlar (2007) and the new two-parameter estimator by Yang and Chang (2010) have been developed to mitigate the problem of multicollinearity for the LRM. In this study, we developed a new estimator with different biasing parameters. We gave the short discussion on the estimation of the biasing parameters. We derived the statistical properties of the proposed estimator and compared it theoretically with the existing estimators. Furthermore, we conducted a simulation study and analyzed a real-life data to evaluate and compare the performance of the estimators. Based on the theoretical results, the simulation study and application, we conclude that the OLS estimator exhibited the least performance as expected. Moreover, the ORR estimator outperforms the Liu estimator. In addition to that, the PTP estimator with different biasing parameter estimators performs better than ORR, Liu, TP and NTP especially PTP3 in most cases. Therefore, we recommend practitioners to use in special PTP3 estimator for the multicollinear dataset.

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