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Transmuted Logistic-Exponential Distribution for Modelling Lifetime Data

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Abstract

Logistic-exponential (LE) distribution is one of the rare distributions in existence for modeling lifetime data due to its unique features. It is the only two-parameter distribution with quintuplet characteristics of hazard failure rates. However, its limitation is inability to model extremely skewed real life situation phenomena appropriately. This study proposed, developed and studied a new transmuted logistic-exponential distribution with three parameters with the aim of increasing the shape flexibility of LE distribution that will be more applicable to skewed lifetime data in various fields. We adopt the cumulative distribution function of LE and the quadratic rank transmutation map (QRTM) function in its development. Its quantile, survival, hazard functions, order statistics, skewness and kurtosis were derived. Its hazard function was found to have increasing, decreasing and constant failure rates while the survival function has a decreasing shape. Again, the estimates of the parameters were obtained using maximum likelihood estimation technique. Its efficiency was examined using real life dataset. The maximum likelihood estimates obtained were compared with the existing similar distributions using Akaike information criteria (AIC) and log-likelihood. The result showed that the newly transmuted LE distribution outperformed other models fitted to the dataset. The fitness of the distributions was also examined using two goodness of fit test. Both Kolmogorov-Smirnov and Anderson-Darling tests statistics confirmed that NTLE has a good fit. Hence, the new transmuted logistic-exponential (NTLE) distribution is more appropriate in modeling skewed lifetime datasets. In future research, we intend to study some other properties of this newly transmuted distribution and compare different estimation procedures for its parameters.

Keywords: Failure rate, skewed data, quadratic rank and transmutation map.

1. Introduction

Distributions have been numerously used in several areas of life such as health, finance, meteorology, insurance and many others. Many new classes of distributions had been introduced and studied recently by adding at least a parameter or extend the existing known families of distributions. Studies have shown that these new distributions are most times better in terms of flexibility and goodness of fit when compared with the existing families of distributions. Gupta et al. (1998) stated that there are different methods of extending or modifying existing distributions to obtain new ones with greater flexibility. Some of these methods include but not limited to differential equation technique pioneered by Pearson (1895), transformation method proposed by Johnson (1949), quantile function techniques by Hasting et. al. (1947), parameter induction method, exponentiation technique by Gompertz (1825), transform the transformer approach credited to Alzaatreh et al. (2013), transmutation method by Shaw and Buckley (2007).

A new generalization of the exponential distribution was proposed by Lan and Leemis (2008) which was named logistic-exponential (LE) distribution. The probability distribution function (pdf) and cumulative distribution function (cdf) of LE are given respectively as:

$$f(z)=\frac{\beta\phi(e^{\phi z}-1)^{\beta-1}e^{\phi z}}{[1+(e^{\phi z}-1)^{\beta}]^{2}}, \tag{1}$$

and

$$F(z)=1-[1+(e^{\phi z}-1)^{\beta}]^{-1}; \quad z>0,\phi>0,\beta>0. \tag{2}$$

These two researchers stated that this LE distribution is the only distribution with two parameters that exhibits quintuplet failure rate shapes. (constant, decreasing, increasing, bathtub and upside–down bathtub). However, as interesting as the properties of LE distribution is, its moments cannot be expressed in a close form but can be computed numerically.

According to Shaw and Buckley (2007), it has been shown that the QRTM distributions obtained from the base distributions performed better than the parent distributions when fitted to datasets, because they tend to have more parameters and they are more flexible in shape. Suppose that Z is a real-valued random variable and $\eta(z)$ is the cumulative distribution of the parent or base distribution the transmutation function is given as:

$$G(z)=\eta(z)+\delta\eta(z)[1-\eta(z)], \tag{3}$$

where $|\delta|\le 1,$ $G(z)$ is the transmuted cdf and δ is the transmuting parameter of the base distribution $\eta(z)$.

Researchers have introduced several generalizations of distributions that are more flexible to investigate the properties of models and their fitness. Khan et al. (2016) proposed the transmuted generalized exponential distribution using the QRTM method. In 2013, Merovci derived the transmuted exponentiated exponential distribution for modelling lifetime phenomenon. Samuel (2019) developed the transmuted logistic distribution and studied its properties. Mansoor et al. (2018) also introduced a three-parameter LE distribution called Marshall-Olkin logistic-exponential distribution (MOLE) using Marshall-Olkin approach but logistic-exponential distribution is yet to be transmuted. Owoloko et al. (2015) obtained the pdf and the cdf of transmuted exponential distributions respectively as

$$f(x)=\frac{1}{\theta}e^{-\left(\frac{x}{\theta}\right)}\left[1-\lambda+2\lambda e^{-\left(\frac{x}{\theta}\right)}\right] \text{ and } F(x)=\left[1-e^{-\left(\frac{x}{\theta}\right)}\right]\left[1+\lambda e^{-\left(\frac{x}{\theta}\right)}\right],$$

for $x>0,\ \theta>0,\ |\lambda\le 1|$, where θ is the scale parameter and λ is the transmuted parameter. Also, the cdf and pdf of transmuted logistic distribution by Samuel (2019) are given as $F(x)=\frac{1+(1+\lambda)e^{-x}}{(1+e^{-x})^2}$ and $f(x)=\frac{e^{-x}\{(1+\lambda)(1+e^{-x})-2\lambda}{(1+e^{-x})^3}$, respectively. Mansoor et al (2018) also gave the pdf and cdf of Marshall-Olkin logistic exponential (MOLE) distribution respectively as

$$f(x)=\frac{\beta\theta\lambda e^{\lambda x}(e^{\lambda x}-1)^{-\beta-1}}{[1+\theta(e^{\lambda x}-1)^{-\beta}]^2} \text{ and } F(x)=[1+\theta(e^{\lambda x}-1)^{-\beta}]^{-1},\ x\ge 0$$

where $\beta>0,\theta>0,\lambda>0$. Recently, Tabassum et al. (2021) proposed and studied the transmuted Burr type X model with application to life time data.

In this research work, a new probability distribution called the transmuted logistic-exponential with three parameters is proposed, developed and studied with the aim of increasing the shape flexibility of LE distribution which will be more applicable to lifetime datasets. We also derived its properties, obtained the estimates of the parameters and compare the new transmuted logistic-exponential (TLE) distribution with the existing similar distributions (logistic-exponential (LE), transmuted exponential (TE) and exponentiated exponential (EE)). This new transmuted LE distribution was found to be more flexible than LE. It also has the best fit when compared with the existing similar distributions. Its hazard failure rate has an increasing, decreasing and constant shapes and survival function tends to decrease with time.

The next section discusses the methodology for the NTLE derivation. The properties and application of the new transmuted distribution are discussed in Sections 3 and 4 respectively. Section 5 gave the concluding part while the codes for the maximum likelihood estimation of the model parameters were given in the appendix.

2. Methodology

2.1. Derivation of the cumulative distribution function of the new transmuted logistic-exponential (NTLE) distributions

This new continuous distribution was developed by transmuting the logistic-exponential distribution using its cumulative distribution function derived by Lan and Leemis (2008) in (2) above and plugging it into the transmutation function developed by Shaw and Buckley (2007) in Equation (3). Then,

$$G(z)=\eta(z)+\delta\eta(z)\left[1-\eta(z)\right]$$

$$=\left[1-\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-1}\right]+\delta\left[1-\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-1}\right]\left[1-\left[1-\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-1}\right]\right]$$

$$=\left[1-\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-1}\right]+\delta\left[1-\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-1}\right]\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-1}$$

$$=\left[1-\frac{1}{1+\left(e^{\phi z}-1\right)^{\beta}}\right]+\delta\left[1-\frac{1}{1+\left(e^{\phi z}-1\right)^{\beta}}\right]\left[\frac{1}{1+\left(e^{\phi z}-1\right)^{\beta}}\right]$$

$$=\left[1-\frac{1}{1+\left(e^{\phi z}-1\right)^{\beta}}\right]+\left[\frac{\delta}{1+\left(e^{\phi z}-1\right)^{\beta}}-\frac{\delta}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}\right]$$

$$=\left[\frac{1+\left(e^{\phi z}-1\right)^{\beta}-1}{1+\left(e^{\phi z}-1\right)^{\beta}}\right]+\left[\frac{\delta\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)-\delta}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{2}}\right]=\left[\frac{\left(e^{\phi z}-1\right)^{\beta}}{1+\left(e^{\phi z}-1\right)^{\beta}}\right]+\left[\frac{\delta\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)-\delta}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{2}}\right]$$

$$=\frac{\left(e^{\phi z}-1\right)^{\beta}\left(1+\left(e^{\phi z}-1\right)^{\beta}+\delta\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}},$$

$$G(z)=\frac{\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}. \tag{4}$$

The CDF of NTLE is given by (4) above.

2.2. Derivation of probability density function of the new transmuted logistic-exponential (NTLE) distributions

To obtain the probability distribution function, we differentiate $G(z)$ with respect to z,

$$g(z)=\frac{\delta\left[G(z)\right]}{\delta z},\ \ G(z)=\frac{\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}$$

Let $u=\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)$ and $v=\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}$. Differentiating u and v with respect to x, we have

$$\frac{\delta u}{\delta z}=\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(1+\delta+2\left(e^{\phi z}-1\right)^{\beta}\right)$$

and

$$\frac{\delta v}{\delta z}=\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(2+2\left(e^{\phi z}-1\right)^{\beta}\right)$$

$$g(z)=\frac{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(1+\delta+2\left(e^{\phi z}-1\right)^{\beta}\right)-\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(2+2\left(e^{\phi z}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{4}}.$$

For ease of simplification, let $\psi=\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}$ and $\xi=\left(e^{\phi z}-1\right)^{\beta}$,

$$g(z)=\frac{\left(1+\xi\right)^{2}\psi\left(1+\delta+2\xi\right)-\xi\left(1+\delta+\xi\right)\psi\left(2+2\xi\right)}{\left(1+\xi\right)^{4}}$$

$$=\frac{\left(1+\xi\right)^{2}\psi\left(1+\delta+2\xi\right)-2\xi\left(1+\delta+\xi\right)\psi\left(1+\xi\right)}{\left(1+\xi\right)^{4}}$$

$$=\frac{\psi\left(1+\xi\right)\left[\left(1+\xi\right)\left(1+\delta+2\xi\right)-2\xi\left(1+\delta+\xi\right)\right]}{\left(1+\xi\right)\left(1+\xi\right)^{3}}=\frac{\psi\left[\left(1+\xi\right)\left(1+\delta+2\xi\right)-2m\left(1+\delta+\xi\right)\right]}{\left(1+\xi\right)^{3}}$$

$$=\frac{\psi\left[1+\delta+2\xi+\xi+\xi\delta+2\xi^{2}-2\xi-2\xi\delta-2\xi^{2}\right]}{\left(1+\xi\right)^{3}}=\frac{\psi\left[1+\delta+\xi+\xi\delta-2\xi\delta\right]}{\left(1+\xi\right)^{3}}$$

$$=\frac{\psi\left[1+\delta+\xi-\xi\delta\right]}{\left(1+\xi\right)^{3}}$$

$$=\frac{\psi\left[1+\xi+\delta\left(1-\xi\right)\right]}{\left(1+\xi\right)^{3}}.$$

Substituting, $\psi=\beta\phi e^{\phi y}\left(e^{\phi z}-1\right)^{\beta-1}$ and $\xi=\left(e^{\phi z}-1\right)^{\beta}$ in $g(z)$ we have

$$g(z)=\frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}. \tag{5}$$

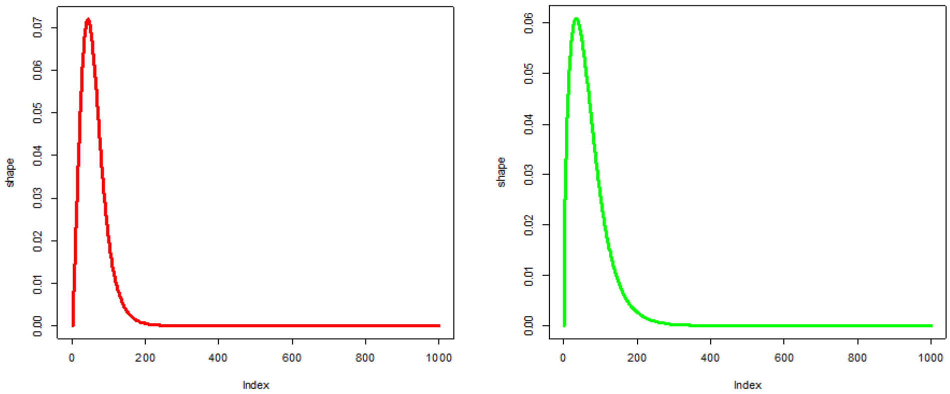
The Equation (5) above can also be expressed as

$$g(z)=\frac{\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{2}}+\frac{\delta\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}, \tag{6}$$

where $|\delta|\le 1, \phi\ge 0, \beta\ge 0, z\ge 0.$ It is worthy to note that, if $\delta=0, \phi\ge 0, \beta=1,$ the distribution $g(z)=\phi e^{-\phi z}$ which is the primary distribution from which both the logistic-exponential (LE) and transmuted logistic-exponential (TLE) distributions were obtained.

Also, if $\delta = 0, \phi \geq 0, \beta \geq 0,$ the function $g(z) = \dfrac{\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{2}}$ which is the LE distribution from which NTLE was obtained.

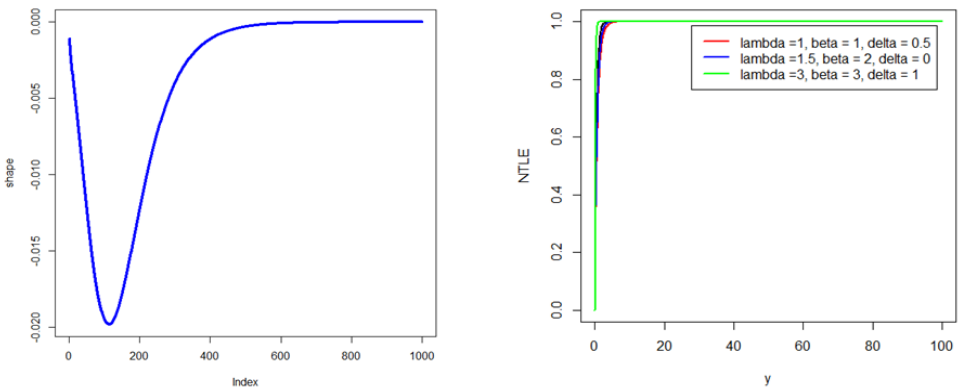
It is to be noted that the NTLE has three parameters–two shape parameters and one scale parameter. The plot of the pdf and cdf at different values of the parameters are shown in Figures 1 and 2.



(a) (b)

Figure 1 Pdf of NTLE distribution

Figure 1 (a) and (b) shows the pdf of NTLE when the parameters are (beta = 2.0, lambda = 0.1, delta = 0.2) and (beta = 1.5, lambda = 0.07, delta = 0.6), respectively.



(a) (b)

Figure 2 Pdf (beta = -1.5, lambda = 0.05, delta = 0.8) and CDF of NTLE distribution, respectively

3. Properties of New Transmuted Logistic-Exponential (NTLE) Distribution

In this section, we discussed some properties of the new transmuted logistic-exponential (NTLE) distribution.

3.1. Survival function of NTLE distribution

The survival function of a distribution is usually used to determine the probability that an object will function or a patient will live beyond a particular time. Let $S(z)$ be the survival function given by $S(z)=1-G(z).$ Then,

$$S(z)=1-G(z)$$

$$=1-\frac{(e^{\phi z}-1)^{\beta}[1+\delta+(e^{\phi z}-1)^{\beta}]}{[1+(e^{\phi z}-1)^{\beta}]^2} = \frac{(1+(e^{\phi z}-1)^{\beta})^2-(e^{\phi z}-1)^{\beta}[1+\delta+(e^{\phi z}-1)^{\beta}]}{[1+(e^{\phi z}-1)^{\beta}]^2}$$

$$=\frac{1+2(e^{\phi z}-1)^{\beta}+(e^{\phi z}-1)^{2\beta}-(e^{\phi z}-1)^{\beta}-\delta(e^{\phi z}-1)^{\beta}-(e^{\phi z}-1)^{2\beta}}{[1+(e^{\phi z}-1)^{\beta}]^2}$$

$$=\frac{1+(e^{\phi z}-1)^{\beta}-\delta(e^{\phi z}-1)^{\beta}}{[1+(e^{\phi z}-1)^{\beta}]^2},$$

$$S(z)=\frac{1+\left(e^{\phi z}-1\right)^{\beta}(1-\delta)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^2}. \tag{7}$$

The survival function of the new transmuted logistic-exponential (NTLE) distribution is given in Equation (7) above. The plot of the survival function shows that the survival rate decreases as the time increases (see Figure 3 below).

3.2. Hazard function of NTLE distribution

One major property of a distribution is the hazard function behavior. The hazard function is also known as failure rate. The term used for this function depends on the area of use. For instance, it is called age-age specific death rate in vital statistics and life sciences. It is the probability that an object will fail or a person will die at a particular time given that the event has not occurred previously. Hazard function is defined given as $h(z)=\frac{g(z)}{1-G(z)}=\frac{g(z)}{S(z)}.$

Hence,

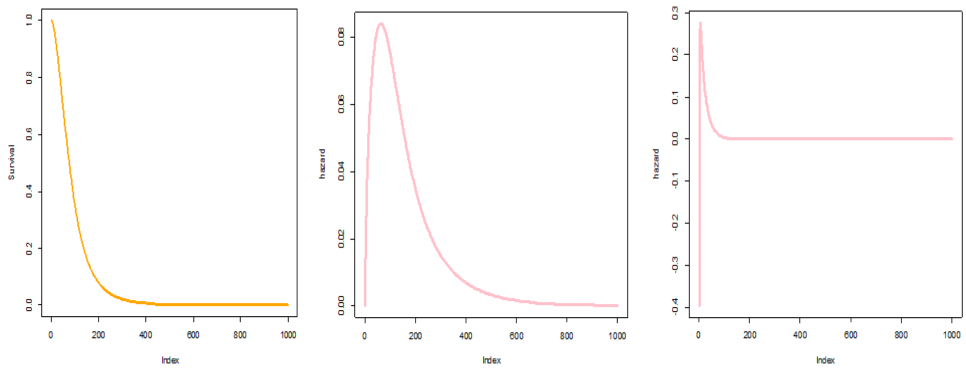
$$h(z)=\frac{\dfrac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^3}}{\dfrac{1+\left(e^{\phi z}-1\right)^{\beta}(1-\delta)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^2}}$$

$$=\frac{\left(\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]\right)\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}\left(1+\left(e^{\phi z}-1\right)^{\beta}(1-\delta)\right)}$$

$$=\frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)\left(1+\left(e^{\phi z}-1\right)^{\beta}(1-\delta)\right)},$$

$$h(z)=\frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{1+2\left(e^{\phi z}-1\right)^{\beta}-\delta\left(e^{\phi z}-1\right)^{\beta}+\left(e^{\phi z}-1\right)^{2\beta}(1-\delta)}. \tag{8}$$

The hazard function of the transmuted logistic-exponential (NTLE) distribution is given in Equation (8) above. The hazard function has three major shapes. It can be increasing, decreasing or constant as shown in Figures 3(a) and 3(b) below.



(a) (b) (c)

Figure 3 Survival and hazard functions of NTLE distribution

The plot of the survival function is given in Figure 3(a) above. Also, Figures 3(b) and 3(c) shows the plots of the hazard function for different values of the parameters and when the parameters are equal, respectively. The hazard function has three major shapes. It can be increasing, decreasing or constant as shown in Figures 3(b) and 3(c).

3.3. Quantile function and random number generation

Given that $G(z)$ is a cdf of a random variable Z, a value of z such that $G^{-1}(z)=P(Z<z)\leq p$ is called a quantile of order p for the distribution and where $G(z)=P(Z\leq z)\geq p.$ The quantile function is defined as

$$G^{-1}(z)=Q(p)=\inf\{z:G(z)\geq p\},\ \forall p\in(0,1)$$

$$Q(z_p) = p, \quad 0 < p < 1.$$

3.3.1. Derivation of quantile function of the new transmuted logistic-exponential distribution (NTLE)

To find the quantile function, we equate the cdf of NTLE to a point p and solve for z_p, where p is any quantile. We have

$$\frac{\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}} = p.$$

Let $u=\left(e^{\phi z}-1\right)^{\beta}$. Therefore,

$$\frac{u\left(1+\delta+u\right)}{\left(1+u\right)^{2}} = p$$

$$u+\delta u+u^{2} = p(1+u)(1+u)$$

$$u+\delta u+u^{2} = p+2pu+pu^{2}$$

$$u+\delta u+u^{2} = p+2pu+pu^{2}$$

$$u^{2}-pu^{2}+u+\delta u-2pu-p=0$$

$$(1-p)u^{2}+u(1+\delta-2p)-p=0.$$

Solving for u quadratically, where

$$a=(1-p), \quad b=(1+\delta-2p) \text{ and } c=-p.$$

We obtain

$$u=\frac{\left(2p-\delta-1\right)\pm\sqrt{\delta^{2}-2\delta(2p-1)+1}}{2(1-p)}.$$

Now,

$$(e^{\phi z_q}-1)^{\beta} = \frac{\left(2p-\delta-1\right)\pm\sqrt{\delta^{2}-2\delta(2p-1)+1}}{2(1-p)}$$

$$(e^{\phi z_q}-1) = \left[\frac{\left(2p-\delta-1\right)\pm\sqrt{\delta^{2}-2\delta(2p-1)+1}}{2(1-p)}\right]^{1/\beta}$$

$$e^{\phi z_q} = 1+\left[\frac{\left(2p-\delta-1\right)\pm\sqrt{\delta^{2}-2\delta(2p-1)+1}}{2(1-p)}\right]^{1/\beta}$$

$$\phi z_q = \ln\left[1+\left(\frac{\left(2p-\delta-1\right)\pm\sqrt{\delta^{2}-2\delta(2p-1)+1}}{2(1-p)}\right)^{1/\beta}\right]$$

$$z_p = \phi^{-1}\ln\left[1+\left(\frac{(2p-\delta-1)\pm\sqrt{\delta^2-2\delta(2p-1)+1}}{2(1-p)}\right)^{1/\beta}\right]. \tag{9}$$

Equation (9) above is the quantile function of the transmuted logistic-exponential distribution where ϕ, δ and β are to be estimated.

3.3.2. Median

To find median, we put $p=\frac{1}{2}$,

$$z_{me} = \phi^{-1}\ln\left[1+\left(\frac{\left(2\left(\frac{1}{2}\right)-\delta-1\right)\pm\sqrt{\delta^2-2\delta(2\left(\frac{1}{2}\right)-1)+1}}{2\left(1-\left(\frac{1}{2}\right)\right)}\right)^{1/\beta}\right],$$

$$z_{me} = \phi^{-1}\ln\left[1+\left(-\delta\pm\sqrt{\delta^2+1}\right)^{1/\beta}\right]. \tag{10}$$

Equation (10) above is the median of NTLE. The median tends to decrease as ϕ and β increases and increases as ϕ and β decreases. Other quantiles can be obtained using Equation (9) above.

3.3.3. Random number generation

If $P\sim U(0,1)$, and $\phi\neq 0$, then we can compute a random number,

$$z_p = \phi^{-1}\ln\left[1+\left(\frac{(2p-\delta-1)\pm\sqrt{\delta^2-2\delta(2p-1)+1}}{2(1-p)}\right)^{1/\beta}\right].$$

We will repeat this procedure until the required amount of random numbers to be completed are obtained.

3.4. Order statistics

Generally, Let $Z_1, Z_2, ..., Z_n$ be independently and identically distributed random variable with pdf $g(z)$ and cdf $G(z)$ the density of the i^{th} order statistics which is given as

$$f_{i:n} = \frac{n!}{(n-1)!(i-1)!}g(z)\left[G(z)\right]^{i-1}\left[1-G(z)\right]^{n-i}.$$

3.4.1. Derivation of the smallest and largest order statistics of the new transmuted logistic-exponential distribution

The smallest order statistics is given by $f_{i:n} = n\,g(z)\left[1-G(z)\right]^{n-i}$. Therefore, the smallest order statistics for the new transmuted logistic-distribution is given as

$$f_{1:n} = n\frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}\left[1-\frac{\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}\right]^{n-1}$$

$$f_{1:n} = n\frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}\left[\frac{1+\left(e^{\phi z}-1\right)^{\beta}(1-\delta)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}\right]^{n-1}$$

$$f_{1:n} = n\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]\left[1+\left(e^{\phi z}-1\right)^{\beta}(1-\delta)\right]^{n-1}$$
$$\times\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-(2n+1)} \tag{11}$$

Equation (11) is the smallest or minimum order statistics of NTLE. Again, the largest order statistics is given by $f_{n:n} = n\, g(z)\left[G(z)\right]^{n-1}$. Therefore, the largest order statistics for the new transmuted logistic-distribution is given as

$$f_{1:n} = n\left[\frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}\right]\left[\frac{\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)}{\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{2}}\right]^{n-1}$$

$$f_{1:n} = n\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]\left[\left(e^{\phi z}-1\right)^{\beta}\left(1+\delta+\left(e^{\phi z}-1\right)^{\beta}\right)\right]^{n-1}$$
$$\times\left[1+\left(e^{\phi z}-1\right)^{\beta}\right]^{-(2n+1)}. \tag{12}$$

Equation (12) is the largest or maximum order statistics of NTLE.

3.5. Moments

The r^{th} moment of a random variable Z is given as $E(Z^r) = \int z^r f(z)dz$. Hence, the r^{th} moment of the new transmuted logistic exponential distribution is given as

$$E(Z^r) = \int z^r \frac{\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{2}}dz + \int z^r \frac{\delta\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}dz,$$

$$E(Y^r) = \text{moments of logistic exponential distribution } + \int z^r \frac{\delta\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^3}.$$

Lan and Leemis (2008) stated that the moments of logistic exponential distribution is finite but cannot be expressed in close form. We also, conclude here that the moments of NTLE cannot also be expressed in closed form since it is a combination of moments of LE distribution and $\int z^r \frac{\delta\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^3}$.

Now, we will investigate the skewness and kurtosis of NTLE distribution using the relation between moments and the quantile function. Kenny and Keeping (1962) gave the Bowley skewness based on quantiles given which is given as

$$B = \frac{G^{-1}(3/4) + G^{-1}(1/4) - 2G^{-1}(2/4)}{G^{-1}(3/4) - G^{-1}(1/4)}. \tag{13}$$

And also, Moors (1988) gave Moor kurtosis based on quantiles as

$$M = \frac{G^{-1}(3/8) - G^{-1}(1/8) + G^{-1}(7/8) - G^{-1}(5/8)}{G^{-1}(6/8) - G^{-1}(2/8)}, \tag{14}$$

where G^{-1} is the quantile function. Therefore, the skewness of NTLE distribution is given as

$$B = \frac{\phi^{-1}\ln\left\{1+\left(1-2\delta\pm2\sqrt{\delta^2-\delta+1}\right)^{1/\beta}\right\} + \phi^{-1}\ln\left[1+\left(\frac{2\left(-\left(\delta+\frac{1}{2}\right)\pm\sqrt{\delta^2+\delta+1}\right)}{3}\right)^{1/\beta}\right] - 2\phi^{-1}\ln\left[1+\left(-\delta\pm\sqrt{\delta^2+1}\right)^{1/\beta}\right]}{\phi^{-1}\ln\left\{1+\left(1-2\delta\pm2\sqrt{\delta^2-\delta+1}\right)^{1/\beta}\right\} - \phi^{-1}\ln\left[1+\left(\frac{2\left(-\left(\delta+\frac{1}{2}\right)\pm\sqrt{\delta^2+\delta+1}\right)}{3}\right)^{1/\beta}\right]} \tag{15}$$

Also, the Moore's kurtosis of NTLE distribution is given as

$$M = \frac{\phi^{-1}\ln\left[1+\left(\frac{4\left(-\left(\delta+\frac{1}{4}\right)\pm\sqrt{\delta^2+\frac{\delta}{2}+1}\right)}{5}\right)^{1/\beta}\right] - \phi^{-1}\ln\left[1+\left(\frac{4\left(-\left(\delta+\frac{3}{4}\right)\pm\sqrt{\delta^2+3\delta+1}\right)}{7}\right)^{1/\beta}\right] + \phi^{-1}\ln\left[1+\left(1.5-2\delta\pm3\sqrt{\delta^2-\frac{3}{2}\delta+1}\right)^{1/\beta}\right] - \phi^{-1}\ln\left[1+\left(\frac{\left(1-4\delta\pm4\sqrt{\delta^2-\frac{\delta}{2}+1}\right)}{3}\right)^{1/\beta}\right]}{\phi^{-1}\ln\left\{1+\left(1-2\delta\pm2\sqrt{\delta^2-\delta+1}\right)^{1/\beta}\right\} - \phi^{-1}\ln\left[1+\left(\frac{2\left(-\left(\delta+\frac{1}{2}\right)\pm\sqrt{\delta^2+\delta+1}\right)}{3}\right)^{1/\beta}\right]} \tag{16}$$

where ϕ, β and δ are parameters of the distribution that needs to be estimated**.**

3.6. Maximum likelihood estimation (MLE)

There are different methods of estimating the parameters of a model. Here, we adopt the maximum likelihood estimation technique. In MLE technique, the parameters are selected to maximize the likelihood that the assumed model results in the observed data. Its estimates are unbiased but it relied on the assumptions of the model and may be sensitive to the choice of the initial values. Given a random samples $Z_1, Z_2, ..., Z_n$ with probability density function of each Z_i as $f(z_i;\theta_i)$, then the joint probability density function of $X_1, X_2, ..., X_n$ is given as

$$L(\theta) = f(z_1;\theta) f(z_2;\theta) \cdots f(z_n;\theta) = \prod_{i=1}^{n} f(z_1;\theta).$$

3.6.1. Derivation of the maximum likelihood estimates of the parameters of the new transmuted logistic-exponential distribution (NTLE)

The probability density function (pdf) of the new transmuted logistic-exponential distribution is given as

$$g(z) = \frac{\left(\beta\phi e^{\phi z}\left(e^{\phi z}-1\right)^{\beta-1}\right)\left[\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z}-1\right)^{\beta}\right)\right]}{\left(1+\left(e^{\phi z}-1\right)^{\beta}\right)^{3}}.$$

The joint pdf is given as

$$L(g(z)) = \frac{(\beta\phi)^n e^{\phi\sum_{i=1}^{n} z_i} \prod_{i=1}^{n}\left(e^{\phi z_i}-1\right)^{\beta-1}\left[\prod_{i=1}^{n}\left[\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z_i}-1\right)^{\beta}\right)\right]\right]}{\prod_{i=1}^{n}\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right)^{3}}.$$

Taking the log of the likelihood, we have

$$\log L(g(z)) = n\log(\beta\phi) + \phi\sum_{i=1}^{n} z_i + (\beta-1)\sum_{i=1}^{n}\log\left(e^{\phi z_i}-1\right) + \sum_{i=1}^{n}\log\left(\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z_i}-1\right)^{\beta}\right)\right) - 3\sum_{i=1}^{n}\log\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right) \quad (17)$$

Now, to obtain the estimates of the parameters, we differentiate the log of the likelihood functions in Equation (17) with respect to each of the parameters. To obtain the estimate for ϕ, we differentiate in Equation (17) with respect to ϕ and we have

$$\frac{\partial \log L(g(z))}{\phi} = \frac{n}{\phi} + \sum_{i=1}^{n} z_i + (\beta-1)\sum_{i=1}^{n}\frac{z_i e^{\phi z_i}}{e^{\phi z_i}-1} + \frac{\sum_{i=1}^{n}(1-\delta)\beta\phi\left(e^{\phi z_i}-1\right)^{\beta-1} e^{\phi z_i}}{\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z_i}-1\right)^{\beta}\right)} - \frac{3\sum_{i=1}^{n}\beta\left(e^{\phi z_i}-1\right)^{\beta-1}\phi e^{\phi z_i}}{1+\left(e^{\phi z_i}-1\right)^{\beta}} \quad (18)$$

To obtain the estimate for $\beta,$ we differentiate in Equation (17) with respect to β and we have

$$\frac{\partial \log L(g(z))}{\beta}=\frac{n}{\beta}+\sum_{i=1}^{n}\log\left(e^{\phi z_i}-1\right)+(1-\delta)\frac{\sum_{i=1}^{n}\left(e^{\phi z_i}-1\right)^{\beta}\log\left(e^{\phi z_i}-1\right)}{\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z_i}-1\right)^{\beta}\right)}-3\frac{\sum_{i=1}^{n}\left(e^{\phi z_i}-1\right)^{\beta}\log\left(e^{\phi z_i}-1\right)}{1+\left(e^{\phi z_i}-1\right)^{\beta}} \tag{19}$$

To obtain the estimate for $\delta,$ we differentiate Equation (17) with respect to β and we have

$$\frac{\partial \log L(g(z))}{\delta}=\frac{\sum_{i=1}^{n}\left(1-\left(e^{\phi z_i}-1\right)^{\beta}\right)}{\left(1+\left(e^{\phi z_i}-1\right)^{\beta}\right)+\delta\left(1-\left(e^{\phi z_i}-1\right)^{\beta}\right)}. \tag{20}$$

Setting in Equations (18), (19) and (20) to zeros and solving them simultaneously, we obtain the estimates of the model parameters. It is to be noted that the estimates of these parameters can be solved numerically. (See Appendix for the codes)

4. Application

In this section, we fitted the new transmuted logistic-exponential distribution to a data set and compare its fitness with existing similar distributions like transmuted exponential (TE), logistic-exponential (LE) and exponentiated exponential (EE) distributions using Akaike information criteria and log-likelihood. We also gave a summary statistics of the data, plotted the density and the QQ-norm plots to examine the skewness in the data set. The data is on the remission times (in months) of a random sample of 128 bladder cancer patients adapted from Lee and Wang (2003). The summary of statistics of the data is given in Table 1 below. Table 2 shows the estimates of the parameters and their corresponding standard errors. Table 3 shows the AICs and the log-likelihood of the models.

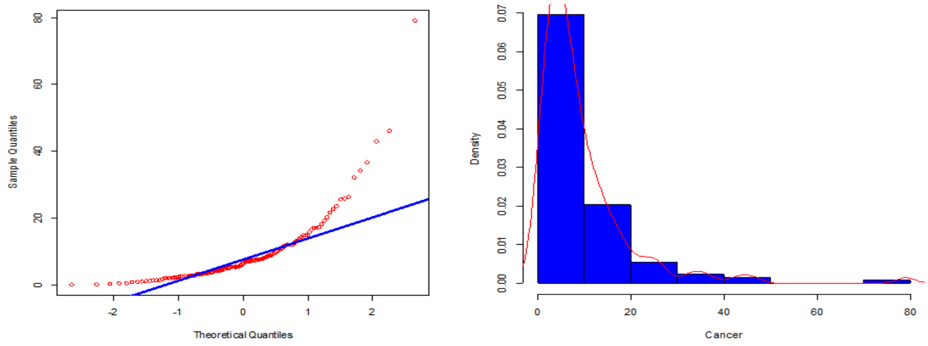
Table 1 Summary of data on cancer patients

Min	Max	Median	Mean	Variance	Std. Dev.	Skewness	Kurtosis
0.08	79.050	6.395	9.366	110.425	10.508	3.287	18.483

The histogram with density and QQ-norm plots of the cancer patients are given in Figures 4(a) and 4(b), respectively.

5. Discussion

It was observed form both the density and QQ-norm plots that the data is skewed in nature. The low standard errors of the estimates of the parameters of the models showed that the estimates are precise and reliable. NTLE distribution has the least AIC and the highest log-likelihood and thereby fits the data better than the similar distributions compared with. Both Kolmogorov-Smirnov and Anderson-Darling tests statistics also confirm the goodness of NTLE distribution. The results validate the flexibility of NTLE distribution in modeling skewed data than the existing similar distributions compared with



(a) (b)

Figure 4 QQ-norm plot and histogram of the cancer patients

Table 2 MLEs and their standard errors (in parentheses) for the cancer patients' data

Models	Estimates and their standard errors		
Transmuted exponential (TE) distribution	$\hat{\theta} = 16.3476$ (4.3768)	$\hat{\lambda} = 0.8617$ (0.2544)	-
Logistic-exponential (LE) distribution	$\hat{\alpha} = 1.1634$ (0.0916)	$\hat{\lambda} = 0.1007$ (0.0085)	-
Exponentiated exponential (EE) distribution	$\hat{\alpha} = 1.2180$ (0.1489)	$\hat{\lambda} = 0.1212$ (0.0136)	-
New transmuted logistic-exponential (NTLE) distribution	$\hat{\beta} = 1.2021$ (0.09090)	$\hat{\phi} = 0.07121$ (0.01216)	$\hat{\delta} = 0.6507$ 0.2544)

Table 3 AICs and Log-likelihoods of the compared models

Models	AICs	Log-likelihoods
Transmuted exponential (TE) distribution	830.9945	–413.4972
Logistic-exponential (LE) distribution	829.2507	–412.6254
Exponentiated exponential (EE) distribution	830.1552	–413.0776
New transmuted logistic-exponential (NTLE) distribution	827.6799	–410.8400

Table 4 Kolmogorov-Smirnov and Anderson-Darling test statistics of the fitted distributions

Models	Kolmogorov-Smirnov test statistics	Anderson-Darling test statistics
Transmuted exponential (TE) distribution	0.9922	44.852
Logistic-exponential (LE) distribution	0.9453	37.652
Exponentiated exponential (EE) distribution	0.9131	33.318
New transmuted logistic-exponential (NTLE) distribution	0.6357	20.352

6. Conclusion

Transmuted logistic-exponential (NTLE) distribution is a new distribution which was obtained using cumulative distribution function (cdf) of logistic-exponential distribution and the quadratic rank transmutation map function. The essence of adding a parameter to a distribution is to improve its flexibility to give room for wider application. Despite the unique feature of logistic-exponential distribution, this distribution provides more shape flexibility and will be more appropriate in modeling real life data sets that are skewed in nature than the existing logistic-exponential (LE) and other similar distributions compared with. It also gives relationship with the other probability models in earlier researches, hence the NTLE will always be found useful in distribution and open room for more researches. It can also be modified or compounded to obtain new distributions.

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Appendix

Here, we present the codes for the estimation of the parameters of the new transmuted distribution and other compared models using their likelihood functions obtained through MLE and the maxlik function in R choosing suitable starting conditions.

```
#Parameter Estimation
> library(maxLik)
> n=length(Cancer)                    #where the data is stored in as Cancer
> TE<-function(p) {n*log(1/p[1])-sum(Cancer/p[1])+sum(log(1-p[2]+2*p[2]*exp(-
(Cancer/p[1]))))}
> Estimates1<-maxLik(TE, start=c(20,1))
> summary(Estimates1)
> AIC(Estimates1)

> NTLE<-function(p){n*log(p[1]*p[2]) +p[2]*sum(Cancer)+(p[1]-
1)*sum(log(exp(p[2]*Cancer)-1))+sum(log((1+ (exp(p[2]*Cancer)-1)^p[1])+p[3]*(1-
(exp(p[2]*Cancer)-1)^p[1])))-3*sum(log(1+ (exp(p[2]*Cancer)-1)^p[1]))}
> Estimates2<-maxLik(NTLE, start=c(2,1,0))
> summary(Estimates2)
> AIC(Estimates2)

> EE<-function(p) {n*log(p[1])+n*log(p[2])+(p[1]-1)*sum(log(1-(exp(-p[2]*Cancer))))-
p[2]*sum(Cancer)}
> Estimates3<-maxLik(EE, start=c(20,1))
> summary(Estimates3)
> AIC(Estimates3)
```

```
> LE<- function(p){n*log(p[1]*p[2])+p[2]*sum(Cancer)+(p[1]-
1)*sum(log(exp(p[2]*Cancer)-1))-2*sum(log(1+(exp(p[2]*Cancer)-1)^p[1]))}
> Estimates4<-maxLik(LE, start=c(2,1))
> summary(Estimates4)
> AIC(Estimates4)
```