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A New Family of Generalized Distributions with an Application to Weibull Distribution

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Abstract

A new method has been introduced to add an extra parameter to a family of distributions to get more flexibility in the new model. A special case namely; two parameter Weibull distribution has been considered. The proposed distribution has a desirable property to model monotone and non monotone hazard rate functions, which are very common in reliability theory. Various properties of the proposed distribution are derived including moments, quantiles, entropy, moment generating function, mean residual life time and stress-strength reliability. A simulation study has been carried out to describe the performance of the model. Two data sets have been analyzed to illustrate how the proposed model works in practice.

Keywords: Weibull distribution, hazard rate function, p-p plot, mean residual life, maximum likelihood estimation

1. Introduction

Statistical distributions are very important in describing and predicting real world phenomena. Although researchers have developed many distributions but still there is scope for developing distributions which are either more flexible or for fitting specific real world scenarios. Due to the importance of statistical distributions, their theory is widely studied and new distributions are developed. This has motivated researchers to develop new and more flexible distributions. As a result, many new distributions have been developed and studied.

Adding an extra parameter to an existing family of distribution functions, is very common in the statistical distribution theory. Often introducing an extra parameter brings more flexibility to a class of distribution functions, and it can be very useful for data analysis purposes. For example, Marshall and Olkin (1997) proposed a general method for generating a new family of life time distributions defined in terms of survival function as

$$\bar{G}(x;\alpha)=\frac{\alpha\bar{F}(x)}{1-\bar{\alpha}\bar{F}(x)}=\frac{\alpha\bar{F}(x)}{F(x)+\alpha\bar{F}(x)} \quad ;x\in\mathbb{R},\quad \alpha\in\mathbb{R}^{+},$$

where $\bar{\alpha}=1-\alpha$ and $\bar{F}(x)=1-F(x)$ is the survival function of the random variable X.

The corresponding cumulative distribution function (cdf) and is given by

$$G(x)=\frac{F(x)}{1-\bar{\alpha}\bar{F}(x)}.$$

Eugene et al. (2002) proposed the beta generated method that uses the beta distribution with parameters α and β as the generator to develop the beta generated distributions. The cdf of a beta-generated random variable X is defined as

$$G(x) = \int_0^{F(x)} b(t)dt,$$

where $b(t)$ is the probability density function (pdf) of a beta random variable and $F(x)$ is the cdf of any random variable X. Alzaatreh et al. (2013) introduced a new method for generating families of continuous distributions called T-X family by replacing the beta pdf with a pdf, $r(t)$, of a continuous random variable and applying a function $W(F(x))$ that satisfies some specific conditions.

Recently, Mahdavi and Kundu (2017) proposed a new method to introduce an extra parameter to a family of distributions for more flexibility. The proposed method is called Alpha Power Transformation (APT) and it is useful to consolidate skewness to a family of distributions. Let $F(x)$ be the cdf of a continuous random variable X, then they define the APT of $F(x)$ for $x \in \mathbb{R}$ as follows:

$$F_{APT}(x) = \begin{cases} \frac{\alpha^{F(x)}-1}{\alpha-1} & ;\alpha \in \mathbb{R}^+, \quad \alpha \neq 1 \\ F(x) & ;\alpha = 1. \end{cases}$$

They applied the proposed method to a one-parameter exponential distribution and generated a two-parameter Alpha Power Exponential distribution. They also studied the various properties of the proposed distribution.

The main objective of this paper is to introduce a new method that adds an extra parameter to a family of distribution functions to bring more flexibility to the given family. We call this new method as MIT method. The proposed MIT method is very easy to use, hence it can be used quite effectively for data analysis purposes. First we discuss some general properties of this class of distribution functions. Then, the MIT method has been specialized to a two-parameter Weibull distribution and generated a three-parameter MIT Weibull (MITW) distribution.

The rest of the paper is organised as follows. In Section 2, we introduce the MIT method and discuss some general properties of this family of distributions. In Section 3, MITW distribution has been introduced and some special cases are presented. Some of its structural properties including quantile function, median, moment generating function, moments, mean residual life function, Rnyi entropy, order statistic and stress-strength reliability have been discussed. In Section 4, Maximum likelihood estimators of unknown parameter as well as simulation study have been carried out. In Section 5, Two real data sets have been analyzed to illustrate the potency of the proposed model. Finally, the paper is concluded in Section 6.

2. General Properties of MIT Method

Let $F(x)$ be the cdf of any continuous random variable X. Then the cdf of MIT method can be obtained by inverting the following equation

$$G(x) = \frac{F(x)}{1-\bar{\alpha}\bar{F}(x)}$$

this implies,

$$G(x) - \bar{\alpha}G(x)\bar{F}(x) = F(x)$$

after solving the above equation, we get

$$F(x) = \frac{\alpha G(x)}{1-\bar{\alpha}G(x)} \tag{1}$$

to avoid the ambiguity, we write (1) as

$$F_{MIT}(x) = \frac{\alpha F(x)}{1-\bar{\alpha}F(x)}.$$

Clearly, $F_{MIT}(x)$ is a proper cdf. If $F(x)$ is an absolute continuous distribution function with the pdf $f(x)$, then $F_{MIT}(x)$ is also an absolute continuous distribution function with the pdf

$$f_{MIT}(x) = \frac{\alpha f(x)}{[1-\bar{\alpha}F(x)]^2} \quad ; \quad \alpha \in \mathbb{R}^+ \quad , \quad \bar{\alpha} = 1-\alpha. \tag{2}$$

It is clear that $f_{MIT}(x)$ is a weighted version of $f(x)$, where the weight function

$$w(x,\alpha) = \frac{1}{[1-\bar{\alpha}F(x)]^2}$$

and $f_{MIT}(x)$ can be written as

$$f_{MIT}(x) = \frac{f(x)w(x;\alpha)}{c}.$$

Here the normalizing constant $c = E[w(X)]$.

The reliability function $R_{MIT}(x)$ for MIT distribution is defined as

$$R_{MIT}(x) = \frac{1-F(x)}{1-\bar{\alpha}F(x)}. \tag{3}$$

If the hazard function of X is denoted by $h_F(x)$, then the hazard function for MIT distribution is defined as

$$h_{MIT}(x) = \frac{\alpha}{1-\bar{\alpha}F(x)}\, h_F(x); \quad x \in \mathbb{R}, \quad \alpha > 0 \tag{4}$$

Thus

$$\lim_{x\to-\infty} h_{MIT}(x) = \alpha \lim_{x\to-\infty} h_F(x)$$

and

$$\lim_{x\to\infty} h_{MIT}(x) = \lim_{x\to\infty} h_F(x)$$

It follows from (4) that

$$\begin{aligned}
h_F(x) \le h_{MIT}(x) \le \ \alpha\, h_F(x) \quad &; \quad x \in \mathbb{R}, \quad \alpha \ge 1\\
h_F(x) \ge h_{MIT}(x) \ge \ \alpha\, h_F(x) \quad &; \quad x \in \mathbb{R}, \quad \alpha \le 1\\
\frac{\bar{F}(x)}{\alpha} \le R_{MIT}(x) \le \bar{F}(x) \quad &; \quad x \in \mathbb{R}, \quad \alpha \ge 1\\
\frac{\bar{F}(x)}{\alpha} \ge R_{MIT}(x) \ge \bar{F}(x) \quad &; \quad x \in \mathbb{R}, \quad \alpha \le 1.
\end{aligned}$$

Obviously, $\frac{h_{MIT}(x)}{h_F(x)}$ is increasing in x for $\alpha > 1$ and decreasing in x for $0 < \alpha < 1$.

The p-th quantile y_p of $F_{MIT}(x)$ can be obtained as

$$y_p = F^{-1}\left(\frac{p}{\alpha+\bar{\alpha}p}\right).$$

If x_p denotes the p-th quantile for $F(x)$, then it follows that

$$y_p \le x_p \quad if \quad \frac{p}{\alpha+\bar{\alpha}p} \le p.$$

Thus it is possible to determine for what values of α, $F_{MIT}(x)$ will be heavier tail than $F(x)$.

$$y_p \le x_p \quad if\ \alpha \ge 1 \quad \text{and} \quad y_p \ge x_p \quad if\ \alpha \le 1.$$

Therefore, if $\alpha > 1$ then $F(x)$ has a heavier tail than $F_{MIT}(x)$, and for $\alpha < 1$, it is the other way.

Theorem 2.1 *If $f(x)$ is a decreasing function, and $\alpha \ge 1$, then $f_{MIT}(x)$ is a decreasing function.*

Proof. We have,

$$\frac{d}{dx} log f_{MIT}(x) = \frac{f^{'}(x)}{f(x)} + \frac{2\bar{\alpha}f(x)}{1-\bar{\alpha}F(x)}.$$

Since, both the terms on the right hand side are negative. Therefore, $f_{MIT}(x)$ is a decreasing function. □

Theorem 2.2 *If $f(x)$ is a decreasing function, and $f(x)$ is log-convex, then for $\alpha \ge 1$, the hazard function $h_{MIT}(x)$ is a decreasing function.*

Proof. We have,

$$\frac{d^2}{dx^2}\ log f_{MIT}(x) = \frac{d^2}{dx^2}\ log f(x)\ +\ 2\left\{\frac{(1-\bar{\alpha}F(x))\ \bar{\alpha}f^{'}(x)+\bar{\alpha}^2 f^2(x)}{(1-\bar{\alpha}F(x))^2}\right\}$$

since, both the terms on the right hand side are positive, it implies that $f_{MIT}(x)$ is log-convex. Hence the result follows from Barlow and Proschan (1975). □

3. MITW Distribution and Its Properties

In this section, the MIT method is specialized to two parameter Weibull distribution and now onwards it is called as the three-parameter MITW distribution.

Definition: A random variable X is said to have a three-parameter MITW distribution denoted by MITW(α,β,λ) with parameters α, β and λ , if the cdf of X for $x > 0$, is

$$F_{MITW}(x) = \frac{\alpha\ (1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}})}{1-\bar{\alpha}\ \left(1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)} \quad ; \quad \alpha, \beta, \lambda > 0 \tag{5}$$

and the corresponding pdf is given by

$$f_{MITW}(x) = \frac{\alpha\ \frac{\beta}{\lambda}\left(\frac{x}{\lambda}\right)^{\beta-1} e^{-\left(\frac{x}{\lambda}\right)^{\beta}}}{\left(1-\bar{\alpha}(1-e^{-\left(\frac{x}{\lambda}\right)^{\beta}})\right)^2} \quad ; \quad \alpha, \beta, \lambda > 0. \tag{6}$$

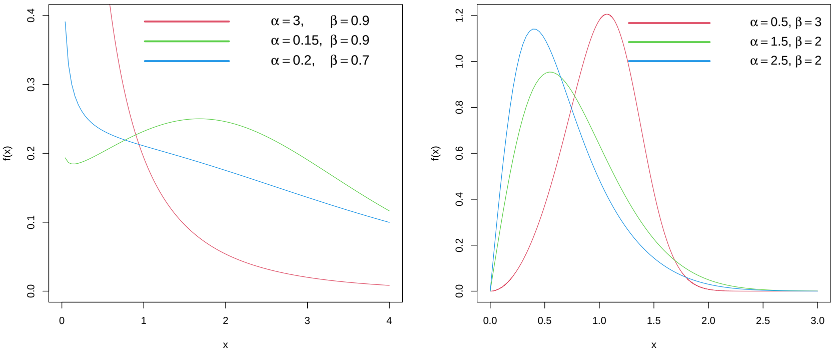

Figure 1 Plots of the MITW density for $\lambda = 1$ and various values of α and β.

The Reliability functon $R_{MITW}(x)$ and the hazard rate function $h_{MITW}(x)$ for $x > 0$ are given by

$$R_{MITW}(x) = \frac{e^{-\left(\frac{x}{\lambda}\right)^{\beta}}}{1 - \bar{\alpha}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)} \quad ; \quad \alpha\,, \beta\,, \lambda\, > 0 \tag{7}$$

$$h_{MITW}(x) = \frac{\alpha\, \frac{\beta}{\lambda} \left(\frac{x}{\lambda}\right)^{\beta-1}}{1 - \bar{\alpha}\left(1 - e^{-\left(\frac{x}{\lambda}\right)^{\beta}}\right)} \quad ; \quad \alpha\,, \beta\,, \lambda\, > 0. \tag{8}$$

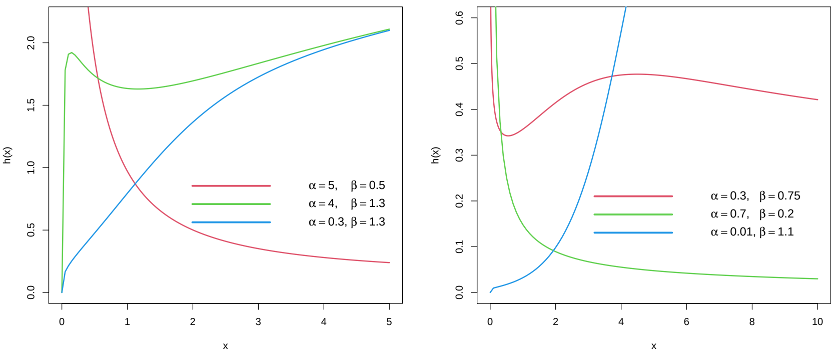

Figure 2 Plots of the MITW hazard rate function for $\lambda = 1$ and various values of α and β.

Note that for all $\alpha, \lambda > 0$, we have

$$h(0) = \begin{cases} \infty & for\; 0 < \beta < 1, \\ \frac{\alpha}{\lambda} & for\; \beta = 1, \\ 0 & for\; \beta > 1, \end{cases} \qquad h(\infty) = \begin{cases} 0 & for\; 0 < \beta < 1, \\ \frac{1}{\lambda} & for\; \beta = 1, \\ \infty & for\; \beta > 1. \end{cases}$$

Theorem 3.1 *If $h(x)$ is the hazard rate function of the MITW distribution.*

(i) *For $\alpha \geq 1$ and $\beta < 1$, then $h(x)$ is decreasing.*

(ii) *For $\alpha \leq 1$ and $\beta > 1$, then $h(x)$ is increasing.*

(iii) *For $\alpha > 1$, $\beta > 1$ and $\psi(\alpha,\beta) = \alpha(\beta-1) + \beta\bar{\alpha}e^{-\frac{1}{\beta}} > 0$, then $h(x)$ is increasing, otherwise, $h(x)$ is increasing-decreasing-increasing.*

(iv) *For $\alpha < 1$, $\beta < 1$ and $\psi(\alpha,\beta) = \alpha(\beta-1) + \beta\bar{\alpha}e^{-\frac{1}{\beta}} < 0$, then $h(x)$ is decreasing, otherwise, $h(x)$ is decreasing-increasing-decreasing.*

Proof. Since λ is a scale parameter, we assume, without loss of generality, that $\lambda = 1$. The first derivative of $h(x)$ with respect to x is given by:

$$h^{'}(x) = s(x)t(x^{\beta}), \qquad x > 0$$

where $s(x) > 0$ and $t(y) = (\beta-1)[1-\bar{\alpha}(1-e^{-y})] + \bar{\alpha}\beta y e^{-y}$, $y = x^{\beta} > 0$.

(i) For $\alpha \geq 1, \beta < 1$, clearly $t(y) < 0$, this implies $h^{'}(x) < 0$. Therefore, $h(x)$ is decreasing.

(ii) By using similar approach as (i).

(iii) For $\alpha > 1, \beta > 1$, the first derivative of $t(y)$ with respect to x is given by

$$t^{'}(y) = \bar{\alpha}e^{-y}(1-\beta y)\ ; \qquad y > 0,$$

which implies that $t(y)$ has a stationary point at $y^* = 1/\beta$. Since $t^{''}(y^*) = -\bar{\alpha}\beta e^{-\frac{1}{\beta}} > 0$. This implies $t(y)$ has the global minimum at y^*. The global minimum value of $t(y)$ is given by $t(y^*) = \alpha(\beta-1) + \beta\bar{\alpha}e^{-\frac{1}{\beta}} = \psi(\alpha,\beta)$, say. Clearly, for $\beta > 1$, $\lim_{y\to 0} t(y) = \beta - 1 > 0$ and $\lim_{y\to\infty} t(y) = \alpha(\beta-1) > 0$.

If $t(y^*) = \psi(\alpha,\beta) > 0$, then $t(y) > 0\ \forall\, y > 0$. Hence, $h^{'}(x) > 0\ \forall\, x > 0$, i.e. $h(x)$ is increasing. If $t(y^*) = \psi(\alpha,\beta) < 0$, then $t(y)$ has exactly two zeros $x_1 < x_2$, such that $h(x)$ increases on $(0,x_1)$, decreases on (x_1,x_2) and finally increases on (x_2,∞). So, $h(x)$ is increasing-decreasing-increasing (see Figure 2).

(iv) By using similar approach as (iii).

□

Remark: When $\alpha = 1$, the MITW distribution reduces to the Weibull distribution. In that case the shapes for hazard rate function are well known in the literature. Table 1 lists seven important special models of the new distribution.

Figure 1 displays some plots of the MITW density for selected parameter values. Plots of the $h(x)$ of the MITW distribution for selected parameter values are given in Figure 2.

Table 1 Sub-models of the MITW Distribution

α	λ	β	Reduced model
-	1	-	MIT one-parameter Weibull distribution
1	-	-	Two-parameter Weibull distribution
1	1	-	One-parameter Weibull distribution
-	-	2	MIT-Rayleigh distribution
1	-	2	Rayleigh distribution
-	-	1	MIT-exponential distribution
1	-	1	Exponential distribution

3.1. Simulation and quantile

The MITW distribution can be simulated using inverse cdf method

$$X = \lambda \left(log \left(\frac{\alpha + \bar{\alpha} U}{\alpha(1-U)} \right) \right)^{\frac{1}{\beta}} \tag{9}$$

where U follows uniform $(0,1)$ distribution. The p-th quantile function of MITW distribution is given by

$$x_p = \lambda \left(log \left(\frac{\alpha + \bar{\alpha} p}{\alpha(1-p)} \right) \right)^{\frac{1}{\beta}}.$$

The median can be obtained as

$$x_{0.5} = \lambda \left(log \left(\frac{1+\alpha}{\alpha} \right) \right)^{\frac{1}{\beta}}.$$

3.2. Moment generating function and moment

Using the series representations

$$(1-x)^{-2} = \sum_{k=0}^{\infty} (k+1) x^k \; ; \qquad |x| < 1, \tag{10}$$

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n}{k} x^k \; ; \qquad |x| < 1, \tag{11}$$

and

$$(1-x)^{n} = \sum_{k=0}^{n} \binom{n}{k} (-1)^k x^k \; ; \qquad |x| < 1, \tag{12}$$

the moment-generating function (mgf) of MITW distribution can be obtained as

$$M_X(t) = \begin{cases} \alpha \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \sum\limits_{l=0}^{k} \frac{(t\lambda)^j \bar{\alpha}^k}{j!} (k+1) \binom{k}{l} (-1)^l \frac{\Gamma(\frac{j}{\beta}+1)}{(l+1)^{\frac{j}{\beta}+1}} \; ; & \alpha < 1, \\ \alpha \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \sum\limits_{l=0}^{\infty} \frac{(t\lambda)^j}{j!} \frac{(k+1)}{\bar{\alpha}^{k+2}} \binom{k+2}{l} \frac{\Gamma(\frac{j}{\beta}+1)}{(l+1)^{\frac{j}{\beta}+1}} \; ; & \alpha > 1. \end{cases}$$

Hence, the rth moment of X becomes

$$E(X^r) = \begin{cases} \alpha \lambda^r \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{j} \bar{\alpha}^j (j+1) \binom{j}{k} (-1)^k \frac{\Gamma(\frac{r}{\beta}+1)}{(k+1)^{\frac{r}{\beta}+1}} \; ; & \alpha < 1, \\ \alpha \lambda^r \sum\limits_{j=0}^{\infty} \sum\limits_{k=0}^{\infty} \frac{(j+1)}{\bar{\alpha}^{j+2}} \binom{j+2}{k} \frac{\Gamma(\frac{r}{\beta}+1)}{(k+1)^{\frac{r}{\beta}+1}} \; ; & \alpha > 1. \end{cases}$$

3.3. Mean residual life and mean waiting time

Suppose that X is a continuous random variable with reliability function $R(x)$, the mean residual life is the expected additional lifetime given that a component has survived until time t. The mean residual life function, say $\mu(t)$, is given by

$$\mu(t)=\frac{1}{R(t)}\left(E(t)-\int_0^t xf(x)dx\right)-t.$$

The mean residual life of MITW distribution is obtained by using (10), (11) and (12) and is given by

For $\alpha<1$

$$\mu(t)=\frac{\left\{1-\bar{\alpha}\left(1-e^{-\left(\frac{t}{\lambda}\right)^{\beta}}\right)\right\}}{e^{-\left(\frac{t}{\lambda}\right)^{\beta}}}\alpha\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\bar{\alpha}^{j}(j+1)}{(k+1)^{\frac{1}{\beta}+1}}\binom{j}{k}(-1)^{k}$$
$$\left\{\Gamma(\frac{1}{\beta}+1)-\gamma\left(\left(\frac{t}{\lambda}\right)^{\beta}(k+1),\frac{1}{\beta}+1\right)\right\}-t.$$

For $\alpha>1$

$$\mu(t)=\frac{\left\{1-\bar{\alpha}\left(1-e^{-\left(\frac{t}{\lambda}\right)^{\beta}}\right)\right\}}{e^{-\left(\frac{t}{\lambda}\right)^{\beta}}}\alpha\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(j+1)\binom{j+2}{k}}{\bar{\alpha}^{j+2}(k+1)^{\frac{1}{\beta}+1}}$$
$$\left\{\Gamma(\frac{1}{\beta}+1)-\gamma\left(\left(\frac{t}{\lambda}\right)^{\beta}(k+1),\frac{1}{\beta}+1\right)\right\}-t,$$

where $\gamma(a,b)=\int_0^a x^{b-1}e^{-x}dx$ is the lower incomplete gamma function.

The mean waiting time represents the waiting time elapsed since the failure of an object on condition that this failure had occurred in the interval [0, t]. The mean waiting time of X, say $\bar{\mu}(t)$, is defined by

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\int_0^t xf(x)dx.$$

For $\alpha<1$,

$$\bar{\mu}(t)=t-\frac{\left\{1-\bar{\alpha}\left(1-e^{-\left(\frac{t}{\lambda}\right)^{\beta}}\right)\right\}}{1-e^{-\left(\frac{t}{\lambda}\right)^{\beta}}}\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{j}\frac{\bar{\alpha}^{j}(j+1)}{(k+1)^{\frac{1}{\beta}+1}}\binom{j}{k}(-1)^{k}$$
$$\gamma\left(\left(\frac{t}{\lambda}\right)^{\beta}(k+1),\frac{1}{\beta}+1\right).$$

For $\alpha>1$,

$$\bar{\mu}(t)=t-\frac{\left\{1-\bar{\alpha}\left(1-e^{-\left(\frac{t}{\lambda}\right)^{\beta}}\right)\right\}}{1-e^{-\left(\frac{t}{\lambda}\right)^{\beta}}}\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\frac{(j+1)\binom{j+2}{k}}{\bar{\alpha}^{j+2}(k+1)^{\frac{1}{\beta}+1}}$$
$$\gamma\left(\left(\frac{t}{\lambda}\right)^{\beta}(k+1),\frac{1}{\beta}+1\right).$$

3.4. Rnyi entropy

The entropy of a random variable measures the variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Rnyi entropy, say$RE_X(u)$ is defined as

$$RE_X(u) = \frac{1}{1-u} log\left(\int_{-\infty}^{\infty} f(x)^u dx\right); \quad u > 0, \quad u \neq 1.$$

The Rnyi entropy of MITW distribution is obtained by using (10), (11) and (12) and is given by

For $\alpha < 1$

$$\begin{aligned} RE_X(u) =& \frac{u}{1-u} log(\alpha) - log(\frac{\beta}{\lambda}) \\ &+ \frac{1}{1-u} log\left(\sum_{j=0}^{\infty}\sum_{k=0}^{j} \bar{\alpha}^j \binom{2u}{j}\binom{j}{k}(-1)^k \frac{\Gamma(u-\frac{(u-1)}{\beta})}{(u+k)^{(u-\frac{(u-1)}{\beta})}}\right). \end{aligned}$$

For $\alpha > 1$

$$\begin{aligned} RE_X(u) =& \frac{u}{1-u} log(\alpha) - log(\frac{\beta}{\lambda}) \\ &+ \frac{1}{1-u} log\left(\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \frac{\binom{2u}{j}\binom{j+2u}{k}}{\bar{\alpha}^{j+2u}} \frac{\Gamma(u-\frac{(u-1)}{\beta})}{(u+k)^{(u-\frac{(u-1)}{\beta})}}\right). \end{aligned}$$

3.5. Order statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n, and let $X_{r:n}$ denote the rth order statistic, then, the pdf of $X_{r:n}$, say $f_{r:n}(x)$ is given by

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} f(x)(1-F(x))^{n-r}.$$

We can write $f_{r:n}(x)$ as

$$f_{r:n}(x) = \frac{\alpha^r \frac{\beta}{\lambda}}{B(r, n-r+1)} \left(\frac{x}{\lambda}\right)^{\beta-1} \frac{\left(1-e^{-\left(\frac{x}{\lambda}\right)^\beta}\right)^{r-1} e^{-\left(\frac{x}{\lambda}\right)^{\beta(n-r+1)}}}{\left(1-\bar{\alpha}(1-e^{-\left(\frac{x}{\lambda}\right)^\beta})\right)^{n+1}},$$

where $B(a,b)$ is the beta function.

3.6. Stress strength reliability

Suppose X_1 and X_2 be independent strength and stress random variables respectively, where $X_1 \sim MITW(\alpha_1, \lambda_1, \beta)$ and $X_2 \sim MITW(\alpha_2, \lambda_2, \beta)$, then the stress strength reliability $\mathbb{P}(X_1 > X_2)$, say *SSR*, is defined as

$$SSR = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx.$$

The stress strength reliability SSR, is obtained by using (5), (6), (10), (11) and (12) and is given by

$$SSR = \begin{cases} \frac{\alpha_1\alpha_2\lambda_2}{\lambda_1^{\beta-1}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{j}\sum_{m=0}^{k+1} (j+1)\binom{j}{l}(-1)^{l+m} \frac{(k+1)\alpha_1^j\alpha_2^k}{m(m\lambda_1+(l+1)\lambda_2)}; & \alpha_1 < 1,\ \alpha_2 < 1 \\ \frac{\alpha_1\alpha_2\lambda_2}{\bar{\alpha_1}^2(\alpha_2-1)\lambda_1^{\beta-1}} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty}\sum_{m=0}^{\infty} \binom{j+2}{l}\binom{k}{m} \frac{j+1}{\bar{\alpha_1}^j\bar{\alpha_2}^k(m\lambda_1+(l+1)\lambda_2)}; & \alpha_1 > 1,\ \alpha_2 > 1. \end{cases}$$

4. Statistical Inference

4.1. Maximum likelihood estimators

Let $x_1, x_2, \ldots, x_n$ be a random sample from MITW distribution, then the logarithm of the likelihood function becomes

$$l = nlog\alpha + nlog\beta - n\beta log\lambda + (\beta-1)\sum_{i=1}^{n} log(x_i) - \frac{\sum_{i=1}^{n} x_i^{\beta}}{\lambda^{\beta}} - 2\sum_{i=1}^{n} log\left(1-\bar{\alpha}\left(1-e^{-(\frac{x_i}{\lambda})^{\beta}}\right)\right). \qquad (13)$$

The MLEs of α, λ and β are obtained by partially differentiating (13) with respect to the corresponding parameters and equating to zero we have

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} - 2\sum_{i=1}^{n}\left(\frac{(1-e^{-(\frac{x_i}{\lambda})^{\beta}})}{1-\bar{\alpha}\left(1-e^{-(\frac{x_i}{\lambda})^{\beta}}\right)}\right) = 0$$

$$\frac{\partial l}{\partial \lambda} = -\frac{n\beta}{\lambda} + \beta\frac{\sum_{i=1}^{n} x_i^{\beta}}{\lambda^{\beta+1}} + 2\frac{\bar{\alpha}\beta}{\lambda^{\beta+1}}\sum_{i=1}^{n}\left(\frac{x_i^{\beta} e^{-(\frac{x_i}{\lambda})^{\beta}}}{1-\bar{\alpha}\left(1-e^{-(\frac{x_i}{\lambda})^{\beta}}\right)}\right) = 0$$

$$\frac{\partial l}{\partial \beta} = \frac{n}{\beta} - nlog\lambda + \sum_{i=1}^{n} logx_i - \frac{\beta}{\lambda^{\beta}}\sum_{i=1}^{n} x_i^{\beta-1} + \frac{log\lambda}{\lambda^{\beta}}\sum_{i=1}^{n} x_i^{\beta} + 2\sum_{i=1}^{n}\left(\frac{\bar{\alpha}\beta(\frac{x_i}{\lambda})^{\beta-1} e^{-(\frac{x_i}{\lambda})^{\beta}}}{1-\bar{\alpha}\left(1-e^{-(\frac{x_i}{\lambda})^{\beta}}\right)}\right) = 0.$$

Since the normal equations are complex in nature and are solved by R software.

4.2. Simulation study

Table 2 Average values of MLEs and the corresponding MSEs ($n = 50$)

Parameter			MLE			MSE		
λ	α	β	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.5	1.5	1.05747	0.75566	1.56545	0.10484	0.78746	0.09050
		3	0.98845	0.67483	3.04906	0.03264	0.44393	0.40167
		5	0.96303	0.55106	4.78708	0.01358	0.20574	1.04465
	1.5	1.5	0.99033	1.70767	1.49440	0.05723	1.49425	0.03295
		3	0.99146	1.74764	3.01389	0.01914	1.17414	0.16121
		5	1.00293	1.78810	5.07979	0.00390	0.93316	0.36411
	3	1.5	0.98824	3.18791	1.48676	0.05300	3.31959	0.02204
		3	0.99084	3.28670	2.97711	0.01503	3.54019	0.10937
		5	0.99197	3.33680	4.91370	0.01241	3.64137	0.11320
2	0.5	1.5	2.00867	0.65565	1.49673	0.50871	0.27872	0.10405
		3	1.98480	0.67958	3.00840	0.11882	0.40014	0.35086
		5	2.01576	0.74818	5.15706	0.04045	0.46083	0.88584
	1.5	1.5	1.96822	1.69830	1.47213	0.30443	1.78864	0.04046
		3	1.97723	1.72749	2.97728	0.07864	0.94075	0.16896
		5	1.98308	1.63773	4.95950	0.01788	1.02682	0.31782
	3	1.5	2.05793	3.27554	1.51815	0.22557	3.84219	0.02552
		3	1.94835	3.18430	2.98372	0.08831	3.10406	0.15714
		5	1.96153	3.09219	4.94148	0.02398	3.18379	0.29821

Table 3 Average values of MLEs and the corresponding MSEs ($n = 100$)

Parameter			MLE			MSE		
λ	α	β	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$
1	0.5	1.5	1.00902	0.58267	1.51253	0.05540	0.09670	0.04514
		3	1.00192	0.59944	3.03499	0.01377	0.12031	0.18619
		5	1.01386	0.65760	5.21095	0.00477	0.12943	0.50492
	1.5	1.5	0.99723	1.61885	1.49753	0.03470	0.75210	0.02021
		3	0.99210	1.58804	2.99378	0.00885	0.50492	0.09625
		5	1.00127	1.54214	4.98210	0.00151	0.39571	0.12154
	3	1.5	1.02508	3.15421	1.50571	0.03497	2.73379	0.01438
		3	0.99922	3.18673	2.99201	0.00762	2.97769	0.04753
		5	1.00254	3.21548	5.01278	0.00421	2.98912	0.04523
2	0.5	1.5	2.08277	0.62822	1.53828	0.24562	0.13707	0.05472
		3	2.01662	0.54063	2.98488	0.06391	0.09677	0.19660
		5	1.97359	0.56653	4.98672	0.03361	0.13124	0.66129
	1.5	1.5	2.01562	1.63873	1.49667	0.12487	0.55183	0.02047
		3	1.98943	1.61163	2.99938	0.02895	0.60416	0.08303
		5	1.99889	1.72353	5.05222	0.02150	0.55681	0.26121
	3	1.5	1.95387	3.04062	1.48687	0.11099	1.58212	0.01051
		3	1.99222	3.12137	3.00127	0.03531	2.05399	0.05250
		5	2.00254	3.27214	5.00246	0.01043	2.18099	0.15539

The simulation study has been performed using R Software to show the behaviour of the MLEs in terms of the sample size n. Two sets of sample ($n = 50, n = 100$) each replicated 100 times with different values of parameters $\lambda = (1,2)$, $\alpha = (0.5, 1.5, 3)$ and $\beta = (1.5, 3, 5)$ were generated from MITW. In each setting, the average values of MLEs and the corresponding empirical mean squared errors (MSEs) were obtained. The simulation results are presented in Table 2 and Table 3. From Tables 2 and 3, it can be seen that the estimates are stable and quite close to the true parameter values. As the sample size increases the MSE decreases in all the cases.

5. Applications

In this section, we analyse two data sets to describe the significance and flexibility of the MITW distribution. The data set one corresponds to intervals in days between 109 successive coal-mining disasters in Great Britain, for the period 1875-1951, reported by Nassar et al. (2017), originally published by Maguire et al. (1952). The sorted data are given as follows:
1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49, 50, 54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120, 123, 124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224, 228, 233, 255, 271, 275, 275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338, 345, 348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613, 1630.

The data set second reported by Nassar et al. (2017), originally published by Smith and Naylor (1987), corresponding to strengths of 1.5 cm glass fibers, measured at the National Physical Laboratory, England. The data are as follows:
0.55, 0.93,1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53,1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69,1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 0.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

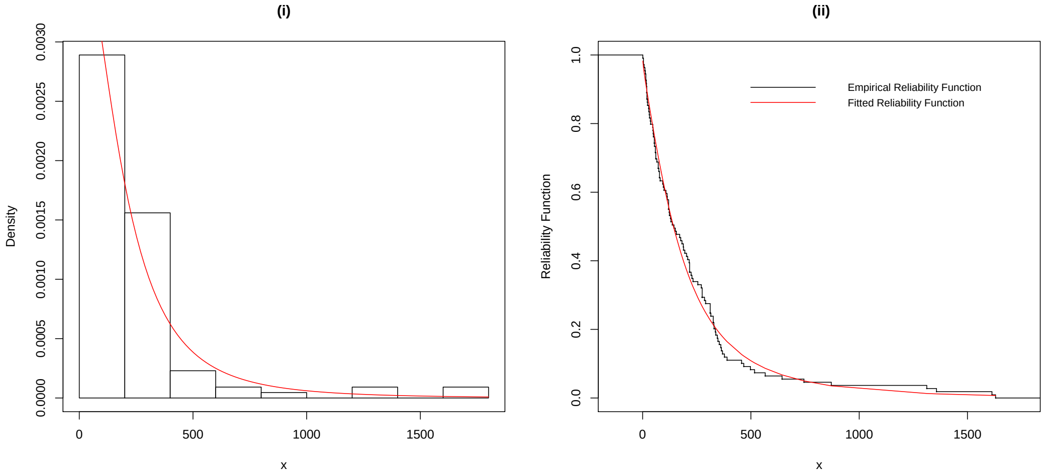

Figure 3 (i) The relative histogram and the fitted MITW distribution (ii) The fitted MITW reliability function and empirical reliability function for first data set

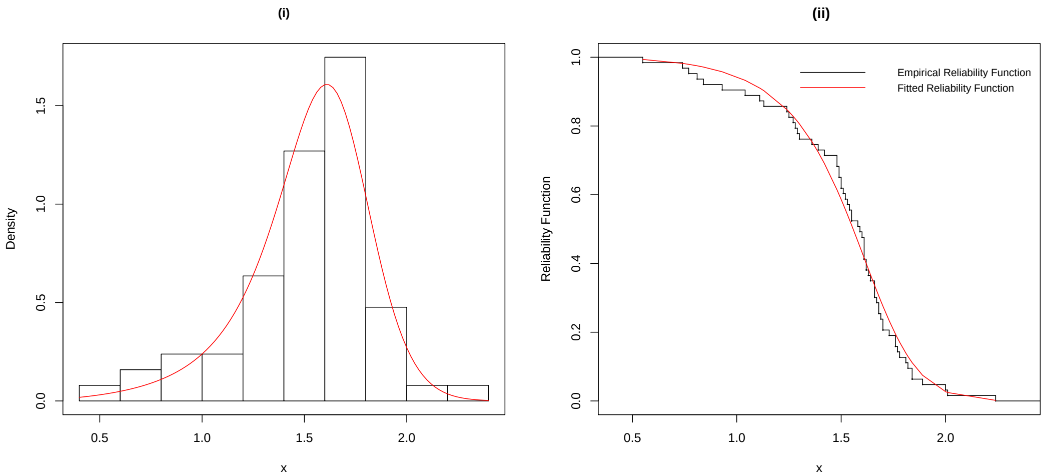

Figure 4 (i) The relative histogram and the fitted MITW distribution (ii) The fitted MITW reliability function and empirical reliability function for second data set

We compare the fit of the proposed MITW with several other models, namely McDonald Weibull (Mc-W) Cordeiro et al. (2014), beta Weibull (BW) Lee et al. (2007), modified Weibull (MW) Sarhan and Zaindin (2009), gamma Lomax (GL) Cordeiro et al. (2015), ZografosBalakrishnan log-logistic (ZBLL) Zografos and Balakrishnan (2009), Inverse Weibull (IW) Johnson et al. (1995).

From Table 4, Table 5, Table 6 and Table 7, it is evident that MITW distribution has lowest $-2l(\hat{\theta})$, AIC, AICC, BIC, K-S values and highest p-value among all the other competitive models. Hence the proposed model yields the better fit than the other models for both data sets.

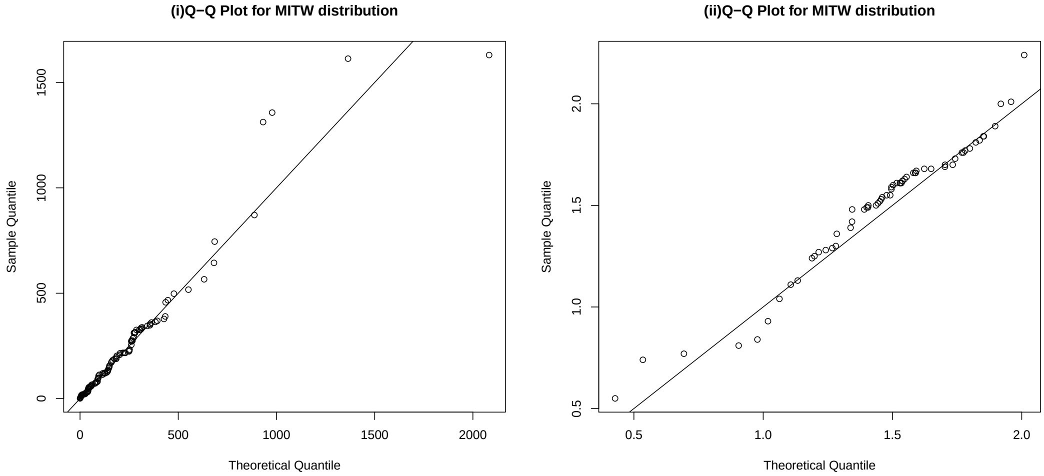

Figure 5 Q-Q plot for the MITW distribution for data set 1 and data set 2, respectively

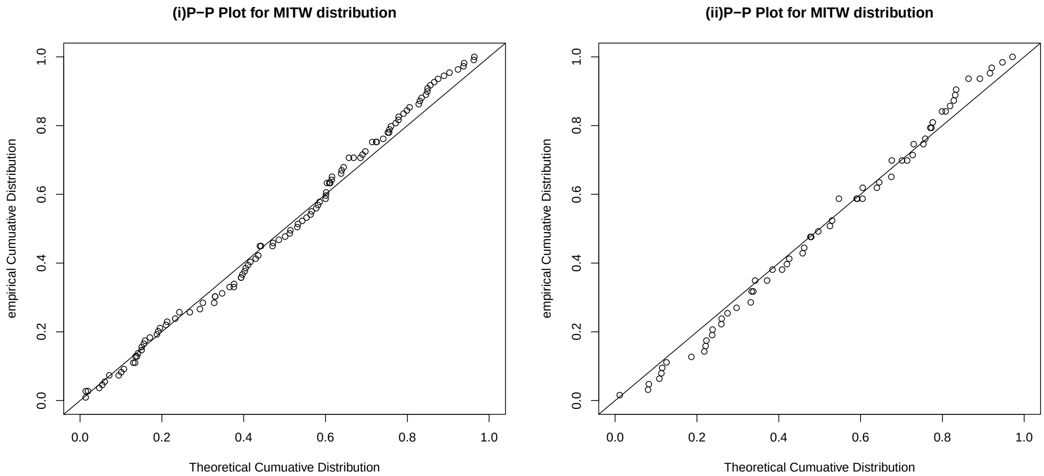

Figure 6 P-P plot for the MITW distribution for data set 1 and data set 2, respectively

The relative histogram and the fitted MITW distribution of the data set first and second are shown in Figures 3(i) and 4(i), respectively. The plots of the fitted MITW reliability function and empirical reliability function of the data set first and second are shown in Figures 3(ii) and 4(ii), respectively. The Q-Q plots for data set first and second are shown in Figure 5(i) and 5(ii) respectively. Also, The P-P plots for data set first and second are shown in Figure 6(i) and 6(ii) respectively that allows us to differentiate between the empirical distribution of the data with the MITW distribution. These graphical goodness of fit measures clearly support the results in Tables 4, Table 5, Table 6 and Table 7.

Table 4 MLEs (standard errors in parentheses), K-S Statistic, and p-values for the first data set

Model	Estimates					Statistics	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{b}$	$\hat{k}$	K-S	p-value
Mc-W	21.73374 (19.74300)	0.05625 (0.01956)	2.26312 (0.99590)	20.26463 (17.73100)	12.35624 (13.06030)	0.07905	0.50359
BW	6.26933 (3.41100)	0.52681 (0.09100)	0.25154 (0.19300)	0.18992 (0.11700)	-	0.08634	0.39084
MW	0.00429 (0.00047)	0.34131 (5.95200)	0.00004 (0.01100)	-	-	0.07827	0.51645
GL	7.41676 (2.27150)	1.85007 (0.10700)	2.24237 (1.73700)	-	-	0.11726	0.09984
ZBLL	1.53280 (0.09952)	1.19260 (0.09100)	58.29380 (0.82900)	-	-	0.09001	0.34030
IW	-	0.64027 (0.04065)	57.89748 (9.21448)	-	-	0.14526	0.02010
MITW	0.00846 (0.00628)	0.28979 (0.04578)	0.64399 (0.87546)	-	-	0.05833	0.85210

Table 5 $-2l(\hat{\theta})$, AIC, AICC, BIC for the first data set

Model	$-2l(\hat{\theta})$	AIC	AICC	BIC
Mc-W	1410.7460	1420.7460	1421.3285	1434.2027
BW	1410.4217	1418.4217	1418.8063	1429.1871
MW	1406.6267	1412.6267	1412.8553	1420.7007
GL	1433.4712	1439.4712	1439.6998	1447.5452
ZBLL	1443.6280	1449.6280	1449.8566	1457.7020
IW	1452.6029	1456.6029	1456.7161	1461.9856
MITW	1404.2839	1410.2839	1410.5125	1418.3580

6. Conclusion

A new family of distributions has been introduced based on the idea of Marshall and Olkin transformation introduced by Marshall and Olkin (1997). MIT method has been specialized on the two parameter Weibull distribution and a new three parameter MITW distribution has been introduced. We have discussed various properties of MITW distribution. It is observed that the three-parameter MITW distribution has more flexibility in the form of hazard and density functions. The effectiveness of the proposed model is compared with other existing models by using goodness of fit measures. The model has been fitted to two different data sets, the figures show that the proposed model provides better fit for both data sets in comparison to all other competitive models.

Table 6 MLEs (standard errors in parentheses), K-S Statistic, and p-values for the second data set

Model	Estimates					Statistics	
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{b}$	$\hat{k}$	K-S	p-value
Mc-W	0.38232 (0.37740)	5.13668 (4.62500)	0.02828 (0.06990)	5.54804 (22.94500)	2.48502 (2.73760)	0.12955	0.24088
BW	0.63515 (0.26217)	7.61488 (2.07370)	0.51284 (0.20970)	2.31224 (6.65120)	-	0.12977	0.23921
MW	0.03108 (0.04375)	6.38083 (0.97580)	0.04071 (0.02495)	-	-	0.11725	0.35187
GL	18.96058 (3.9454)	128.71263 (89.38365)	9.47211 (7.98425)	-	-	0.20380	0.01067
ZBLL	1.64144 (0.13178)	6.26230 (0.63220)	1.28259 (0.00390)	-	-	0.16794	0.05723
IW	-	2.88755 (0.23443)	1.26434 (0.05885)	-	-	0.24443	0.00107
MITW	0.06040 (0.07440)	3.20558 (0.93824)	1.12147 (0.24436)	-	-	0.10002	0.55410

Table 7 $-2l(\hat{\theta})$, AIC, AICC, BIC for the second data set

Model	$-2l(\hat{\theta})$	AIC	AICC	BIC
Mc-W	28.6496	38.6496	39.7022	49.3653
BW	29.2396	37.2396	37.9293	45.8121
MW	29.7893	35.7893	36.1961	42.2187
GL	49.7569	55.7569	56.1637	62.1863
ZBLL	74.3721	80.3721	80.7789	86.8015
IW	93.7066	97.7066	97.9066	101.9929
MITW	24.0672	30.0672	30.4740	36.4966

Appendix

The ”R” code which is used to obtain the maximum likelihood estimates of the parameters. Here ”a” is used for alpha, ”b” is used for beta and ”l” is used for lambda.

```
rm(list=ls(all=TRUE))
data=c(1, 4, 4, 7, 11, 13, 15, 15, 17, 18, 19, 19, 20, 20, 22, 23, 28, 29, 31, 32, 36, 37, 47, 48, 49, 50,
54, 54, 55, 59, 59, 61, 61, 66, 72, 72, 75, 78, 78, 81, 93, 96, 99, 108, 113, 114, 120, 120, 120, 123,
124, 129, 131, 137, 145, 151, 156, 171, 176, 182, 188, 189, 195, 203, 208, 215, 217, 217, 217, 224,
228, 233, 255, 271, 275,275, 275, 286, 291, 312, 312, 312, 315, 326, 326, 329, 330, 336, 338, 345,
348, 354, 361, 364, 369, 378, 390, 457, 467, 498, 517, 566, 644, 745, 871, 1312, 1357, 1613, 1630)
hist(data , prob = T,col = 3, angle = c(45), density = 20, main = ”Fig. 1.1: MITW Model Fitting”,
cex.main = 1)
mean(data)
length(data)
library(MASS)
MITW = function(x, a , b , l) ((a * (b/l)* (x/l)**(b-1)) * exp(- (x/l)**b)) / (1-(1-a)*(1-exp(-(x/l)**b)))**2
mle = fitdistr(x =data,densfun = MITW,start = list(a=.02,b=.4,l=1.5),lower=c(0.001,0.001,0.001),upper=c(Inf,Inf,Inf))
mle
```

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