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On a New Two-parameter Weighted Exponential Distribution and Its Application to the COVID-19 and Censored Data

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Abstract

Simple and flexible lifetime distributions are appreciated by practitioners in all applied fields. They allow the construction of fairly manageable statistical models. In this article, a new simple lifetime distribution involving two parameters is proposed. It is based on a simple modification of the construction of the so-called weighted exponential distribution, by replacing the exponential distribution with the Erlang distribution. This choice is motivated by solid mathematical and physical interpretations. The new distribution is naturally named the new two-parameter weighted distribution. In the first part, we present the main mathematical properties of this distribution, with an emphasis on the flexibility of the probabilistic functions, the closed forms of various moments, and the analysis of the skewness and kurtosis coefficients. The remaining part is devoted to the associated model, showing how it can be applied in a real statistical scenario dealing with data. In this regard, four data sets are considered, two complete and two censored data sets; one on the survival times of guinea pigs injected with a certain bacteria, one on the COVID-19 daily death rate in Israel, one on censored data about survival times of patients infected with HIV, and one on censored data about remission times for leukemia patients treated with a special drug. The performance of the new model is compared with that of the weighted exponential, two-parameter weighted exponential and extended weighted exponential models. The obtained comparison results are quite favorable to the proposed methodology.

Keywords: Lifetime distributions, Azzalini technique, moments, maximum likelihood estimation, applications

1. Introduction

The weighted exponential (WE) distribution introduced by Gupta and Kundu (2009) is a flexible two-parameter lifetime distribution that finds numerous applications in reliability, finance, econometrics, engineering, and biology. Concerning the main point of interest, it provides a genuine alternative to the Weibull and gamma distributions; it is more pliant on certain functional and probabilistic aspects, as detailed in (Gupta and Kundu, 2009, Table 2). From the mathematical point of view, the construction of the WE distribution is based on the Azzalini technique [see Azzalini (1985)] applied to the exponential distribution. It was further generalized and extended by the two-parameter weighted exponential (TWE) distribution by Shakhatreh (2012), generalized weighted exponential (GWE) distribution by Kharazmi *et al.* (2015), new weighted exponential (NWE) distribution by Kharazmi and Jabbari (2017) and the (three-parameter) extended weighted exponential (EWE) distribution by Mahdavi and Jabbari (2017). All these distributions have demonstrated a high ability for several statistical purposes, and in data fitting in particular.

In this paper, inspired by the construction of the WE and EWE distributions, we introduce a new two-parameter weighted exponential distribution defined by the following probability density function (pdf):

$$f(x;\alpha,\beta)=\frac{\alpha(\beta+1)^2}{\beta^2}\exp(-\alpha x)\left[1-(1+\alpha\beta x)\exp(-\alpha\beta x)\right],\quad x>0. \tag{1}$$

As with any lifetime distribution, we set $f(x;\alpha,\beta)=0$ for $x\leq 0$. This distribution is named the new two-parameter weighted exponential (NTWE) distribution. Some motivational facts behind its construction are discussed below. To begin, the NTWE distribution comes from the Azzalini technique under a special configuration involving the Erlang distribution. More precisely, we can express $f(x;\alpha,\beta)$ as

$$f(x;\alpha,\beta)=\frac{1}{P(V\leq\beta U)}g_1(x;\alpha)G_2(\beta x;\alpha),$$

where $g_1(x;\alpha)=\alpha\exp(-\alpha x)$ for $x>0$ and $g_1(x;\alpha)=0$ for $x\leq 0$, which is the pdf of the exponential distribution with parameter α, $G_2(y;\alpha)=1-(1+\alpha y)\exp(-\alpha y)$ for $y>0$ and $G_2(y;\alpha)=0$ for $y\leq 0$, which is the cdf of the Erlang distribution with parameters 2 and α [also known under the name of length-biased exponential distribution, see Dara and Ahmad (2012)], U and V are two independent random variables, U following the exponential distribution with parameter α, and V following the Erlang distribution with parameters 2 and α. Thus, in comparison to the construction of the WE and EWE distributions, the Erlang distribution takes the place of the former exponential distribution and the extended exponential distribution, respectively. Consequently, the WE distribution is not a special case of the NTWE distribution. In addition, as noticed in (Gupta and Kundu, 2009, Interpretation 4), the WE distribution is a sub-case of the beta exponential distribution introduced by Jones (2004), whereas the NTWE distribution is not (due to the polynomial term in the pdf). Thus, there is no immediate connection between the EWE and NTWE distributions, making them complementary in the modeling sense.

The above remarks are encouraging for a more in-depth study of the NTWE distribution, which is the scope of this paper. We begin by giving important and attractive mathematical interpretations of the NTWE distribution. Then, we examine its main mathematical characteristics. Specifically, we analyze the possible shapes of $f(x;\alpha,\beta)$, as well as those of the corresponding hazard rate function (hrf). We show that the pdf is exclusively unimodal with a varying tail weight, and that the hrf increases with concave or convex features. They are revealed to be sufficiently pliant to produce a competitor model to the EWE and WE models. Theoretical results and graphics support this claim. Also, we highlight the moment properties of the NTWE distribution, illustrated by a numerical study. After these mathematical developments, the NTWE model is the subject of a complete statistical study. We apply the maximum likelihood method to estimate the parameters α and β. The NTWE model is then used to fit two complete data sets and two censored data sets. We show that it is competitive in this regard, with the TWE, EWE, and WE models in particular achieving higher results in terms of reference statistical criteria.

We structure the rest of the paper as follows. More mathematical interpretations of the NTWE distribution are given in Section 2. Section 3 highlights its modeling capabilities by conducting a functional analysis. Its mathematical properties are developed in Section 4. The two other sections emphasize the NTWE model: Section 5 investigates the estimation of the model parameters, and Section 6 provides applications to real data sets. Some concluding notes are given in Section 7.

2. Mathematical Interpretations

Following the spirit of (Gupta and Kundu, 2009, Section 2), some mathematical interpretations on the NTWE distribution are presented below.

- After some algebra, we can express $f(x;\alpha,\beta)$ as the convolution product:

$$f(x;\alpha,\beta) = \int_{-\infty}^{+\infty} g_1(x-t;\alpha)g_2(t;\alpha(\beta+1))dt, \qquad (2)$$

  where $g_1(t;\alpha) = \alpha\exp(-\alpha t)$ for $t > 0$ and $g_1(t;\alpha) = 0$ for $t \leq 0$, which is the pdf of the exponential distribution with parameter α, and $g_2(t;\alpha(\beta+1)) = \alpha^2(\beta+1)^2 t\exp(-\alpha(\beta+1)t)$ for $t > 0$ and $g_2(t;\alpha(\beta+1)) = 0$ for $t \leq 0$, which is the pdf of the Erlang distribution with parameters 2 and $\alpha(\beta+1)$. In other words, the NTWE distribution corresponds to the distribution of

$$X = U + W,$$

  where U and W are independent random variables, U follows the exponential distribution with parameter α, and W follows the Erlang distribution with parameters 2 and $\alpha(\beta+1)$.

  - Because the exponential and Erlang distributions are well-known, a number of features of the NTWE distribution would be accessible.
  - In the distribution sense, we can write W as $W = Y + Z$, where Y and Z are two independent random variables which follow the exponential distribution with parameter $\alpha(\beta+1)$. As a result, we have $X = U + W = U + Y + Z \geq T$, where $T = U + Y$ is a random variable which follow the WE distribution. This inequality implies that the NTWE distribution first order stochastically dominates the WE distribution, showing an interesting hierarchy between them on this stochastic aspect.
  - Values from the exponential and Erlang distributions can be generated via standard computer programs. From them and the sum representation, we easily generate values from the NTWE distribution. This can be useful for diverse computational investigations.

- As another remark, for $x > 0$, we can express $f(x;\alpha,\beta)$ as the following three-term linear combination:

$$f(x;\alpha,\beta) = \frac{(\beta+1)^2}{\beta^2} f_1(x;\alpha) - \frac{\beta+1}{\beta^2} f_2(x;\alpha,\beta) - \frac{1}{\beta} f_3(x;\alpha,\beta), \qquad (3)$$

  where $f_1(x;\alpha) = \alpha\exp(-\alpha x)$, which is the pdf of the exponential distribution with parameter α, $f_2(x;\alpha,\beta) = \alpha(\beta+1)\exp(-\alpha(\beta+1)x)$, which is the pdf of the exponential distribution with parameter $\alpha(\beta+1)$, and $f_3(x;\alpha,\beta) = \alpha^2(\beta+1)^2 x\exp(-\alpha(\beta+1)x)$ which is the pdf of the Erlang distribution with parameters 2 and $\alpha(\beta+1)$. Since $(\beta+1)^2/\beta^2 - (\beta+1)/\beta^2 - 1/\beta = 1$, the NTWE distribution is a generalized mixture of the three above mentioned distributions. Again, we can use this manageable mixture to deduce some properties of the NTWE distribution.

- Finally, the NTWE distribution belongs to the family of the hidden truncation distributions as developed by Arnold and Beaver (2000). Indeed, let Y and Z be two random variables such that (Y,Z) has the following joint pdf:

$$f(y,z;\alpha) = \alpha^3 y^2 z\exp(-\alpha y(z+1)), \quad y,z > 0,$$

  and $f(y,z;\alpha) = 0$ otherwise. Then, the conditional random variable $Y \mid \{Z \leq \beta\}$ follows the NTWE distribution.

All of these interpretations illustrate that the NTWE distribution is mathematically approachable and offers a distinct alternative to the WE distribution. When we see the impact of the WE distribution in Statistics, we are motivated to do a more in-depth analysis of the NTWE distribution, hoping for the same outcome for it.

3. Functional Analysis

We recall that the pdf of the NTWE distribution is determined in Eqn. (1). We now study this pdf analytically to identify its main functional properties. At the limit points, the following equivalence and limit results hold. When x tends to 0, we have $f(x;\alpha,\beta) \sim \alpha^3(\beta+1)^2x^2/2 \to 0$, and when x tends to $+\infty$, we have $f(x;\alpha,\beta) \sim [\alpha(\beta+1)^2/\beta^2]\exp(-\alpha x) \to 0$. For this last case, the exponential term is dominant in the convergence, and the parameter β plays a major role in the decay rate, compared to α.

In order to complete the previous asymptotic study, let us investigate the possible mode(s) of the NTWE distribution, given as the point(s) x making $f(x;\alpha,\beta)$ maximal. Since $\lim_{x\to 0 \text{ or } x\to+\infty} f(x;\alpha,\beta) = 0$ with $f(x;\alpha,\beta) > 0$, we already know that $f(x;\alpha,\beta)$ is "at least" unimodal. More information is given in the next proposition.

Proposition 1 The NTWE distribution is unimodal, with a mode given as

$$x_o = -\frac{1}{\alpha\beta}\left[\frac{1}{\beta+1} + \mathscr{W}\left(-\frac{1}{\beta+1}\exp\left(-\frac{1}{\beta+1}\right)\right)\right], \tag{4}$$

where $\mathscr{W}(x)$ denotes the Lambert function.

Proof. After some simplification, the derivative of $f(x;\alpha,\beta)$ with respect to x is given as a main tool by

$$\frac{d}{dx}f(x;\alpha,\beta) = \frac{\alpha^2(\beta+1)^2}{\beta^2}\exp(-\alpha(\beta+1)x)\left[\alpha\beta(\beta+1)x - \exp(\alpha\beta x)+1\right].$$

Therefore, a critical point for $f(x;\alpha,\beta)$ is a solution of the following equation: $\alpha\beta(\beta+1)x - \exp(\alpha\beta x)+1 = 0$. For $x > 0$, owing to (Jodrá, 2010, Lemma 1), the unique positive solution is given by Eqn. (4), and it is a maximum point since $\lim_{x\to 0} f(x;\alpha,\beta) = 0$ with $f(x;\alpha,\beta) > 0$. Thus, x_0 is the unique mode of the NTWE distribution, making it unimodal. This ends the proof of Proposition 1. □

As a precision, we can mention that the negative branch of the Lambert function is considered in the definition of x_0 in Eqn. (4); it is more mathematically correct to adopt the notation: $\mathscr{W}(x) = \mathscr{W}_{-1}(x)$.

For a direct graphical illustration, Figure 1 shows some plots of $f(x;\alpha,\beta)$ for selected values of α and β.

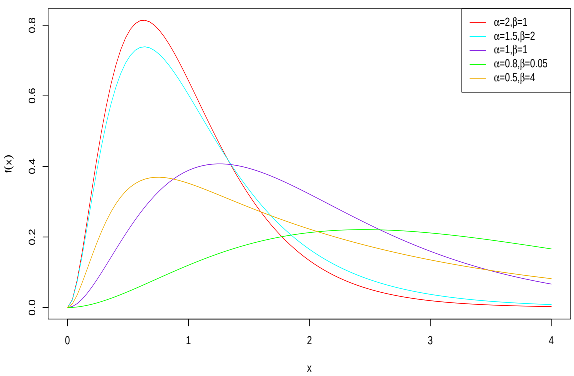

Figure 1 Illustrations of the pdf of the NTWE distribution for various values of α and β.

From Figure 1, we can see that the pdf has great mode-shape flexibility, and it can be "almost symmetric" and "right skewed". Diverse degrees of platedness are also observed.

By integrating $f(t;\alpha,\beta)$ for $t \in (0,x)$ with $x > 0$, the cdf of the NTWE distribution is given as

$$F(x;\alpha,\beta) = 1 - \frac{1}{\beta^2}\exp\left(-\alpha(\beta+1)x\right)\left[\exp(\alpha\beta x)(\beta+1)^2 - \alpha\beta(\beta+1)x - 2\beta - 1\right], \quad x > 0.$$

We naturally take $F(x;\alpha,\beta) = 0$ for $x \leq 0$.

The hrf is specified by $h(x;\alpha,\beta) = f(x;\alpha,\beta)/[1-F(x;\alpha,\beta)]$, wich yields

$$h(x;\alpha,\beta) = \alpha(\beta+1)^2 \frac{\exp(\alpha\beta x) - (1+\alpha\beta x)}{\exp(\alpha\beta x)(\beta+1)^2 - \alpha\beta(\beta+1)x - 2\beta - 1}, \quad x > 0$$

and $h(x;\alpha,\beta) = 0$ for $x \leq 0$.

A functional study of this hrf is now being conducted. The following equivalence and limit results hold at the limit points. When x tends to 0, we have $h(x;\alpha,\beta) \sim \alpha^3(\beta+1)^2x^2/2 \to 0$, and when x tends to $+\infty$, we have $h(x;\alpha,\beta) \sim \alpha$. So, for large values of x, a constant shape at $y = \alpha$ is expected. The following result is about an important monotonicity property of this hrf.

Proposition 2 The hrf of the NTWE distribution is increasing; the increasing failure rate property is satisfied.

Proof. After non-trivial developments, the derivative of $h(x;\alpha,\beta)$ with respect to x can be expressed as

$$\frac{d}{dx}h(x;\alpha,\beta) = \alpha^2\beta^2(\beta+1)^2 \frac{\exp(\alpha\beta x)\left[\alpha\beta(\beta+1)x + \exp(-\alpha\beta x) - 1\right]}{\left[\exp(\alpha\beta x)(\beta+1)^2 - \alpha\beta(\beta+1)x - 2\beta - 1\right]^2}.$$

By virtue of the following exponential inequality: $\exp(y) \geq 1 + y$ for $y \in \mathbb{R}$, for $x > 0$, we get

$$\alpha\beta(\beta+1)x + \exp(-\alpha\beta x) - 1 \geq \alpha\beta(\beta+1)x - \alpha\beta x = \alpha\beta^2 x > 0.$$

Therefore, $dh(x;\alpha,\beta)/dx > 0$, implying that $h(x;\alpha,\beta)$ is an increasing function in x, with a minimum attains when x tends to 0, and a maximum attains when $x \to +\infty$. This concludes the proof of Proposition 2. □

In Figure 2, we plot the hrf of the NTWE distribution for selected values of α and β.

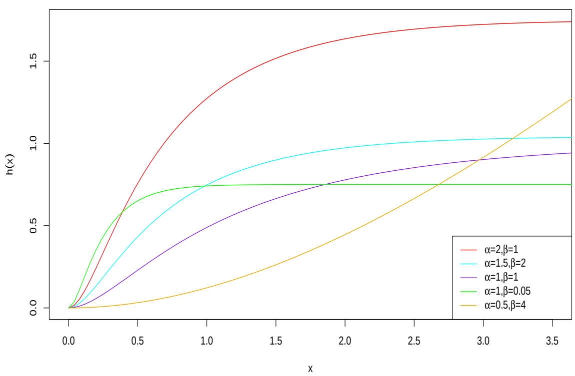

Figure 2 Illustrations of the hrf of the NTWE distribution for various values of α and β

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From Figure 2, we have visual confirmation that the hrf is increasing. As for new facts of importance, convex or concave shapes are observed.

The previous investigations show that the NTWE distribution has a wide range of possible applications in lifetime modeling. This is especially true when it comes to portraying wear-out or aging phenomena over time. The research continues with a moment analysis in the next part.

4. Moment Analysis

A moment analysis on the NTWE distribution is now conducted. The standard moments are the subject of the next result.

Proposition 3 Let s be a positive integer, and X be a random variable following the NTWE distribution, with the pdf specified by Eqn. (1). Then, the s^{th} standard moment of X is given as

$$m_s = E(X^s) = \frac{s!}{\alpha^s} \frac{1}{\beta^2(\beta+1)^s} \left((\beta+1)^{s+2} - (\beta+1) - \beta(s+1)\right).$$

Here, E refers to the expecation operator.

Proof. For this result, we propose two different proofs based on two interpretations discussed in Section 2.

- **Proof 1.** In the distribution sense, based on Eqn. (2), we can write $X = U + W$, where U and W are independent random variables, U following the exponential distribution with parameter α, and W following the Erlang distribution distribution with parameters 2 and $\alpha(\beta+1)$.

  As a first result, the s^{th} standard moment of X is obtained using the standard binomial formula and the well-known standard moments of the exponential and Erlang distributions. Precisely, we have

$$\begin{aligned} m_s &= E((U+W)^s) = \sum_{k=0}^{s} \binom{s}{k} E(U^{s-k})E(W^k) = \sum_{k=0}^{s} \binom{s}{k} \frac{(s-k)!}{\alpha^{s-k}} \frac{(k+1)!}{\alpha^k(\beta+1)^k} \\ &= \frac{s!}{\alpha^s} \sum_{k=0}^{s} (k+1)(\beta+1)^{-k} = \frac{s!}{\alpha^s} \frac{1}{\beta^2(\beta+1)^s} \left((\beta+1)^{s+2} - (\beta+1) - \beta(s+1)\right). \end{aligned}$$

- **Proof 2.** A more direct but more calculated approach is based on the mixture representation given as Eqn. (3). Precisely, we have

$$\begin{aligned} m_s &= \int_{-\infty}^{+\infty} x^s f(x;\alpha,\beta)dx \\ &= \frac{(\beta+1)^2}{\beta^2} \int_0^{+\infty} x^s f_1(x;\alpha)dx - \frac{\beta+1}{\beta^2} \int_0^{+\infty} x^s f_2(x;\alpha,\beta)dx - \frac{1}{\beta} \int_0^{+\infty} x^s f_3(x;\alpha,\beta)dx \\ &= \frac{(\beta+1)^2}{\beta^2} \frac{s!}{\alpha^s} - \frac{\beta+1}{\beta^2} \frac{s!}{\alpha^s(\beta+1)^s} - \frac{1}{\beta} \frac{(s+1)!}{\alpha^s(\beta+1)^s} \\ &= \frac{s!}{\alpha^s} \frac{1}{\beta^2(\beta+1)^s} \left((\beta+1)^{s+2} - (\beta+1) - \beta(s+1)\right). \end{aligned}$$

In each proof, the stated result is obtained, concluding the proof of Proposition 3. □

Hereafter, for further moment analysis, X is a random variable following the NTWE distribution. From Proposition 3, we deduce the mean and variance of X are

$$m = m_1 = \frac{\beta+3}{\alpha(\beta+1)} \quad \text{and} \quad \sigma^2 = m_2 - m^2 = \frac{\beta^2+2\beta+3}{\alpha^2(\beta+1)^2}, \quad \text{respectively.}$$

The standard binomial formula and Proposition 3 yield the s^{th} central moment of X. Indeed, we have

$$m_s^* = E((X-m)^s) = \sum_{k=0}^{s} \binom{s}{k} (-1)^{s-k} m^{s-k} m_k$$
$$= \frac{1}{\beta^2} \frac{(-1)^s s!(\beta+3)^s}{\alpha^s(\beta+1)^s} \sum_{k=0}^{s} \frac{1}{(s-k)!} (-1)^k (\beta+3)^{-k} \left((\beta+1)^{k+2} - (\beta+1) - \beta(k+1)\right).$$

Thus, m_s^* is simply calculable as a finite sum of coefficients. The skewness coefficient of X is derived as $S = m_3^*/\sigma^3$ and the kurtosis coefficient of X is derived as $K = m_4^*/\sigma^4$. Traditionally, the sign of S indicates the direction "left-symmetric-right" of the skewness, whereas the comparison of K with the value 3 indicates the nature of the kurtosis of the distribution "platykurtic-mesokurtic-leptokurtic"

Table 1 indicates numerical values for the quantities above, i.e., m, σ^2, m_3^*, m_4^*, S and K, for selected values of α and β. In some senses, Table 1 confirms what we have observed in Figure 1; For

Table 1 Numerical values of m, σ^2, m_3^*, m_4^*, S and K for various values of α and β

Parameter	m	σ^2	m_3^*	m_4^*	S	K
$\alpha = 0.5, \beta = 3$	3.0000	4.5000	16.5000	157.5000	1.7285	7.7778
$\alpha = 1.6, \beta = 2$	1.0417	0.4774	0.5245	1.6219	1.5898	7.1157
$\alpha = 3.65, \beta = 0.01$	0.8165	0.2222	0.1209	0.2469	1.1547	5.0002
$\alpha = 30, \beta = 10$	0.0394	0.0012	0.0000	0.0000	1.9543	8.8072
$\alpha = 3.5, \beta = 10$	0.3377	0.0829	0.0467	0.0606	1.9543	8.8072
$\alpha = 1, \beta = 1$	2.0000	1.5000	2.5000	13.5000	1.3608	6.0000
$\alpha = 3.4901, \beta = 10$	0.5730	0.1231	0.0588	0.0909	1.3608	6.0000
$\alpha = 3.6, \beta = 0.05$	0.8069	0.2171	0.1169	0.2359	1.1556	5.0043
$\alpha = 5, \beta = 1$	0.4000	0.0600	0.0200	0.0216	1.3608	6.0000

the considered values, it is clear that $S > 0$ and $K > 3$ meaning that the NTWE distribution is right skewed and leptokurtic.

The next finding is about the incomplete moments of the NTWE distribution.

Proposition 4 Let s be a positive integer, X be a random variable following the NTWE distribution, $y > 0$ be a certain threshold variable, and $I(X \leq y)$ be a binary random variable such that $I(X \leq y) = 1$ if $\{X \leq y\}$ is realized, and $I(X \leq y) = 0$ otherwise. Then, the s^{th} incomplete moment of X is given as

$$m_s(y) = E(X^s I(X \leq y)) = \frac{(\beta+1)^2}{\beta^2} \frac{1}{\alpha^s} \gamma(s+1, \alpha y) - \frac{\beta+1}{\beta^2} \frac{1}{\alpha^s(\beta+1)^s} \gamma(s+1, \alpha(\beta+1)y)$$
$$- \frac{1}{\beta} \frac{1}{\alpha^s(\beta+1)^s} \gamma(s+2, \alpha(\beta+1)y),$$

where $\gamma(a,b)$ denotes the lower incomplete gamma function: $\gamma(a,x) = \int_0^x t^{a-1} \exp(-t) dt$ for $x > 0$.

Proof. Thanks to the mixture representation in Eqn. (3) and some integral developments, we get

$$\begin{aligned}
m_s(y) &= \int_{-\infty}^{y} x^s f(x;\alpha,\beta)dx \\
&= \frac{(\beta+1)^2}{\beta^2}\int_0^y x^s f_1(x;\alpha)dx - \frac{\beta+1}{\beta^2}\int_0^y x^s f_2(x;\alpha,\beta)dx - \frac{1}{\beta}\int_0^y x^s f_3(x;\alpha,\beta)dx \\
&= \frac{(\beta+1)^2}{\beta^2}\frac{1}{\alpha^s}\gamma(s+1,\alpha y) - \frac{\beta+1}{\beta^2}\frac{1}{\alpha^s(\beta+1)^s}\gamma(s+1,\alpha(\beta+1)y) \\
&- \frac{1}{\beta}\frac{1}{\alpha^s(\beta+1)^s}\gamma(s+2,\alpha(\beta+1)y).
\end{aligned}$$

The stated result is obtained. □

In particular, from Proposition 4, the incomplete mean of X is obtained by putting $s=1$, and it is given by

$$m_1(y) = \frac{(\beta+1)^2}{\beta^2}\frac{1}{\alpha}\gamma(2,\alpha y) - \frac{\beta+1}{\beta^2}\frac{1}{\alpha(\beta+1)}\gamma(2,\alpha(\beta+1)y) - \frac{1}{\beta}\frac{1}{\alpha(\beta+1)}\gamma(3,\alpha(\beta+1)y),$$

with $\gamma(2,z) = 1-(1+z)\exp(-z)$ and $\gamma(3,z) = 2-(2+2z+z^2)\exp(-z)$.

In full generality, the incomplete mean naturally appears in the definition of important quantities, such as the Lorenz and Bonferroni curves. These curves may have applications in insurance, medical science, demography, economics, and reliability [see Bonferroni (1930) and Zenga (2007)]. In the context of the NTWE distribution, the Lorenz and Bonferroni curves are defined a given probability p by $L(p) = m_1(q)/m$ and $B(p) = L(p)/p$, respectively, where $q = F^{-1}(p;\alpha,\beta)$ and $p \in (0,1)$. We can investigate it numerically, thanks to the expression of $m_1(y)$, and the evaluation of q. Other residual life functions also depend on the incomplete moment, and can be examined in the setting of the NTWE distribution.

We now end this moment analysis by presenting the moment generating function of the NTWE distribution.

Proposition 5 Let X be a random variable following the NTWE distribution. Then,

- for $t < \alpha$, the moment generating function of X is given as

$$M(t) = \frac{\alpha^3(\beta+1)^2}{(\alpha-t)[\alpha(\beta+1)-t]^2}.$$

- for $t \in \mathbb{R}$, the characteristic function of X is given as

$$\varphi(t) = \frac{\alpha^3(\beta+1)^2}{(\alpha-it)[\alpha(\beta+1)-it]^2}.$$

Proof. Let us prove the two points in turn. Both points using the fact that, in the distribution sense, we can write $X = U+W$, where U and W are independent random variables, U following the exponential distribution with parameter α, and W following the Erlang distribution distribution with parameters 2 and $\alpha(\beta+1)$.

- For $t < \alpha$, by using the well-known expressions of the moment generating functions of the exponential and Erlang distributions, we get

$$M(t) = E(e^{tX}) = E(e^{tU})E(e^{tW}) = \left(\frac{\alpha}{\alpha-t}\right)\left(\frac{\alpha(\beta+1)}{\alpha(\beta+1)-t}\right)^2 = \frac{\alpha^3(\beta+1)^2}{(\alpha-t)[\alpha(\beta+1)-t]^2}.$$

- Similarly, for $t \in \mathbb{R}$, by using the well-known expressions of the characteristic functions of the exponential and Erlang distributions, we obtain

$$\varphi(t) = E(e^{itX}) = E(e^{itU})E(e^{itW}) = \frac{\alpha^3(\beta+1)^2}{(\alpha - it)[\alpha(\beta+1) - it]^2}.$$

The proof of Proposition 5 ends. □

From Proposition 5, we see that the moment generating and characteristic functions have manageable expressions, which can be of interest for the use of the NTWE distribution in several branches of applied statistics using such functions, as in time series models (autoregressive (AR) model, ...), queuing theory, and so on.

The rest of the study is about the practical side of the NTWE distribution; the NTWE distribution is thus turned out to be a statistical model, and estimation techniques are examined to make it suitable for the fitting of lifetime data of interest.

5. Estimation Method

In the NTWE model, the parameters α and β are now supposed to be unknown. As a result, they can be approximated using a variety of estimation methods. In this section, we will look at the maximum likelihood estimation method, which is described in detail in Casella and Berger (1990). Some basics are recalled below.

To begin, let $x_1, x_2, \ldots, x_n$ represent n observations of a random variable X with the NTWE distribution. These observations are supposed to be independent of each other and represent possible data for a lifetime phenoma whose value distribution is in adequation with the NTWE distribution. Then, from Eqn. (1), the log-likelihood function based on these observations is obtained as

$$\ell(x_1, \ldots, x_n; \alpha, \beta) = \sum_{i=1}^{n} \log[f(x_i; \alpha, \beta)] = n \log \alpha + 2n \log(\beta+1) - 2n \log(\beta) - \alpha \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \log[1 - (1 + \alpha\beta x_i)\exp(-\alpha\beta x_i)].$$

The maximum likelihood estimates (MLEs) of α and β, say $\hat{\alpha}$ and $\hat{\beta}$, are obtained by maximizing the function $\ell(x_1, \ldots, x_n; \alpha, \beta)$ with respect to α and β. They should ideally satisfy the following equations: $\partial \ell(x_1, \ldots, x_n; \alpha, \beta)/\partial \alpha = 0$ and $\partial \ell(x_1, \ldots, x_n; \alpha, \beta)/\partial \beta = 0$, where

$$\frac{\partial \ell(x_1, \ldots, x_n; \alpha, \beta)}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^{n} x_i + \alpha\beta^2 \sum_{i=1}^{n} \frac{x_i^2 \exp(-\alpha\beta x_i)}{1 - (1 + \alpha\beta x_i)\exp(-\alpha\beta x_i)}$$

and

$$\frac{\partial \ell(x_1, \ldots, x_n; \alpha, \beta)}{\partial \beta} = \frac{2n}{\beta+1} - \frac{2n}{\beta} + \alpha^2\beta \sum_{i=1}^{n} \frac{x_i^2 \exp(-\alpha\beta x_i)}{1 - (1 + \alpha\beta x_i)\exp(-\alpha\beta x_i)}.$$

Clearly, $\hat{\alpha}$ and $\hat{\beta}$ cannot be determined explicitly, but efficient numerical methods exist to allow a precise numerical evaluation of them. As main asymptotic distribution result, when n is large enough and under precise but technical regularity hypotheses, the subjacent distribution of the random vector giving $(\hat{\alpha}, \hat{\beta})$ can be approximate by the bivariate Gaussian distribution $\mathcal{N}_2\left((\alpha, \beta), J(\hat{\alpha}, \hat{\beta})^{-1}\right)$, where $J(\hat{\alpha}, \hat{\beta})^{-1}$ denotes the inverse of the following matrix taken at $(\alpha, \beta) = (\hat{\alpha}, \hat{\beta})$:

$$J(\alpha, \beta) = -\begin{pmatrix} \dfrac{\partial^2 \ell(x_1, \ldots, x_n; \alpha, \beta)}{\partial \alpha^2} & \dfrac{\partial^2 \ell(x_1, \ldots, x_n; \alpha, \beta)}{\partial \alpha \partial \beta} \\ \dfrac{\partial^2 \ell(x_1, \ldots, x_n; \alpha, \beta)}{\partial \beta \partial \alpha} & \dfrac{\partial^2 \ell(x_1, \ldots, x_n; \alpha, \beta)}{\partial \beta^2} \end{pmatrix},$$

with

$$\frac{\partial^2 \ell(x_1,\ldots,x_n;\alpha,\beta)}{\partial \alpha^2} = -\frac{n}{\alpha^2} - \beta^2 \sum_{i=1}^{n} \frac{x_i^2\left((\alpha\beta x_i - 1)\exp(\alpha\beta x_i) + 1\right)}{(\exp(\alpha\beta x_i) - (1+\alpha\beta x_i))^2},$$

$$\frac{\partial \ell(x_1,\ldots,x_n;\alpha,\beta)}{\partial \beta^2} = -\frac{2n}{(\beta+1)^2} + \frac{2n}{\beta^2} - \alpha^2 \sum_{i=1}^{n} \frac{x_i^2\left((\alpha\beta x_i - 1)\exp(\alpha\beta x_i) + 1\right)}{(\exp(\alpha\beta x_i) - (1+\alpha\beta x_i))^2}$$

and

$$\frac{\partial \ell(x_1,\ldots,x_n;\alpha,\beta)}{\partial \alpha \partial \beta} = -\alpha\beta \sum_{i=1}^{n} \frac{x_i^2\left((\alpha\beta x_i - 2)\exp(\alpha\beta x_i) + \alpha\beta x_i + 2\right)}{(\exp(\alpha\beta x_i) - (1+\alpha\beta x_i))^2}.$$

On the basis of this asymptotic result, asymptotic confidence intervals for α and β, and likelihood statistical tests can be established, which remain undeniable advantages of considering the maximum likelihood method in comparison to other methods. Also, criteria of goodness-of-fit of the NTWE model with concrete data are simply defined with the estimated log-likelihood function given as $\hat{\ell} = \ell(\hat{\alpha},\hat{\beta})$, such as the Akaike information criterion (AIC), Bayesian information criterion (BIC), correct Akaike information criterion (AICc), and Hannan-Quinn information criterion (HQIC). These statistical objects are concretely defined as AIC $= 2k - 2\hat{\ell}$, BIC = $k\log(n) - 2\hat{\ell}$, AICc = AIC $+2k(k+1)/(n-k-1)$, and HQIC $= -2\hat{\ell} + 2k\log(\log(n))$, where k denotes the number of parameters in the model under consideration. These definitions can be adapted to other models by substituting their log-likelihood function and MLEs in the appropriate places. The interpretation of these statistical objects is simple: When several models are considered, the model with the minimum values of AIC, BIC, AICc, and HQIC is considered as the best to fit the considered data.

In the (right) censoring scheme, the log-likelihood function must be modified as follows:

$$\ell(x_1,\ldots,x_n;\alpha,\beta) = \sum_{i=1}^{n} \delta_i \log[f(x_i;\alpha,\beta)] + \sum_{i=1}^{n} (1-\delta_i)\log[1 - F(x_i;\alpha,\beta)],$$

with $\delta_i = 1$ if uncensored and $\delta_i = 0$, if censored, and the main theory of the MLEs can be transposed with a slight adaptation.

6. Applications

In this section, we illustrate the usefulness of the NTWE model. We fit the NTWE model to four different data sets, two complete and two censored, and compare the results with those of the fitted WE, TWE and EWE models. The details of these competitor models can be found in Gupta and Kundu (2009), Shakhatreh (2012) and Mahdavi and Jabbari (2017), respectively. As a main point, the TWE and EWE models are more complex than the NTWE model in the sense that they have one more parameter. We will see that the NTWE model can, however, outperform them for the considered data sets.

Guinea pigs data set Bjerkedal (1960) observed and reported the survival periods (in days) of 72 guinea pigs injected with various levels of tubercles. This constitutes a data set provided below.

{12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376}

By applying our methodology to this data set, the essential numerical requirements for the four fits based on the guinea pig data set are listed in Table 2.

The graphs of the estimated pdfs and cdfs of the considered models for the guinea pig data set are shown in Figure 3.

Table 2 Estimated parameter values, $\hat{\ell}$, AIC, BIC,AICc and HQIC for the guinea pigs data set

Model	Estimates	$-\hat{\ell}$	AIC	BIC	AICc	HQIC
NTWE	$\hat{\alpha} = \mathbf{0.0141}$, $\hat{\beta} = \mathbf{3.9615}$	**391.3671**	**786.7341**	**791.2875**	**786.908**	**788.5469**
WE	$\hat{\alpha} = 1.6241$, $\hat{\lambda} = 0.0138$	393.5689	791.1379	795.6912	791.3118	792.9505
TWE	$\hat{\alpha}_1 = 2.8464$, $\hat{\alpha}_2 = 2.8464$, $\hat{\lambda} = 0.0141$	391.5069	789.0138	795.8438	789.3667	791.7328
EWE	$\hat{\alpha} = 3.9035$, $\hat{\beta} = 3.0313$, $\hat{\lambda} = 0.0141$	391.3828	788.7657	795.5957	789.1186	791.4846

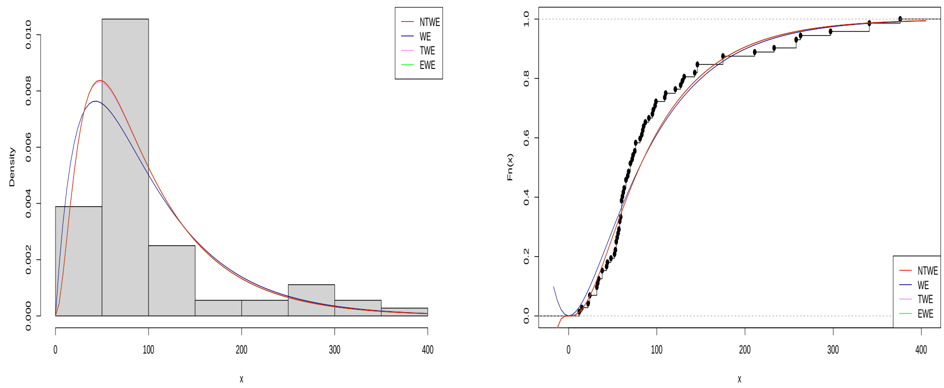

Figure 3 Estimated pdfs (left) and cdfs (right) of the fitted models for the guinea pig data set

COVID-19 death data set This data set represents an actual daily death number because of COVID-19 in Israel in a short period of the summer: $\{29^{th}$ july 2021 to 17^{th} august 2021$\}$. This data is taken from the given website https://www.worldometers.info/coronavirus/country/israel/, which indicates the daily number of deaths due to COVID-19 in Israel. It is given below.

$\{4, 3, 4, 4, 10, 8, 8, 6, 7, 19, 7, 17, 12, 16, 17, 7, 11, 46, 19, 17\}$

Table 3 gives the relevant statistical criteria summaries for the four fits based on COVID-19 death data sets.

Table 3 Estimated parameter values, $\hat{\ell}$, AIC, BIC, AICc and HQIC for the COVID-19 death data set

Model	Estimates	$-\hat{\ell}$	AIC	BIC	AICc	HQIC
NTWE	$\hat{\alpha} = \mathbf{0.1223}$, $\hat{\beta} = \mathbf{3.1731}$	**66.0618**	**136.1237**	**138.1151**	**136.8296**	**136.5124**
WE	$\hat{\alpha} = 0.9379$, $\hat{\lambda} = 0.1253$	66.7164	137.4328	141.9862	138.1387	137.8216
TWE	$\hat{\alpha}_1 = 2.2270$, $\hat{\alpha}_2 = 2.2270$, $\hat{\lambda} = 0.1234$	66.0808	138.1616	144.9916	139.6616	138.7447
EWE	$\hat{\alpha} = 3.1981$, $\hat{\beta} = 622.6428$, $\hat{\lambda} = 0.1219$	66.06211	138.1242	141.1114	139.6242	138.7074

The graphs of the estimated pdfs and cdfs of the evaluated models for the COVID-19 death data set are shown in Figure 4.

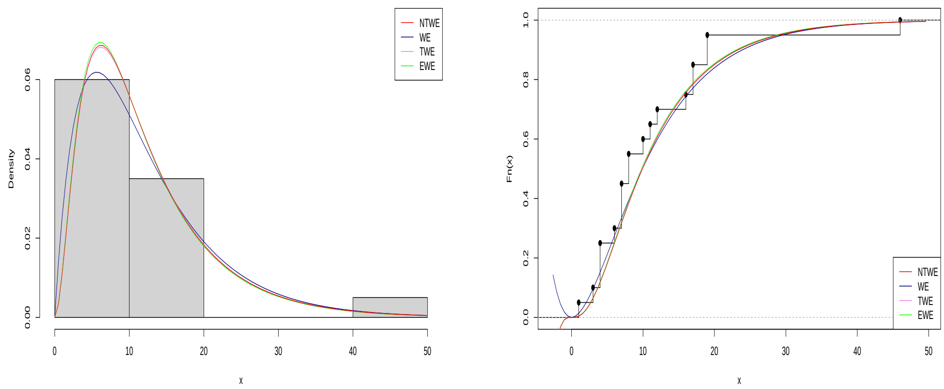

Figure 4 Estimated pdfs (left) and cdfs (right) of the fitted models for the COVID-19 death data set

Censored HIV data set This data set represents the survival times (in months) of 100 patients infected with HIV, observed and reported by Hosmer and Lemeshow (1999), where the + sign indicates a right-censored time. It is given below.

{5, 6+, 8, 3, 22, 1+, 7, 9, 3, 12, 2+, 12, 1, 15, 34, 1, 4, 19+, 3+, 2, 2+, 6, 60+, 7+, 60+, 11, 2+, 5, 4+, 1+, 13, 3+, 2+, 1+, 30, 7+, 4+ ,8+, 5+, 10, 2+, 9+, 36, 3+, 9+, 3+, 35, 8+, 1+, 5+ ,11, 56+ ,2+, 3+, 15, 1+, 10, 1+, 7+, 3+ ,3+ ,2+, 32, 3+, 10+, 11 ,3+, 7+, 5+, 31, 5+, 58, 1+, 2+, 1, 3+, 43, 1+, 6+, 53, 14, 4+, 54, 1+, 1+, 8+, 5+, 1+, 1+ ,2+, 7+, 1+, 10, 24+, 7+, 12+, 4+, 57, 1+, 12+}.

The essential numerical criteria for the four fits based on the censored HIV data set are listed in Table 4.

Table 4 Estimated parameter values, $\hat{\ell}$, AIC, BIC, AICc and HQIC for the censored HIV data set

Model	Estimates	$-\hat{\ell}$	AIC	BIC	AICc	HQIC
NTWE	$\hat{\alpha} = \mathbf{0.0351}$, $\hat{\beta} = \mathbf{65.8195}$	**162.1595**	**328.319**	**333.5294**	**328.4427**	**330.4277**
WE	$\hat{\alpha} = 15.8139$, $\hat{\lambda} = 0.03714$	162.1599	328.3198	333.5302	328.4435	330.4285
TWE	$\hat{\alpha}_1 = 16.5029$, $\hat{\alpha}_2 = 67.0532$, $\hat{\lambda} = 0.0375$	162.1321	330.2642	338.0797	330.5116	333.4273
EWE	$\hat{\alpha} = 65.7086$, $\hat{\beta} = 33.6869$, $\hat{\lambda} = 0.0351$	162.1599	330.3198	338.1353	330.5672	333.4829

Censored leukemia data set This data set represents the remission time (in weeks) for a set of 21 leukemia patients treated with a drug 6-mercaptopurine, observed and reported by Freireich *et al.* (1963), where the + sign indicates a right-censored time. It is given below.

{ 6, 6, 6, 7, 10, 13, 16, 22, 23, 6+, 9+, 10+, 11+, 17+, 19+, 20+, 25+, 32+, 32+, 34+, 35+}.

Table 5 displays the numerical criteria for the four fits based on the censored leukemia data set.

Table 5 Estimated parameter values, $\hat{\ell}$, AIC, BIC, AICc and HQIC for the censored leukemia data set

Model	Estimates	$-\hat{\ell}$	AIC	BIC	AICc	HQIC
NTWE	$\hat{\alpha} = \mathbf{0.0323}$, $\hat{\beta} = \mathbf{14.9627}$	**40.7519**	**85.5039**	**87.593**	**86.1706**	**85.9573**
WE	$\hat{\alpha} = 7.4091$, $\hat{\lambda} = 0.0318$	41.0916	86.18331	88.2723	86.8499	86.6366
TWE	$\hat{\alpha}_1 = 10.8087$, $\hat{\alpha}_2 = 10.7799$, $\hat{\lambda} = 0.0325$	40.7813	87.5626	90.6961	88.89591	88.24264
EWE	$\hat{\alpha} = 14.2838$, $\hat{\beta} = 1.9506$, $\hat{\lambda} = 0.0327$	40.7625	87.52501	90.6586	88.8583	88.2051

As a result, the MLEs of the parameters for the fitted models, as well as $-\hat{\ell}$, AIC, BIC, AICc, and HQIC, are presented in Tables 2 and 3 for two different complete data sets. Based on the lowest values of the AIC, BIC, AICc, and HQIC, the NTWE model turns out to be a better model than the WE, TWE and EWE models. Visual comparisons of the closeness of the estimated pdfs with the observed histogram of the data and estimated cdfs with empirical cdfs for different complete data sets are presented in Figures 3 and 4, respectively. These plots indicate that the proposed model provides a closer fit to these data.

The MLEs of the parameters for the fitted models, as well as $-\hat{\ell}$, AIC, BIC, AICc, and HQIC values, are presented in Tables 4 and 5 for two different censored data sets. Based on the lowest values of the AIC, BIC, AICc and HQIC, the NTWE model turns out to be a better model than the WE, TWE and EWE models.

7. Conclusion

In this paper, a new two-parameter lifetime distribution based on the Azzalini technique under a special configuration involving the Erlang distribution, called the new two-parameter weighted exponential (NTWE) distribution, is proposed. A consequent list of accessible mathematical interpretations is provided. Among other functional properties, we show that the pdf is exclusively unimodal with a varying tail weight, and that the hrf increases with concave or convex features. In addition, the pdf can be expressed as a linear combination of the exponential and Erlanf pdfs. Based on this result, we derive some structural properties of the NTWE distribution and provide the moments, incomplete moments, skewness, kurtosis, moment generating function and characteristic function. Then, the NTWE model is examined from the statistical viewpoint. The model parameters are estimated by the maximum likelihood method. Four different data sets, including two complete and two censored data sets, are applied to demonstrate that the NTWE model can provide a better fit than some reference models, namely the weighted exponential (WE) model, two-parameter weighted exponential (TWE) model and (three-parameter) extended weighted exponential (EWE) model. Future research includes the construction of extended NTWE models involving one more parameter via weighted or powered techniques, a NTWE discrete model for the analysis of count data, and a bivariate NTWE model for the analysis of bivariate data.

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