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The Accurate Results of Average Run Length on Modified EWMA Control Chart for the First-Order Moving Average Process with Exogenous Variables Models

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Abstract

In this study, we derive explicit formulas for the average run length (ARL) of a first-order moving average process with exogenous variables (MAX(1,1)) with exponential white noise and compare their performance on standard and modified and exponentially weighted moving average (EWMA) control charts based on the absolute percentage relative error (APRE) and the relative mean index (RMI). Moreover, we compare the accuracy and CPU time of the explicit formulas with the ARL based on the numerical integral equation (NIE) method derived by using the Gauss-Legendre quadrature rule for the same process on the two control charts. To demonstrate the capability of our explicit formulas approach, we applied it to two real datasets: the closing stock prices for the PTT public company limited with the THB/USD daily foreign exchange rate as the exogenous variable and the monthly gold futures price with the crude oil futures price as the exogenous variable. The results of applying the ARL based on the explicit formulas with the two real datasets show that the modified EWMA control chart performed better than the EWMA control chart under these circumstances.

Keywords: ARL, MAX process, explicit formulas, monitoring process, explanatory variable

1. Introduction

Statistical process control (SPC) is a technique for monitoring and controlling a process using statistical methodologies. The objective of statistical process control is to identify and reduce sources of variation as part of a strategy to enhance output quality. SPC involves analyzing process data with statistical tools such as control charts, histograms, and Pareto charts to identify patterns of variation. Control charts are the most frequently employed SPC instrument for plotting process data over time and identifying trends or patterns that could indicate a process is out of control. SPC can be utilized in manufacturing, healthcare, and service industries. Implementing SPC benefits include improved product and process quality, increased efficiency and productivity, and reduced costs associated with defects and waste.

The initial control chart proposed by Shewhart (1931) is appropriate for significant changes in the process mean or variance when the observations have a normal distribution. Later, Page (1954) introduced the cumulative sum (CUSUM) control chart, and Roberts (1959) introduced the exponentially weighted moving average (EWMA) control chart based on the Shewhart control chart, both of which can detect minor variations in the process mean. Lucas and Saccucci (1990), Serel (2009), and Barbeito et al. (2017) all report the benefits of utilizing the EWMA control chart. The modified EWMA control chart presented by Alpaben and Jyoti (2011) is more effective than the EWMA control chart at detecting minor and rapid changes in the process mean. Later, Khan et al. (2017) enhanced the modified EWMA control chart by introducing an exponential smoothing parameter value that outperformed the two existing EWMA control charts concerning the average run length (ARL).

The ARL is a measure of the performance of the control chart, the limits of which are defined as follows: ARL_0 is the average number of observations when the process is in control, which ought to be as large as possible, and ARL_1 is the average number of observations until the process becomes out-of-control, which ought to be as small as possible. Besides, several methods have been used to evaluate the ARL, such as the Markov Chain, martingale, Monte Carlo simulation, and numerical integration equation (NIE) approaches. Mastrangelo and Montgomery (1995) evaluated the EWMA control chart's performance running serially-correlated processes using the Monte Carlo simulation technique. Vanbrack and Reynold (1997) studied an AR(1) process with additional random error running on EWMA and CUSUM control charts by evaluating the ARL via the NIE and Markov Chain approaches.

The observations in practical situations are always collected from stochastic processes contingent on time-space or time series. Moreover, the models established under the econometric models were specified in time-series models. The observations in econometrics as a time series enclose of autoregressive (AR) model and moving average (MA) model. However, the moving averages process is sometimes unpredictable in identifying the movement patterns of time series due to several factors embedded in observations as a seasonal moving average (SMA) model. Furthermore, the errors caused differences between accurate and predictable values; more minor errors indicate better accuracy. The error of time series is called white noise, and for seasonal factors model with auto-correlated which is called exponential white noise (see Andel 1988, Turkman 1990, and Ibazizen 2003). The exogenous variable is not influenced by other variables, which is favored in econometric models. For example, saving rates, diminishing returns to capital, and technological variables determine economic growth. A time series model with exogenous variables is a statistical model that attempts to explain the behavior of a time series variable, such as a stock price or temperature measurement, by incorporating the influence of one or more external factors, known as exogenous variables. The time series variable is the dependent variable in this model, while the exogenous variables are the independent variables. The model attempts to encapsulate the relationships between these variables to predict the future values of the time series variable. The model can be estimated using various methods, including maximum likelihood estimation and least squares estimation. Once the model has been estimated, it can be used to predict future values of the dependent variable based on the importance of the exogenous variables and the initial values of the dependent variable. Incorporating exogenous variables into a time series model can increase its precision by considering external factors that may influence the behavior of the time series variable. However, selecting the appropriate exogenous variables is crucial and ensuring they are not affected by the time series variable. When forecasting process with exogenous variables that is usually X variables at least one variable for more accuracy. Normally, for established element of AR, MA or SMA with exogenous variable able to specify as ARX, MAX and SMAX, respectively. The numerical integration equation method which is popularly used in continuous distribution in observations. Moreover, there is observation researching that utilize same method compare with explicit formulas by using the performance of a modified EWMA control chart (see Supharakonsakun et al. 2020 and Silpakob et al. 2021).

Several researchers have derived explicit formulations for the ARL of EWMA control chart-based processes. Sukparungsee and Areepong (2009) derived explicit formulations for the ARL of a process operating on an EWMA control chart and contrasted the precision of numerical results using a Monte Carlo simulation study. Areepong (2012) provided explicit formulations for the ARL of a process operating on a moving average (MA) control chart when the observations have a binomial distribution. Petcharat et al. (2015) developed explicit formulations based on the Fredholm integral equation of the second kind for the ARL of an MA(q) process with exponential white noise operating on a CUSUM control chart. Exogenous variables, such as exchange rates, interest rates, and inflation rates, can influence the outcome of forecasts, such as economic growth forecasts, and must be included in the model. Paichit (2016) utilized NIEs to determine the precise expression for the ARL of a p-order AR process with an exogenous variable (ARX(p)) operating on an EWMA control chart.

Thus, our objective is to derive explicit formulas or the ARL and compare them with the NIE method show that explicit formulas was more quickly evaluated based on the Gauss-Legendre quadrature rule for a MAX(1,1) process with exponential white noise running on a modified EWMA control chart. For real data that use for observation process which is component with two set of data. Firstly, closing stock price for PTT public company limited with the THB/USD daily foreign exchange rate as the exogenous variable. Secondly, daily highest silver price with the highest crude oil price as the exogenous variable. Moreover, we compared its performance on modified and EWMA control charts based on the absolute percentage relative error (APRE) and the relative mean index (RMI).

2. Materials and Methods

The EWMA control chart used to monitor and detect small changes in the process mean by Roberts (1959) can be derived using the recursive equation.

$$E_t = (1-\lambda)E_{t-1} + \lambda Y_t\,,\ t = 1, 2, 3, ... \tag{1}$$

where E_t is the EWMA statistic, $0 < \lambda < 1$ is an exponential smoothing parameter, and Y_t is the sequence of the $\text{MAX}(1,1)$ process with exponential white noise. The mean and variance of the EWMA control chart are $E(E_t)$ and $\text{Var}(E_t) = \sigma^2\left(\frac{\lambda}{2-\lambda}\right)$, respectively. Therefore, the general upper control limit (UCL) and lower control limit (LCL) to detect the sequence are respectively given by

$$UCL = \mu_0 + L\sigma\sqrt{\frac{\lambda}{2-\lambda}} \tag{2}$$

$$LCL = \mu_0 - L\sigma\sqrt{\frac{\lambda}{2-\lambda}}, \tag{3}$$

where μ_0 is the target mean, σ is the process standard deviation, and L is an appropriate control width limit. The stopping time for the EWMA control chart for one-sided is given by

$$\zeta_h = \inf\{t > 0 : E_t < l \text{ or } E_t > h\}, \tag{4}$$

where ζ_h is the stopping time, l is the LCL, and h is the UCL.

Khan et al. (2017) proposed a new structure for the control statistics of the modified EWMA control chart by using the following recursive equation

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t + k(Y_t - Y_{t-1}), \tag{5}$$

where $0 < \lambda < 1$ is an exponential smoothing parameter and k is a constant. The mean and variance of the modified EWMA control chart are $E(Z_t) = \mu_0$ and $\operatorname{Var}(Z_t) = \sigma^2\left(\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}\right)$, respectively. Therefore, the general UCL and LCL to detect the sequence are respectively given by

$$UCL = \mu_0 + H\sigma\sqrt{\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}} \tag{6}$$

$$LCL = \mu_0 - H\sigma\sqrt{\frac{\lambda + 2\lambda k + 2k^2}{2-\lambda}}, \tag{7}$$

where μ_0 is the target mean, σ is the process standard deviation, H is an appropriate control width limit, Y_t is the sequence of observations, $Z_0 = u$ and $Y_0 = v$ are the initial values, and $0 < \lambda \le 1$ is an exponential smoothing parameter. The stopping time for the modified EWMA control chart is given by

$$\zeta_b = \inf\{t > 0 : Z_t < a = 0 \text{ or } Z_t > b\}, \tag{8}$$

where ζ_b is the stopping time, a is the LCL, and b is the UCL.

3. The ARL of Modified EWMA Control Chart

3.1. The exact solution of ARL the modified EWMA control chart for MAX(1,1) process with exponential white noise

A MAX (1,1) process with exponential white noise can be derived as

$$Y_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1} + \beta X_t, \tag{9}$$

where μ is a constant, ε_t is the white noise process $\varepsilon_t \sim Exp(\alpha)$, θ is the MA coefficient with an initial value of $\varepsilon_0 = s$, X_t is an exogenous variable, and β is the coefficient of X_t. Therefore, modified EWMA statistic (Z_t) can be written as

$$Z_t = (1-\lambda)Z_{t-1} + (\lambda + k)(\mu + \varepsilon_t - \theta\varepsilon_{t-1} + \beta X) - kY_{t-1},$$

where $0 < \lambda \le 1$ and initial values $Z_0 = u$, $Y_0 = v$, $\varepsilon_0 = s$, $X_t = X$, $LCL = a = 0$, and

$$Z_1 = (1-\lambda)u + (\lambda + k)(\mu + \varepsilon_1 - \theta s + \beta X) - kv.$$

If ε_1 give an in-control process for Z_1, then $0 < Z_1 \le b$. So,

$$0 < (1-\lambda)u + (\lambda + k)(\mu + \varepsilon_1 - \theta s + \beta X) - kv < b. \tag{10}$$

The inequality in (10) can be rewritten as following

$$\frac{-\big((1-\lambda)u + (\lambda + k)(\mu - \theta s + \beta X) - kv\big)}{(\lambda + k)} < \varepsilon_1 < \frac{b - \big((1-\lambda)u + (\lambda + k)(\mu - \theta s + \beta X) - kv\big)}{(\lambda + k)}.$$

Thus, the integral equations where $L(u)$ denotes the ARL on the modified EWMA control chart can be derived as follows

$$L(u) = 1 + \int_{\frac{-((1-\lambda)u + (\lambda+k)(\mu - \theta s + \beta X) - kv)}{(\lambda+k)}}^{\frac{b-((1-\lambda)u + (\lambda+k)(\mu - \theta s + \beta X) - kv)}{(\lambda+k)}} L\{(1-\lambda)u - kv + (\lambda + k)(\mu - \theta s + \beta X) + (\lambda + k)y\} f(y)dy$$

By changing the integral variable, we obtain the integral equation as

$$L(u) = 1+\frac{1}{\lambda+k}\int_0^b L(w)\, f\left\{\frac{w-(1-\lambda)u+kv}{\lambda+k}-\mu+\theta s-\beta X\right\}dw, \tag{11}$$

where $w=(1-\lambda)u-kv+(\lambda+k)(\mu-\theta s)+(\lambda+k)(\beta X+y)$. Let $C(I)$ be a nonempty and closed set in a Banach space. Assume that $T:Y\to Y$ is a contraction mapping, with contraction constant $0\le g<1$ such that $\|T(L_1)-T(L_2)\|\le g\|L_1-L_2\|$ for all $L_1,L_2\in Y$. Then there exists a unique $L(.)\in Y$ such that $T(L(u))=L(u)$, i.e., has a unique-fixed point in Y.

The integral equations for the ARL by using explicit formulas can be shown to uniquely exist by applying Banach's fixed-point theorem.

$$T(L(u)) = 1+\frac{1}{\lambda+k}\int_0^b L(w)\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left\{\frac{w-(1-\lambda)u+kv}{\lambda+k}-\mu+\theta s-\beta X\right\}}dw. \tag{12}$$

Proof: First, to show that T is a contraction for any $u\in I$, and $L_1,L_2\in C(I)$. We will show that $\|T(L_1)-T(L_2)\|\le g\|L_1-L_2\|$ for all $L_1,L_2\in C(I)$ with $0\le g<1$, then

$$\begin{aligned}
\|T(L_1)-T(L_2)\|_\infty &= \sup_{u\in[0,b]}\left|\begin{array}{l} L_1(w)-L_2(w)\dfrac{1}{\alpha(\lambda+k)}e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}} \\ \times\displaystyle\int_0^b L(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw\end{array}\right| \\
&\le \sup_{u\in[0,b]}\left|\begin{array}{l} \|L_1-L_2\|\dfrac{1}{\alpha(\lambda+k)}e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}} \\ \times(-\alpha(\lambda+k))\left(e^{\frac{-b}{\alpha(\lambda+k)}}-1\right)\end{array}\right| \\
&= \|L_1-L_2\|_\infty\left|1-e^{\frac{-b}{\alpha(\lambda+k)}}\right|\sup_{u\in[0,b]}\left|e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}}\right| \quad \le g\|L_1-L_2\|_\infty,
\end{aligned}$$

where $g=\left|1-e^{\frac{-b}{\alpha(\lambda+k)}}\right|\sup_{u\in[0,b]}\left|e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}}\right|$ and $C(u)=e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}}$; $0\le g<1$. Therefore, the uniqueness of the solution are guaranteed by the Banach's fixed point theorem. Therefore, $L(u)$ can be written as

$$\begin{aligned}
L(u) &= 1+\frac{1}{\lambda+k}\int_0^b L(w)\times\frac{1}{\alpha}e^{-\frac{1}{\alpha}\left(\frac{w-(1-\lambda)u+kv}{\lambda+k}-\mu+\theta s\right)-\frac{1}{\alpha}(-\beta X)}dw \\
&= 1+\frac{1}{\alpha(\lambda+k)}\int_0^b L(w)e^{\frac{-w}{\alpha(\lambda+k)}}\ e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}}dw
\end{aligned}$$

where $C(u)=e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)}+\frac{\mu}{\alpha}-\frac{\theta s}{\alpha}+\frac{\beta X}{\alpha}}$. Therefore,

$$L(u) = 1+\frac{C(u)}{\alpha(\lambda+k)}\int_0^b L(w)e^{\frac{-w}{\alpha(\lambda+k)}}dw = 1+\frac{C(u)}{\alpha(\lambda+k)}\cdot d \tag{13}$$

From (13), let $d = \int_0^b L(w) e^{\frac{-w}{\alpha(\lambda+k)}} dw = \int_0^b (1 + \frac{dC(w)}{\alpha(\lambda+k)}) e^{\frac{-w}{\alpha(\lambda+k)}} dw$

$$= \int_0^b e^{\frac{-w}{\alpha(\lambda+k)}} dw + \frac{d}{\alpha(\lambda+k)} \int_0^b e^{\frac{-w}{\alpha(\lambda+k)}} e^{(\frac{(1-\lambda)w-kv}{\alpha(\lambda+k)} + \frac{\mu}{\alpha} - \frac{\theta s}{\alpha} + \frac{\beta X}{\alpha})} dw$$

$$= -\alpha(\lambda+k)(e^{\frac{-b}{\alpha(\lambda+k)}} - 1) - \frac{d}{\lambda} e^{(-\frac{kv}{\alpha(\lambda+k)} + \frac{\mu}{\alpha} - \frac{\theta s}{\alpha} + \frac{\beta X}{\alpha})} (e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1),$$

$$d = \frac{-\alpha(\lambda+k)(e^{\frac{-b}{\alpha(\lambda+k)}} - 1)}{1 + \frac{1}{\lambda} e^{(-\frac{kv}{\alpha(\lambda+k)} + \frac{\mu}{\alpha} - \frac{\theta s}{\alpha} + \frac{\beta X}{\alpha})} \times (e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1)}. \tag{14}$$

After that, (14) is substituted into (13) can be written as

$$L(u) = 1 + \frac{e^{\frac{(1-\lambda)u-kv}{\alpha(\lambda+k)} + \frac{\mu}{\alpha} - \frac{\theta s}{\alpha} + \frac{\beta X}{\alpha}} (-\alpha(\lambda+k)(e^{\frac{-b}{\alpha(\lambda+k)}} - 1))}{\alpha(\lambda+k) 1 + \frac{1}{\lambda} e^{(-\frac{kv}{\alpha(\lambda+k)} + \frac{\mu}{\alpha} - \frac{\theta s}{\alpha} + \frac{\beta X}{\alpha})} \times (e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1)} = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}} (e^{\frac{-b}{\alpha(\lambda+k)}} - 1)}{\lambda e^{-(-\frac{kv}{\alpha(\lambda+k)} + \frac{\mu}{\alpha} - \frac{\theta s}{\alpha} + \frac{\beta X}{\alpha})} + e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1}.$$

Therefore,

$$L(u) = 1 - \frac{\lambda e^{\frac{(1-\lambda)u}{\alpha(\lambda+k)}} (e^{\frac{-b}{\alpha(\lambda+k)}} - 1)}{\lambda e^{\frac{1}{\alpha}(\frac{kv}{\lambda+k} - \mu + \theta s - \beta X)} + e^{\frac{-\lambda b}{\alpha(\lambda+k)}} - 1}. \tag{15}$$

4. Numerical Results

Here, we compare the results for ARL_0 and ARL_1 derived by using explicit formulas and the NIE method for a MAX(1,1) process with exponential white noise running on a modified EWMA chart. The numerical results were computed by using MATHEMATICA with the number of division points set as 1,000. The performances are reported as the absolute percentage relative error, which is derived as

$$APRE(\%) = \frac{|ARL_{Explicit\ Formula} - ARL_{NIE}|}{ARL_{Explicit\ Formula}} \times 100.$$

For comparison, the performance measure for the ARL of a MAX(1,1) process with exponential white noise on the EWMA and modified EWMA control chart is the RMI, which is computed as

$$RMI = \frac{1}{n} \sum_{i=1}^{n} \left(\frac{ARL_{shift,i} - Min[ARL_{shift,i}]}{Min[ARL_{shift,i}]} \right),$$

where $ARL_{shift,i}$ is the ARL of the control chart when a shift in the process mean is detected and $Min[ARL_{shift,i}]$ is the minimum value of the ARL at the same level. The results for the one-sided ARL when using the explicit formulas and the NIE method for ARL_0 = 370, $\beta = 0.5$, $X = 1$, $k = 1$, and $\lambda = 0.05, 0.10$ reported in Tables 1-2 when $\theta = -0.20, 0.20$ and $\theta = -0.10, 0.10$ are in good agreement. Nevertheless, the CPU time for the explicit formulas was infinitesimal that for the NIE method was around 8-13 seconds references from Tables 1-2, respectively. The APRE value to compare the difference between the ARL values of explicit formulas and NIE found that the ARL values are similar.

Table 1 The one-sided ARL for a MAX(1,1) process running on the modified EWMA chart when $ARL_0 = 370$, $\beta = 0.5$, $X = 1$, $k = 1$ and $\theta = -0.20, 0.20$

θ	Shift size	$\lambda = 0.05$			$\lambda = 0.10$		
		Explicit	NIE	APRE	Explicit	NIE	APRE
-0.20	0.00	370.000173 (<0.001)	370.000168[a] (8.078)[b]	1.382×10^{-6}	370.000701 (<0.001)	370.000691 (8.188)	2.645×10^{-6}
	0.01	87.389348 (<0.001)	87.389347 (8.109)	1.017×10^{-6}	81.848629 (<0.001)	81.848628 (8.171)	1.237×10^{-6}
	0.02	49.572115 (<0.001)	49.572115 (8.188)	9.465×10^{-7}	46.092216 (<0.001)	46.092215 (8.219)	1.040×10^{-6}
	0.03	34.619165 (<0.001)	34.619164 (8.281)	9.038×10^{-7}	32.122186 (<0.001)	32.122186 (8.406)	9.495×10^{-7}
	0.04	26.612056 (<0.001)	26.612056 (8.328)	8.699×10^{-7}	24.679726 (<0.001)	24.679726 (8.016)	8.906×10^{-7}
-0.20	0.05	21.6257288 (<0.001)	21.625728 (8.156)	8.407×10^{-7}	20.058248 (<0.001)	20.058248 (8.172)	8.460×10^{-7}
	0.10	11.235685 (<0.001)	11.235685 (8.172)	7.227×10^{-7}	10.459286 (<0.001)	10.459286 (8.312)	6.999×10^{-7}
	0.20	5.868751 (<0.001)	5.8687517 (8.141)	5.487×10^{-7}	5.5135575 (<0.001)	5.5135575 (8.109)	5.187×10^{-7}
	0.30	4.087304 (<0.001)	4.0873040 (8.141)	4.233×10^{-7}	3.871362 (<0.001)	3.871362 (8.297)	3.978×10^{-7}
	0.40	3.213929 (<0.001)	3.2139292 (8.343)	3.329×10^{-7}	3.065059 (<0.001)	3.065059 (8.235)	3.099×10^{-7}
	0.50	2.702314 (<0.001)	2.7023143 (8.203)	2.664×10^{-7}	2.591905 (<0.001)	2.591905 (8.218)	2.431×10^{-7}
0.20	0.00	370.000310 (<0.001)	370.000298 (7.953)	3.220×10^{-6}	370.000069 (<0.001)	370.000045 (8.047)	6.430×10^{-6}
	0.01	98.9327752 (<0.001)	98.9327729 (8.203)	2.354×10^{-6}	93.273414 (<0.001)	93.273411 (8.047)	3.042×10^{-6}
	0.02	57.1599543 (<0.001)	57.1599530 (8.141)	2.174×10^{-6}	53.472128 (<0.001)	53.472126 (8.172)	2.509×10^{-6}
	0.03	40.225431 (<0.001)	40.225430 (8.000)	2.070×10^{-6}	37.537469 (<0.001)	37.537468 (8.140)	2.266×10^{-6}
	0.04	31.055902 (<0.001)	31.055901 (8.172)	1.991×10^{-6}	28.9566262 (<0.001)	28.956625 (8.110)	2.113×10^{-6}
	0.05	25.3087543 (<0.001)	25.308753 (8.110)	1.924×10^{-6}	23.5950727 (<0.001)	23.595072 (8.250)	1.999×10^{-6}
	0.10	13.234958 (<0.001)	13.234958 (8.359)	1.659×10^{-6}	12.3711938 (<0.001)	12.371193 (8.343)	1.647×10^{-6}
	0.20	6.930608 (<0.001)	6.930608 (7.875)	1.280×10^{-6}	6.5280909 (<0.001)	6.528090 (8.235)	1.233×10^{-6}
	0.30	4.817282 (<0.001)	4.817282 (8.172)	1.007×10^{-6}	4.5693969 (<0.001)	4.569396 (8.234)	9.586×10^{-7}
	0.40	3.772474 (<0.001)	3.772474 (8.203)	8.032×10^{-7}	3.599814 (<0.001)	3.599814 (8.094)	7.584×10^{-7}
	0.50	3.155557 (<0.001)	3.155557 (7.891)	6.496×10^{-7}	3.026406 (<0.001)	3.026405 (8.172)	6.113×10^{-7}

a. The ARL value for explicit formula and NIE method respectively.

b. The central processing unit (CPU) time (System: AMD Ryzen 7 5700U with Radeon Graphics@1.8GHz. Processor, 16GB RAM. 64-bit Operating System).

Table 2 The one-sided ARL for a MAX(1,1) process running on the modified EWMA chart when $ARL_0 = 370$, $\beta = 0.5$, $X = 1$, $k = 1$ and $\theta = -0.10, 0.10$

θ	Shift size	$\lambda = 0.05$ Explicit	$\lambda = 0.05$ NIE	$\lambda = 0.05$ APRE	$\lambda = 0.10$ Explicit	$\lambda = 0.10$ NIE	$\lambda = 0.10$ APRE
-0.10	0.00	370.003520[a] (<0.001) [b]	370.00351 (13.182)	1.704×10^{-6}	370.003854 (<0.001)	370.00384 (13.510)	3.292×10^{-6}
	0.01	89.975530 (<0.001)	89.975528 (13.307)	1.253×10^{-6}	84.392943 (<0.001)	84.392942 (13.353)	1.523×10^{-6}
	0.02	51.249644 (<0.001)	51.249643 (13.681)	1.164×10^{-6}	47.714379 (<0.001)	47.714378 (13.619)	1.292×10^{-6}
	0.03	35.852361 (<0.001)	35.852360 (13.619)	1.111×10^{-6}	33.306679 (<0.001)	33.306679 (13.712)	1.177×10^{-6}
	0.04	27.587091 (<0.001)	27.587090 (13.541)	1.063×10^{-6}	25.612938 (<0.001)	25.612937 (13.494)	1.103×10^{-6}
	0.05	22.432670 (<0.001)	22.432670 (13.510)	1.033×10^{-6}	20.828910 (<0.001)	20.828910 (13.854)	1.047×10^{-6}
	0.10	11.672781 (<0.001)	11.672781 (13.401)	8.884×10^{-7}	10.875032 (<0.001)	10.875032 (13.260)	8.652×10^{-7}
	0.20	6.100916 (<0.001)	6.100916 (13.603)	6.769×10^{-7}	5.734192 (<0.001)	5.7341924 (13.462)	6.435×10^{-7}
	0.30	4.246945 (<0.001)	4.246945 (13.401)	5.274×10^{-7}	4.023209 (<0.001)	4.023209 (13.463)	4.921×10^{-7}
	0.40	3.336069 (<0.001)	3.336069 (13.556)	4.167×10^{-7}	3.181383 (<0.001)	3.181383 (13.478)	3.866×10^{-7}
	0.50	2.801393 (<0.001)	2.801393 (13.790)	3.320×10^{-7}	2.686393 (<0.001)	2.686393 (13.790)	3.090×10^{-7}
0.10	0.00	370.008789 (<0.001)	370.008779 (13.166)	2.601×10^{-6}	370.003621 (<0.001)	370.003602 (13.214)	5.131×10^{-6}
	0.01	95.709440 (<0.001)	95.70943778 (13.260)	1.906×10^{-6}	90.065950 (<0.001)	90.065948 (13.182)	2.419×10^{-6}
	0.02	55.015084 (<0.001)	55.01508312 (13.775)	1.764×10^{-6}	51.375346 (<0.001)	51.375345 (13.712)	2.007×10^{-6}
	0.03	38.633470 (<0.001)	38.63346903 (13.682)	1.681×10^{-6}	35.992098 (<0.001)	35.992098 (14.103)	1.818×10^{-6}
	0.04	29.791222 (<0.001)	29.79122102 (13.431)	1.617×10^{-6}	27.733511 (<0.001)	27.733511 (13.713)	1.698×10^{-6}
	0.05	24.259322 (<0.001)	24.25932165 (13.900)	1.562×10^{-6}	22.582425 (<0.001)	22.582425 (13.978)	1.609×10^{-6}
	0.10	12.664445 (<0.001)	12.66444446 (13.853)	1.347×10^{-6}	11.823008 (<0.001)	11.823008 (13.791)	1.327×10^{-6}
	0.20	6.627863 (<0.001)	6.627862789 (13.588)	1.035×10^{-6}	6.237453 (<0.001)	6.237453 (13.228)	9.908×10^{-7}
	0.30	4.609344 (<0.001)	4.609344191 (13.759)	8.114×10^{-7}	4.369604 (<0.001)	4.369604 (13.259)	7.667×10^{-7}
	0.40	3.613446 (<0.001)	3.613446005 (13.712)	6.448×10^{-7}	3.446831 (<0.001)	3.446830 (13.401)	6.064×10^{-7}
	0.50	3.026531 (<0.001)	3.02653086 (13.292)	5.187×10^{-7}	2.902125 (<0.001)	2.902125 (13.603)	4.859×10^{-7}

a. The ARL value for explicit formula and NIE method respectively.

b. The central processing unit (CPU) time (System: Windows 7 Professional, i5-4210U@1.7GHz. Processor, 4GB RAM. 64-bit Operating System).

Regarding about $\lambda = 0.05$ there is comparison between EWMA and modified EWMA control charts. The results found that performance of EWMA chart is greater than modified EWMA chart for k=0.5 when shift size more than or equal to initiative 0.2. Furthermore, when k =1, k =5 and k =10 the results found that performance of EWMA chart better than modified EWMA chart at shift size from 0.3, 0.4 and 0.5 respectively. For $\lambda = 0.10$, the performance of modified EWMA chart was better than EWMA chart for all of shift size as shown in Table 3.

Table 3 Comparison of the ARL values for a MAX(1,1) process running on EWMA and modified EWMA control charts when ARL_0 = 370, $\beta = 0.5$, $X = 1$, and $\theta = 0.2$

λ	Shift size	EWMA	Modified EWMA			
			k=0.5	k=1	k=5	k=10
		$h = 2.8172\times10^{-8}$	b=0.374197733	b=0.75137524	b = 3.76490011	b = 7.5318573
0.05	0.00	370	370	370	370	370
	0.01	299.82446	163.69696	98.93275	57.72783	53.53997
	0.02	243.9741	104.28756	57.15995	31.74972	29.33016
	0.03	199.33861	76.10240	40.22543	22.10222	20.41501
	0.04	163.52091	59.66509	31.05590	17.06892	15.77976
	0.05	134.66541	48.90842	25.30875	13.97827	12.93882
	0.1	53.98621	25.13876	13.23496	7.63061	7.11578
	0.2	11.43475	**12.20066**	6.93061	4.36863	4.12736
	0.3	3.62168	**7.87695**	**4.81728**	3.26630	3.11644
	0.4	1.79798	**5.78465**	**3.77247**	**2.71144**	2.60653
	0.5	1.28328	**4.58060**	**3.15556**	**2.37691**	**2.29834**
	RMI	3.36882	2.15501	0.84847	0.17678	0.11279
λ	Shift size	h=0.001193827	b=0.384888125	b=0.76681129	b=3.8368715	b=7.67713767
0.1	0.00	370	370	370	370	370
	0.01	330.60438	143.05614	93.27341	58.26078	54.43988
	0.02	296.04428	88.11827	53.47213	32.06220	29.85151
	0.03	265.65622	63.39710	37.53747	22.32236	20.78066
	0.04	238.87662	49.35237	28.95663	17.23865	16.06099
	0.05	215.22552	40.30378	23.59507	14.11629	13.16715
	0.1	131.43874	20.68433	12.37119	7.70170	7.23296
	0.2	55.48112	10.18918	6.52809	4.40447	4.18626
	0.3	26.87782	6.69234	4.56940	3.29015	3.15555
	0.4	14.60194	4. 99436	3.59981	2.72927	2.63570
	0.5	8.75491	4.01235	3.02641	2.39112	2.32155
RMI		7.14484	1.43654	0.57210	0.05354	0.00000

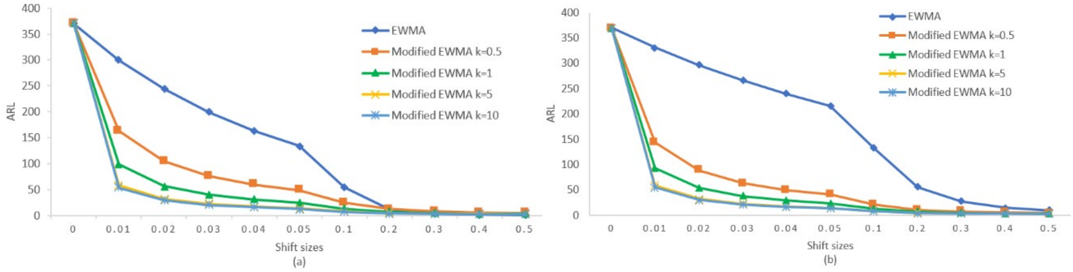

Figure 1 The ARL of the EWMA and modified EWMA control charts simulation data for (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$

5. Application

5.1. Example 1

This time, we used the closing stock price for PTT public company limited with the THB/USD daily foreign exchange rate as the exogenous variable from 3 August to 30 October 2020 (Fusion media limited 2022), as reported in Table 4. The performance of the modified EWMA was better than that of the EWMA control chart except for shift size = 0.3 and $k = 0.5$ or 1 and shift sizes ≥ 0.4 for all k. Furthermore, the results show that the modified EWMA control chart provided smaller RMI values than the EWMA control chart. Finally, the ARL and the RMI values tended to decrease when k increased.

Explanation from Figures 1-2 show that the ARL of the EWMA and modified EWMA control charts observation researching found that efficiency of modified EWMA was better than EWMA when k increasing.

Table 4 Comparison of the ARL values for a MAX(1,1) process running on EWMA and modified EWMA control charts when $ARL_0 = 370$, $\beta = 1.106954$, and $\theta = 0.964217$

λ	Shift size	EWMA	Modified EWMA			
			k=0.5	k=1	k=5	k=10
		$h = 9.93996 \times 10^{-8}$	b=0.46987581	b=0.94050032	b=4.70220093	b=9.40445122
0.05	0.00	370	370	370	370	370
	0.01	306.54208	168.18525	105.61759	64.18584	59.85607
	0.02	254.86481	108.15900	61.70486	35.60047	33.04879
	0.03	212.63137	79.36484	43.64003	24.84581	23.05197
	0.04	177.99637	62.47315	33.79313	19.20446	17.82724
	0.05	149.49698	51.37674	27.59721	15.72997	14.61576
	0.10	65.53330	26.71440	14.51417	8.56816	8.01071
	0.20	15.79851	13.15676	7.63445	4.87254	4.60809
	0.30	5.19273	8.57279	5.31234	3.61995	3.45414
	0.40	2.40559	6.33149	4.15749	2.98799	2.87099
	0.50	1.53994	5.02951	3.47181	2.60612	2.51788
	RMI	4.30751	1.91953	0.75435	0.13548	0.07532

Table 4 (Continued)

λ	Shift size	EWMA	Modified EWMA			
			k=0.5	k=1	k=5	k=10
		h=0.00245004	b=0.48651376	b=0.96514197	b=4.81177694	b=9.62322079
0.10	0.00	370	370	370	370	370
	0.01	334.54313	148.61376	100.34754	65.12772	61.18546
	0.02	303.04093	92.53707	58.19201	36.16366	33.83412
	0.03	274.99708	66.96636	41.05379	25.24553	23.60692
	0.04	249.98393	52.34134	31.76103	19.51385	18.25584
	0.05	227.63225	42.87923	25.93125	15.98216	14.96468
	0.10	145.97365	22.23949	13.66397	8.69873	8.19100
	0.20	66.73545	11.09158	7.23281	4.93855	4.69925
	0.30	34.32936	7.33700	5.06269	3.66392	3.51490
	0.40	19.46903	5.49654	3.98237	3.02086	2.91645
	0.50	11.99731	4.42290	3.34010	2.63232	2.55414
RMI		8.92183	1.31929	0.53568	0.05103	0.00000

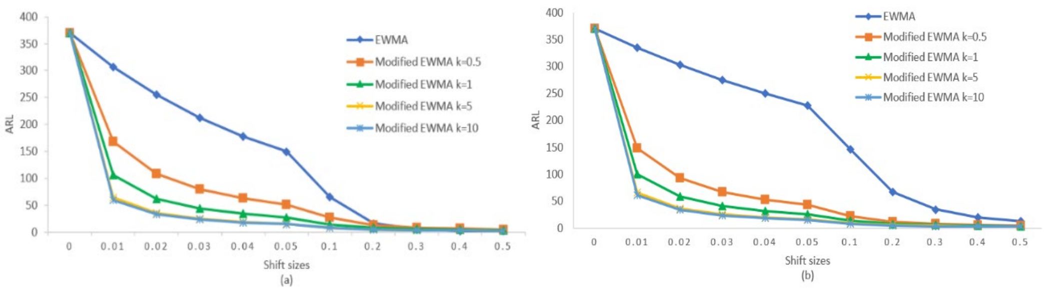

Figure 2 The ARL of the EWMA and modified EWMA control charts real data for (a) $\lambda = 0.05$ and (b) $\lambda = 0.10$

5.2. Example 2

The ARL derived by using explicit formulas was applied to a MAX(1,1) process running on modified and EWMA control charts involving real data comprising the daily highest silver price with the highest crude oil price as the exogenous variable from 1 July to 31 August 2022 (Fusion media limited, 2022). The results in Table 5 show that for $\lambda = 0.05, 0.10$ the results showed that the modified EWMA control chart had good performance the EWMA control chart except for shift size ≥ 0.20 and shift sizes ≥ 0.5 respectively. For $\lambda = 0.20$, the performance of the modified EWMA was better than the EWMA control chart for all shift sizes since the ARL of the modified EWMA is less than the EWMA. The RMI values of the modified EWMA control chart for all values of λ were less than those of the EWMA control chart.

$h = 2.61161 \times 10^{-10}$

Table 5 Comparison of the ARL values for a MAX(1,1) process running on EWMA and modified EWMA control charts when $ARL_0 = 370$, $\beta = 0.2$, $\alpha = 0.8095$ and $\theta = -0.851$

Shift size	λ =0.05		λ =0.10		λ =0.20	
	EWMA	Modified EWMA	EWMA	Modified EWMA	EWMA	Modified EWMA
	$h = 2.61161\times10^{-10}$	$b =$ 0.765273	$h =$ 0.000120464	$b =$ 0.776284925	$h =$ 0.0334232	$b =$ 0.801663934
0.00	370	370	370	370	370	370
0.01	270.41886	**107.13498**	314.2075	**99.63878**	314.54988	**87.55602**
0.02	199.17955	**62.51937**	267.86513	**57.53806**	270.21077	**49.74664**
0.03	147.82657	**44.09167**	229.21658	**40.44065**	234.19905	**34.79539**
0.04	110.53582	**34.03270**	196.85856	**31.17769**	204.56287	**26.78857**
0.05	83.26388	**27.70059**	169.66493	**25.37131**	179.89636	**21.80200**
0.07	48.29624	**20.17976**	127.31343	**18.49869**	141.58644	**15.93056**
0.10	22.56104	**14.34266**	84.76126	**13.18103**	102.47387	**11.40839**
0.20	**3.18737**	7.36908	26.23364	**6.84077**	43.17088	**6.03182**
0.30	**1.33222**	5.05032	10.35412	**4.73133**	22.48644	**4.24043**
0.40	**1.06840**	3.91469	5.0558	**3.69589**	13.44729	**3.35762**
0.50	**1.01781**	3.25003	**2.98542**	3.08833	8.88897	**2.83730**
0.70	**1.00203**	2.51699	**1.62273**	2.41588	4.81328	**2.25785**
1.00	**1.00019**	1.99711	**1.17218**	1.93636	2.71953	**1.84071**
RMI	0.87734	0.81919	2.65588	0.08394	4.26570	0.00000

6. Conclusions

Explicit formulas were derived for the ARL of a MAX(1,1) process with exponential white noise operating on a modified EWMA control chart. Notably, they were simple to compute and implement and required significantly less CPU time than the NIE method. The results of explicit formulas precision were presented as the absolute percentage relative error compared to the NIE method. The results indicate that a modified EWMA control chart effectively detects minor changes in the process mean. When the k value was increased, and the shift size was smaller, the modified EWMA control chart monitoring performed significantly better than the EWMA control chart. Finally, the performances of the modified and EWMA control charts running MAX(1,1) processes based on real datasets were assessed by using the explicit formulas for the ARL, in which the modified EWMA control chart outperformed the EWMA control chart when k and the exponential smoothing parameter increases.

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References

Alpaben KP, Jyoti D. Modified exponentially weighted moving average (EWMA) control chart for an analytical process data. J Chem Eng Mater Sci. 2011; 2(1): 12-20.

Andel J. On AR(1) processes with exponential white noise. Commun Stat - Theory Methods. 1988; 17(5): 1481-1495.

Areepong Y. Explicit formulas of average run length for a moving average control chart for monitoring the number of defective products. Int J Pure Appl Math. 2012; 80(3): 331-343.

Barbeito I, Zaragoza S, Tarrío-Saavedra J, Naya S. Assessing thermal comfort and energy efficiency in buildings by statistical quality control for autocorrelated data. Appl Energy. 2017; 190: 1-7.

Fusion media limited. (2022, September 9). World Financial Markets. Retrieved from https://www.investing.com/markets.

Ibazizen M, Fellag H. Bayesian estimation of an AR(1) process with exponential white noise. Statistics. 2003; 37(5): 365-72.

Khan N, Aslam M, Jun CH. Design of a control chart using a modified EWMA statistic. Qual Reliab Eng Int. 2017; 33(5): 1095-1104.

Lucas JM, Saccucci MS. Exponentially weighted moving average control schemes: properties and enhancements. Technometrics. 1990; 32(1): 1-2.

Mastrangelo CM, Montgomery DC. SPC with correlated observations for the chemical and process industries. Qual Reliab Eng Int. 1995; 11(2): 79-89.

Page ES. Continuous inspection schemes. Biometrika. 1954; 41(1/2): 100-105.

Paichit P. An integral equation procedure for average run length of control chart of ARX(p) processes. Far East J Math Sci. 2016; 99(3), https://doi.10.17654/MS099030359.

Petcharat K, Sukparungsee S, Areepong Y. Exact solution of the average run length for the cumulative sum chart for a moving average process of order q. ScienceAsia. 2015; 41: 141-147.

Roberts SW. Control chart tests based on geometric moving averages. Technometrics. 1959; 239-250.

Serel DA. Economic design of EWMA control charts based on loss function. Math Comput Model. 2009; 49(3-4): 745-759.

Shewhart WA. Economic control of quality of manufactured product. Macmillan and Co Ltd, London; 1931.

Silpakob K, Areepong Y, Sukparungsee S, Sunthornwat R. Explicit analytical solutions for the average run length of modified EWMA control chart for ARX(p, r) processes. Songklanakarin J Sci Technol. 2021; 43(5): 1414-1427.

Sukparungsee S, Areepong Y. A study of the performance of EWMA chart with transformed Weibull observations. Thail Stat. 2009; 7(2): 179-191.

Supharakonsakun Y, Areepong Y, Sukparungsee S. The performance of a modified EWMA control chart for monitoring autocorrelated PM2.5 and carbon monoxide air pollution data. PeerJ. 2020; 8: e10467.

Turkman M. Bayesian analysis of an autoregressive process with exponential white noise. Statistics. 1990; 21(4): 601-608.

Vanbrackle III LN, Reynolds Jr MR. EWMA and CUSUM control charts in the presence of correlation. Commun Stat - Simul Comput. 1997; 26(3): 979-1008.