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On Designing of Extended EWMA Control Chart for Detecting Mean Shifts and its Application

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Abstract

Extended exponentially weighted moving average (Extended EWMA) control chart is one of the control charts for efficient monitoring and detecting changes in the process mean. The average run length (ARL) is a metric commonly used to evaluate and quantify the performance of control charts. This study aims to propose the explicit formulas of ARL on the extended EWMA control chart for moving average with exogenous variables model (MAX(q,r)) with exponential white noise. The accuracy of the solution from the extended EWMA control chart was compared with that from the numerical integral equation (NIE) method. The results show that the ARLs obtained from the explicit formula and the NIE method are not different. After obtaining the ARL values from the explicit formula method of the extended EWMA control chart, it was compared with the exponentially weighted moving average (EWMA) and cumulative sum (CUSUM) control charts. In this research, the standard deviation run length (SDRL) and median run length (MRL) values are also presented which are the important performance metrics used for evaluating the effectiveness of control charts in detecting out-of-control conditions. Both MRL and SDRL were calculated in addition to the ARL values to assess the performance of control charts. The results show that the performance of the extended EWMA control chart is better than the EWMA and CUSUM control charts in all scenarios. In addition, this explicit formula of the ARL is demonstrated to be used in practical applications.

Keywords: ARL, MAX process, explicit formulas, explanatory variable.

1. Introduction

Statistical process control (SPC) charts, also known as control charts, are graphical tools used in quality control to monitor and analyze process data over time. SPC charts help identify and distinguish between common cause variation and special cause variation. The primary objective of SPC charts is to detect deviations or shifts in a process to ensure that it operates within acceptable limits and remains in a state of statistical control. This allows organizations to make informed decisions, take corrective actions, and continuously improve their processes. SPC charts offer several benefits, including early detection of process shifts, improved process understanding, reduction in defects, and data-driven decision-making. By monitoring and analyzing process performance over time, organizations can proactively manage and improve their processes, leading to enhanced quality and efficiency.

The cumulative sum control chart, proposed by Page (1954) is a statistical tool used in quality control to monitor and detect small and moderate shifts or changes in process mean. The exponentially weighted moving average (EWMA) control chart initially proposed by Roberts (1959) is commonly used as a statistical process control tool to monitor the stability of a process over time. The extended exponentially weighted moving average (Extended EWMA) control chart proposed by Neveed et al. (2018) is an extension of the basic EWMA control chart that incorporates additional information about the process being monitored. The main difference between the EWMA and the extended EWMA control charts lies in the calculation of the smoothing parameter or weight. In the extended EWMA control chart, the smoothing parameter is adjusted based on the length of the time series, allowing it to adapt more effectively to different data patterns and trends. This can result in improved accuracy and responsiveness to changes in the underlying data. When dealing with correlated data, the traditional control charts such as the Shewhart control chart may not be appropriate because independent observations were assumed. Extended exponentially weighted moving average (Extended EWMA) is a statistical technique used for smoothing time series data, where more recent observations are given higher weights. When it comes to correlated data, some common types of control charts may not be suitable because it assumes independence between observations. However, there are methods to address correlated data in the context of extended EWMA.

For instance, the autoregressive and moving average (ARMA) control chart is a method that uses a combination of time-series models and control charts to monitor a process with autocorrelated data. Besides, the exogenous variables can include any external factors that may influence dependent variables, such as economic indicators. An example of a time series model is a moving average with exogenous variables model (MAX(q,r)). MAX model is a time series forecasting model that incorporates both a moving average component and exogenous variables. This model is an extension of the traditional moving average model, which uses only past observations to make predictions.

The average run length (ARL) is a statistical measure used in statistical process control (SPC) to evaluate the performance of a control chart. It is defined as the average number of samples or observations that are taken from a process before a control chart signals that the process is out of control, comprises two components: The ARL_0 refers to the average run length when a process is in - control state. In an ideal scenario, where the control chart is properly designed and the process is under control, the ARL_0 should be relatively large. The ARL_1 refers to the average run length when a process is out of control. When an out-of-control condition occurs, such as a shift in the process mean, the control chart is expected to detect it and signal the need for investigation and corrective actions. The ARL_1 measures the efficiency of the control chart in detecting and signaling such out-of-control conditions.

Control charts with shorter ARL_1 values are more effective in detecting process shifts and minimizing the time between detection and corrective action. Several approaches have been used to estimate the ARL, such as the Markov chain (Chananet et al., 2015), Monte Carlo simulation (Mastrangelo and Montgomery, 1995), martingale (Areepong and Novikov, 2008), and the numerical integral equation (NIE) method (Phanthuna and Areepong, 2021). Several researchers have computed the ARL with explicit formulas and checked their accuracy by using these methods. For instance, Phanyaem (2022) proposed the explicit formulas for the ARL on the CUSUM control chart under the observations based on the seasonal ARX model. From the study, Chanannet and Phanyaem (2022) derive the explicit formulas for the ARL of the CUSUM chart under trend stationary seasonal autocorrelated data. Recently, Petcharat (2022) presented the explicit formulas of ARL for the EWMA control chart for stationary moving average process with exogenous variables. Moreover, Peerajit

(2023) solved the explicit formula for the ARL for the one-sided CUSUM control chart for the FIMAX process.

From previous works of literature, the derivation of an explicit formula for ARL on an extended EWMA control chart for observations from MAX(q,r) model in the case of exponential white noise has not been studied. In addition, the goal of this research is to derive the ARL explicit formula on the extended EWMA control chart for MAX(q,r) model. Moreover, a comparison of the performance of extended EWMA, EWMA and CUSUM control charts have been conducted.

2. Materials and Methods

2.1. Autoregressive moving average process with exogenous variable process (MAX(q,r))

Let $Y_t, t = 1,2,3,...$ be a sequence of observations from moving average with the exogenous variable model (MAX(q,r)) described in the form of the backward operator with constant as follows,

$$Y_t = \mu + \theta(B)\varepsilon_t + \sum_{j=1}^{r} \beta_j X_{jt},$$

where $\theta(B) = 1 - \theta_1 B - \theta_2 B^2 - ... - \theta_q B^q$ and $B^q \varepsilon_t = \varepsilon_{t-q}$ or can be written in equation

$$Y_t = \mu + \varepsilon_t - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i} + \sum_{j=1}^{r} \beta_j X_{jt}, \tag{1}$$

where μ is a constant, θ_i is a moving average coefficient, ε_t is an exponential white noise, X_{jt} is an exogenous variables, and β_j is a coefficient parameter of X_{jt}.

2.2. Control charts

2.2.1. Exponentially weighted moving average (EWMA) control chart

The EWMA control chart was initially proposed by Roberts (1959). It is designed to detect small to moderate shifts in the process mean by giving more weight to recent data points. It is widely used in quality control and process monitoring to identify out-of-control conditions. The EWMA statistics can be control chart can be expressed by

$$Z_t = (1-\lambda)Z_{t-1} + \lambda Y_t, t = 1,2,... \tag{2}$$

where Y_t is a MAX process, λ is an exponential smoothing parameter with $0 < \lambda < 1$, and Z_0 is the initial value of EWMA statistics, $Z_0 = \mu$. The upper control limit (UCL) and lower control limit (LCL) of EWMA control charts are given by

$$UCL = \mu_0 + L_1 \sigma \sqrt{\frac{\lambda}{2-\lambda}},$$

$$LCL = \mu_0 - L_1 \sigma \sqrt{\frac{\lambda}{2-\lambda}},$$

where μ_0 is the target mean, σ is the process standard deviation, and L_1 is suitable control limit width. The stopping time of the EWMA control chart (τ_b) is given by

$$\tau_b = \{t \geq 0 : Z_t > b\},$$

where τ_b is the stopping time and b is UCL.

2.1.2. Extended exponentially weighted moving average (Extended EWMA) control chart

The extended EWMA control chart was presented by Neveed et al. (2018). It is designed to quickly monitor and detect small to moderate shifts in the process mean by giving more weight to recent data points. The extended EWMA statistic is given by

$$E_t = \lambda_1 Y_t - \lambda_2 Y_{t-1} + (1-\lambda_1+\lambda_2)E_{t-1}, \tag{3}$$

where λ_1 and λ_2 are exponential smoothing parameters with $(0<\lambda_1 \le 1)$ and $(0 \le \lambda_2 \le \lambda_1)$ and the initial value is a constant, $E_0 = \mu.$ The upper control limit (UCL) and lower control limit (LCL) of the extended EWMA control charts are given by

$$UCL = \mu_0 + L_2\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

$$LCL = \mu_0 - L_2\sigma\sqrt{\frac{\lambda_1^2+\lambda_2^2-2\lambda_1\lambda_2(1-\lambda_1+\lambda_2)}{2(\lambda_1-\lambda_2)-(\lambda_1-\lambda_2)^2}},$$

where μ_0 is the target mean, σ is the process standard deviation, and L_2 is suitable control limit width. The stopping time of the extended EWMA control chart (τ_h) is given by

$$\tau_h = \{t \ge 0 : E_t > h\},$$

where τ_h is the stopping time and h is UCL.

2.1.3. Cumulative sum (CUSUM) control chart

Page (1954) designed the CUSUM control chart for quality control, which can be used to spot small differences in process mean. The CUSUM statistics can be expressed using the algorithm in (4) as follows

$$C_t = C_{t-1} + Y_t - a, t = 1,2,3,... \tag{4}$$

where a is non-zero constant, $C_0 = \Theta$ is the initial value of CUSUM; $\Theta \in [0,s]$ and the CUSUM chart's stopping time is described as

$$\tau_s = \inf\{t > 0; C_t > UCL\},$$

where τ_s is the stopping time and s is UCL.

3. Average Run Length on Extended EWMA Control Chart

3.1. Explicit formula of ARL

From the recursion of extended EWMA statistics,

$$E_t = \left(1-\lambda_1+\lambda_2\right)E_{t-1} + \lambda_1 Y_t - \lambda_2 Y_{t-1}.$$

Therefore, the extended EWMA control chart for the MAX process can be written as,

$$E_t = \left(1-\lambda_1+\lambda_2\right)E_{t-1} + \lambda_1(\mu+\varepsilon_t - \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} + \sum_{j=1}^{r}\beta_j X_{jt}) - \lambda_2 Y_{t-1}.$$

Consider the in-control process, given $LCL = 0,\ UCL = h,$ and initial value $E_0 = \mu,\ \ 0 < E_t < h,$

$$0 < \left(1-\lambda_1+\lambda_2\right)u + \lambda_1(\mu - \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} + \sum_{j=1}^{r}\beta_j X_{jt}) + \lambda_1\varepsilon_t - \lambda_2 Y_{t-1} < h,$$

$$\frac{-\left(1-\lambda_1+\lambda_2\right)u + \lambda_2 Y_{t-1}}{\lambda_1} - \mu + \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt} < \varepsilon_t < \frac{h-\left(1-\lambda_1+\lambda_2\right)u + \lambda_2 Y_{t-1}}{\lambda_1} - \mu + \sum_{i=1}^{q}\theta_i\varepsilon_{t-i} - \sum_{j=1}^{r}\beta_j X_{jt}.$$

The stopping time $\tau_h = \{t > 0; E_t > h\}$ and then the ARL is defined as

$$ARL = M(u) = E_{\infty}(\tau_h).$$

We study the change-point time at $t = 1,$ then we set $Y_0 = v$ and $\varepsilon_0 = e.$ Therefore $M(u)$ can be expressed by Fredholm integral equation of the second kind as follows,

$$M(u) = 1 + \int_{\frac{-(1-\lambda_1+\lambda_2)u+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}}^{\frac{h-(1-\lambda_1+\lambda_2)u+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}} M(E_1) f(\varepsilon_1) d\varepsilon_1. \tag{5}$$

Let

$$w = E_1 = (1-\lambda_1+\lambda_2)u - \lambda_2 v + \lambda_1(\mu - \theta_1 e - \sum_{i=2}^{q}\theta_i\varepsilon_{1-i} + \sum_{j=1}^{r}\beta_j X_{j1}) + \lambda_1\varepsilon_1, \text{ then } \frac{dw}{d\varepsilon_1} = \lambda_1, d\varepsilon_1 = \frac{1}{\lambda_1}dw.$$

After changing the variable, (5) can be rewritten as $M(u) = 1 + \frac{1}{\lambda_1}\int_0^h M(w) f(\varepsilon_1) dw,$

$$M(u) = 1 + \frac{1}{\lambda_1}\int_0^h M(w) f(\frac{w-(1-\lambda_1+\lambda_2)u+\lambda_2 v}{\lambda_1} - \mu + \theta_1 e + \sum_{i=2}^{q}\theta_i\varepsilon_{1-i} - \sum_{j=1}^{r}\beta_j X_{j1}) dw. \tag{6}$$

Since we determine $\varepsilon_1 \sim Exp(\alpha)$ then $f(x) = \frac{1}{\alpha}e^{-\frac{x}{\alpha}}.$ Thus,

$$M(u) = 1 + \frac{1}{\lambda_1}\int_0^h M(w)\frac{1}{\alpha} e^{\frac{-w+(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}} dw$$

$$M(u) = 1 + \frac{e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}}{\alpha\lambda_1}\int_0^h M(w) e^{\frac{-w}{\alpha\lambda_1}} dw.$$

Let $$D(u) = e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}, G = \int_0^h M(w) e^{\frac{-w}{\alpha\lambda_1}} dw.$$

Then, $$M(u) = 1 + \frac{D(u)}{\alpha\lambda_1}G. \tag{7}$$

Consider

$$G = \int_0^h M(w) e^{\frac{-w}{\alpha\lambda_1}} dw \; = \int_0^h \left(1 + \frac{D(w)}{\alpha\lambda_1}G\right) e^{\frac{-w}{\alpha\lambda_1}} dw$$

$$= -\alpha\lambda_1 e^{\frac{-w}{\alpha\lambda_1}}\Big|_0^h + \frac{G}{\alpha\lambda_1}\int_0^h e^{\frac{(1-\lambda_1+\lambda_2)w-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}} \cdot e^{\frac{-w}{\alpha\lambda_1}} dw$$

$$G = -\alpha\lambda_1(e^{\frac{-h}{\alpha\lambda_1}} - 1) - \frac{G}{(\lambda_1-\lambda_2)} e^{\frac{-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}} (e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha\lambda_1}} - 1)$$

$$= \frac{-\alpha\lambda_1(\lambda_1-\lambda_2)(e^{\frac{-h}{\alpha\lambda_1}}-1)}{(\lambda_1-\lambda_2)+e^{\frac{-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}(e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha\lambda_1}}-1)}. \quad (8)$$

Substituting G in (7), we have

$$M(u) = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\alpha\lambda_1}}(e^{\frac{-h}{\alpha\lambda_1}}-1)}{(\lambda_1-\lambda_2)e^{\frac{\lambda_2 v}{\alpha\lambda_1}-(\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha})}+(e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha\lambda_1}}-1)}. \quad (9)$$

The in-control process $(\alpha = \alpha_0)$, the ARL of the extended EWMA control chart can be expressed as follows

$$ARL_0 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\alpha_0\lambda_1}}(e^{\frac{-h}{\alpha_0\lambda_1}}-1)}{(\lambda_1-\lambda_2)e^{\frac{\lambda_2 v}{\alpha_0\lambda_1}-(\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha_0})}+(e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha_0\lambda_1}}-1)}. \quad (10)$$

Meanwhile, the out-of-control process $(\alpha = \alpha_1)$ as well as $\alpha_1 = (1+\delta)\alpha_0$, the ARL of the extended EWMA control chart can be expressed as follows

$$ARL_1 = 1 - \frac{(\lambda_1-\lambda_2)e^{\frac{(1-\lambda_1+\lambda_2)u}{\alpha_1\lambda_1}}(e^{\frac{-h}{\alpha_1\lambda_1}}-1)}{(\lambda_1-\lambda_2)e^{\frac{\lambda_2 v}{\alpha_1\lambda_1}-(\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha_1})}+(e^{\frac{-(\lambda_1-\lambda_2)h}{\alpha_1\lambda_1}}-1)}. \quad (11)$$

3.2. Existence and uniqueness of ARL

The Banach's fixed-point theorem provides theoretical support for the validity of the ARL equation, ensuring that there is a unique solution to the integral equation for explicit formulas. Let T be an operation on the class of all continuous functions defined by

$$T(M(u)) = 1 + \frac{1}{\lambda_1}\int_0^h M(w) f(\frac{w-(1-\lambda_1+\lambda_2)u+\lambda_2 v}{\lambda_1} - \mu + \theta_1 e + \sum_{i=2}^{q}\theta_i\varepsilon_{1-i} - \sum_{j=1}^{r}\beta_j X_{j1})dw. \quad (12)$$

According to Banach's fixed-point theorem, if an operator T is a contraction, and then the fixed-point equation $T(M(u)) = M(u)$ has a unique solution. To show that (12) exists and has a unique solution, theorem can be used as follows below.

Theorem 1. Banach's fixed-point theorem

Let (X,d) *defined on a complete metric space and* $T: X \to X$ *satisfies the conditions of a contraction mapping with contraction constant* $0 \le r < 1$ *such that* $\|T(M_1)-T(M_2)\| \le r\|M_1-M_2\|$, $\forall M_1, M_2 \in X$. *Then there exists a unique* $M(\cdot) \in X$ *such that* $T(M(u)) = M(u)$, *i.e., a unique fixed-point in* X.

Proof: Let T defined in (12) is a contraction mapping for $M_1, M_2 \in F[0,h]$, such that

$$\left\|T(M_1)-T(M_2)\right\| \le r\left\|M_1-M_2\right\|,$$

$$\left\|T(M_1)-T(M_2)\right\| \le r\left\|M_1-M_2\right\|,\ \forall M_1, M_2 \in F[0,h],$$

with $0 \le r < 1$ under the norm $\|M\|_\infty = \sup_{u\in[0,h]}|M(u)|,$ so

$$\left\|T(M_1)-T(M_2)\right\|_\infty = \sup_{u\in[0,h]}\left|\frac{1}{\alpha\lambda_1}e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\int_0^h (M_1(w)-M_2(w))e^{-\frac{w}{\alpha\lambda_1}}dw\right|$$

$$\le \sup_{u\in[0,h]}\left|\left\|M_1-M_2\right\|\frac{1}{\alpha\lambda_1}e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\cdot(-\alpha\lambda_1)(e^{-\frac{h}{\alpha\lambda_1}}-1)\right|$$

$$= \left\|M_1-M_2\right\|_\infty \sup_{u\in[0,h]}\left|e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\right|\left|1-e^{-\frac{h}{\alpha\lambda_1}}\right| \le r\left\|M_1-M_2\right\|_\infty,$$

where $r = \sup_{u\in[0,h]}\left|e^{\frac{(1-\lambda_1+\lambda_2)u-\lambda_2 v}{\alpha\lambda_1}+\frac{\mu-\theta_1 e-\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}+\sum_{j=1}^{r}\beta_j X_{j1}}{\alpha}}\right|\left|1-e^{-\frac{h}{\alpha\lambda_1}}\right|;\ \ 0 \le r < 1.$

3.3. Numerical integral equation (NIE) of ARL

The advantage of using the NIE method with quadrature rules is that it provides a computationally efficient and accurate way to estimate the ARL. Different quadrature rules can be employed to obtain similar ARL estimates, and the results obtained from these rules are generally very close to each other (Phanthuna et al. 2018). In the present study, we use the Gauss-Legendre rule to evaluate the ARL.

$$M(u) = 1+\frac{1}{\lambda_1}\int_0^h M(w)f\Big(\frac{w-(1-\lambda_1+\lambda_2)u+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\Big)dw.$$

The approximation for an integral is evaluated by the quadrature rule as follows

$$\int_0^h f(x)dx \approx \sum_{k=1}^{n} w_k f(a_k),$$

where a_k is a point and w_k is a weight that is determined by the different rules. Using the quadrature formula, we obtain

$$\tilde{M}(a_b) = 1+\frac{1}{\lambda_1}\sum_{k=1}^{n}\frac{w_k M(a_k)f\Big(\frac{a_k-(1-\lambda_1+\lambda_2)a_b+\lambda_2 v}{\lambda_1}}{-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\Big)},\quad b=1,2,\ldots,n$$

The system of n linear equations is as follows:

$$\tilde{M}(a_1) = 1+\frac{1}{\lambda_1}\sum_{k=1}^{n} w_k M(a_k)f\Big(\frac{a_k-(1-\lambda_1+\lambda_2)a_1+\lambda_1 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\Big)$$

$$\tilde{M}(a_2) = 1+\frac{1}{\lambda_1}\sum_{k=1}^{n} w_k M(a_k)f\Big(\frac{a_k-(1-\lambda_1+\lambda_2)a_2+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\Big)$$

$$\tilde{M}\left(a_n\right)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k M(a_k)f\left(\frac{a_k-\left(1-\lambda_1+\lambda_2\right)a_n+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\right).$$

This system can be shown as

$$\mathbf{M}_{n\times 1}=1_{n\times 1}+\mathrm{R}_{n\times n}\mathrm{L}_{n\times 1}\quad\text{or}\quad \mathrm{I}_n-\mathrm{R}_{n\times n}=1_{n\times 1}\quad\text{or}\quad \mathbf{M}_{n\times 1}=\left(\mathrm{I}_n-\mathrm{R}_{n\times n}\right)^{-1}1_{n\times 1},$$

where $\mathbf{M}_{n\times 1}=\begin{bmatrix}\tilde{M}\left(a_1\right)\\ \tilde{M}\left(a_2\right)\\ \vdots\\ \tilde{M}\left(a_n\right)\end{bmatrix}$, $\mathrm{I}_n=diag\left(1,1,...,1\right)$ and $1_{n\times 1}=\begin{bmatrix}1\\1\\ \vdots\\1\end{bmatrix}$. Let $\mathrm{R}_{n\times n}$ is a matrix and define the n to n^{th} as an element of the matrix R as follows

$$\left[R_{bk}\right]\approx\frac{1}{\lambda_1}w_k f\left(\frac{a_k-\left(1-\lambda_1+\lambda_2\right)a_b+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\right).$$

If $\left(\mathrm{I}-\mathrm{R}\right)^{-1}$ exists, the numerical approximation for the integral equation is the term of the matrix,

$$\mathbf{M}_{n\times 1}=\left(\mathrm{I}_{n\times 1}-\mathrm{R}_{n\times n}\right)^{-1}1_{n\times 1}.$$

Finally, we substitute a_b by u in $\tilde{M}\left(a_b\right)$, the approximation of numerical integral for the function $M\left(u\right)$ is

$$\tilde{M}\left(u\right)=1+\frac{1}{\lambda_1}\sum_{k=1}^{n}w_k M(a_k)f\left(\frac{a_k-\left(1-\lambda_1+\lambda_2\right)u+\lambda_2 v}{\lambda_1}-\mu+\theta_1 e+\sum_{i=2}^{q}\theta_i\varepsilon_{1-i}-\sum_{j=1}^{r}\beta_j X_{j1}\right). \quad (13)$$

4. Numerical Results

In this section, a simulation study comparing the efficacies of the explicit formulas $(M(u))$ and the NIE method $\left(\tilde{M}(u)\right)$ for the ARL of an MAX(q,r) process on the Extended EWMA control chart are presented.

The numerical procedure for ARL calculating for the MAX process can be summarized as follows:

Step 1: Given the value of the parameters of the MAX(q,r) process, the parameter of exponential white noise (α_0) for the in-control process, and the smoothing parameters; λ_1,λ_2.

Step 2: Determine the initial value of the MAX(q,r) process, the initial value of the Extended EWMA statistic.

Step 3: Select an acceptable in-control value of ARL_0 and decide shift sizes (δ).

Step 4: Compute the upper control limit (h) that yields the desired ARL for in control process by (10).

Step 5: Compute ARL_1 for shift sizes in the monitoring process where $\alpha_1=(1+\delta)\alpha_0$ by (11) when given the upper control limit (h) from Step 4.

Step 6: Approximate the ARL by the NIE method using (12).

Step 7: Compare the ARL of the explicit formula and the NIE method.

To compare the ARL solution between the explicit formulas and NIE method for the ARL results, the absolute relative change (ARC) is computed as follows

$$ARC(\%) = \frac{\left| M(u) - \tilde{M}(u) \right|}{M(u)} \times 100. \quad (14)$$

Next, the efficiency of the extended EWMA control chart is compared with the EWMA and CUSUM control charts. Several performance measurements are commonly used to assess their overall effectiveness. These measurements can evaluate the chart's ability to detect process variations. First, the relative mean index (RMI) (Tang et al. 2018) is used to assess the ability of a control chart to detect shifts in the process mean. Based on the lowest RMI values it can be concluded that the control chart performs best. The RMI is calculated using the formula shown below

$$RMI = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{ARL_i(r) - ARL_i(s)}{ARL_i(s)} \right], \quad (15)$$

where $ARL_i(r)$ is the ARL of each control chart for the determined shift sizes of the row i, $ARL_i(s)$ and is the lowest ARL of the row i from all control charts.

In addition, the standard deviation run length (SDRL) and median run length (MRL) (Fonseca et al. 2021) are performance measurements used to evaluate the effectiveness of control charts in detecting out-of-control conditions. For in-control process, SDRL and MRL are calculated as follow.

$$ARL_0 = \frac{1}{\alpha}, \; SDRL_0 = \sqrt{\frac{1-\alpha}{\alpha^2}}, \; MRL_0 = \frac{\log(0.5)}{\log(1-\alpha)}, \quad (16)$$

where α represents type I error. In this study, ARL_0 was fixed at 370 and it can be calculated as $SDRL_0$ and MRL_0 by (16) at approximately 370 and 256, respectively. On the other hand, for out-of-control situations, $SDRL_1$, and MRL_1 are calculated by

$$ARL_1 = \frac{1}{1-\beta}, \; SDRL_1 = \sqrt{\frac{\beta}{(1-\beta)^2}}, \; MRL_1 = \frac{\log(0.5)}{\log \beta}, \quad (17)$$

where β represents type II error. By considering the SDRL, MRL, and ARL together, the control chart's effectiveness in detecting various types of process variations can be evaluated and informed decisions about its performance. A lower the $SDRL_1$, MRL_1, and ARL_1 suggest better performance in quickly identifying shifts in the process mean.

Table 1 Control limits of extended EWMA control chart with MAX processes

Models	Coefficients						$\lambda_1 = 0.05$			
	μ	θ_1	θ_2	β_1	β_2	β_3	$\lambda_2 = 0.015$	$\lambda_2 = 0.02$	$\lambda_2 = 0.025$	$\lambda_2 = 0.03$
MAX(1,1)	1	0.1		0.2			0.00403760	0.00244620	0.00148260	0.00089885
MAX(1,2)	1	0.1		0.2	0.3		0.00270460	0.00163910	0.00099365	0.00060242
MAX(1,3)	1	0.1		0.2	0.3	0.4	0.00181215	0.00109845	0.00066597	0.00040379
MAX(2,1)	1	0.1	0.2	0.1	0.15		0.00493390	0.00298850	0.00181115	0.00109790
MAX(2,2)	1	0.1	0.2	0.1	0.15		0.00424500	0.00257170	0.00155870	0.00094495
MAX(2,3)	1	0.1	0.2	0.1	0.15	0.2	0.00347420	0.00210510	0.00127600	0.00077360
Models	Coefficients						$\lambda_1 = 0.1$			
	μ	θ_1	θ_2	β_1	β_2	β_3	$\lambda_2 = 0.03$	$\lambda_2 = 0.05$	$\lambda_2 = 0.07$	$\lambda_2 = 0.09$
MAX(1,1)	1	0.1		0.2			0.0081780	0.00300130	0.00110325	0.000405740
MAX(1,2)	1	0.1		0.2	0.3		0.0054763	0.00201125	0.00073948	0.000271970
MAX(1,3)	1	0.1		0.2	0.3	0.4	0.0036683	0.00134795	0.00049567	0.000182307
MAX(21)	1	0.1	0.2	0.1	0.15		0.0099955	0.00366650	0.00134756	0.000495570
MAX(2,2)	1	0.1	0.2	0.1	0.15		0.0085987	0.00315533	0.00115983	0.000426540
MAX(2,3)	1	0.1	0.2	0.1	0.15	0.2	0.0070358	0.00258295	0.00094955	0.000349220

Table 2 The ARL of extended EWMA control chart for MAX(1,1) using explicit formula against NIE method given $\beta_1 = 0.2$, $\mu = 1$ and $\alpha_0 = 1$

θ_1	δ	$\lambda_1 = 0.05$ ($\lambda_2 = 0.025$) $h = 0.001098155$		*ARC* (%)	$\lambda_1 = 0.1$ ($\lambda_2 = 0.09$) $h = 0.0003005712$		*ARC* (%)
		Explicit Formula	NIE		Explicit Formula	NIE	
−0.1	0.000	370.080722174	370.080722141	8.917×10^{-9}	370.01325070	370.01325088	0.000
	0.001	293.36940999	293.369409964	8.863×10^{-9}	190.740826149	290.4733788	0.001
	0.003	206.90090424	206.90090422	9.666×10^{-9}	96.745269592	202.5126834	0.003
	0.005	159.41295577	159.41295576	6.273×10^{-9}	64.735757952	155.0444434	0.005
	0.010	100.615736918	100.615736910	7.951×10^{-9}	35.341912795	97.07659626	0.010
	0.030	39.161615960	39.1616159571	7.405×10^{-9}	12.429484244	37.42737913	0.030
	0.050	23.486826866	23.4868268647	5.535×10^{-9}	7.50543590507	22.36993112	0.050
	0.100	10.9976513024	10.9976513018	5.456×10^{-9}	3.79980011805	10.4295365	0.100
	0.300	3.03598472101	3.03598472091	3.294×10^{-9}	1.53568970997	2.877006863	0.300
	0.500	1.8236436306	1.82364363058	1.097×10^{-9}	1.20466204206	1.745456329	0.500

Table 2 (Continued)

θ_1	δ	$\lambda_1 = 0.05$ ($\lambda_2 = 0.025$) $h = 0.00134145$ Explicit Formula	NIE	ARC (%)	$\lambda_1 = 0.1$ ($\lambda_2 = 0.09$) $h = 0.000367125$ Explicit Formula	NIE	ARC (%)
0.1	0.000	370.3843939	370.3843939	1.251×10^{-8}	370.6278702	370.6278701	0.000
	0.001	295.9701872	295.9701871	1.151×10^{-8}	193.6357408	290.4733788	0.001
	0.003	210.6563177	210.6563177	1.184×10^{-8}	98.91673027	202.5126834	0.003
	0.005	163.1410832	163.1410832	1.227×10^{-8}	66.35239390	155.0444434	0.005
	0.010	103.646232	103.6462320	1.172×10^{-8}	36.30907530	97.07659626	0.010
	0.030	40.65833559	40.65833559	1.101×10^{-8}	12.79515651	37.42737913	0.030
	0.050	24.45352072	24.45352071	1.049×10^{-8}	7.729382308	22.36993112	0.050
	0.100	11.49117474	11.49117474	9.063×10^{-9}	3.911296771	10.42953650	0.100
	0.300	3.175582451	3.175582451	4.868×10^{-9}	1.567575802	2.877006863	0.300
	0.500	1.893042542	1.893042542	2.516×10^{-9}	1.220335375	1.745456329	0.500

Table 3 The ARL of Extended EWMA control chart for MAX(2,3) using explicit formula against NIE method given $\theta_1 = 0.1, \beta_1 = 0.1, \beta_2 = 0.15, \beta_3 = 0.2, \mu = 1$ and $\alpha_0 = 1$

θ_2	δ	$\lambda_1 = 0.05$ ($\lambda_2 = 0.025$) $h = 0.00085516$ Explicit Formula	NIE	ARC (%)	$\lambda_1 = 0.1$ ($\lambda_2 = 0.09$) $h = 0.000234085$ Explicit Formula	NIE	ARC (%)
–0.2	0.000	370.2056052	370.2056052	4.184×10^{-9}	370.2056052	370.2056052	3.769×10^{-8}
	0.001	290.4733788	290.4733788	4.493×10^{-9}	290.4733788	290.4733788	9.352×10^{-9}
	0.003	202.5126834	202.5126834	4.812×10^{-9}	202.5126834	202.5126834	6.767×10^{-9}
	0.005	155.0444434	155.0444434	4.610×10^{-9}	155.0444434	155.0444434	1.167×10^{-9}
	0.010	97.07659627	97.07659626	4.582×10^{-9}	97.07659627	97.07659626	1.585×10^{-9}
	0.030	37.42737914	37.42737913	4.613×10^{-9}	37.42737914	37.42737913	2.446×10^{-9}
	0.050	22.36993112	22.36993112	4.272×10^{-9}	22.36993112	22.36993112	7.54×10^{-10}
	0.100	10.42953650	10.42953650	3.647×10^{-9}	10.42953650	10.42953650	1.728×10^{-10}
	0.300	2.877006863	2.877006863	1.876×10^{-9}	2.877006863	2.877006863	2.936×10^{-11}
	0.500	1.745456329	1.745456329	9.224×10^{-10}	1.745456329	1.745456329	7.578×10^{-12}
θ_2	δ	$\lambda_1 = 0.05$ ($\lambda_2 = 0.025$) $h = 0.001276$ Explicit Formula	NIE	ARC (%)	$\lambda_1 = 0.1$ ($\lambda_2 = 0.09$) $h = 0.00034922$ Explicit Formula	NIE	ARC (%)
0.2	0.000	370.52747111	370.52747111	0	370.69144757	370.69144747	2.698×10^{-8}
	0.001	295.45723513	295.45723509	1.354×10^{-8}	96331011.192	192.96331009	1.036×10^{-8}
	0.003	209.77849985	209.77849983	9.534×10^{-9}	380562037.98	98.380562034	3.049×10^{-9}
	0.005	162.23843059	162.23843058	6.164×10^{-9}	94795568.654	65.947955684	0
	0.010	102.89164657	102.89164656	9.719×10^{-9}	36.064644354	36.064644354	0
	0.030	40.277947634	40.277947630	9.931×10^{-9}	12.702085769	12.702085769	0
	0.050	24.206723993	24.206723991	8.262×10^{-9}	7.6722952087	7.6722952087	0
	0.100	11.364686175	11.364686173	1.760×10^{-8}	3.8828297057	3.8828297057	0
	0.300	3.1396137202	3.1396137201	3.185×10^{-9}	1.5593998017	1.5593998017	0
	0.500	1.8750924599	1.8750924599	0	1.2162987947	1.2162987947	0

In addition, the performance measurements can be used to assess a control chart's success throughout a variety of changes $(\delta_{\min} \le \delta \le \delta_{\max})$. The average extra quadratic loss (AEQL) may refer to the average extra loss incurred due to an out-of-control condition. It could be calculated as the average difference between the observed values and the target or desired values during out-of-control periods. AEQL can be calculated as follows (Alevizakos et al. 2021).

$$AEQL = \frac{1}{\Delta} \sum_{\delta_i = \delta_{\min}}^{\delta_{\max}} \left(\delta_i^2 \times ARL(\delta_i) \right), \tag{18}$$

where δ represents the particular change in the process, and Δ represents the sum of number of divisions from $\delta_{\min}$ to $\delta_{\max}$. In this study, $\Delta = 9$ is determined from $\delta_{\min} = 0.001$ to $\delta_{\max} = 0.05$. The control chart with the lowest AEQL values perform the best.

Additionally, the performance comparison index (PCI) is a measurement used to compare the performance of different control charts. It is a statistical index that helps evaluate and rank the effectiveness of control charts based on their ability to detect process variations and maintain control. The PCI measurement is the ratio between the AEQL of the control chart and the most efficient control chart, which is shown as the lowest AEQL. The mathematical formula for the PCI is

$$PCI = \frac{AEQL}{AEQL_{lowest}}. \tag{19}$$

Generally, the PCI value equal to 1 indicates better performance in terms of quickly detecting out-of-control conditions and minimizing false alarms. For the results, a simulation of the in-control process is given with $ARL_0 = 370$, and then the initial parameter value was studied $\alpha_0 = 1$. The out-of-control process $\alpha_1 = (1+\delta)\alpha_0$ is computed by determining shift sizes (δ) to be 0.001, 0.003, 0.005, 0.01, 0.03, 0.05, 0.1, 0.3, and 0.5. The upper control limits of extended EWMA control chart with MAX(q,r) processes when $ARL_0 = 370$ are provided in Table 1. For example, if the parameter values were set as $\lambda_1 = 0.5$, $\lambda_2 =$ 0.015 the upper control limit will be equal to 0.0040376 for MAX(1,1) for $\mu = 1, \theta_1 = 0.1$ and $\beta_1 = 0.2$. Additionally, in Tables 2-3, the ARL of the extended EWMA control chart for the MAX(1,1) and MAX(2,3) models are computed by using two techniques such that the explicit formula and NIE method with various $\lambda_1 =$ 0.05, 0.1. The results of the analysis showed that the ARL from explicit formula and the NIE methods gave very similar results for the ARL values. In addition, the ARC value was found to be very low, it could be concluded that the ARL values of both methods were not different.

According to Tables 4-5, the comparison of the ARL for MAX(1,1) and MAX(2,3) processes on Extended EWMA, EWMA and CUSUM control charts are presented. The parameter values were set as $ARL_0 = 370$, $\lambda_1 =$ 0.05, 0.1, 0.2. (10) and (11) were used to evaluate the ARL on extended EWMA control chart of the MAX(q,r) processes. It is evident that the ARL values derived from the explicit formulas for the extended EWMA control chart were less than those for the EWMA and CUSUM control charts for all shift sizes and all values of λ_2. In addition, as λ_2 increases, the result shown that the ARL_1 values decreased accordingly. And when considering the SDRL and MRL values, the results are the same as the ARL values. Moreover, when the ARL values obtained from each control chart shown in Tables 4-5 were used to find the RMI, AEQL and PCI values to see the performance of each chart. It was found that the extended EWMA control chart had the best performance because it gave the lowest RMI, AEQL and PCI equal to 1. Therefore, it also can be concluded that the extended EWMA control chart performs better than the EWMA and CUSUM control charts.

Table 4 The ARL of extended EWMA control chart for MAX(1,1) using explicit formula against EWMA and CUSUM control charts given $\theta_1 = -0.1, \beta_1 = 0.2, \mu = 1$ and $\alpha_0 = 1$

λ_i		$\lambda_1 = 0.05$			$\lambda_1 = 0.1$			$\lambda_1 = 0.2$		
δ	Control Chart	Extended EWMA ($\lambda_2 = 0.025$)	EWMA	CUSUM	Extended EWMA ($\lambda_2 = 0.09$)	EWMA	CUSUM	Extended EWMA ($\lambda_2 = 0.15$)	EWMA	CUSUM
	UCL	0.001098155	0.013503	2.249	0.0003005712	0.02746689	2.249	0.001276526	0.05584	2.249
0.000	ARL_0	370.0807	370.1052	370.5310	370.0133	370.0402	370.5310	370.0482	370.3482	370.5310
	$SDRL_0$	369.5804	369.6049	370.0307	369.5130	369.5399	370.0307	369.5479	369.8479	370.0307
	MRL_0	256.1737	256.1907	256.4858	256.1270	256.1456	256.4858	256.1511	256.3591	256.4858
0.001	ARL_1	293.3694	322.5541	368.3080	190.7408	4044.266	368.3080	154.5897	5827.227	368.3080
	$SDRL_1$	292.8690	322.0537	367.8077	190.2401	9039.265	367.8077	154.0889	227.0822	367.8077
	MRL_1	203.0014	223.2307	254.9449	131.8646	3107.184	254.9449	106.8065	157.4015	254.9449
0.003	ARL_1	206.9009	3458.256	363.9130	96.7453	7428.170	363.9130	71.5614	6832.128	363.9130
	$SDRL_1$	206.4003	8453.255	363.4127	96.2440	2421.170	363.4127	71.0596	1822.128	363.4127
	MRL_1	143.0659	177.3386	251.8986	66.7116	118.0030	251.8986	49.2552	88.84937	251.8986
0.005	ARL_1	159.4130	4437.212	359.5880	64.7358	6212.125	359.5880	46.6371	8163.89	359.5880
	$SDRL_1$	158.9122	9431.211	359.0877	64.2339	1202.125	359.0877	46.1344	3149.89	359.0877
	MRL_1	110.1497	9079.146	248.9007	44.5240	86.7270	248.9007	31.9786	61.9087	248.9007
0.010	ARL_1	100.6157	148.2347	349.0700	35.3419	75.6332	349.0700	25.0332	3376.51	349.0700
	$SDRL_1$	100.1145	147.7339	348.5696	34.8383	75.1315	348.5696	24.5281	50.8351	348.5696
	MRL_1	69.3943	102.4015	241.6102	24.1489	52.0776	241.6102	17.0028	35.2368	241.6102
0.030	ARL_1	39.1616	65.6557	310.8730	12.4295	29.1766	310.8730	8.9706	2371.19	310.8730
	$SDRL_1$	38.6584	65.15378	310.3726	11.9190	28.6722	310.3726	8.4558	7304.18	310.3726
	MRL_1	26.7967	45.16160	215.1340	8.2641	19.8751	215.1340	5.8646	12.9845	215.1340
0.050	ARL_1	23.4868	41.2850	278.0700	7.5054	18.0892	278.0700	5.5936	032.120	278.0700
	$SDRL_1$	22.9814	40.7819	277.5696	6.9875	17.5821	277.5696	5.0690	5212.11	277.5696
	MRL_1	15.9307	28.26859	192.3967	4.8475	12.1886	192.3967	3.5192	7.9884	192.3967
0.100	ARL_1	10.9977	20.5272	214.156	3.7998	9.3308	214.1560	3.0520	4438.6	214.1560
	$SDRL_1$	10.4858	20.0210	213.6554	3.2617	8.8166	213.6554	2.5025	9227.5	213.6554
	MRL_1	7.2710	13.8789	148.0948	2.2696	6.1145	148.0948	1.7460	4.1102	148.0948
0.300	ARL_1	3.0360	6.0179	92.02240	1.5357	3.3965	92.02240	1.4511	6676.2	92.02240
	$SDRL_1$	2.4862	5.4952	91.5210	0.9070	2.8530	91.5210	0.8091	1091.2	91.5210
	MRL_1	1.7348	3.8142	63.4379	0.6581	1.9876	63.4379	0.5933	1.4754	63.4379
0.500	ARL_1	1.8236	3.4251	49.5367	1.2047	2.2762	49.5367	1.1927	933.10	49.5367
	$SDRL_1$	1.2255	2.8821	49.0342	0.4966	1.7043	49.0342	0.4794	1.3429	49.0342
	MRL_1	0.8720	2.0076	33.9885	0.3911	1.1979	33.9885	0.3803	0.9516	33.9885
RMI		0	0.590	0.730	0	1.062	0.868	0	0.901	0.899
AEQL		0.105	0.199	2.648	0.057	0.117	2.648	0.054	0.094	2.648
PCI		1	1.884	25.103	1	2.049	46.403	1	1.737	49.052

Table 5 The ARL of extended EWMA control chart for MAX(2,3) using explicit formula against EWMA and CUSUM control charts give $\theta_1 = 0.1, \theta_2 = 0.2, \beta_1 = 0.1, \beta_2 = 0.15, \beta_3 = 0.2, \mu = 1$ and $\alpha_0 = 1$

λ_i		$\lambda_1 = 0.05$			$\lambda_1 = 0.1$			$\lambda_1 = 0.2$		
δ	Control Chart	Extended EWMA ($\lambda_2 = 0.025$)	EWMA	CUSUM	Extended EWMA ($\lambda_2 = 0.09$)	EWMA	CUSUM	Extended EWMA ($\lambda_2 = 0.15$)	EWMA	CUSUM
	UCL	0.001276	0.0157105	2.0899	0.00034922	0.0319878	2.0899	0.00148316	0.0651815	2.0899
0.000	ARL_0	370.5275	370.2570	370.0860	370.6910	370.8186	370.0860	370.4476	370.2184	370.0860
	$SDRL_0$	370.0271	369.7567	369.5857	370.1911	370.3183	369.5857	369.9473	369.7181	369.5857
	MRL_0	256.4833	256.2959	256.1773	256.5970	256.6852	256.1773	256.4280	256.2691	256.1773
0.001	ARL_1	295.4572	324.6316	367.8740	9633.192	270.4905	367.8740	156.8796	232.1480	367.8740
	$SDRL_1$	294.9568	324.1313	367.3737	192.4627	269.9901	367.3737	156.3788	231.6475	367.3737
	MRL_1	204.4486	224.6708	254.6441	133.4051	187.1430	254.6441	108.3937	160.5659	254.6441
0.003	ARL_1	209.7785	260.1901	363.5010	3806.98	175.5061	363.5010	73.0051	133.1460	363.5010
	$SDRL_1$	209.2779	259.6896	363.0007	97.8793	175.0054	363.0007	72.5034	132.6451	363.0007
	MRL_1	145.0605	180.0032	251.6130	67.8451	121.3047	251.6130	50.2559	91.9428	251.6130
0.005	ARL_1	162.2384	216.8605	359.1980	9480.65	129.8871	359.1980	47.6522	93.4564	359.1980
	$SDRL_1$	161.7377	216.3599	358.6977	65.4461	129.3862	358.6977	47.1495	92.9551	358.6977
	MRL_1	112.1082	149.9694	248.6304	45.3642	89.6839	248.6304	32.6822	64.4319	248.6304
0.010	ARL_1	102.8917	152.6087	348.7330	36.0646	78.7211	348.7330	25.6110	53.7169	348.7330
	$SDRL_1$	102.3904	152.1079	348.2326	35.5611	78.2195	348.2326	25.1060	53.2146	348.2326
	MRL_1	70.9719	105.4334	241.3766	24.6499	54.2180	241.3766	17.4033	36.8861	241.3766
0.030	ARL_1	40.2780	68.3949	310.7160	12.7021	30.5665	310.7160	9.1827	20.2186	310.7160
	$SDRL_1$	39.7748	67.8931	310.2156	12.1918	30.0623	310.2156	8.6683	19.7123	310.2156
	MRL_1	27.5705	47.0603	215.0252	8.4531	20.8386	215.0252	6.0117	13.6650	215.0252
0.050	ARL_1	24.2067	43.1891	278.054	7.6723	18.9852	278.0540	5.7238	12.6553	278.0540
	$SDRL_1$	23.7015	42.6862	277.5536	7.1549	18.4784	277.5536	5.1998	12.1449	277.5536
	MRL_1	16.4298	29.5885	192.3856	4.9634	12.8098	192.3856	3.6098	8.4206	192.3856
0.100	ARL_1	11.3647	21.5799	214.3700	3.8828	9.8089	214.3700	3.1182	6.7763	214.3700
	$SDRL_1$	10.8532	21.0740	213.8694	3.3457	9.2954	213.8694	2.5700	6.2564	213.8694
	MRL_1	7.5255	14.6088	148.2431	2.3276	6.4462	148.2431	1.7925	4.3412	148.2431
0.300	ARL_1	3.1396	6.36385	92.4299	1.5594	3.5668	92.4299	1.4720	2.7916	92.4299
	$SDRL_1$	2.5918	5.8425	91.9285	0.9340	3.0258	91.9285	0.8336	2.2364	91.9285
	MRL_1	1.8076	4.0546	63.7203	0.6761	2.1068	63.7203	0.6094	1.5629	63.7203
0.500	ARL_1	1.8751	3.6182	49.8698	1.2163	2.3801	49.8698	1.2040	2.0120	49.8698
	$SDRL_1$	1.2810	3.0778	49.3673	0.5129	1.8124	49.3673	0.4955	1.4270	49.3673
	MRL_1	0.9095	2.1427	34.2194	0.4014	1.2718	34.2194	0.3904	1.0087	34.2194
RMI		0	0.353	0.726	0	0.511	0.866	0	0.477	0.820
AEQL		0.109	0.210	2.661	0.058	0.122	2.661	0.055	0.098	2.661
PCI		1	1.927	24.413	1	2.103	45.879	1	1.782	48.382

5. Practical Applications with Real Data

The explicit formulas for the ARL of an MAX(q,r) process on extended EWMA control chart is applied and compared the performance with CUSUM and EWMA control charts using 65 real-world data observations of the gold price. The exogenous variable is USD/THB exchange rate from March 2023 to May 2023. The model has an improvement pattern with two MAX processes i.e. MAX(1,1) and MAX(2,1) so that these two model should be included in the model estimation as shown in Table 7. Consequently, the MAX(2,1) has the lowest RMSE, MAPE and normalized Bayesian information criteria (Normalized BIC) value, implying that the best model is the MAX(2,1) as shown in Table 8. Based on the final result of coefficient parameter in Table 9, the MAX(2,1) coefficient parameters are obtained as follows: $\hat{\theta}_1 = -1.274, \hat{\theta}_2 = -0.758, \hat{\beta} = 57.549.$ The in-control parameter equal to 21.5387 as shown in Table 9. Through the parameter of this prediction model it can be assigned as follows

$$\hat{Y}_t = -1.274\varepsilon_{t-1} + 0.758\varepsilon_{t-2} + 57.549X_t.$$

Table 6 Comparison of the ARL values for a MAX(2,1) process running on Extended EWMA, EWMA and CUSUM control charts when $ARL_0 = 370,\ \theta_1 = -1.274, \theta_2 = -0.758,$ $\beta_1 = 57.549$ and $\alpha_0 = 21.5387$

λ_i		$\lambda_1 = 0.05$			$\lambda_1 = 0.1$		
δ	Control Chart	Extended EWMA ($\lambda_2 = 0.025$)	EWMA	CUSUM	Extended EWMA ($\lambda_2 = 0.09$)	EWMA	CUSUM
	UCL	0.0601830	0.06765	8.065	0.1096570	0.13553	8.065
0.00	ARL_0	370.0864	370.3077	370.5260	370.1295	370.0693	370.5260
	$SDRL_0$	369.5860	369.8074	370.0257	369.6291	369.5689	370.0257
	MRL_0	256.1776	256.3310	256.4823	256.2075	256.1657	256.4823
0.001	ARL_1	348.3381	349.0554	370.4240	346.9820	347.9393	370.4240
	$SDRL_1$	347.8378	348.5550	369.9237	346.4816	347.4389	369.9237
	MRL_1	241.1029	241.6000	256.4116	240.1628	240.8264	256.4116
0.003	ARL_1	311.7149	313.1266	370.2210	308.4203	310.7836	370.2210
	$SDRL_1$	311.2145	312.6262	369.7207	307.9199	310.2832	369.7207
	MRL_1	215.7176	216.6960	256.2709	213.4339	215.0720	256.2709
0.005	ARL_1	282.0725	283.9161	370.0170	277.5866	280.8114	370.0170
	$SDRL_1$	281.5721	283.4156	369.5167	277.0861	280.3109	369.5167
	MRL_1	195.1710	196.4488	256.1295	192.0616	194.2968	256.1295
0.010	ARL_1	227.9295	230.2591	369.5100	222.1211	226.2980	369.5100
	$SDRL_1$	227.4290	229.7585	369.0097	221.6205	225.7974	369.0097
	MRL_1	157.6419	159.2566	255.7781	153.6158	156.5110	255.7781
0.030	ARL_1	129.0979	131.2925	367.4890	123.6271	127.5565	367.4890
	$SDRL_1$	128.5969	130.7915	366.9887	123.1261	127.0555	366.9887
	MRL_1	89.1368	90.6580	254.3772	85.3448	88.0684	254.3772
0.050	ARL_1	90.1779	91.9522	365.4840	85.7829	88.9411	365.4840
	$SDRL_1$	89.6765	91.4508	364.9837	85.2815	88.4396	364.9837
	MRL_1	62.1593	63.3892	252.9875	59.1129	61.3020	252.9875
0.100	ARL_1	51.6014	52.7518	360.5330	48.7822	50.8151	360.5330
	$SDRL_1$	51.0989	52.2494	360.0327	48.2796	50.3127	360.0327
	MRL_1	35.4197	36.2171	249.5557	33.4655	34.8747	249.5557
0.300	ARL_1	19.3813	19.8511	341.6040	18.2595	19.0839	341.6040
	$SDRL_1$	18.8746	19.3446	341.1036	17.7524	18.5772	341.1036
	MRL_1	13.0844	13.4102	236.4351	12.3067	12.8783	236.4351
0.500	ARL_1	12.1412	12.4379	179.7610	11.4479	11.9674	179.7610
	$SDRL_1$	11.6304	11.9274	179.2603	10.9365	11.4565	179.2603
	MRL_1	8.0641	8.2699	124.2539	7.58321	7.94357	124.2539
RMI		0	0.015	4.722	0	0.027	5.033
AEQL		0.630	0.645	8.954	0.595	0.621	8.954
PCI		1	1.024	14.213	1	1.044	15.049

Table 7 MAX estimate for gold price with USD/THB exchange rate as exogenous variable

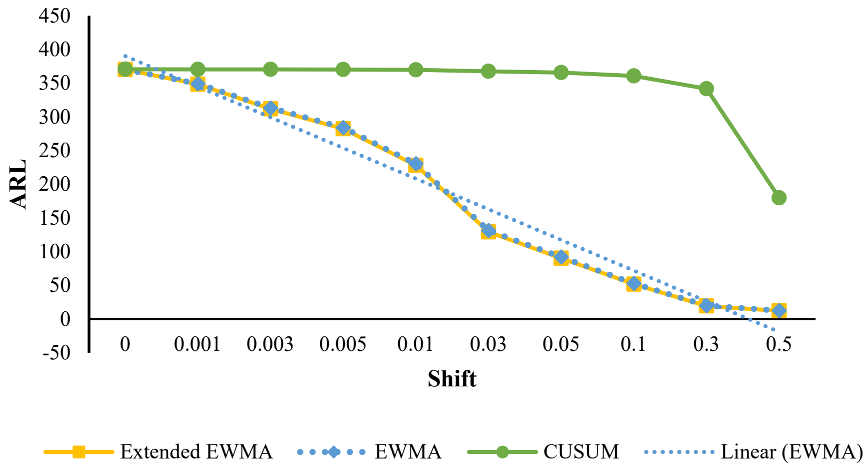
Process	Variable	Coefficient	Std. Error	t	Sig.
MAX(1,1)	MA(1) $(\hat{\theta})$	–0.896	0.064	-14.024	0.00
	USD/THB exchange rate $(\hat{\beta})$	57.577	0.305	188.513	0.00
MAX(2,1)	MA(1) $(\hat{\theta}_1)$	–1.274	0.092	–13.894	0.00
	MA(2) $(\hat{\theta}_2)$	0.758	0.096	–7.874	0.00
	USD/THB exchange rate $(\hat{\beta})$	57.549	0.360	160.061	0.00

Table 8 Model fit

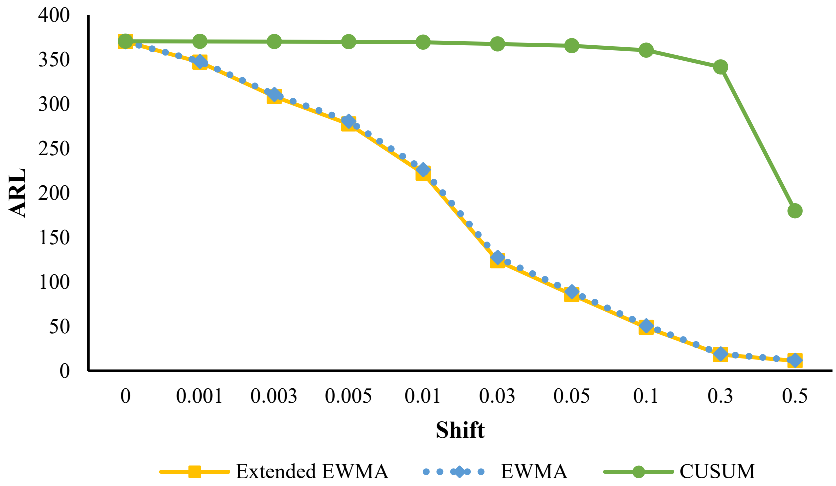
Process	RMSE	MAPE	Normalized BIC
MAX(1,1)	48.718	1.771	7.901
MAX(2,1)	39.203	1.401	7.530

Table 9 Exponential white noise of residual using the Kolmogorov-Smirnov goodness-of-fit test

Process	Mean (α_0)	Kolmogorov-Smirnov	Sig.
MAX(2,1)	21.5387	0.830	0.496

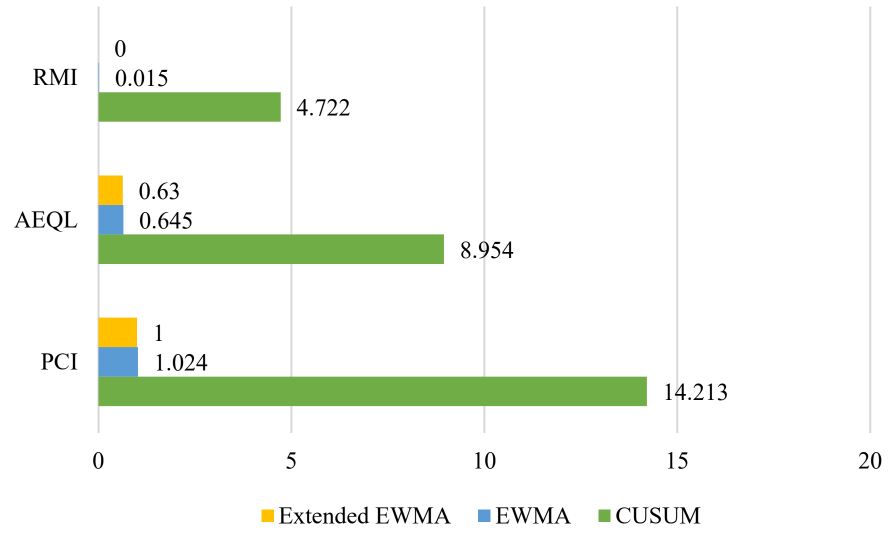


(a)

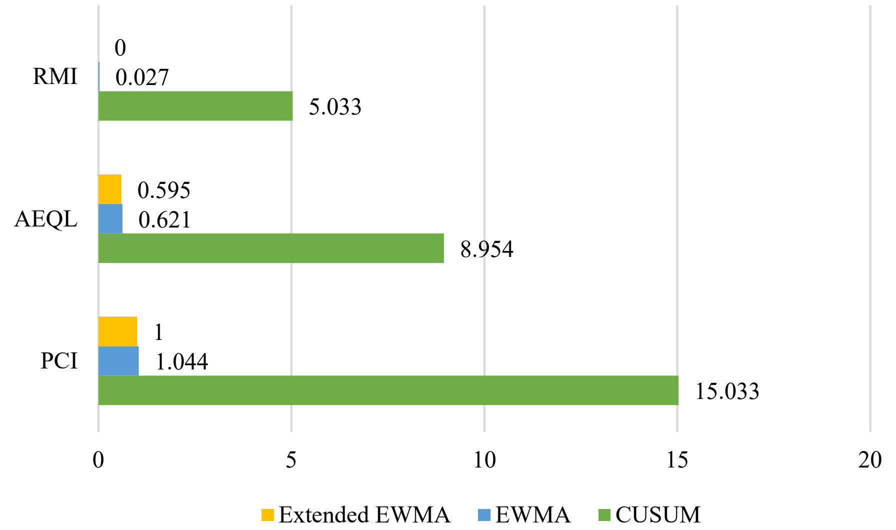


(b)

Figure 1 The ARL of the one-sided extended EWMA chart for MAX(2,1) against the EWMA and CUSUM charts for; (a) $\lambda_1 = 0.05$ and (b) $\lambda_1 = 0.1$



(a)



(b)

Figure 2 The RMI, AEQL and PCI values on the control charts for MAX(2,1) when (a) $\lambda_1 = 0.05$ and (b) $\lambda_1 = 0.1$

The ARL values for the extended EWMA control chart, EWMA, and CUSUM control charts are displayed. According to the findings of the control chart comparison, the extended EWMA control chart had a lower ARL_1 and performed better than the EWMA and CUSUM control charts in every scenario, as illustrated in Figure 1. In addition, to verify performance, the RMI, AEQL and PCI values were also used in the same way as the simulated results above. The results show that the RMI and AEQL values of the extended EWMA chart is lower than the AEQL values of the EWMA and CUSUM control charts, and the PCI value of the extended EWMA control chart equal to 1, as in the simulated data above. As a result, the results indicate that the outcomes of this application is similar to simulated data, as illustrated in Figure 1. And then, the RMI, AEQL and PCI values supported the control chart's effectiveness by using ARL_1 values in the formulas mentioned above, as shown in Figure 2.

6. Conclusions

In this research, the ARL explicit formulas on the extended EWMA control chart for MAX(q,r) process was presented. The explicit formula is a method for finding the exact value of the ARL and is very useful in terms of decreased computational time. The numerical integral equation (NIE) method is used to compare the explicit formula by measuring the ARC (%).Therefore, both methods show that the ARL values are close but the explicit formula method can be computed with a small amount of time. For comparing the performance of the control charts, the extended EWMA, EWMA, and CUSUM control charts were presented to assess their effectiveness in detecting process shifts. The results found that the extended EWMA control chart had the best performance because it gave the lowest RMI, AEQL values and PCI values equal to 1. Therefore, it also can be concluded that the extended EWMA control chart performs better than the EWMA and CUSUM control charts. From the study of both simulation and its application to real data, it is known that the proposed ARL explicit formulas can be used as a criterion to measure the efficiency of control charts accurately, and quickly, and to give results in the same direction. Comparing the performance of the extended EWMA control chart with other control charts can also be extended by considering the optimal parameters for MAX(q,r) processes in future research. Moreover, it is also possible to develop formulas for ARL values on extended EWMA control chart for new control charts or other interesting models.

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