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Efficient Estimation of the Burr XII Distribution in Presence of Progressive Censored Samples with Binomial Random Removal

Amal S. Hassan, Salwa M. Assar* and Kareem A. Ali
Department of Mathematical Statistics, Faculty of Graduate Studies for Statistical Research (FGSSR), Cairo University, Egypt.
*Corresponding author; e-mail: salwaassar@yahoo.com

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Abstract

A progressive type II (PTII) censoring schemes has a widely application in lifetime and reliability studies. This work investigates the optimal estimator of the probability density and the cumulative distribution functions of the Burr type XII distribution based on PTII censoring samples. The uniformly minimum variance unbiased, maximum likelihood, maximum product spacing, least squares, and weighted least squares estimators are obtained. The closed form expressions for maximum likelihood and uniform minimum variance unbiased estimator, associated mean squared errors and τ^{th} moment are derived. A simulation study is used to demonstrate theoretical achievements. The outcomes of simulation study showed that the maximum product spacing estimates are preferred over all the other estimates. We examine one real data set to demonstrate the applicability and relevance of the proposed estimators. The results of a real-life analysis reveal that proposed estimators outperform some other competitive models.

Keywords: Burr type XII distribution, least squares estimator, uniform minimum variance unbiased estimator, maximum product spacing estimator, progressive type II censored samples.

1. Introduction

In 2015, the United Nations Development Programme announced a sustainable development goal (SDG) target 3.6 was aimed at halving the number of global deaths and injuries from road traffic accidents (RTAs) by the 2020 (UN 2016). In the same year, a report from the WHO on road safety showed 1.25 million deaths, globally, due to road accidents (WHO 2015). It is a primary cause of deaths worldwide among people aged 15-29 years old. RTAs have become the top 10 causes of death. It moved from the ninth in 2000 to the seventh in 2019 (WHO 2020).

In life testing experiments, it is a common practice to cease testing before the failure of all items. This is due to the lack of funds and/or time constrains. Samples that result from such situations are called censored samples. Several designs are applied based on minimizing cost and time. One of these designs is progressive censored scheme. Progressive censoring allows the manufacturer to remove active units with the failure units during the test. This process leads to time and cost reduction more than ordinary censored samples and without losing the information from the survival units. Recently, the progressive type-II censoring (PTIIC) scheme has received considerable interest among the statisticians. In PTIIC, pre-specified numbers of stages (m), such that m less than the sample size (n), is fixed before putting the units under a test. At every stage, a number of units, denoted by $R_i, i=1,...,m$ are removed from the experiment with the failure unit in the stage and the remainders of survival units, denoted by $\overline{R}_i = n-i-R_1-R_2\ldots-R_{i-1}$, are progressed to the next stage. When $R_1=R_2=\ldots=R_m=0$ and $n=m$ the PTIIC scheme reduces to the complete sample case. If $R_1=R_2=\ldots=R_{m-1}=0$ and $R_m=n-m$ then the type II censoring will be obtained. Note that, in this scheme, $r_1, r_2,..., r_m$ are all pre-determined. However, in some practical situations, these numbers may occur at random. Yuen and Tse (1996) indicated that, for example, the number of patients that drop out from a clinical test at each stage is random and cannot be pre-determined. In some reliability experiments, an experimenter may decide that it is inappropriate or too dangerous to carry on testing some of the tested units even though these units have not failed. In these cases, the pattern of removal at each failure is random. Suppose that any test unit being dropped out from the life test is independent of the others but with the same removal probability p. Then, Tse et al. (2000) indicated that the number of test units removed at each failure time has a binomial distribution. In recent years the feature of random removal has been adopted by many researchers in designing various kinds of progressive censoring schemes, such as, Wu and Chang (2002), Wu et al. (2007), Dey and Dey (2014), Gunasekera (2018), Qin and Gui (2020), and Bantan et al. (2021). So, in the current work, we consider R_i to be a binomial random removal rule.

Burr type XII distribution (BXII-D) is very compatible for modelling lifetime data and phenomena with distinct failure rates. Various values of its parameters cover a broad set of skewness and kurtosis. Hence, it is used in various fields to model a variety of data types. The cumulative distribution function (CDF) and probability density function (PDF) of BXII-D are defined by

$$G(x) = 1-\left(1+x^c\right)^{-k}, \; x,k,c>0, \tag{1}$$

and

$$g\left(x\right) = ckx^{c-1}\left(1+x^c\right)^{-k-1}, \tag{2}$$

where k and c are shape parameters. The BXII-D includes as the sub-models the logistic and Weibull distributions (see Tadikamalla 1980).

There have been numerous studies in the literature that deal with estimating the BXII-D's unknown parameters. Hossain and Nath (1997) investigated how to estimate Burr's parameters using various estimating approaches. Watkins (1999) has given a technique for maximum likelihood (ML) estimation in three-parameter BXII-D. Chou et al. (2000) developed a cost-effective control chart statistical design for non-normally distributed data using the BXII-D. Ghitany and Al-Awadhi (2002) had given a BXII-D parameter estimator. Using type I censoring, Abd-Elfattah et al. (2008) investigated estimation in step-stress partially accelerated life tests for the BXII-D. Soliman et al. (2013) used progressive first-failure censored data to estimate BXII-D. Hassan et al. (2015) studied the estimation of a stress strength model using several methods of ranked set sampling. The B-robust estimators for the parameters of the BXII-D were studied by Dogru and Arslan (2016). Maurya et al. (2017) looked on parameter estimators for BXII-D under PTIIC. Gunasekera (2018) proposed the reliability function of BXII-D by the concept of generalized variable method viz PTIIC with random removals. Qin and Gui (2020) examined BXII-D using the competing risks model via PTIIC, in which the amount of units withdrawn at each stage is assumed to be random and subject to a binomial distribution. Hassan et al. (2020) developed a generalized Bayesian shrinkage estimator of BXII-D parameters under various loss functions.

The PDF and CDF estimation problem is vital for a variety of reasons. For example, the PDF can be employed for estimating; differential entropy, Rényi entropy, Fisher information and Kullback-Leibler divergence. Also, the CDF can be utilized for estimating; cumulative residual entropy, the quantile function, Bonferroni and Lorenz curves. We recall some previous studies that have been conducted about the parametric estimation of the PDF and CDF for some models based on complete samples. For instance; Pareto (Asrabadi 1990, Dixit and Nooghabi 2010), generalized exponential Poisson (Bagheri et al. 2014), exponentiated Weibull and generalized exponential (Alizadeh et al. 2015a, 2015b), exponentiated Gumbel (Bagheri et al. 2016), generalized logistic (Tripathi et al. 2017), Frechet (Maleki and Deiri 2017) and Lindley (Maiti and Mukherjee 2018) distributions.

To the best of our knowledge, there are hardly any studies on the PTII-based estimate problem for PDF and CDF. So, our goal is to find the best estimators for the PDF and CDF of BXII-D that rely on PTIIC. Parametric methods of estimation; such as, ML, uniformly minimum variance unbiased (UMVU), maximum product spacing (MPS), least squares (LS) and weighted least squares (WLS) are considered. The simulation study and data concerns are offered to clarify overall conclusions. The following sections can be put together in the following manner. The ML and UMVU estimators of the PDF and CDF of BXII-D are worked out in Sections 2 and 3. In addition, mean squared errors (MSEs) and τth moment for estimators are attained. The LS, WLS, and MPS estimators for our problem are provided in Sections 4 and 5. Simulation studies and data analysis are addressed to clarify the overall results in Sections 6 and 7, respectively. Conclusions are at the end of the paper.

2. Maximum Likelihood Estimators

This section deals with estimating the shape parameter k, PDF, and CDF of the BXII-D, where the shape parameter c is assumed known using ML method. Furthermore, we provide the analytical expressions for the bias and the mean squared error for the PDF and CDF estimators.

Let X denote the lifetime of a product and has the BXII-D with the density function in (2). With PTIIC censoring, n units are placed on test. Let $(X_1,R_1),(X_2,R_2),...,(X_m,R_m)$ denote a PTIIC sample, $X_i = X_{i:m:n}, i = 1,2,...,m$ denotes the i^{th} order statistics between failure units from BXII-D, with pre-determined number of removals, say $R_1 = r_1, R_2 = r_2, = ... =, R_m = r_m$ where m denote the number of failures, where the shape parameter c is assumed to be known. The conditional likelihood function can be written, (see Tse et al. 2000), as follows

$$L(k;\underline{x} \mid \underline{R}) = B\prod_{i=1}^{m} g(x_i)(1-G(x_i))^{r_i}, \tag{3}$$

where $B = n(n-r_1-1)(n-r_1-r_2-2)\ldots(n-r_1-r_2\ldots-r_{m-1}-m+1)$, for $i = 1,2,...,m-1$. Inserting (1) and (2) in (3), we obtain

$$L(k;\underline{x} \mid \underline{R}) = B\prod_{i=1}^{m} ckx_i^{c-1}\left(1+x_i^{c}\right)^{-k-1}\left(1+x_i^{c}\right)^{-r_i k}. \tag{4}$$

We assume that R_i is independent of X_i for all i. The number of units removed at each failure time assumed to follow a binomial distribution with the following probability mass function

$$P(R_1 = r_1) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1},$$

while

$$P(R_i = r_i \,|\, R_{i-1} = r_{i-1}, ..., R_1 = r_1) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=1}^{i} r_j}, \text{ for } i = 1,2,...,m-1.$$

The joint probability mass function of $r_1, r_2, ..., r_n$ is given by

$$\begin{aligned} P(\underline{R} = \underline{r}) &= P(R_1 = r_1, R_2 = r_2, ..., R_{m-1} = r_{m-1}) \\ &= P(R_{m-1} = r_{m-1} \,|\, R_{m-2} = r_{m-2}, ..., R_1 = r_1) \times P(R_{m-2} = r_{m-2} \,|\, R_{m-3} = r_{m-3}, ..., R_1 = r_1) \times ... \times \\ &\quad P(R_2 = r_2 \,|\, R_1 = r_1) P(R_1 = r_1) \\ &= \frac{(n-m)!}{(n-m-\sum_{i=1}^{m-1} r_i)! \prod_{i=1}^{m-1} r_i} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i) r_i}. \end{aligned} \tag{5}$$

The unconditional likelihood function can be written as

$$L(k; \underline{x}, \underline{R}) = L(k; \underline{x} \,|\, \underline{R} = \underline{r}) P(\underline{R} = \underline{r}). \tag{6}$$

Using (3) and (5) in (6), we can write the full likelihood function as

$$L(k; \underline{x}, \underline{R}) = B \prod_{i=1}^{m} ck x_i^{c-1} \left(1 + x_i^c\right)^{-k-1} \left(1 + x_i^c\right)^{-r_i k} \frac{(n-m)!}{(n-m-\sum_{i=1}^{m-1} r_i)! \prod_{i=1}^{m-1} r_i} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i) r_i}. \tag{7}$$

It is clear that $P(R = r)$ does not depend on the parameter k and, hence the ML estimator of k, assuming the shape parameter c is assumed to be known, can be derived by maximizing (4) directly. The first derivative of the log conditional likelihood function is, from Equation (3),

$$\frac{dL(k; \underline{x}, \underline{R})}{dk} = \frac{m}{k} - \sum_{i=1}^{m} (1 + r_i) ln(1 + x_i^c).$$

The ML estimator of k can be obtained by solving $dL(k; \underline{x}, \underline{R}) / dk = 0.$ Then, we find immediately

$$\hat{k}_{ML} = \frac{m}{\sum_{i=1}^{m} (1 + r_i) ln(1 + x_i^c)} = \frac{m}{T},$$

where $T = \sum_{i=1}^{m} (1 + r_i) ln(1 + x_i^c).$ Similarly, since $L(k; \underline{x}, \underline{R})$ does not involve the binomial parameter $p,$ the ML estimator of p can be derived by maximizing (5) directly. Hence the ML estimator of p is obtained by solving the following equation

$$\hat{p} = \left(\sum_{i=1}^{m-1} r_i\right) \Big/ \left[(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i\right].$$

Note that: The CDF in (1) can be rewritten as follows

$$-\ln(1 - G(x)) = k V,$$

where $V = ln(1 + x^c).$ It is seen that V has an exponential distribution (ExD) with scale parameter $k,$ then the PDF of V is given as

$$f(v; k) = k e^{-kv}, v \,\text{and}\, k > 0.$$

Lemma. *If* $V_1, V_2, \ldots, V_m$ *are identically independent random variables (iid) from ExD with mean* $(1/k)$ *and assuming in order with the likelihood function under PTIIC is given by*

$$L(v_1, v_2, \ldots, v_m) = Bk^m e^{-k\sum_{i=1}^{m}(1+r_i)v_i}, \tag{8}$$

where $V_1 < V_2 < \ldots < V_m$ *and* $V_i = V_{i:m:n}$, $i=1,2,\ldots,m$ *be the* i^{th} *order statistics between failure units. Then, the random variables* $Y_1, \ldots, Y_m$, *which are obtained from the following transformation,*

$$Y_1 = nV_1, \quad Y_i = (n-i+1-r_1-r_2\ldots-r_{m-1})(V_i - V_{i-1}), i = 2,3,\ldots m,$$

are iid from ExD with mean $(1/k)$.

Proof: To derive the distribution of $V_i, i=1,2,...,m,$ let' s consider the following inverse transformation

$$V_i = \sum_{j=1}^{i} \frac{Y_j}{n - \sum_{l=1}^{j} r_{l-1} - j + 1}, r_0 = 0, i = 1\ldots m.$$

Thus, the Jacobean of the transformation is given by

$$J = \begin{vmatrix} \frac{1}{n} & 0 & \cdots & 0 \\ \frac{1}{n} & \frac{1}{n-r_1-1} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \frac{1}{n} & \frac{1}{n-r_1-1} & \cdots & \frac{1}{n-\sum_{j=1}^{m} r_{j-1} - m + 1} \end{vmatrix} = \frac{1}{B}, \tag{9}$$

where B defined in (3). The summation of the random variables $Y_i, i=1,2,...,m,$ is given by

$$T = \sum_{i=1}^{m}(1+r_i) ln(1+x_i^c) = \sum_{i=1}^{m} Y_i = \sum_{i=1}^{m}(1+r_i)V_i. \tag{10}$$

Substituting (9) and (10) in (8), that is proving the lemma (see Balakrishnan and Aggarwala 2000).

Remark. We can see that T is distributed as a gamma (m,k) with the following PDF

$$f(t) = \frac{k^m}{\Gamma(m)} t^{m-1} e^{-kt}, \ t > 0, \ k > 0, \ m > 0. \tag{11}$$

Therefore, $\hat{k} = \frac{m}{T} = S$ distributed as an inverted gamma distribution with parameters (m, mk), where

$$f(s) = \frac{(mk)^m}{\Gamma(m)} s^{-m-1} e^{\frac{-mk}{s}}, \quad s > 0.$$

The required PDF and CDF estimators are obtained from the invariance property as follows

$$\hat{g}(x) = c\hat{k}x^{c-1}(1+x^c)^{-\hat{k}-1} \text{ and } \hat{G}(x) = 1 - (1+x^c)^{-\hat{k}}.$$

In the next theorems, $E(\hat{g}(x)^\tau), E(\hat{G}(x)^\tau),$ and the MSEs are deduced.

Theorem 1. *Let* $\hat{g}(x)$ *and* $\hat{G}(x)$ *are the estimators of the probability density function and the cumulative distribution function of the BXII-D respectively. Then,*

$$E((\hat{g}(x))^{\tau}) = 2c^{\tau}b^{\tau}d^{-\tau}\frac{(mk)^{m}}{\Gamma(m)}\left(\frac{mk}{\tau ln(d)}\right)^{\frac{\tau-m}{2}}\mathrm{K}_{\tau-m}\left(2\sqrt{mk\tau ln(d)}\right), \tag{12}$$

and,

$$E((\hat{G}(x)^{\tau})) = 1+2\frac{(mk)^{m}}{\Gamma(m)}\sum_{i=1}^{\tau}\binom{\tau}{i}(-1)^{i}\left(\frac{mk}{iln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{mkiln(d)}\right), \tag{13}$$

where $b = x^{c-1}$ *and* $d = 1 + x^{c}$.

Proof: First by using $b = x^{c-1}$, $d = 1 + x^{c}$, then $\hat{g}(x)$ and $\hat{G}(x)$ can be simplified as follows

$$\hat{g}(x) = csbd^{-s-1} \text{ and } \hat{G}(x) = 1 - d^{-s}.$$

Thus,

$$E\left((\hat{g}(x))^{\tau}\right) = \int_{0}^{\infty} c^{\tau}s^{\tau}b^{\tau}\,\mathrm{d}^{-\tau s-\tau}\frac{(mk)^{m}}{\Gamma(m)}s^{-m-1}e^{\frac{-mk}{s}}ds = c^{\tau}b^{\tau}d^{-\tau}\frac{(mk)^{m}}{\Gamma(m)}\int_{0}^{\infty}s^{\tau-m-1}e^{-s(\tau ln(d))}e^{\frac{-mk}{s}}ds,$$

which leads to

$$E\left((\hat{g}(x))^{\tau}\right) = 2c^{\tau}b^{\tau}d^{-\tau}\frac{(mk)^{m}}{\Gamma(m)}\left(\frac{mk}{\tau ln(d)}\right)^{\frac{\tau-m}{2}}\mathrm{K}_{\tau-m}\left(2\sqrt{mk\tau ln(d)}\right),$$

where $\mathrm{K}_{\nu}(\cdot)$ is defined in Gradshteyn and Ryzhik (2000). Similarly, we can prove the second part of the theorem as follows

$$E\left(\left(\widehat{G}(x)\right)^{\tau}\right) = \int_{0}^{\infty}(1-d^{-s})^{\tau}\frac{(mk)^{m}}{\Gamma(m)}s^{-m-1}e^{\frac{-mk}{s}}ds = \int_{0}^{\infty}\sum_{i=0}^{\tau}\binom{\tau}{i}(-1)^{i}d^{-si}\frac{(mk)^{m}}{\Gamma(m)}s^{-m-1}e^{\frac{-mk}{s}}ds.$$

Hence,

$$E\left(\left(\widehat{G}(x)\right)^{\tau}\right) = \int_{0}^{\infty}\left[1+\sum_{i=1}^{\tau}\binom{\tau}{i}(-1)^{i}d^{-si}\right]\frac{(mk)^{m}}{\Gamma(m)}s^{-m-1}e^{\frac{-mk}{s}}ds = I_{1}+I_{2},$$

where I_{1} is the pdf of inverse gamma distribution, so the previous equation can be written as

$$E\left(\left(\widehat{G}(x)\right)^{\tau}\right) = 1+\frac{(mk)^{m}}{\Gamma(m)}\sum_{i=1}^{\tau}\binom{\tau}{i}(-1)^{i}\int_{0}^{\infty}e^{-si\ln(d)}s^{-m-1}e^{\frac{-mk}{s}}ds.$$

Using the Bessel function, hence $E\left(\left(\widehat{G}(x)\right)^{\tau}\right)$ takes the following for

$$E\left(\left(\widehat{G}(x)^{\tau}\right)\right) = 1+2\frac{(mk)^{m}}{\Gamma(m)}\sum_{i=1}^{\tau}\binom{\tau}{i}(-1)^{i}\left(\frac{mk}{iln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{mkiln(d)}\right).$$

Theorem 2. *The MSEs of* $\hat{g}(x)$ *and* $\hat{G}(x)$ *respectively are given by*

$$MSE\left(\hat{g}(x)\right)=2c^2b^2d^{-2}\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{2ln(d)}\right)^{\frac{2-m}{2}}\mathrm{K}_{2-m}\left(2\sqrt{2mkln(d)}\right)-$$
$$4kc^2b^2d^{-s-2}\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{ln(d)}\right)^{\frac{1-m}{2}}\mathrm{K}_{1-m}\left(2\sqrt{mkln(d)}\right)+k^2c^2b^2d^{-s-2}, \tag{14}$$

and

$$MSE\left(\left(\hat{G}(x)\right)\right)=1-4\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{ln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{mkln(d)}\right)+$$
$$2\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{2ln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{2mkln(d)}\right)- \tag{15}$$
$$2\left(1-2\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{ln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{mkln(d)}\right)\right)\left(1-d^{-s}\right)+\left(1-d^{-s}\right)^2.$$

Proof: Since

$$MSE\left(\hat{g}(x)\right)=E\left(\left(\hat{g}(x)\right)^2\right)-2g(x)E\left(\hat{g}(x)\right)+\left(g(x)\right)^2. \tag{16}$$

Setting $\tau=1$ and $\tau=2$ in (12), then

$$E(\hat{g}(x))=2cbd^{-1}\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{ln(d)}\right)^{\frac{1-m}{2}}\mathrm{K}_{1-m}\left(2\sqrt{m\,k\,ln(d)}\right), \tag{17}$$

$$E((\hat{g}(x))^2)=2c^2b^2d^{-2}\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{2ln(d)}\right)^{\frac{2-m}{2}}\mathrm{K}_{2-m}\left(2\sqrt{2mk\,ln(d)}\right). \tag{18}$$

Substituting (17) and (18) in (16), we obtain the result provided in (14). Similarly, since

$$MSE\left(\hat{G}(x)\right)=E\left(\left(\hat{G}(x)\right)^2\right)-2G(x)E\left(\hat{G}(x)\right)+\left(G(x)\right)^2. \tag{19}$$

Then by putting $\tau=1$ and $\tau=2$ in (13), we obtain $E\left(\hat{G}(x)\right)$ and $E\left(\left(\hat{G}(x)\right)^2\right)$ as follows

$$E(\hat{G}(x))=1-2\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{ln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{mkln(d)}\right), \tag{20}$$

$$E\left(\left(\hat{G}(x)\right)^2\right)=1-4\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{ln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{mkln(d)}\right)+2\frac{(mk)^m}{\Gamma(m)}\left(\frac{mk}{2ln(d)}\right)^{\frac{-m}{2}}\mathrm{K}_{-m}\left(2\sqrt{2mkln(d)}\right). \tag{21}$$

Substituting (20) and (21) in (19), we obtain the result provided in (15).

3. Uniformly Minimum Variance Unbiased Estimators

The UMVU estimators of the PDF and CDF of the BXII-D are considered under PTIIC. Furthermore; the τ^{th} moment and the MSEs of these estimators are determined.

Let $X_1,\ldots,X_m$ be a PTIIC from BXII-D with number of stages m. Then, $T=\sum_{i=1}^{m}\left(1+r_i\right)\ln\left(1+x_i^c\right)$ is the complete sufficient statistic for the parameter k (assumed c is a known parameter), and the PDF of T is provided in (11). According to Lehmann-Scheffe theorem, if $g\left(x_1|t\right)=g^*\left(t\right)$ is the conditional PDF of $X_1\,|\,T,$ we have

$$E\left(g^*\left(T\right)\right)=\int g\left(x_1|t\right)f\left(t\right)dt=\int g\left(x_1,t\right)dt=g\left(x_1\right),$$

where $g(x_1,t)$ is the joint PDF of X_1 and T. Therefore, $g^*(t)$ is the UMVU estimator of $g(x)$. In the following theorem, the UMVU estimators for $g(x)$ and $G(x)$ are deduced.

Theorem 3. *The UMVU estimators for* $g(x)$ *and* $G(x)$ *respectively are given by*

$$\tilde{g}\left(x\right)=g^*(t)=\frac{\left(m-1\right)\left(t-\ln\left(1+x^c\right)\right)^{m-2}}{t^{m-1}}\times\frac{cx^{c-1}}{1+x^c},\ \ln\left(1+x^c\right)<t<\infty, \tag{22}$$

and

$$\tilde{G}\left(x\right)=1-\left\{\frac{t-\ln\left(1+x^c\right)}{t}\right\}^{m-1}. \tag{23}$$

Proof:

The estimator $\tilde{g}(x)$ is the UMVU estimator for $g(x)$ is proved through the following steps:

1. Obtain the joint PDF of V and $T-V$ as follows:

Since $V=\ln\left(1+x_1^{\,c}\right)$ and $T-V=-\sum_{i=2}^{m}\left(1+r_i\right)\ln\left(1+x_i^c\right),$ where the distribution of V has a gamma $(1,k),$ and the distribution of $T-V$ has a gamma $(m-1,k).$ The joint distribution of V and $T-V$ is given by

$$f\left(v,t-v\right)=f\left(v\right)f\left(t-v=\sum_{i=2}^{\infty}(1+r_i\ln(1+x_i^c)\right)=(ke^{-kv})\left(\frac{k^{m-1}}{\Gamma(m-1)}(t-v)^{m-2}e^{-k(t-v)}\right)$$
$$=\frac{k^m}{\Gamma(m-1)}(t-v)^{m-2}e^{-kt},\ \ v<t<\infty.$$

2. Obtain the conditional distribution of $V\big|T$ as follows:

$$g_{V|T}\left(v|t\right)=\frac{g\left(v,t-v\right)}{f\left(t\right)}=\frac{k^m\left(t-v\right)^{m-2}e^{-kt}\Gamma(\mathrm{m})}{\Gamma\left(m-1\right)k^m t^{m-1}e^{-kt}}=\frac{\left(m-1\right)\left(t-v\right)^{m-2}}{t^{m-1}},v<t<\infty.$$

3. Obtain the conditional distribution of $X_1\big|T$

Since $V=\ln\left(1+x_1^{\,c}\right)$ so, V is a function of $X_1,$ then we can determine the distribution of $X_1\big|T$ using the transformation method. Obtain the Jacobean of the transformation

$$\left|J\right|=\left|\frac{dV}{dX_1}\right|=\left|\frac{cx^{c-1}}{1+x^c}\right|.$$

We obtain the following pdf $g\left(x_1|t\right)=f\left(v\left(x_1\right)|t\right)\left|J\right|$ as follows

$$g\left(x_1|t\right)=\frac{\left(m-1\right)\left(t-\ln\left(1+x_1^c\right)\right)^{m-2}}{t^{m-1}}\times\frac{cx_1^{c-1}}{1+x_1^{c-1}},\ \ln\left(1+x_1^{c-1}\right)<t<\infty.$$

Then by applying the Lehmann-Scheffe theorem, then the UMVU of $g(x)$ is as follows:

$$\tilde{g}\left(x\right)=\frac{\left(m-1\right)\left(t-\ln\left(1+x^c\right)\right)^{m-2}}{t^{m-1}}\times\frac{cx^{c-1}}{1+x^c},\ln\left(1+x^{c-1}\right)<t<\infty.$$

The estimator $\tilde{G}\left(x\right)$ is the UMVU estimator for $G(x)$ is proved by noting that (6) is the derivative of (7) with respect to x, that is,

$$\frac{d}{dx}\tilde{G}\left(x\right)=\frac{d}{dx}\left[1-\left\{\frac{t-\ln\left(1+x^c\right)}{t}\right\}^{m-1}\right]=\frac{(m-1)}{t^{m-1}}\left\{t-\ln\left(1+x^c\right)\right\}^{m-2}\frac{cx^{c-1}}{(1+x^c)}=\tilde{g}\left(x\right),$$

which is the same as (22).

Theorem 4. *The MSEs for* $\tilde{g}\left(x\right)$ *and* $\tilde{G}\left(x\right)$, *respectively, are given by*

$$MSE\left(\tilde{g}\left(x\right)\right)=\sum_{i=0}^{2m-4}\frac{M^2}{\Gamma(m)}\binom{2m-4}{i}\left(-1\right)^i k^{i+2}p(x)^i\Gamma(m-i-2,p(x)k)-(g(x))^2, \tag{24}$$

and,

$$MSE\left(\tilde{G}\left(x\right)\right)=\sum_{j=0}^{2}\sum_{i=0}^{mj-j}\frac{(-1)^{i+j}p(x)^ik^i}{\Gamma(m)}\binom{2}{j}\binom{mj-j}{i}\Gamma(m-i,p(x)k)-(G(x))^2. \tag{25}$$

Proof: It is known that the

$$MSE\left(\tilde{g}\left(x\right)\right)=E\left(\left(\tilde{g}\left(x\right)\right)^2\right)-g\left(x\right)^2. \tag{26}$$

To obtain (26), we must obtain $E\left(\left(\tilde{g}\left(x\right)\right)^\tau\right)$ as follows

$$E\left(\left(\tilde{g}(x)\right)^\tau\right)=\int_{p(x)}^{\infty}\left(\tilde{g}\left(x\right)\right)^\tau f\left(t\right)dt=\frac{k^m}{\Gamma(m)}\int_{\ln\left(1+x^c\right)}^{\infty}\left(\frac{\left(m-1\right)cx^{c-1}}{1+x^c}\right)^\tau\frac{\left(t-p(x)\right)^{m\tau-2\tau}}{t^{m\tau-\tau}}t^{m-1}e^{-kt}dt. \tag{27}$$

Put $M=\frac{\left(m-1\right)cx^{c-1}}{1+x^c}$ and $p(x)=\ln\left(1+x^c\right)$, in (27) leads to

$$E\left(\tilde{g}\left(x\right)^\tau\right)=M^\tau\frac{k^m}{\Gamma(m)}\int_{p(x)}^{\infty}(1-\frac{p(x)}{t})^{m\tau-2\tau}t^{m-\tau-1}e^{-kt}dt$$

$$=M^\tau\frac{k^m}{\Gamma(m)}\sum_{i=0}^{m\tau-2\tau}\binom{m\tau-2\tau}{i}(-1)^i p(x)^i\int_{p(x)}^{\infty}t^{m-i-\tau-1}e^{-kt}dt.$$

Let $u=kt$, we obtain

$$E\left(\tilde{g}\left(x\right)^\tau\right)=\frac{M^\tau}{\Gamma(m)}\sum_{i=0}^{m\tau-2\tau}\binom{m\tau-2\tau}{i}\left(-1\right)^i p(x)^i k^{i+\tau}\int_{p(x)k}^{\infty}u^{m-i-\tau-1}e^{-u}du,$$

which $\int_{p(x)k}^{\infty}u^{m-i-\tau-1}e^{-u}du$ is the upper incomplete gamma function, so $E\left(\left(\tilde{g}\left(x\right)\right)^\tau\right)$ can be formulated as follows

$$E\left(\left(\tilde{g}(x)\right)^{\tau}\right)=\sum_{i=0}^{m\tau-2\tau}\frac{k^{i+\tau}}{\Gamma(m)}\binom{m\tau-2\tau}{i}(-1)^{i}M^{\tau}p(x)^{i}\Gamma(m-i-\tau,p(x)k). \tag{28}$$

Setting $\tau=2$ in (28) and then substituting the result in (26), we prove the first part in Theorem 4. Also, since

$$MSE\left(\tilde{G}(x)\right)=E\left(\left(\tilde{G}(x)\right)^{2}\right)-\left(G(x)\right)^{2}. \tag{29}$$

To obtain (29), we must obtain $E\left(\left(\tilde{G}(x)\right)^{\tau}\right)$ as follows

$$\begin{aligned}E\left(\left(\tilde{G}(x)\right)^{\tau}\right)&=\int_{p(x)}^{\infty}\left(\tilde{G}(x)\right)^{\tau}f(t)dt=\frac{k^{m}}{\Gamma(m)}\int_{p(x)}^{\infty}(1-\frac{\left(t-p(x)\right)^{m-1}}{t^{m-1}})^{\tau}t^{m-1}e^{-kt}dt\\&=\frac{k^{m}}{\Gamma(m)}\sum_{j=0}^{\tau}(-1)^{j}\binom{\tau}{j}\int_{p(x)}^{\infty}\frac{\left(t-p(x)\right)^{mj-j}}{t^{mj-j}}t^{m-1}e^{-kt}dt,\\&=\frac{k^{m}}{\Gamma(m)}\sum_{j=0}^{\tau}\sum_{i=0}^{mj-j}(-1)^{i+j}\binom{\tau}{j}\binom{mj-j}{i}p(x)^{i}\int_{p(x)}^{\infty}t^{m-i-1}e^{-kt}dt.\end{aligned}$$

After some simplification, we obtain

$$E\left(\left(\tilde{G}(x)\right)^{\tau}\right)=\sum_{j=0}^{\tau}\sum_{i=0}^{mj-j}\frac{(-1)^{i+j}}{\Gamma(m)}\binom{\tau}{j}\binom{mj-j}{i}p(x)^{i}k^{i}\int_{p(x)k}^{\infty}u^{m-i-1}e^{-u}du,$$

which $\int_{p(x)k}^{\infty}u^{m-i-1}e^{-u}du$ is the upper incomplete gamma function, so $E\left(\left(\tilde{G}(x)\right)^{\tau}\right)$ can be formulated as follows

$$E\left(\left(\tilde{G}(x)\right)^{\tau}\right)=\sum_{j=0}^{\tau}\sum_{i=0}^{mj-j}\frac{(-1)^{i+j}p(x)^{i}k^{i}}{\Gamma(m)}\binom{\tau}{j}\binom{mj-j}{i}\Gamma(m-i,p(x)k). \tag{30}$$

Setting $\tau=2$ in (30) and then substituting the result in (29), we prove the second part in Theorem 4.

4. Least Squares and Weighted Least Squares Estimators

The basic idea of LS and WLS estimation is the procedure of minimizing the weighted squared distance between the theoretical CDF and a non-parametric estimation of the CDF with respect to the unknown parameters. Then, the LS and WLS estimators can be obtained by minimizing the following quantity with respect to k,

$$\psi(k)=\sum_{i=1}^{m}w_{i}\left(G\left(x_{i:m:n}\right)-\bar{G}\left(x_{i:m:n}\right)\right)^{2}, \tag{31}$$

where $\bar{G}\left(x_{i:m:n}\right)$ is a non-parametric estimator for the $G\left(x_{i:m:n}\right)$, and $w_{i}=1/var\left(\bar{G}\left(x_{i:m:n}\right)\right)$ the weighted value. The Kaplan-Meier estimator is utilized to estimate $G\left(x_{i:m:n}\right)$ in this work. As well as Greenwood's formula is employed to estimate the variance of $G\left(x_{i:m:n}\right)$, (see Ng et al. (2012)).

Unfortunately, (31) has no closed form solution to find the estimate value of WLS estimator, say k'', so the numerical method is employed. The WLS estimators of the CDF and PDF are attained as follows

$$G''(x)=1-\left(1+x^{c}\right)^{-k''},\ g''(x)=ck''x^{c-1}\left(1+x^{c}\right)^{-k''-1}.$$

In addition, the LS estimators can be obtained by substituting $w_{i}=\underline{1}$ $,i=1,2,\ldots m$ in (31).

5. Maximum Product of Spacing Estimators

A new method of estimation; named as MPS; was presented by Cheng and Amin (1983). They mentioned that this method is suitable to estimate the curves with J-shaped. Ng et al. (2012) and Singh et al. (2016) gave the MPS estimators for the parameters of three-parameter Weibull distribution and generalized inverted ExD under PTIIC. The MPS can be obtained by maximizing the following quantity with respect to k,

$$D(k)=\prod_{i=1}^{m+1}\left(G(x_{i:m:n})-G(x_{i-1:m:n})\right)\prod_{i=1}^{m}(1-G(x_{i:m:n}))^{r_i},$$

where $G(x_{0:m:n})=0,\ G(x_{m+1:m:n})=1,$ or equivalent to maximizing

$$D(k)=\sum_{i=1}^{m+1}\ln(\left(G(x_{i:m:n})-G(x_{i-1:m:n})\right))+\sum_{i=1}^{m}r_i\ln(1-G(x_{i:m:n})). \tag{32}$$

A numerical method is employed in (32) to find the estimate value of k, say k'. The MPS estimators of the CDF and PDF are obtained as follows

$$G'(x)=1-\left(1+x^c\right)^{-k'},\ g'(x)=ck'x^{c-1}\left(1+x^c\right)^{-k'-1}.$$

6. Simulation Study

A simulation study is included in this section to determine the efficient estimators of the PDF and CDF for BXII-D when the available data are considered as PTIIC. The removal scheme is regarded as binomial removals. ML, UMVU, LS, WLS and MPS estimators are used to estimate the PDF and the CDF of BXII-D under PTIIC. The MSEs of $g(0.2)$ and $G(0.2)$ are computed to compare between these estimators at different sets of parameters (k,c) = (0.25, 2), (0.5, 2) and (1.5, 2). One thousand random samples with sizes $n=50$ and 100 at different stages are generated from BXII-D. The binomial parameter (p) is regarded as p = 0.2 and 0.7. The MSEs corresponding to each estimator are listed in Tables 1-4.

In addition, Figures 1-3 display the MSEs values for the PDF and CDF of BXII-D by proposed methods. The left hand side graph in each figure is related to the PDF estimates and the corresponding right hand side graph is the CDF estimates. Generally the efficiency of all estimates improves as sample size increases.

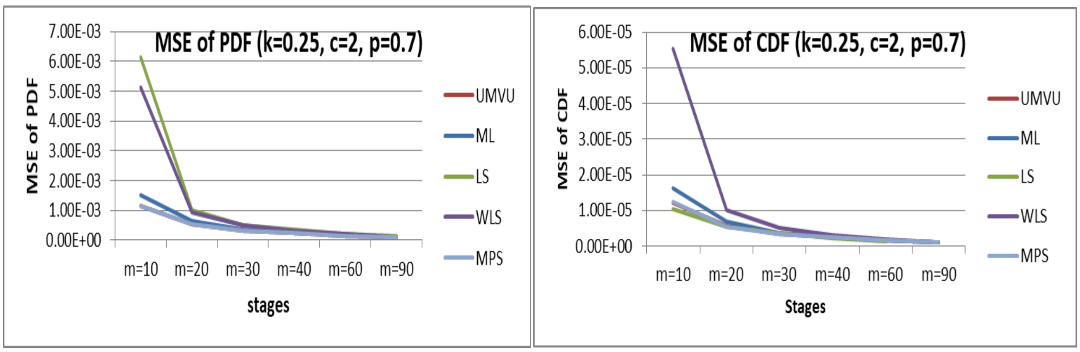


Figure 1 The MSEs of the PDF and CDF for the parameter set $(k,c,p)=(0.25,2,0.7)$ at $x=0.2$ and $n=100$

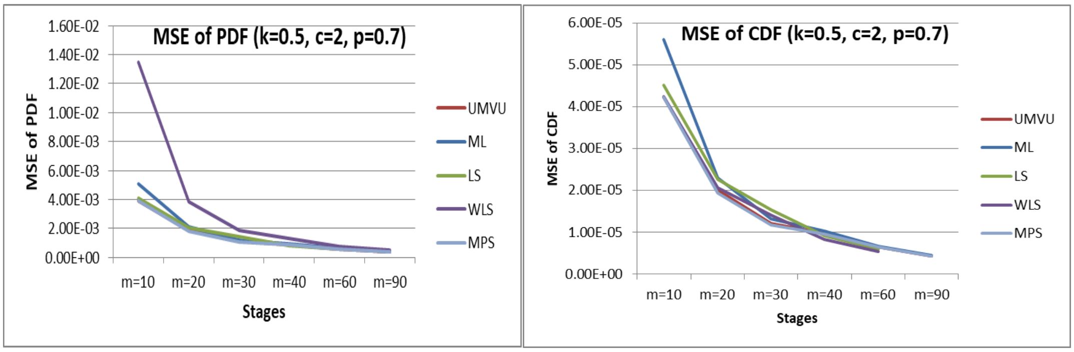


Figure 2 The MSEs of the PDF and CDF for the parameter set $(k, c, p) = (0.5, 2, 0.7)$ at $x = 0.2$ and $n = 100$

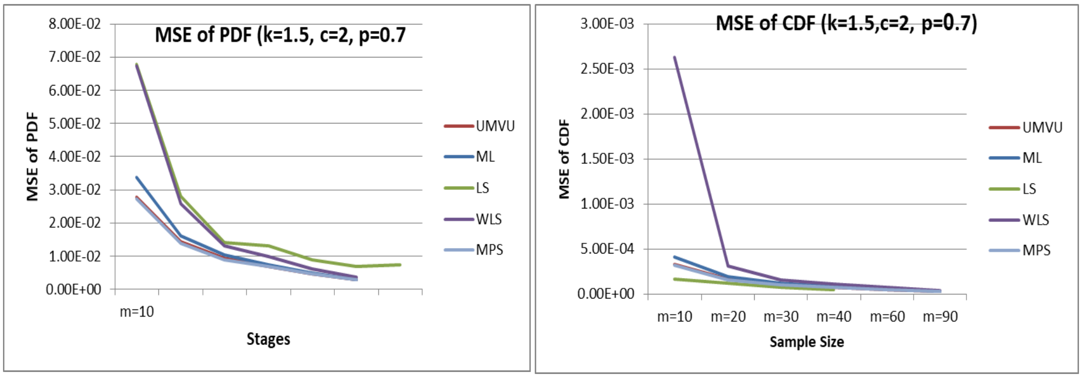


Figure 3 The MSEs of the PDF and CDF for the parameter set $(k, c, p) = (1.5, 2, 0.7)$ at $x = 0.2$ and $n = 100$

It can be realized from Tables 1-4 that:

- The MSEs of MPS estimates for the PDF and CDF of BXII-D for $p = 0.7$ are smaller than the MSEs of MPS estimates for PDF and CDF of BXII-D for $p = 0.2$ when $n = 100$ and for different values of m.
- The MSEs of ML estimates are decreasing as m increases for $p = 0.2$ and 0.7 when $n = 50$ and 100.
- The MSEs for all estimates for PDF and CDF of BXII-D are increasing as the true value of k increases for all sample sizes.
- For the parameter sets under consideration, the MSEs of MPS estimations for PDF and CDF of BXII-D yield the least values.
- The MPS estimates are competitive methods for studying the underlying function, followed by UMVU and ML estimators.
- The MSEs of LS estimates for PDF and CDF of BXII-D get the largest value compared to the other estimates.

Table 1 The MSEs of the PDF and CDF estimates of BXII-D at $x = 0.2$ and $p = 0.2$ when $n = 50$

m	Methods	$k = 0.25, c = 2$		$k = 0.5, c = 2$		$k = 1.5, c = 2$	
		PDF	CDF	PDF	CDF	PDF	CDF
10	UMVU	0.001501	0.000016	0.005791	0.000063	0.040700	0.000490
	ML	0.002139	0.000023	0.008155	0.000090	0.053700	0.000671
	LS	0.004325	0.000988	0.020400	0.000228	0.081000	0.001916
	WLS	0.002211	0.000024	0.015400	0.000172	0.054500	0.000684
	MPS	0.001881	0.000020	0.007205	0.000080	0.047700	0.000593
20	UMVU	0.000514	0.000005	0.002057	0.000022	0.016200	0.000193
	ML	0.000591	0.000006	0.002384	0.000026	0.018600	0.000225
	LS	0.001323	0.000994	0.003761	0.001002	0.021000	0.000252
	WLS	0.001247	0.000013	0.003414	0.000037	0.020500	0.000246
	MPS	0.000507	0.000005	0.002012	0.000022	0.015800	0.000189
30	UMVU	0.000335	0.000004	0.001348	0.000015	0.008523	0.000100
	ML	0.000368	0.000004	0.001493	0.000016	0.009325	0.000111
	LS	0.000594	0.000006	0.002242	0.000024	0.016500	0.001085
	WLS	0.000556	0.000006	0.002099	0.000023	0.015400	0.001072
	MPS	0.000325	0.000003	0.001291	0.000014	0.008228	0.000097
40	UMVU	0.000247	0.000003	0.000992	0.000011	0.007110	0.000084
	ML	0.000265	0.000003	0.001062	0.000012	0.007459	0.000088
	LS	0.000314	0.000003	0.001244	0.000014	0.010000	0.000119
	WLS	0.000296	0.000003	0.001177	0.000013	0.009244	0.000110
	MPS	0.000241	0.000003	0.000974	0.000011	0.007009	0.000082

7. Applications to Real Data

A data set is employed to compare between ML, LS, WLS, and MPS estimators. The data are listed in Murthy et al. (2004) and represent failure times (by 1000 h) for 50 components. We conduct a data analysis to evaluate the BXII-D's goodness-of-fit by comparing it to various some well-known distributions such as Lomax, generalized logistic, and inverse Lomax. The BXII-D parameters are estimated by ML, MPS, LS and WLS methods based on complete sample case. The ML and MPS estimates can be obtained by maximizing Equations (4) and (32), respectively, with respect to k. The LS and WLS estimates are obtained by minimizing Equation (31) with respect to k. Table 5 gives the numerical values of ML, LS, WLS and MPS estimates, Akaike information criterion (AIC) and Bayesian information criterion (BIC) based on the complete sample case for the four distributions. The mathematical forms of the AIC and BIC are defined, respectively, as follows

$$AIC = -2\ln L(\Theta) + 2q \quad \text{and} \quad BIC = -2\ln L(\Theta) + q\ln(m),$$

where Θ denotes to the parameter space, $L(\Theta)$ is the maximized value of log likelihood function, q is the number of the estimated parameters in the distribution, and m is the number of stages in the progressive censored data set. The model with the minimum AIC and BIC is chosen as the best model to fit the data. In left hand of Figure 4, we provide the histogram for the complete data with the estimated curves of BXII-D based on the complete sample case and in right hand of the same figure, the emperical CDF plots for the complete data with the estimated CDFs of BXII-D based on the complete sample are sighted. The results in Table 5 show that the BXII-D provides a significantly better fit than the other three distributions.

Table 2 The MSEs of the PDF and CDF estimates of BXII-D at $x = 0.2$ and $p = 0.2$ when $n = 100$

m	Methods	$k = 0.25, c = 2$		$k = 0.5, c = 2$		$k = 1.5, c = 2$	
		PDF	CDF	PDF	CDF	PDF	CDF
10	UMVU	0.001343	0.000014	0.006742	0.000074	0.035500	0.000427
	ML	0.001921	0.000021	0.009422	0.000105	0.047400	0.000592
	LS	0.003983	0.000043	0.029800	0.000336	0.235300	0.003095
	WLS	0.002744	0.000029	0.018200	0.000203	0.071600	0.000907
	MPS	0.001796	0.000019	0.008820	0.000098	0.044400	0.000553
20	UMVU	0.000526	0.000006	0.002136	0.000023	0.017000	0.000202
	ML	0.000631	0.000007	0.002483	0.000027	0.019800	0.000239
	LS	0.001411	0.000015	0.003139	0.003879	0.034600	0.001310
	WLS	0.001172	0.000013	0.002656	0.000029	0.030800	0.000379
	MPS	0.000530	0.000006	0.002154	0.000023	0.017000	0.000204
30	UMVU	0.000292	0.000003	0.001326	0.000014	0.009267	0.000109
	ML	0.000319	0.000003	0.001448	0.000016	0.010100	0.000121
	LS	0.000424	0.000005	0.001937	0.002904	0.016300	0.000197
	WLS	0.000404	0.000004	0.001793	0.000020	0.015400	0.000185
	MPS	0.000288	0.000003	0.001295	0.000014	0.008939	0.000105
40	UMVU	0.000228	0.000002	0.000918	0.000010	0.006569	0.000077
	ML	0.000243	0.000003	0.000986	0.000011	0.006989	0.000083
	LS	0.000316	0.000003	0.001697	0.000019	0.008865	0.000105
	WLS	0.000293	0.000003	0.001555	0.000017	0.008415	0.000100
	MPS	0.000224	0.000002	0.000895	0.000010	0.006419	0.000075
60	UMVU	0.000150	0.000002	0.000557	0.000006	0.004154	0.000049
	ML	0.000155	0.000002	0.000580	0.000006	0.004296	0.000051
	LS	0.000232	0.000002	0.000774	0.000008	0.006118	0.000073
	WLS	0.000209	0.000002	0.000706	0.000008	0.005560	0.000066
	MPS	0.000149	0.000002	0.000554	0.000006	0.004139	0.000049
90	UMVU	0.000104	0.000001	0.000379	0.000004	0.003142	0.000037
	ML	0.000107	0.000001	0.000392	0.000004	0.003207	0.000038
	LS	0.000136	0.000001	0.000535	0.000006	0.003948	0.000046
	WLS	0.000124	0.000001	0.000478	0.000005	0.003629	0.000043
	MPS	0.000104	0.000001	0.000374	0.000004	0.003138	0.000037

Table 3 The MSEs of the PDF and CDF estimates of BXII-D at $x = 0.2$ and $p = 0.7$ when $n = 50$

m	Methods	$k = 0.25, c = 2$		$k = 0.5, c = 2$		$k = 1.5, c = 2$	
		PDF	CDF	PDF	CDF	PDF	CDF
10	UMVU	0.001040	0.000011	0.004530	0.000049	0.029500	0.000351
	ML	0.001390	0.000015	0.005930	0.000065	0.035900	0.000442
	LS	0.004660	0.014600	0.009870	0.014500	0.092900	0.009190
	WLS	0.003280	0.003920	0.009530	0.004910	0.092000	0.005640
	MPS	0.001040	0.000011	0.004480	0.000049	0.028600	0.000345
20	UMVU	0.000484	0.000005	0.001800	0.000020	0.014100	0.000166
	ML	0.000550	0.000006	0.002070	0.000023	0.016100	0.000193
	LS	0.001350	0.000956	0.003150	0.003880	0.023100	0.000280
	WLS	0.000897	0.000989	0.002940	0.001950	0.021300	0.000258
	MPS	0.000470	0.000005	0.001740	0.000019	0.013400	0.000159
30	UMVU	0.000311	0.000003	0.001190	0.000013	0.008510	0.000100
	ML	0.000345	0.000004	0.001280	0.000014	0.009150	0.000109
	LS	0.000862	0.000951	0.001790	0.000020	0.013200	0.000158
	WLS	0.000825	0.000950	0.001630	0.000018	0.012400	0.001030
	MPS	0.000300	0.000003	0.001160	0.000013	0.008310	0.000098
40	UMVU	0.000234	0.000002	0.000917	0.000010	0.006270	0.000074
	ML	0.000253	0.000003	0.000983	0.000011	0.006680	0.000079
	LS	0.000343	0.000004	0.001260	0.000014	0.009620	0.000114
	WLS	0.000316	0.000003	0.001160	0.000013	0.008630	0.000102
	MPS	0.000226	0.000002	0.000898	0.000010	0.006090	0.000072

Table 4 The MSEs of the PDF and CDF estimates of BXII-D at x = 0.2 and p = 0.7 when n = 100

m	Methods	$k=0.25, c=2$		$k=0.5, c=2$		$k=1.5, c=2$	
		PDF	CDF	PDF	CDF	PDF	CDF
10	UMVU	0.001150	0.000012	0.003890	0.000042	0.027900	0.000330
	ML	0.001520	0.000016	0.005100	0.000056	0.033900	0.000415
	LS	0.006130	0.004930	0.014300	0.015500	0.067800	0.015900
	WLS	0.005120	0.000055	0.013500	0.001110	0.067400	0.002630
	MPS	0.001160	0.000012	0.003880	0.000042	0.027200	0.000326
20	UMVU	0.000544	0.000006	0.001850	0.000020	0.014400	0.000170
	ML	0.000637	0.000007	0.002110	0.000023	0.016000	0.000193
	LS	0.000987	0.000011	0.004110	0.000045	0.028000	0.000342
	WLS	0.000937	0.000010	0.003860	0.000042	0.025800	0.000315
	MPS	0.000520	0.000006	0.001790	0.000019	0.013800	0.000164
30	UMVU	0.000325	0.000003	0.001120	0.000012	0.009390	0.000111
	ML	0.000354	0.000004	0.001220	0.000013	0.010300	0.000123
	LS	0.000514	0.000005	0.002070	0.000023	0.014100	0.000169
	WLS	0.000478	0.000005	0.001890	0.000021	0.013100	0.000157
	MPS	0.000318	0.000003	0.001080	0.000012	0.008990	0.000106
40	UMVU	0.000238	0.000003	0.000893	0.000010	0.007000	0.000082
	ML	0.000254	0.000003	0.000949	0.000010	0.007450	0.000088
	LS	0.000335	0.000004	0.001410	0.000015	0.010700	0.000127
	WLS	0.000306	0.000003	0.001290	0.000014	0.009790	0.000117
	MPS	0.000234	0.000002	0.000878	0.000010	0.006840	0.000080
60	UMVU	0.000151	0.000002	0.000599	0.000006	0.004720	0.000056
	ML	0.000158	0.000002	0.000620	0.000007	0.004920	0.000058
	LS	0.000218	0.000002	0.000840	0.000009	0.006970	0.000083
	WLS	0.000196	0.000002	0.000762	0.000008	0.006270	0.000074
	MPS	0.000150	0.000002	0.000596	0.000006	0.004600	0.000054
90	UMVU	0.000100	0.000001	0.000397	0.000004	0.002920	0.000034
	ML	0.000104	0.000001	0.000410	0.000004	0.003000	0.000035
	LS	0.000120	0.000001	0.000547	0.000006	0.004060	0.000048
	WLS	0.000111	0.000001	0.000492	0.000005	0.003650	0.000043
	MPS	0.000100	0.000001	0.000394	0.000004	0.002910	0.000034

Table 5 Estimates of the parameters, AIC and BIC for complete sample $(m = n)$

Distribution	Methods	Estimates	Estimates	AIC	BIC
BXII (c, k)	ML	0.9092	0.9944	215.7757	219.5998
	LS	0.7614	0.9386	218.0715	221.8955
	WLS	0.8728	0.9712	215.3711	219.1951
	MPS	1.0409	0.8822	215.6100	219.0944
Lomax (θ, β)	ML	1.0770	1.2997	220.1301	223.9541
	LS	0.5201	0.3679	220.0453	223.8693
	WLS	0.7578	0.6796	217.9861	221.8102
	MPS	1.0447	1.1515	215.7181	219.5421
Generalized- Logistic (μ, γ)	ML	2.6215	0.4299	273.9075	277.7315
	LS	2.1411	0.4215	275.7395	279.5636
	WLS	2.8103	0.5018	279.5574	279.5574
	MPS	2.5469	0.4423	274.0367	277.8608
Inverse Lomax(λ, α)	ML	0.7584	1.8301	230.2151	234.0392
	LS	0.5083	3.7471	287.5980	291.4220
	WLS	0.6797	2.1195	236.9670	240.7911
	MPS	0.7059	1.9271	229.0349	235.3638

Figure 4 show that MPS estimator has the best fit for BXII-D for these data.

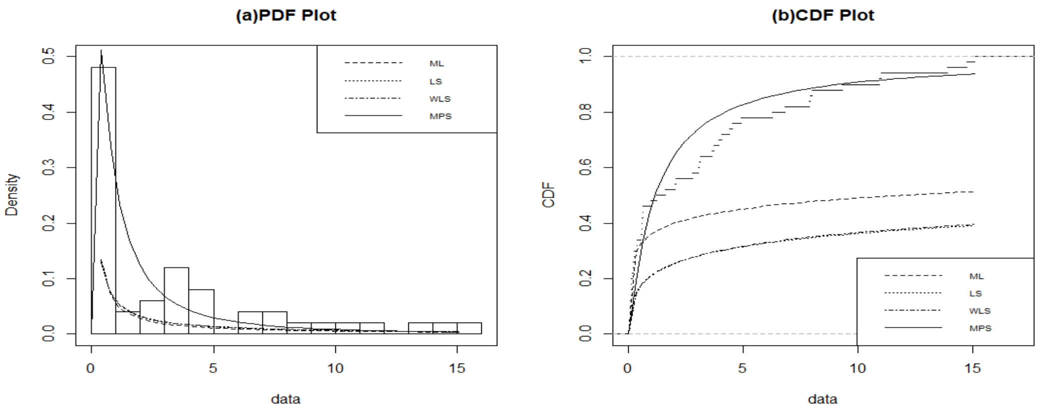


Figure 4 PDF plots (left hand side) and CDF plots (right hand side) for failure time's data fitted by different methods of estimation in complete sample case

We investigate these data by BXII-D in complete sample case once and PTIIC case once again to attain the validity of BXII-D for these data. As we mentioned before, the censored samples must reduce the time and the cost without losing in much information, so we set the data as PTIIC with binomial random removals in every stage. Our objective here is investigated our results in PTIIC case. The binomial parameter p is considered as $p = 0.7$ with number of stages $m = 15, 25$ and 40. The data in PTIIC are obtained as follows:

m = 15: 0.036, 0.058, 0.078, 0.116, 0.254, 0.538, 0.57, 2.054, 2.804, 3.147, 3.704, 4.393, 6.816, 8.022, 11.02;

$m = 25$: 0.036, 0.058, 0.103, 0.114, 0.148, 0.183, 0.192, 0.254, 0.262, 0.379, 0.59, 0.618, 0.645, 0.961, 1.228, 1.6, 2.054, 3.704, 4.534, 6.274, 7.8967.904, 8.022, 11.02, 15.08;

$m = 40$: 0.036, 0.074, 0.078, 0.086, 0.102, 0.103, 0.116, 0.148, 0.183, 0.254, 0.262, 0.379, 0.381, 0.538, 0.57, 0.59, 0.618, 0.645, 0.961, 1.228, 1.6, 2.006, 2.054, 2.804, 3.058, 3.076, 3.625, 3.704, 3.931, 4.393, 4.534, 4.893, 6.274, 6.816, 7.896, 7.904, 8.022, 11.02, 14.73, 15.08.

The data set under PTIIC are estimated by the same methods and the results are listed in Table 6. As seen from Table 6., the MPS estimator gives the smallest AIC and BIC compared with the other estimators based on PTIIC schemes.

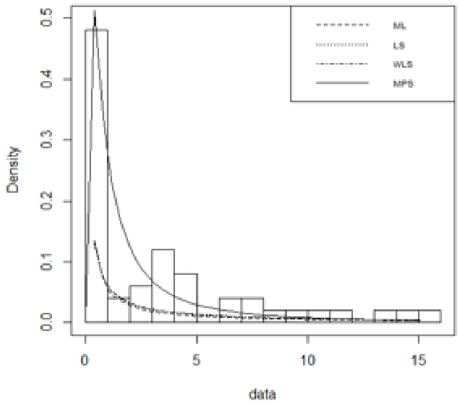
Table 6 Estimates of the parameters, AIC and BIC for PTIIC sample

m	Methods	k Estimates	c Estimates	AIC	BIC
15	ML	0.6459	0.2652	103.8198	105.2359
	LS	0.3464	0.4277	96.3404	97.7565
	WLS	0.3413	0.4473	95.4248	96.8409
	MPS	0.8977	1.1226	66.3112	67.7273
25	ML	0.9851	0.3352	136.8762	139.314
	LS	0.5462	0.5485	124.5836	127.0214
	WLS	0.5218	0.5998	122.7931	125.2309
	MPS	1.0116	1.0718	100.4508	102.8886
40	ML	1.0616	0.775	173.8247	177.2025
	LS	0.7215	0.7471	181.4116	184.7893
	WLS	0.7124	0.8164	179.444	182.8218
	MPS	0.9442	1.0448	171.1182	174.496

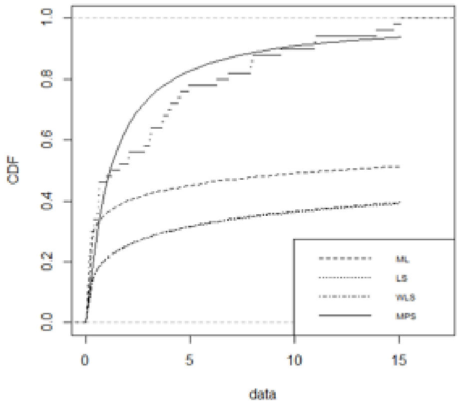
The left hand of Figure 5., represents the histogram for the considered data with the estimated curves of BXII-D based on the PTIIC case, the stages are considered as m =15, 25 and 40, respectively. The right hand of Figure 5 represents the emperical CDF plots for the data with the estimated CDFs of BXII-D based on the PTIIC. The figures indicate the superiority of the MPS method than the other methods. Now, we can confirm that the MPS estimator is the best fit to this data.

8. Conclusions

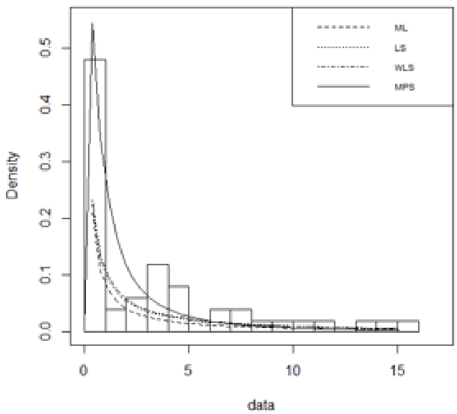
In lifetime and reliability studies, progressive type II censoring designs are extensively used. Based on PTII censoring samples, this study analyses the best estimator of the probability density function and cumulative distribution function of the Burr type XII distribution. We obtain the maximum likelihood, maximum product spacing, least squares, uniformly minimum variance unbiased, and weighted least squares estimators for the PDF and CDF of Burr XII distribution. The ML and UMVU estimators for the density and distribution functions of Burr XII distribution are provided in explicitly forms with their mean squared errors. To illustrate theoretical results, a simulation study is conducted. According to the results of the study, the maximum product spacing estimations are chosen over all other estimates. To demonstrate the applicability and significance of the suggested estimators, we look at an actual data set. The proposed estimators outperform some other competitive models, according on the findings of a real-life analysis. The simulation study as well as the real data example showed that the MPS estimates are the best for the PDF and the CDF of Burr XII distribution in PTIIC case.



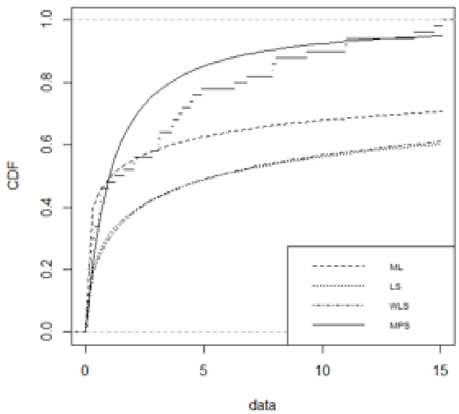
(a) PDF plot for failure time of 50 components



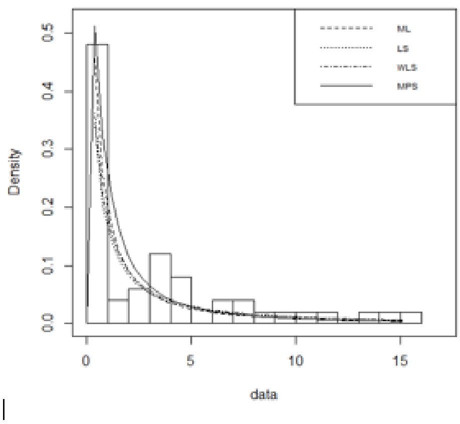
(b) CDF plot for failure time of 50 components



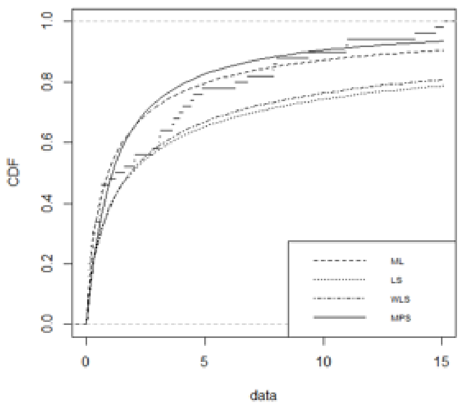
(c) PDF plot for failure time of 50 components



(d) CDF plot for failure time of 50 components



(e) PDF plot for failure time of 50 components



(f) CDF plot for failure time of 50 components

Figure 5 (a), (c), (e): PDF plots under PTIIC samples at stages $m = 15, 25, 40$
and (b), (d), (f): CDF plots at the same stages

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