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Modelling Veterinary Medical Data Utilizing a New Generalized Marshall-Olkin Transmuted Generator of Distributions with Statistical Properties

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Abstract

This paper introduces a new family of continuous distributions called the generalized Marshall-Olkin transmuted-G family to extend the transmuted family that is proposed by Shaw and Buckley (2009). Some of its mathematical properties including hazard rate function, quantile, asymptotes, stochastic orderings, moment generating function, and entropy are derived. A special sub-model of the proposed family is discussed. The maximum likelihood, least squares and weighted least squares estimation methods are adopted to estimate the model parameters. We present a simulation study to explore the biases and mean square errors of these estimators. The superiority of the proposed family over other existing distributions is proved by modelling a veterinary medical data.

Keywords: Statistical model, generalized-Marshall-Olkin family, transmuted-G family, stochastic orderings, simulation, statistics and numerical data.

1. Introduction

Recently, several generalized families of continuous distributions have been proposed to model various phenomena. There has been an increased interest in defining new generated families of univariate continuous distributions by adding extra shape parameters to the baseline model, where statistical models can be utilized to extract all the information from the data to make them more useful. One of the most notable families is the Marshall-Olkin (MO) family (Marshall and Olkin 1997) which has been utilized by many authors to propose new generalized models as well as newly generated families. For example, the generalized Marshall-Olkin-G (GMO-G) (Jayakumar and Mathew 2008), unified beta Marshall-Olkin-G and Marshall-Olkin-Kumaraswamy-G (Chakraborty et al. 2018), exponentiated generalized Marshall-Olkin (Handique et al. 2019), families, among others. The generalized Marshall-Olkin-H (GMO-G) family (Jayakumar and Mathew 2008) is specified by the survival function (sf) and probability density function (pdf)

$$\overline{F}^{\mathrm{GMO}}(x;\theta,\alpha)=\left[\frac{\alpha\,\overline{H}(x)}{1-\overline{\alpha}\,\overline{H}(x)}\right]^{\theta} \text{ and } f^{\mathrm{GMO}}(x;\theta,\alpha)=\frac{\theta\,\alpha^{\theta}\,h(x)\,\overline{H}(x)^{\theta-1}}{[1-\overline{\alpha}\,\overline{H}(x)]^{\theta+1}}, \tag{1}$$

where $-\infty < x < \infty$, $\alpha > 0$ $(\overline{\alpha} = 1-\alpha)$ and $\theta > 0$ is an additional shape parameter. For $\theta = 1$, the GMO-H family reduces to the MO family, and the baseline model follows for $\alpha = \theta = 1$.

Shaw and Buckley (2009) proposed the quadratic rank transmutation map that is also known as transmuted-G (T-G) class. This class of distributions has been received considerable attention over the last two decades.

The cumulative distribution function (cdf) and pdf of the T-G family are given by

$$H^{\mathrm{TG}}(x;\lambda) = G(x)\,[1+\lambda-\lambda\,G(x)] \tag{2}$$

and

$$h^{\mathrm{TG}}(x;\lambda) = g(x)\,[1+\lambda-2\lambda\,G(x)], \tag{3}$$

where $g(x)$ and $G(x)$ denote to the respective pdf and cdf of a baseline model whereas the parameter lambda between -1 and 1 and its boundaries. In this paper, we introduce and study a new extension of the T-G family by adding an extra shape parameter in (1) to provide more flexibility to the generated family. In fact, based on the GMO-H family, we construct a new generator called the generalized Marshall-Olkin transmuted-G (GMOT-G) family of distribution and give a comprehensive description of some of its mathematical properties. We hope that the new model will attract wider applications in reliability, engineering, and other areas of research. The resulting expression for the sf of proposed GMOT-G family is given by

$$\overline{F}^{\mathrm{GMOTG}}(x;\theta,\alpha,\lambda)=\left[\frac{\alpha\,[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}{1-\overline{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}\right]^{\theta}. \tag{4}$$

The corresponding pdf and hrf of the GMOT-G family

$$f^{\mathrm{GMOTG}}(x;\theta,\alpha,\lambda)=\frac{\theta\,\alpha^{\theta}\,g(x)\,[1+\lambda-2\lambda\,G(x)]\,[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]^{\theta-1}}{[1-\overline{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]]^{\theta+1}} \tag{5}$$

and

$$h^{\mathrm{GMOTG}}(x;\theta,\alpha,\lambda)=\frac{\theta\,g(x)\,[1+\lambda-2\lambda\,G(x)]\,[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]^{-1}}{[1-\overline{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]]}.$$

The distribution defined by (5) is referred to as $\mathrm{GMOT-G}(\theta,\alpha,\lambda)$. In particular, the GMOT-G family includes the following special cases:

(i) For $\theta = 1$, we obtain the $\mathrm{MOT\text{-}G}(\alpha,\lambda)$ family,

(ii) For $\theta = \alpha = 1$, we obtain the $\mathrm{T-G}(\lambda)$ family,

(iii) For $\lambda = 0$, we obtain the $\mathrm{GMO}(\theta,\alpha)$ family,

(iv) For $\theta = 1$, $\lambda = 0$, we obtain the $\mathrm{MO}(\alpha)$ family,

(v) For $\alpha = \theta = 1$, $\lambda = 0$, we obtain the baseline model.

The important motivations behind the GMOT-G family are to improve the overall adaptability of the T-G family using the GMO-G class at the same time an extension of GMO family. In addition, the new family ensures a high level of flexibility for important distributional characteristics, such as mean, variance, skewness, and kurtosis. We illustrate this aspect by discussing a special four-parameter distribution of the family called, GMOT-exponential (GOMT-E) model defined with the exponential distribution as baseline model as an example. A further physical motivation of the GMOT-G family is illustrated by the following proposition.

Proposition 1 *Let* $X_{i1}, X_{i2}, ..., X_{iN}$ *be a sequence of* θN *i.i.d. random variables from T-G distribution and* $W_i = \min(X_{i1}, X_{i2}, ..., X_{iN})$ *and* $V_i = \max(X_{i1}, X_{i2}, ..., X_{iN})$ *for* $i = 1, 2, ..., \theta$. *Then*

(i) $\min_i W_i$ *follows* $\mathrm{GMOT-G}(\theta, \alpha, \lambda)$ *if* $N \sim Geometric(\alpha)$ *and*

(ii) $\max_i V_i$ *follows* $\mathrm{GMOT-G}(\theta, \alpha, \lambda)$ *if* $N \sim Geometric(1/\alpha)$.

Case (i) For $0 < \alpha \le 1$, *considering N has a geometric distribution with parameter* α, *we get*

$$\begin{aligned} \mathrm{P}[\min\{W_1, W_2, ..., W_\theta\} > x] &= \mathrm{P}[W_1 > x]\,\mathrm{P}[W_2 > x]\,...\mathrm{P}[W_\theta > x] \\ &= \prod_{i=1}^{\theta} \mathrm{P}[W_i > x] = [\bar{F}^{\mathrm{MOT\text{-}G}}(x; \alpha, \lambda)]^\theta \\ &= \left[\frac{\alpha\,[1 - G(x)\,\{1 + \lambda - \lambda\,G(x)\}]}{1 - \bar{\alpha}[1 - G(x)\,\{1 + \lambda - \lambda\,G(x)\}]}\right]^\theta. \end{aligned}$$

Case (ii) *For* $\alpha > 1$, *considering N has a geometric distribution with parameter* $1/\alpha$, *we get*

$$\begin{aligned} \mathrm{P}[\min\{V_1, V_2, ..., V_\theta\} > x] &= \mathrm{P}[V_1 > x]\,\mathrm{P}[V_2 > x]\,...\mathrm{P}[V_\theta > x] \\ &= \prod_{i=1}^{\theta} P[V_i > x] = [\bar{F}^{\mathrm{MOT\text{-}G}}(x; \alpha, \lambda)]^\theta \\ &= \left[\frac{\alpha\,[1 - G(x)\,\{1 + \lambda - \lambda\,G(x)\}]}{1 - \bar{\alpha}[1 - G(x)\,\{1 + \lambda - \lambda\,G(x)\}]}\right]^\theta. \end{aligned}$$

The T-G family has been extended by many authors to improve its flexibility. For example, Afify et al. (2017) introduced the beta transmuted-H, Yousof et al. (2018) studied the generalized transmuted Poisson-G, Alizadeh et al. (2018, 2020) studied the complementary generalized transmuted Poisson-G family and odd log-logistic Lindley-G family, respectively, Mansour et al. (2019) introduced the transmuted transmuted-G family, El-Morshedy and Eliwa (2019) proposed the odd flexible Weibull-H family, Eliwa et al. (2020a, 2020b) proposed the exponentiated odd Chen-G families and discrete Gompertz generator, respectively, Tahir et al. (2020) proposed new Kumaraswamy generalized family, El-Morshedy et al. (2020, 2021a, 2021b) proposed the odd Chen generator, Poisson generalized exponential-G family and type I half-logistic odd Weibull generator, respectively, Altun et al. (2021) reported the additive odd-G family, Almazah et al. (2021) proposed an application based on the Topp-Leone exponentiated-G, among others. Our motivations to introduce the new generator are:

1. To provide special distributions which are capable of modelling various types of hrfs.

2. The special distributions are a proper to model asymmetric data and can also be used in a variety of applied problems in many areas.

3. To construct heavy-tailed models that are not longer-tailed for modelling real data.

4. To provide consistently better fits than other generated distributions under the same baseline model.

The rest of the paper is outlined as follows. In Section 2, we introduce the GMOT-E distribution. We obtain some general mathematical properties of the GMOT-G family in Section 3. In Section 4, the maximum likelihood, lest squares and weighted lest squares estimators are obtained. Further, a simulation study is carried out to explore the performance of these estimators in the same section. The analysis of a failure time data is presented in Section 5. Finally, we give some conclusions in Section 6.

2. The GMOT-E distribution

In this section, we derive the GMOT-E model as a special sub-model of the proposed family by taking $G(x)=1-e^{-\beta x}$ to be exponential (E) distribution. Hence, we can write pdf and hrf of the GMOT-E distribution as

$$f^{\mathrm{GMOTE}}(x;\theta,\alpha,\lambda,\beta)=\frac{\theta\alpha^{\theta}\beta e^{-\beta x}[1+\lambda-2\lambda(1-e^{-\beta x})][1-(1-e^{-\beta x})\{1+\lambda-\lambda(1-e^{-\beta x})\}]^{\theta-1}}{[1-\bar{\alpha}[1-(1-e^{-\beta x})\{1+\lambda-\lambda(1-e^{-\beta x})\}]]^{\theta+1}}$$

and

$$h^{\mathrm{GMOTE}}(x;\theta,\alpha,\lambda,\beta)=\frac{\theta\alpha^{\theta}\beta e^{-\beta x}[1+\lambda-2\lambda(1-e^{-\beta x})][1-(1-e^{-\beta x})\{1+\lambda-\lambda(1-e^{-\beta x})\}]^{-1}}{[1-\bar{\alpha}[1-(1-e^{-\beta x})\{1+\lambda-\lambda(1-e^{-\beta x})\}]]}.$$

Figures 1 and 2 depicted some possible shapes for the GMOT-E pdf and hrf, respectively.

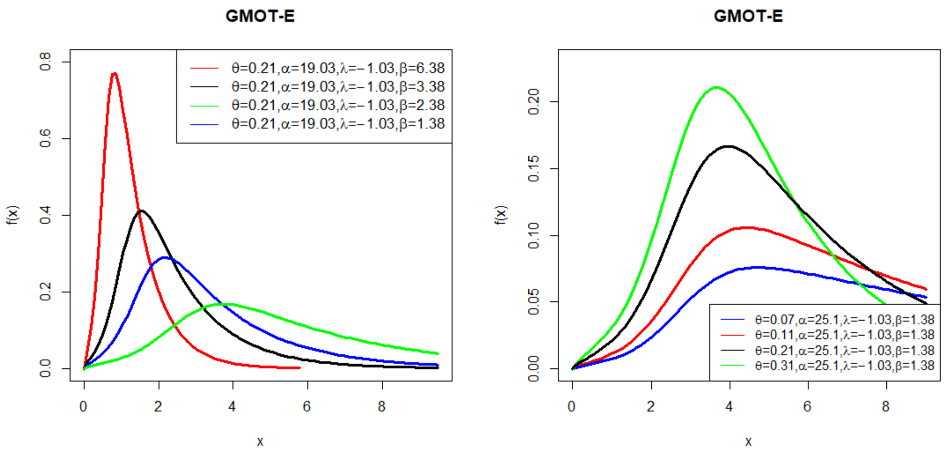


Figure 1. The pdf plots of the GMOT-E distribution.

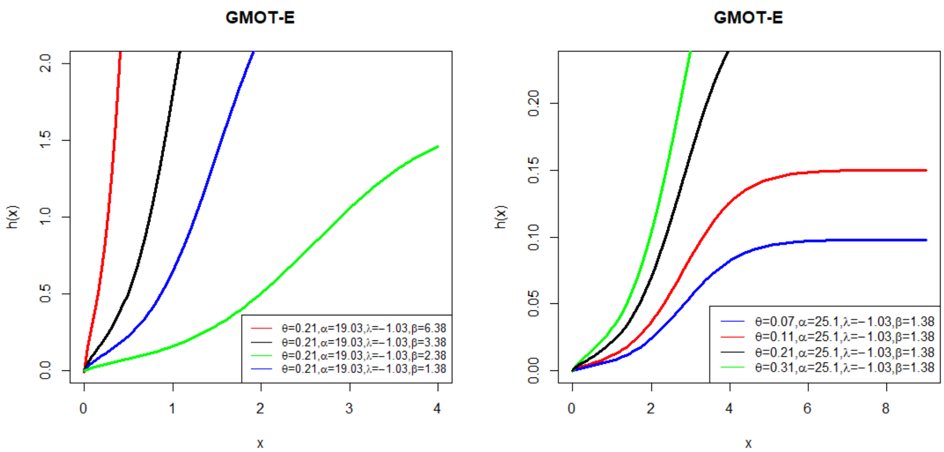


Figure 2. The hrf plots of the GMOT-E distribution.

3. General Properties

3.1. Quantile function

The quantile function (qf) of X, say $Q(u)=F^{-1}(u)$, can be obtained by inverting (4) numerically and it is given by

$$Q(u)=G^{-1}\left[\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda T}}{2\lambda}\right] \text{ where } T=1-\frac{(1-u)^{1/\theta}}{\alpha+\bar{\alpha}(1-u)^{1/\theta}}. \quad (6)$$

For example, the p^{th} quantile, t_p, of the GMOT-E distribution reduces to

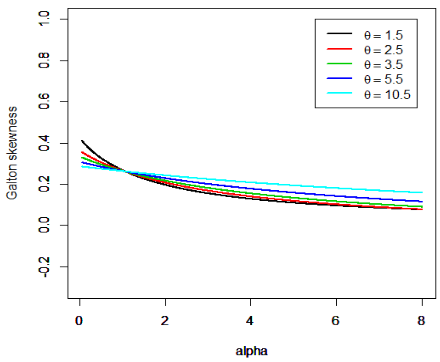
$$t_p=-\frac{1}{\beta}\log\left[1-\frac{1+\lambda-\sqrt{(1+\lambda)^2-4\lambda T}}{2\lambda}\right] \text{ where } T=1-\frac{(1-u)^{1/\theta}}{\alpha+\bar{\alpha}(1-u)^{1/\theta}}.$$

It may be noted that for given uniform random number 'u' corresponding random numbers 'x' from $\text{GMOT}-\text{G}(\theta,\alpha,\lambda)$ can be easily obtained using (6).

The flexibility of skewness and kurtosis of the $\text{GMOT}-\text{G}(\theta,\alpha,\lambda)$ is checked by plotting Galton skewness (S) that measures the degree of the long tail and Moors (1988) kurtosis (K) that measures the degree of tail heaviness. The S and K measures are defined by

$$S=\frac{Q(6/8)-2Q(4/8)+Q(2/8)}{Q(6/8)-Q(2/8)} \text{ and } K=\frac{Q(7/8)-Q(5/8)+Q(3/8)-Q(1/8)}{Q(6/8)-Q(2/8)}.$$

Figure 3 shows the Galton skewness, and the Moor kurtosis plots for different parametric values of α and θ with $\beta=0.5$ and $\lambda=0.9$ for GMOT-E model. One can note that the GMOT-E model can be used to model positive skewness and platykurtic data sets.



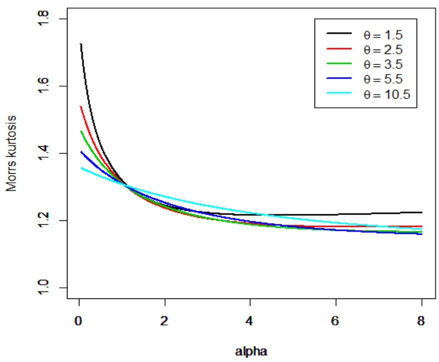


Figure 3. The Galton skewness, *S*, and the Moor kurtosis, *K*, for the GMOT-E distribution.

3.2. Asymptotes and shapes

Two propositions regarding asymptotes of the proposed family are discussed here.

Proposition 2 *The asymptotes of pdf, sf and hrf of* $\text{GMOT}-\text{G}(\theta,\alpha,\lambda)$ *as* $x\to 0$ *are given by*

$$f^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim\theta\,(1+\lambda)g(x)/\alpha,\quad \bar{F}^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim 1,$$

$$h^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim\theta\,(1+\lambda)g(x)/\alpha.$$

Proposition 3 *The asymptotes of pdf, sf and hrf of* GMOT − G (θ,α,λ) *as* $x\to\infty$ *are given by*

$$f^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim\theta\,\alpha^{\theta}(1-\lambda)g(x)\,\bar{G}(x)^{\theta-1}(1-\bar{\alpha}\bar{G}(x))^{-(\theta+1)},$$

$$\bar{F}^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim\left[\frac{\alpha\,\bar{G}(x)}{1-\bar{\alpha}\,\bar{G}(x)}\right]^{\theta},\quad f^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim\theta(1-\lambda)g(x)\,\bar{G}(x)^{-1}(1-\bar{\alpha}\bar{G}(x))^{-1}.$$

The shapes of the density and hazard functions can be described analytically. The critical points of the pdf of the GMOT − G (θ,α,λ) family are the roots of the following equation,

$$\frac{g'(x)}{g(x)}-\frac{2\lambda\,g(x)}{1+\lambda-2\lambda\,G(x)}+\frac{(\theta-1)g(x)\,[1+\lambda-2\lambda\,G(x)]}{1-G(x)[1+\lambda-2\lambda\,G(x)]}+\frac{(\theta+1)\bar{\alpha}\,g(x)\,[1+\lambda-2\lambda\,G(x)]}{1-\bar{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}=0. \quad (7)$$

The critical point of GMOT − G (θ,α,λ) family hazard rate are the roots of the equation,

$$\frac{g'(x)}{g(x)}-\frac{2\lambda\,g(x)}{1+\lambda-2\lambda\,G(x)}-\frac{g(x)\,[1+\lambda-2\lambda\,G(x)]}{1-G(x)[1+\lambda-2\lambda\,G(x)]}+\frac{\bar{\alpha}\,g(x)\,[1+\lambda-2\lambda\,G(x)]}{1-\bar{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}=0. \quad (8)$$

There may be more than one roots of (7) and (8). If $x=x_0$ is a root then it is a local maximum, a local minimum or a point of inflexion if $\psi(x_0)<0,\ \psi(x_0)>0$ *or* $\psi(x_0)=0$ and for (8) if $\omega(x_0)<0,$ $\omega(x_0)>0,$ or $\omega(x_0)=0$ where $\psi(x)=(d^2/dx^2)\log[f(x)]$ and $\omega(x)=(d^2/dx^2)\log[h(x)].$

3.3. Stochastic orderings

Let X and Y be two random variables with respective cdfs F and $G,$ and corresponding pdf's f and $g.$ Then X is said to be smaller than Y in the likelihood ratio order ($X\le_{lr}Y$) if $f(x)/g(x)$ is decreasing in $x\ge 0.$

Theorem 1 *Let* $X\sim$ GMOT − G $(\theta,\alpha_1,\lambda)$ *and* $Y\sim$ GMOT − G $(\theta,\alpha_2,\lambda)$. *If* $\alpha_1<\alpha_2,$ *then* $X\le_{lr}Y.$

Proof $\dfrac{f(x)}{g(x)}=\left(\dfrac{\alpha_1}{\alpha_2}\right)^{\theta}\left[\dfrac{1-\bar{\alpha}_2[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}{1-\bar{\alpha}_1[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}\right]^{\theta+1}$

$$\frac{d}{dt}\left[\frac{f(x)}{g(x)}\right]=(\theta+1)(\alpha_1/\alpha_2)^{\theta}(\alpha_1-\alpha_2)\frac{U^{\theta}V^{\theta}\,g(x)(1+\lambda-2\lambda G(x))}{V^{\theta+2}},$$

where $U=1-\bar{\alpha}_2[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]$ and $V=1-\bar{\alpha}_1[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}].$

Now this is always less than 0 since $\alpha_1<\alpha_2.$ Hence, $f(x)/g(x)$ is decreasing in $x.$ That is, $X\le_{lr}Y.$

3.4. Linear mixture representation

Here, we express the sf and pdf of the GMOT − G (θ,α,λ) as an infinite linear mixture of the corresponding functions of exponentiated− T − G (λ) distribution. Consider the series representation

$$(1-z)^{-k}=\sum_{j=0}^{\infty}\frac{(j+k-1)!}{(k-1)!\,j!}z^{j}. \quad (9)$$

This is valid for $|z|<1$ and $k>0,$ where $\Gamma(.)$ is the gamma function. Applying the expansion in (9) on (4), for $\alpha\in(0,1),$ we obtain

$$\bar{F}^{\text{GMOTG}}(x;\theta,\alpha,\lambda) = \alpha^{\theta}\{\bar{F}^{\text{TG}}(x;\lambda)\}^{\theta}\sum_{j=0}^{\infty}\frac{(j+\theta-1)!}{(\theta-1)!\,j!}(1-\alpha)^{j}\{\bar{F}^{\text{TG}}(x;\lambda)\}^{j}$$

$$=\sum_{j=0}^{\infty}\xi_j'[\bar{F}^{\text{TG}}(x;\lambda)]^{j+\theta}. \quad (10)$$

Differentiating (10), with respect to x, we get

$$f^{\text{GMOTG}}(x;\theta,\alpha,\lambda) = f^{\text{TG}}(x;\lambda)\sum_{j=0}^{\infty}\xi_j[\bar{F}^{\text{TG}}(x;\lambda)]^{j+\theta-1}$$

$$=-\sum_{j=0}^{\infty}\xi_j'\frac{d}{dx}[\bar{F}^{\text{TG}}(x;\lambda)]^{j+\theta}, \quad (11)$$

where $\xi_j'=\xi_j'(\alpha)=\binom{j+\theta-1}{j}(1-\alpha)^{j}\alpha^{\theta}$, $\xi_j=\xi_j(\alpha)=(j+\theta)\xi_j'$.

3.5. Moment generating function

The moment generating function (mgf) of the GMOT-G family can be easily expressed in terms of the exponentiated T-G family using (11) as follows

$$M_X^{\text{GMOTG}}(s)=E[e^{sX}]=\int_{-\infty}^{\infty}e^{sx}f(x)\,dx=-\int_{-\infty}^{\infty}e^{st}\sum_{j=0}^{\infty}\xi_j'\frac{d}{dx}[\bar{F}^{\text{TG}}(x;\lambda)]^{j+\theta}\,dx$$

$$=-\sum_{j=0}^{\infty}\xi_j'\int_{-\infty}^{\infty}e^{sx}\frac{d}{dx}[\bar{F}^{\text{TG}}(x;\lambda)]^{j+\theta}\,dx=\sum_{j=0}^{\infty}\xi_j\,M_X^{\text{TG}}(s),$$

where $M_X^{\text{TG}}(s)$ is the mgf of the exponentiated-T-G family with parameter λ.

Tables 1-4 report some numerical results of mean, variance, skewness, and kurtosis of the GMOT-E distribution using the R software.

Table 1 Some descriptive statistics using GMOT-E distribution as θ grows

θ	α	λ	β	Mean	Variance	Skewness	Kurtosis
1.0	0.9	0.1	0.8	1.31944	1.47087	2.10904	9.59080
1.5				0.77462	0.59650	2.14974	10.03396
2.0				0.54382	0.32277	2.18777	10.33269
3.5				0.26886	0.09609	2.28403	10.98801
5.0				0.16600	0.04309	2.39771	11.74260
10.0				0.05643	0.00780	2.90103	15.61318
30.0				0.00497	0.00019	5.83444	57.72020

Table 2 Some descriptive statistics using GMOT-E distribution as α grows

α	θ	λ	β	Mean	Variance	Skewness	Kurtosis
0.01	1.5	0.1	0.8	0.06520	0.01008	10.39448	215.2991
0.1				0.34735	0.10024	4.81696	37.85083
1.5				0.80174	0.83932	1.94573	8.40369
3.5				0.74260	1.25012	2.07130	8.20236
5.0				0.68522	1.37579	2.26648	8.97715
10.0				0.54351	1.46747	2.86067	12.26209
30.0				0.31507	1.21417	4.46215	25.74435

Table 3 Some descriptive statistics using GMOT-E distribution as β grows

β	θ	λ	α	Mean	Variance	Skewness	Kurtosis
0.01	1.5	0.1	0.8	60.95107	3499.791	2.23055	10.60618
0.1				6.095107	34.99791	2.23055	10.60618
1.5				0.406340	0.15554	2.23055	10.60618
3.5				0.174140	0.02856	2.23055	10.60618
5.0				0.121902	0.01399	2.23055	10.60618
10.0				0.060950	0.00349	2.23055	10.60618
30.0				0.020317	0.0003888	2.23052	10.60636

Table 4 Some descriptive statistics using GMOT-E distribution as λ grows

λ	θ	α	β	Mean	Variance	Skewness	Kurtosis
–0.9	1.5	0.1	0.8	0.860774	0.054419	32.24346	153.5608
–0.5				0.578402	0.142456	5.015745	30.43173
0.0				0.375905	0.110500	4.694646	35.27879
0.5				0.258777	0.061466	5.321890	49.34551
0.9				0.199477	0.032793	5.024815	43.86325

From Tables 1-4, it is observed that the GMOT-E distribution can be utilized to model positively skewed and leptokurtic data. Moreover, it can be used as a flexible model for analysing over dispersion and under dispersion data where in some arbitrary choice of the parameters, the variance becomes greater than the mean or smaller than it.

3.6. Rényi entropy

The entropy of a random variable is a measure of uncertainty variation, and it has been used in various situations in science and engineering. The Rényi entropy (see Song 2001) is defined by

$$I_R(\delta)=(1-\delta)^{-1}\log\left(\int_{-\infty}^{\infty} f(x)^{\delta}\,dx\right),\ \text{where } \delta>0 \text{ and } \delta\neq 1.$$

Thus, the Rényi entropy of the GMOT-G family can be obtained as

$$I_R(\delta)=(1-\delta)^{-1}\log\left(\sum_{j=0}^{\infty}\omega_j\int_{-\infty}^{\infty}[f^{\mathrm{TG}}(x;\lambda)\bar{F}^{\mathrm{TG}}(x;\lambda)^{\theta-1}]^{\delta}[\bar{F}^{\mathrm{TG}}(x;\lambda)]^{j}\,dx\right),$$

where $\omega_j=\dfrac{\theta^{\delta}\alpha^{\delta\theta}(1-\alpha)^{j}\,\Gamma[\delta(\theta+1)+j]}{\Gamma[\delta(\theta+1)]\ \ j!}.$

4. Estimation

4.1. Maximum likelihood method

Let $X=(x_1,x_2,...,x_n)$ be a random sample of size n from $\mathrm{GMOT-G}(\theta,\alpha,\lambda)$ with parameter vector $\boldsymbol{\rho}=(\theta,\alpha,\lambda,\boldsymbol{\xi})$, where $\boldsymbol{\xi}=(\xi_1,\xi_2,...,\xi_q)$ is the parameter vector of any baseline model. Then the log-likelihood function for $\boldsymbol{\rho}$ is given by

$$\ell=\ell(\boldsymbol{\rho})=n\log(\theta\alpha^{\theta})+\sum_{i=1}^{n}\log[g(x_i,\boldsymbol{\xi})]+(\theta-1)\sum_{i=1}^{n}\log\left\{1-G(x_i,\boldsymbol{\xi})\left[1+\lambda-\lambda G(x_i,\boldsymbol{\xi})\right]\right\}$$
$$+\sum_{i=1}^{n}\log\left[1+\lambda-2\lambda G(x_i,\boldsymbol{\xi})\right]-(\theta+1)\sum_{i=1}^{n}\log\left(1-\bar{\alpha}\left\{1-G(x_i,\boldsymbol{\xi})\left[1+\lambda-\lambda G(x_i,\boldsymbol{\xi})\right]\right\}\right).$$

This log-likelihood function cannot be solved analytically because of its complex form but it can be maximized numerically by employing global optimization methods available with the R software©. By taking the partial derivatives of the log- likelihood function with respect to $\theta,\ \alpha,\lambda$ and $\boldsymbol{\xi}$, we obtain the components of the score vector $U_{\boldsymbol{\rho}}=\left(U_{\theta},U_{\alpha},U_{\lambda},U_{\boldsymbol{\xi}}\right).$

The asymptotic variance-covariance matrix of the MLEs of parameters can obtained by inverting the Fisher information matrix $\mathrm{I}(\boldsymbol{\rho})$ which can be derived using the second partial derivatives of the log-likelihood function with respect to each parameter. The $i\,j^{\text{th}}$ elements of $\mathrm{I}_n(\boldsymbol{\rho})$ are given by

$$\mathrm{I}_{ij}=-E[\partial^2 l(\boldsymbol{\rho})/\partial\rho_i\,\partial\rho_j],\ i,j=1,\ 2,\ldots,3+q.$$

The exact evaluation of the above expectations may be cumbersome. In practice one can estimate $\mathrm{I}_n(\boldsymbol{\rho})$ by the observed Fisher's information matrix $\hat{\mathrm{I}}_n(\hat{\boldsymbol{\rho}})=(\hat{\mathrm{I}}_{ij})$ defined as

$$\hat{\mathrm{I}}_{ij}\approx\left(-\partial^2 l(\boldsymbol{\rho})/\partial\rho_i\,\partial\rho_j\right)_{\boldsymbol{\eta}=\hat{\boldsymbol{\eta}}},\ i,j=1,\ 2,\ldots,3+q.$$

Using the general theory of MLEs under some regularity conditions on the parameters as $n\to\infty$ the asymptotic distribution of $\sqrt{n}\,(\hat{\boldsymbol{\rho}}-\boldsymbol{\rho})$ is $N_k(0,V_n)$ where $V_n=(v_{jj})=\mathrm{I}_n^{-1}(\boldsymbol{\rho}).$ The asymptotic behaviour remains valid if V_n is replaced by $\hat{V}_n=\hat{\mathrm{I}}^{-1}(\hat{\boldsymbol{\rho}}).$ Using this result large sample standard errors of j^{th} parameter ρ_j is given by $\sqrt{\hat{v}_{jj}}$.

4.2. Least squares method

Let $X=(x_1,x_2,\ldots,x_n)$ be a random sample of size n from $\mathrm{GMOT-G}(\theta,\alpha,\lambda).$ Least squares estimators (LSE) can be obtained by minimizing the following expression

$$l_{LS}=\sum_{i=1}^{n}\left(1-\left[\frac{\alpha\,[1-G(x_i,\boldsymbol{\xi})\ \{1+\lambda-\lambda\,G(x_i,\boldsymbol{\xi})\}]}{1-\bar{\alpha}[1-G(x_i,\boldsymbol{\xi})\ \{1+\lambda-\lambda\,G(x_i,\boldsymbol{\xi})\}]}\right]^{\theta}-\frac{i}{1+n}\right)^2.$$

This l_{LS} function cannot be minimizing analytically because of its complex form but it can be minimizing numerically by employing global optimization methods available with the R software©.

4.3. Weighted least squares method

Let $X=(x_1,x_2,\ldots,x_n)$ be a random sample of size n from $\mathrm{GMOT-G}(\theta,\alpha,\lambda).$ The weighted least squares estimators (WLSE) can be obtained by minimizing the following expression

$$l_{WLS}=\sum_{i=1}^{n}\frac{(n+2)(n+1)^2}{(n-i+1)\,i}\left(1-\left[\frac{\alpha\,[1-G(x_i,\boldsymbol{\xi})\ \{1+\lambda-\lambda\,G(x_i,\boldsymbol{\xi})\}]}{1-\bar{\alpha}[1-G(x_i,\boldsymbol{\xi})\ \{1+\lambda-\lambda\,G(x_i,\boldsymbol{\xi})\}]}\right]^{\theta}-\frac{i}{1+n}\right)^2.$$

This l_{WLS} function cannot be minimizing analytically because of its complex form but it can be minimizing numerically by employing global optimization methods available with the R software©.

4.4. Simulations results

In order to explore the performance of the MLE, LSE, and WLSE, we conduct a simulation study using the statistical R software© through the package (stats 4). We generate 10,000 samples of size $n =$ 5, 10, 15, …, 100 from the $\text{GMOT}-\text{E}(\theta,\alpha,\beta,\lambda)$ distribution for $\theta = 1.2$, $\alpha = 1.6$, $\beta = 1.3$, $\lambda = 0.36$. The number of Monte Carlo (MC) replications was 1,000. The evaluation of the estimates was performed based on some quantities such as the empirical biases, and mean squared errors (MSEs) that are calculated for each sample size using the R package from MC replications, where

$$\text{Bias}(\alpha) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\alpha}_j - \alpha) \text{ and } \text{MSE}(\theta) = \frac{1}{1000}\sum_{j=1}^{1000}(\hat{\theta}_j - \theta)^2.$$

The empirical results are given in Figures 4-6. From Figures 4-6, the bias and MSE decrease as the sample size n grows. This shows the consistency of the estimates. Thus, MLE, LSE, and WLSE can be used effectively to estimate the parameters of the GMOT-E distribution.

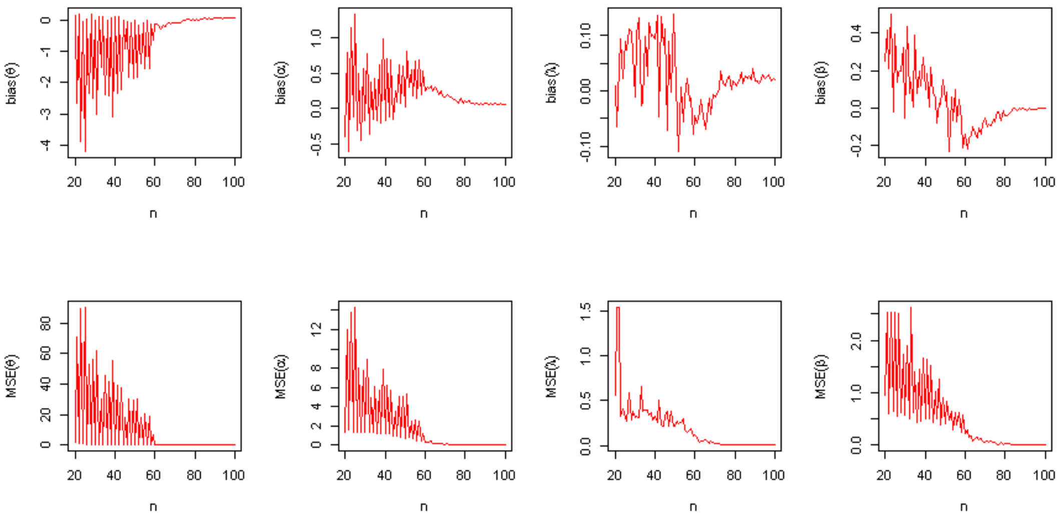


Figure 4 The biases and MSEs of the model parameter versus *n* based on MLE

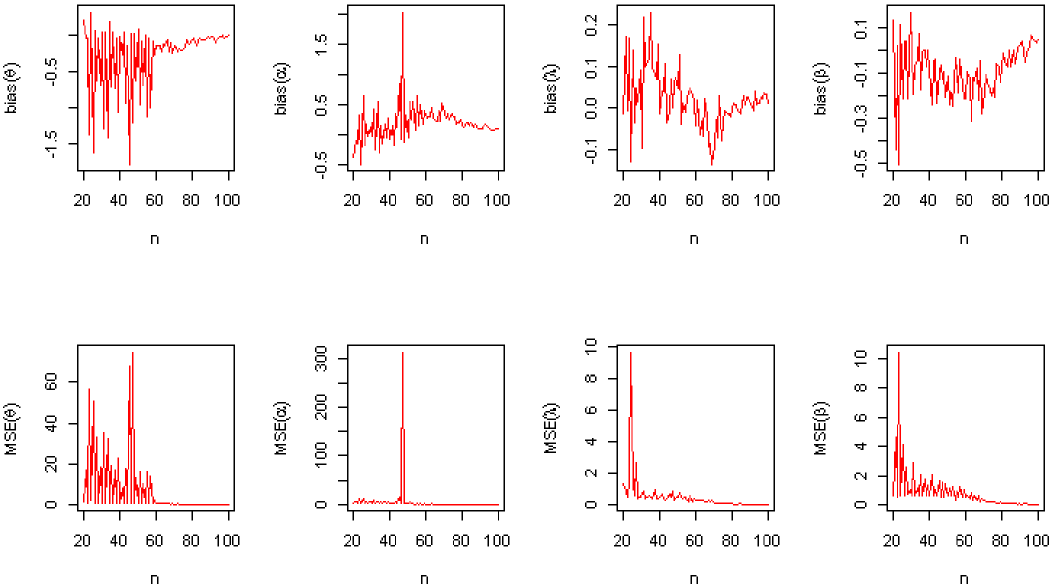


Figure 5 The biases and MSEs of the model parameter versus n based on LSE

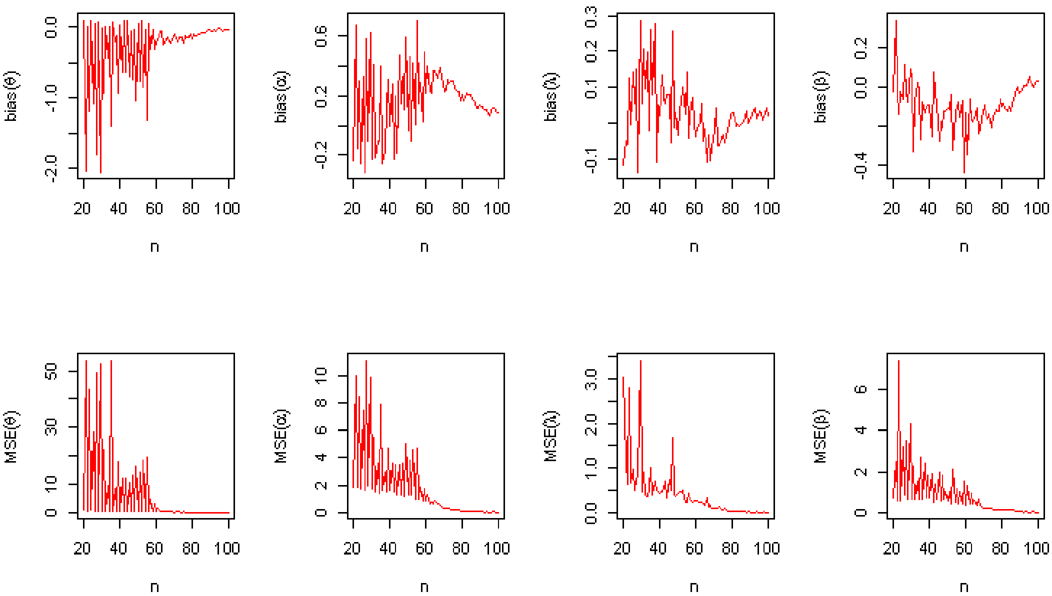


Figure 6 The biases and MSEs of the model parameter versus n based on WLSE

5. Application

5.1. Modelling failure time of infected Guinea pig's data

In Section 5, we consider modelling of one failure time data set to illustrate the suitability of the $\text{GMOT}-\text{E}(\theta,\alpha,\lambda,\beta)$ distribution in comparison to some existing distributions by estimating the parameters by numerical maximization of log-likelihood function. We have considered one failure time data set of 72 Guinea pigs infected with virulent tubercle bacilli which observed and reported by

Bjerkedal (1960). First we have compared the GMOT-E distribution with some of its sub models such as exponential (Exp), moment exponential (ME), transmuted exponential (T-E), Marshall-Olkin exponential (MO-E) (Marshall and Olkin 1997), generalized Marshall-Olkin exponential (GMO-E) (Jayakumar and Mathew 2008) and Marshall-Olkin transmuted exponential (MOT-E) models and also compared the proposed GMOT-E distribution with other competing models such as Kumaraswamy exponential (Kw-E) (Cordeiro and de Castro 2011), beta exponential (BE) (Eugene et al. 2002), Marshall-Olkin Kumaraswamy exponential (MOKw-E) (Handique et al. 2017) and Kumaraswamy Marshall-Olkin exponential (KwMO-E) (Alizadeh et al. 2015), beta Poisson exponential (BP-E) (Handique et al. 2020) and Kumaraswamy Poisson exponential (KwP-E) (Chakraborty et al. 2020) distributions.

The best model is chosen as the one having lowest AIC (Akaike Information Criterion), BIC (Bayesian Information Criterion), CAIC (Consistent Akaike Information Criterion), and HQIC (Hannan-Quinn Information Criterion). Further, we apply formal goodness-of-fit tests to verify which distribution fits better to this data. Particularly, we consider the Anderson-Darling (A), Cramér-von Mises (W) and Kolmogorov-Smirnov (K-S) statistics to compare the fitted models. We have also provided the asymptotic standard errors of the MLEs of the parameters for each competing model. For visual comparisons, the fitted density and fitted cdf are plotted with the corresponding observed histograms and ogives in Figure 8. These plots indicate that the proposed distribution provide a good fit to this data set. The descriptive statistics for the data are shown in Table 5. It is noted that the data set is positively skewed as expected from the nature of lifetime data and the data have a higher kurtosis.

Table 5 Descriptive statistics for the analysed data

n	Min.	Mean	Median	s.d.	Skewness	Kurtosis	1st Qu.	3rd Qu.	Max.
72	0.100	1.851	1.56	1.200	1.788	4.157	1.08	2.303	7.000

We extract the shape of the hazard function from the observed data using the total time on test (TTT) plot (see, Aarset 1987). The TTT plot is a known technique to extract information about hazard function shapes. A straight diagonal line indicates constant hazard for the data set, whereas a convex (concave) shape implies decreasing (increasing) hazard shape. The TTT plot for the given data is portrayed in Figure. 7 and it indicates that the data set have increasing hazard rate. We also provide the box plot of the data to summarise the minimum, first quartile, median, third quartile, and maximum where a box is shown from the first quartile to the third quartile with a vertical line going through the box at the median.

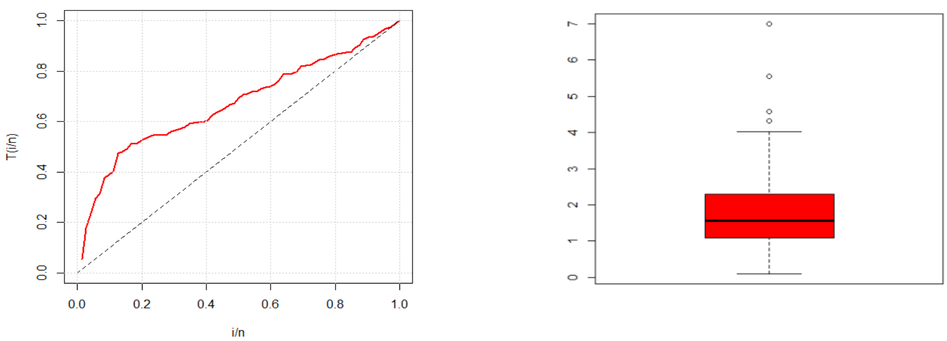


Figure 7 The TTT and box plot for the analyzed data

Tables 6 reports the MLEs with standard errors of the parameters for all the fitted models, and Table 7 lists the AIC, BIC, CAIC, HQIC, A, W and K-S statistic with its p-value for all models. From these findings, the GMOT-E distribution has lowest value of AIC, BIC, CAIC, HQIC, A, W and highest p-value of K-S statistics, and hence it can be utilized to model the given data accurately than other nested and non-nested distributions. These findings are further validated from the plots of fitted density with histogram of the observed data and fitted cdf with ogive of observed data in Figure 8. These plots clearly indicate that the proposed distribution provides close fit to the data set considered here. Further, we utilize the MLE, LSE and WLSE methods to estimate the GMOT-E parameters from real data as reported in Table 8, that also list the values of KS and corresponding p-values. One can note, from Table 8, that the two methods can be used to estimate the GMOT-E parameters. The plots of estimated cdfs and probability-probability plots are displayed in Figures 9 and 10, respectively, for the three estimation methods.

Table 6 MLEs and standard errors (in parentheses) for the analysed data

Models	$\hat{\theta}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{\lambda}$	$\hat{\beta}$
Exp (β)	-	-	-	-	-	0.540 (0.063)
ME (β)	-	-	-	-	-	0.925 (0.077)
T-E (λ,β)	-	-	-	-	–0.812 (0.038)	1.041 (0.105)
MO-E (α,β)	-	8.778 (3.555)	-	-	-	1.379 (0.193)
GMO-E (θ,α,β)	0.179 (0.070)	47.635 (44.901)	-	-	-	4.465 (1.327)
MOT-E (α,λ,β)	-	3.245 (1.863)	-	-	-0.696 (0.137)	1.354 (0.125)
Kw-E (a,b,β)	-	-	3.304 (1.106)	1.100 (0.764)	-	1.037 (0.614)
B-E (a,b,β)	-	-	0.807 (0.696)	3.461 (1.003)	-	1.331 (0.855)
MOKw-E (α,a,b,β)	-	0.008 (0.002)	2.716 (1.316)	1.986 (0.784)	-	0.099 (0.048)
KwMO-E (α,a,b,β)	-	0.373 (0.136)	3.478 (0.861)	3.306 (0.779)	-	0.299 (1.112)
BP-E (a,b,λ,β)	-	-	3.595 (1.031)	0.724 (1.590)	0.014 (0.010)	1.482 (0.516)
KwP-E (a,b,λ,β)	-	-	3.265 (0.991)	2.658 (1.984)	4.001 (5.670)	0.177 (0.226)
GMOT-E ($\theta,\alpha,\lambda,\beta$)	0.215 (0.031)	21.003 (57.072)	-	-	–0.778 (4.699)	4.311 (0.094)

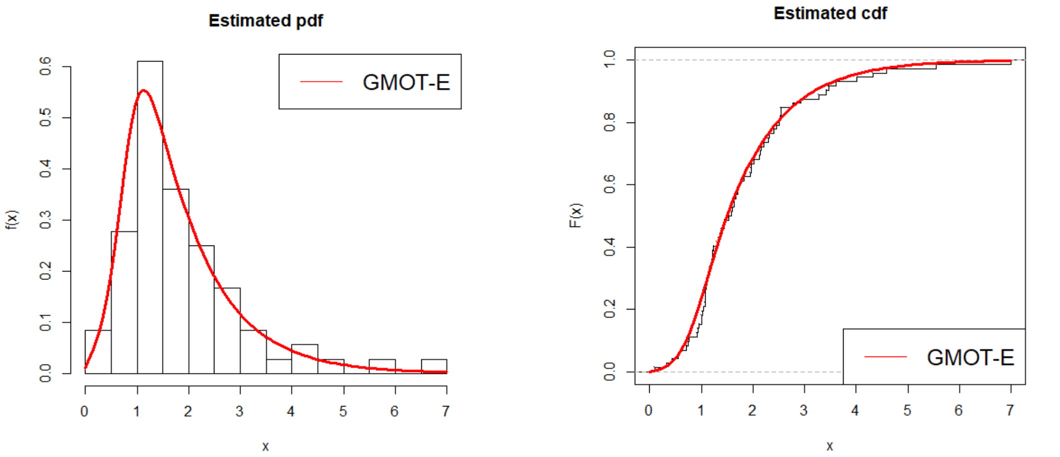


Figure 8 Plots of the observed histogram and estimated GMOT-E pdf (left) and the observed ogive and estimated cdf of the GMOT-E model (right) for the analysed data

Table 7 Analytical measures for the analysed data set

Models	AIC	BIC	CAIC	HQIC	A	W	K-S (p-value)
Exp (β)	234.63	236.91	234.68	235.54	6.53	1.25	0.27 (0.06)
ME (β)	210.40	212.68	210.45	211.3	1.52	0.25	0.14 (0.13)
T-E (λ,β)	209.94	214.50	210.11	211.74	0.98	0.19	0.10 (0.17)
MO-E (α,β)	210.36	214.92	210.53	212.16	1.18	0.17	0.10 (0.43)
GMO-E (θ,α,β)	210.54	217.38	210.89	213.24	1.02	0.16	0.09 (0.51)
MOT-E (α,λ,β)	208.26	215.10	208.61	210.96	0.86	0.15	0.10 (0.47)
Kw-E (a,b,β)	209.42	216.24	209.77	212.12	0.74	0.11	0.08 (0.50)
B-E (a,b,β)	207.38	214.22	207.73	210.08	0.98	0.15	0.11 (0.34)
MOKw-E (α,a,b,β)	209.44	218.56	210.04	213.04	0.79	0.12	0.10 (0.44)
KwMO-E (α,a,b,β)	207.82	216.94	208.42	211.42	0.61	0.11	0.08 (0.73)
BP-E (a,b,λ,β)	205.42	214.50	206.02	209.02	0.55	0.08	0.09 (0.81)
KwP-E (a,b,λ,β)	206.63	215.74	207.23	210.26	0.48	0.07	0.09 (0.79)
GMOT-E $(\theta,\alpha,\lambda,\beta)$	202.68	211.78	203.38	206.31	0.28	0.04	0.06 (0.88)

Table 8 The estimates, K-S and p-values for the analysed data

Model and Methods	θ	α	λ	β	K-S	p-value
GMOT-E (LSE)	0.2271	25.1536	−0.3962	4.2521	0.0561	0.9774
GMOT-E (WLSE)	0.2207	13.2098	−0.8704	4.0792	0.0559	0.9779

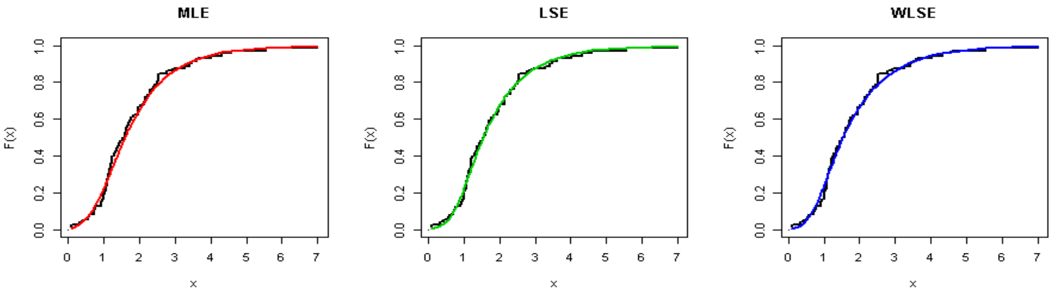


Figure 9 The empirical cdfs for the analyzed data

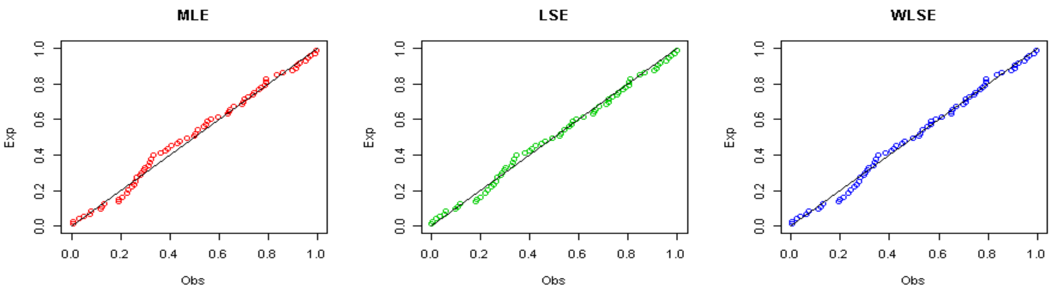


Figure 10 The probability-probability plots for the analyzed data

5.2. Likelihood ratio (LR) test

In this Section, we have carried out the LR test for nested models. The $\text{GMOT-E}(\theta,\alpha,\lambda,\beta)$ distribution reduces to $\text{MOT-E}(\theta,\lambda,\beta)$ if $\theta=1$, to $\text{GMO-E}(\theta,\alpha,\beta)$ if $\lambda=1$, to $\text{MO-E}(\alpha,\beta)$ if $\theta=\lambda=1$, to $\text{T-E}(\lambda,\beta)$ if $\alpha=\theta=1$, to $\text{E}(\beta)$ if $\theta=\alpha=\lambda=1$, so here we have employed the LR test where null hypothesis can be reported as:

(i) $H_0:\theta=1$, that is the sample is from $\text{MOT-E}(\theta,\lambda,\beta)$ vs
$H_1:\theta\neq 1$, that is the sample is from $\text{GMOT-E}(\theta,\alpha,\lambda,\beta)$.

(ii) $H_0:\lambda=1$, that is the sample is from $\text{GMO-E}(\theta,\alpha,\beta)$ vs
$H_1:\lambda\neq 1$, that is the sample is from $\text{GMOT-E}(\theta,\alpha,\lambda,\beta)$.

(iii) $H_0:\theta=\lambda=1$, that is the sample is from $\text{MO-E}(\alpha,\beta)$ vs
$H_1:\theta\neq\lambda\neq 1$, that is the sample is from $\text{GMOT-E}(\theta,\alpha,\lambda,\beta)$.

(iv) $H_0:\alpha=\theta=1$, that is the sample is from $\text{T-E}(\lambda,\beta)$ vs
$H_1:\alpha\neq\theta\neq 1$, that is the sample is from $\text{GMOT-E}(\theta,\alpha,\lambda,\beta)$.

(v) $H_0:\theta=\alpha=\lambda=1$, that is the sample is from $\text{E}(\beta)$ vs
$H_1:\theta\neq\alpha\neq\lambda\neq 1$, that is the sample is from $\text{GMOT-E}(\theta,\alpha,\lambda,\beta)$.

Writing $\rho=(\theta,\alpha,\lambda,\beta)$ the LR test statistic is given by $\text{LR}=-2(\ell(\hat{\rho}^*;x)-\ell(\hat{\rho};x))$, where $\hat{\rho}^*$ is the restricted ML estimates under the null hypothesis H_0, and $\hat{\rho}$ is the unrestricted ML estimates under the alternative (alt) hypothesis H_1. Under the null hypothesis H_0 the LR criterion follows Chisquare distribution with degrees of freedom (df) $(\text{df}_{alt} - \text{df}_{null})$. The null hypothesis is rejected for p-value less than 0.05. Table 7 lists the analytical measures for real data.

Table 7 Analytical measures for the analysed data set

Test / Model	Exp	T-E	MO-E	GMO-E	MOT-E
LR	18.97	5.63	5.84	4.93	3.79
p-value	0.0002	0.05	0.05	0.02	0.05

6. Conclusions

A new extension of the transmuted-G class, called the generalized Marshall-Olkin-transmuted-G family, which encompasses some important sub-families, is proposed. We study a new four-parameter sub-model called the generalized Marshall-Olkin-transmuted-exponential distribution to serve as an alternative to several existing distributions. It yields better fits as compared to existing models, and it can serve in many cases as good alternative to them. Some mathematical properties of the proposed family are obtained. The model parameters are estimated by three estimation methods. Comparative evaluation using failure time data in terms of different model selection, goodness of fit criteria is conducted to the proposed distribution and other competing alternatives, proving that the proposed model provides better fit those other extensions of the three and four-parameter distributions.

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Appendix A

Proof of Proposition 2

As $x \to 0$ then $G(x) \to 0$, the asymptotes of pdf, sf and hrf of $\text{GMOT-G}(\theta,\alpha,\lambda)$ are given by

$$f^{\text{GMOTG}}(x;\theta,\alpha,\lambda) = \frac{\theta\alpha^{\theta} g(x)\,[1+\lambda-2\lambda G(x)]\,[1-G(x)\,\{1+\lambda-\lambda G(x)\}]^{\theta-1}}{[1-\bar{\alpha}[1-G(x)\,\{1+\lambda-\lambda G(x)\}]]^{\theta+1}}$$

$$\sim \frac{\theta\alpha^{\theta} g(x)\,(1+\lambda)}{[1-\bar{\alpha}]^{\theta+1}}, \text{ as } G(x) \to 0$$

$$\sim \theta\,(1+\lambda)g(x)/\alpha,\ 1-\bar{\alpha} = \alpha.$$

$$\bar{F}^{\text{GMOTG}}(x;\theta,\alpha,\lambda) = \left[\frac{\alpha\,[1-G(x)\,\{1+\lambda-\lambda G(x)\}]}{1-\bar{\alpha}[1-G(x)\,\{1+\lambda-\lambda G(x)\}]}\right]^{\theta}$$

$$\sim \left[\frac{\alpha}{1-\bar{\alpha}}\right]^{\theta} \text{ as } G(x) \to 0$$

$$\sim 1.$$

$$h^{\text{GMOTG}}(x;\theta,\alpha,\lambda) \sim \theta\,(1+\lambda)g(x)/\alpha.$$

Proof of Proposition 3

As $x \to \infty \Rightarrow G(\infty) = 1$ then $G(x) \to 1$, the asymptotes of pdf, sf and hrf of $\text{GMOT-G}(\theta,\alpha,\lambda)$ are given by

$$f^{\text{GMOTG}}(x;\theta,\alpha,\lambda)=\frac{\theta\alpha^{\theta}\,g(x)\,[1+\lambda-2\lambda\,G(x)]\,[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]^{\theta-1}}{[1-\bar{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]]^{\theta+1}}$$

$$\sim\theta\alpha^{\theta}\,(1-\lambda)g(x)\,\bar{G}(x)^{\theta-1}(1-\bar{\alpha}\bar{G}(x))^{-(\theta+1)}.$$

$\because G(x)\to 1 \Rightarrow \{1+\lambda-\lambda\,G(x)\}\to 1.$

$$\bar{F}^{\text{GMOTG}}(x;\theta,\alpha,\lambda)=\left[\frac{\alpha\,[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}{1-\bar{\alpha}[1-G(x)\,\{1+\lambda-\lambda\,G(x)\}]}\right]^{\theta}$$

$$\sim\left[\frac{\alpha\;\bar{G}(x)}{1-\bar{\alpha}\;\bar{G}(x)}\right]^{\theta}$$

$$h^{\text{GMOTG}}(x;\theta,\alpha,\lambda)\sim\theta(1-\lambda)g(x)\,\bar{G}(x)^{-1}\,(1-\bar{\alpha}\bar{G}(x))^{-1}.$$

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