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Improved Estimation of Population Mean in Simple Random Sampling Using Attribute

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Abstract

This article discusses the problem of estimation of population mean using the information on an auxiliary attribute under simple random sampling. An improved class of estimator is suggested and the expression of the mean square error is determined up to the first order of approximation using Taylor series method. The usual mean estimator, classical ratio estimator envisaged by Naik and Gupta (1996) and Abd-Elfattah et al. (2010) estimators are identified as the members of the proposed class of estimator. The theoretical conditions are obtained by comparing the mean square error of the proposed estimators with the mean square error of the existing estimators under which the proposed class of estimator dominates the existing estimators suggested till date. A numerical study using real populations and a simulation study using artificially generated populations are conducted to enhance the theoretical findings.

Keywords: Auxiliary attribute, bias, efficiency, mean square error.

1. Introduction

In sample survey methodology, it is well known that the information on an auxiliary variable help to meliorate the efficiency of the estimators. Literature contains a wide range of improved and modified ratio, regression, product and exponential type estimators based on simple random sampling (SRS). Some recent contribution in this direction such as Bhushan and Kumar (2022a, 2022b, 2022c), Bhushan et al. (2022a, 2023a, 2023b) and Bhushan et al. (2022b, 2022c) can be viewed. In real life applications, situations may arise when instead of existence of an auxiliary variable x there exist some auxiliary attribute. which are highly correlated with the study variable y, like, sex (ϕ) and height of persons (y), amount of yield of paddy crop (y) and a certain variety of paddy (ϕ), amount of milk produced (y) and a certain breed of buffalo (ϕ) etc. Naik and Gupta (1996) introduced the classical ratio, product and regression estimators of population mean using the knowledge of attribute. Abd-Elfattah et al. (2010) investigated ratio type estimators and linear combination of different estimators by utilizing the information of an auxiliary attribute. Singh et al. (2008) suggested a class of estimators based on attribute. Grover and Kaur (2011) examined regression ϕ cum exponential type estimator utilizing auxiliary attribute. Singh and Solanki (2012) proffered an attribute based improved estimation procedure of population mean under SRS. Koyuncu (2012) considered an exponential type estimator of population mean using the information on an attribute under SRS. Haq and Shabbir (2014) suggested an improved estimator of population mean consist of attribute. Zaman and Kadilar (2019) suggested an improved exponential family of estimators of population mean using attribute. Bhushan and Gupta (2020) suggested an improved log type family of estimators of population mean using the knowledge of auxiliary attribute. In this article, we suggest an improved class of estimator of population mean using the information on an auxiliary attribute.

Let a sample of size n be quantified from a finite population $\omega=\left(\omega_1,\omega_2,\ldots,\omega_N\right)$ using simple random sampling without replacement (SRSWOR) scheme. Let and ϕ_i be the total number of units on the study variable y and auxiliary attribute ϕ for unit i of the population ω. It is to be noted that ϕ_i = 1 if the unit i possess the attribute ϕ and ϕ_i = 0, otherwise. Let $A=\sum_{i=1}^{N}\phi_i$ and $a=\sum_{i=1}^{n}\phi_i$ be the total number of units in the population ω and sample respectively possessing attribute ϕ whereas $P=\left(A/N\right)$ and $p=\left(a/n\right)$ respectively denote the population proportion and sample proportion with attribute ϕ.

Considering the advantage of point bi-serial correlation between study variable and auxiliary attribute, Naik and Gupta (1996) suggested the classical ratio and regression estimators for the population mean $\overline{Y}$ as

$$t_r=\overline{y}\left(\frac{P}{p}\right),\quad t_{lr}=\overline{y}+\beta_\phi(P-p),$$

where β_ϕ is the regression coefficient of y on ϕ. Following Srivastava (1967), one may suggest respectively the ratio type estimators using the knowledge of an auxiliary attribute as

$$t_{sr}=\overline{y}\left(\frac{P^*}{p^*}\right)^{\beta_1}, \tag{1}$$

where β_1 is a suitably chosen scalar to be determined later. Also, $p^*=\eta p+\lambda$ and $P^*=\eta P+\lambda$, such that η and λ are any real values or function of some known parameters of auxiliary attribute ϕ namely population standard deviation S_ϕ, population coefficient of variation C_ϕ, population coefficient of skewness $\beta_1\left(\phi\right)$, population coefficient of kurtosis $\beta_2\left(\phi\right)$ and population point biserial correlation coefficient ρ between study variable y and attribute ϕ.

The logarithmic function has some useful properties and play prominent role in different fields of science and non-science disciplines. We develop the logarithmic relationships between study variable Y and auxiliary attribute ϕ for efficiently estimating the population mean of study variable. The logarithmic estimator would work in situation when the study variable Y is logarithmically related to the auxiliary attribute ϕ. Therefore, on the lines of Bhushan and Gupta (2019), we develop following logarithmic type estimator using auxiliary attribute as

$$t_g = \bar{y}\left\{1+\log\left(\frac{p^*}{P^*}\right)\right\}^{\beta_2}, \tag{2}$$

where β_2 is a suitably chosen scalar. Singh et al. (2008) introduced the following class of estimators using auxiliary attribute as

$$t_{s_1} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P}{p}\right),\ t_{s_2} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P+\beta_2(\phi)}{p+\beta_2(\phi)}\right),$$

$$t_{s_3} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P+C_\phi}{p+C_\phi}\right),\ t_{s_4} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P\beta_2(\phi)+C_\phi}{p\beta_2(\phi)+C_\phi}\right),$$

$$t_{s_5} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{PC_\phi+\beta_2(\phi)}{pC_\phi+\beta_2(\phi)}\right),\ t_{s_6} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P+\rho}{p+\rho}\right),$$

$$t_{s_7} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{PC_\phi+\rho}{pC_\phi+\rho}\right),\ t_{s_8} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P\rho+C_\phi}{p\rho+C_\phi}\right),$$

$$t_{s_9} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P\beta_2(\phi)+\rho}{p\beta_2(\phi)+\rho}\right),\ t_{s_{10}} = \left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P\rho+\beta_2(\phi)}{p\rho+\beta_2(\phi)}\right).$$

Abd-Elfattah et al. (2010) envisaged the following class of estimators utilizing the information on an auxiliary attribute ϕ as

$$t_{a_2} = \bar{y}\left(\frac{P+\beta_2(\phi)}{p+\beta_2(\phi)}\right),\ t_{a_3} = \bar{y}\left(\frac{P+C_\phi}{p+C_\phi}\right),$$

$$t_{a_4} = \bar{y}\left(\frac{P\beta_2(\phi)+C_\phi}{p\beta_2(\phi)+C_\phi}\right),\ t_{a_5} = \bar{y}\left(\frac{PC_\phi+\beta_2(\phi)}{pC_\phi+\beta_2(\phi)}\right),\ t_{a_6} = \bar{y}\left(\frac{P+\rho}{p+\rho}\right).$$

We would like to note that the mean square error (MSE) of the estimators t_{sr}, t_g and t_{a_i}, i = 2, 3, …, 6 attain the minimum MSE of the classical regression estimator t_{lr}.

Following the procedure of Kadilar and Cingi (2006), and Abd-Elfattah et al. (2010) also suggested the following classes of estimators by combining different estimators as

$$t_{a_7} = m_1\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P}{p}\right)+m_2\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P+\beta_2(\phi)}{p+\beta_2(\phi)}\right),$$

$$t_{a_8} = m_1\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P}{p}\right)+m_2\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P+C_\phi}{p+C_\phi}\right),$$

$$t_{a_9} = m_1\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P}{p}\right)+m_2\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P\beta_2(\phi)+C_\phi}{p\beta_2(\phi)+C_\phi}\right),$$

$$t_{a_{10}} = m_1\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{P}{p}\right)+m_2\left\{\bar{y}+\beta_\phi\left(P-p\right)\right\}\left(\frac{PC_\phi+\beta_2(\phi)}{pC_\phi+\beta_2(\phi)}\right),$$

where m_1 and m_2 are duly opted scalars to be determined later.

The exponential functions model a relationship in which a constant change in the independent variable gives the same proportional change in the dependent variable. Therefore, Grover and Kaur (2011) suggested an exponential type estimator utilizing the knowledge of an auxiliary attribute as

$$t_{gk} = \alpha\bar{y}+\beta\left(P-p\right)\exp\left(\frac{P-p}{P+p}\right),$$

where α and β are duly opted scalars. Singh and Solanki (2012) suggested the following class of estimators for the estimation of population mean utilizing attribute as

$$t_{ss} = \bar{y}\left[d_1+d_2\left(P-p\right)\right]\left(\frac{\eta P+\delta\lambda}{\eta p+\delta\lambda}\right)^g,$$

where δ, d_1 and d_2 are suitably chosen scalars, g assumes values -1 and $+1$ to generate ratio and product type estimators. Koyuncu (2012) suggested the following estimator for the estimation of population mean as

$$t_k = \left[\lambda_1\bar{y}+\lambda_2\left(\frac{p}{P}\right)^\delta\right]\exp\left[\frac{\eta\left(P-p\right)}{\eta\left(P-p\right)+2\lambda}\right],$$

where λ_1 and λ_2 are suitably chosen scalars and δ is suitably chosen real number.

Haq and Shabbir (2014) suggested following class of estimator as

$$t_{hs} = \left[h_1\bar{y}+h_2\left(\frac{P}{p}\right)^\delta\right]\exp\left(\frac{\eta\left(P-p\right)}{\eta\left(P+p\right)+2\lambda}\right)\left(\frac{\eta P+\lambda}{2\left(\eta p+\lambda\right)}+\frac{\eta p+\lambda}{2\left(\eta P+\lambda\right)}\right)^2,$$

where k_1, k_2, h_1 and h_2 are suitably opted scalars. For different values of η and λ, few members of Koyuncu (2012) estimator t_k and Haq and Shabbir (2014) estimator t_{hs} are given in Table 1 for ready reference.

Table 1 Members of Koyuncu (2012) estimator t_k and Haq and Shabbir (2014) estimator t_{hs}

Members of t_k	Members of t_{hs}	η	λ
$t_{k(1)}$	$t_{hs(1)}$	C_ϕ	$\beta_2(\phi)$
$t_{k(2)}$	$t_{hs(2)}$	$\beta_2(\phi)$	C_ϕ
$t_{k(3)}$	$t_{hs(3)}$	1	C_ϕ
$t_{k(4)}$	$t_{hs(4)}$	1	$\beta_2(\phi)$

Zaman and Kadilar (2019) suggested a family of ratio exponential estimator of population mean using attribute as

$$t_{zc} = \bar{y}\, exp\left[\frac{(\eta P+\lambda)-(\eta p+\lambda)}{(\eta P+\lambda)+(\eta p+\lambda)}\right].$$

Using the advantage of logarithmic function, Bhushan and Gupta (2020) suggested the following family of estimator for the estimation of population mean $\bar{Y}$ as

$$t_{bg} = \left[w_1\bar{y} + w_2\left(\frac{p}{P}\right)\right]\left[1+\alpha\log\left(\frac{p^*}{P^*}\right)\right],$$

where w_1, w_2 and α are suitably chosen scalars. The MSE of these estimators are readily available in Appendix A.

2. Proposed estimators

Adapting the procedure of Kadilar and Cingi (2006), we suggest the following class of estimator by combining the Srivastava estimator t_{sr} and the log type estimator t_g given respectively in (1) and (2) as

$$t_{bk} = \zeta_1\bar{y}\left(\frac{P^*}{p^*}\right)^{\beta_1} + \zeta_2\bar{y}\left\{1+\log\left(\frac{p^*}{P^*}\right)\right\}^{\beta_2},$$

where ζ_i and β_i, $i=1,2$ are the suitably chosen optimizing scalars. For different values of scalars $\zeta_1, \zeta_2, \beta_1, \beta_2, \eta$ and λ, the class of estimator t_{bk} reduces to the

i. Usual mean estimator $\bar{y}$, for $(\zeta_1, \zeta_2, \beta_1, \beta_2) = (1, 0, 0, 0)$,

ii. Classical ratio estimator t_r, for $(\zeta_1, \zeta_2, \beta_1, \beta_2, \eta, \lambda) = (1, 0, 1, 0, 1, 0)$,

iii. Abd-Elfattah et al. (2010) estimators t_{a_i}; $i = 2, 3, 4, 5, 6$, for $(\zeta_1, \zeta_2, \beta_1, \beta_2, \eta, \lambda) = (1, 0, 1, 0, \eta, \lambda)$.

Several other estimators can be generated for different values of scalars. Further, to obtain the bias and MSE of the proposed class of estimators t_{bk}, let $\bar{y} = \bar{Y}(1+e_0)$ and $p = P(1+e_1)$ such that $E(e_0) = E(e_1) = 0$ and $E(e_0^2) = \gamma C_y^2$, $E(e_1^2) = \gamma C_\phi^2$, $E(e_0 e_1) = \gamma\rho C_y C_\phi$. Now, express the estimator t_{bk} in terms of e's, we get

$$t_{bk} - \bar{Y} = \bar{Y}\left[\begin{array}{l}\zeta_1\left\{1+e_0-\beta_1 v e_1 + \dfrac{\beta_1(\beta_1+1)}{2}v^2 e_1^2 - \beta_1 v e_0 e_1\right\} + \\ \zeta_2\left\{1+e_0+\beta_2 v e_1 - \beta_2 v^2 e_1^2 + \dfrac{\beta_2^2}{2}v^2 e_1^2 + \beta_2 v e_0 e_1\right\} - 1\end{array}\right]. \quad (3)$$

Squaring and taking expectation both the sides of (3), we get the MSE of the proposed class of estimator to the first order of approximation as

$$MSE\left(t_{bk}\right)=$$
$$\bar{Y}^2\left[\begin{array}{l}1+\zeta_1^2\left\{1+\gamma(C_y^2+(2\beta_1^2\nu^2+\beta_1\nu)C_\phi^2-4\beta_1\nu\rho C_yC_\phi)\right\}\\+\zeta_2^2\left\{1+\gamma(C_y^2+(2\beta_2^2-2\beta_2)\nu^2C_\phi^2+4\beta_2\nu\rho C_yC_\phi)\right\}\\+2\zeta_1\zeta_2\left\{1+\gamma\left\{C_y^2+\left(\frac{\beta_2^2}{2}-\beta_2-\beta_1\beta_2+\frac{\beta_1(\beta_1+1)}{2}\right)\nu^2C_\phi^2-2(\beta_1-\beta_2)\nu\rho C_yC_\phi\right\}\right\}\\-2\zeta_1\left\{1+\gamma\left(\frac{\beta_1(\beta_1+1)}{2}\nu^2C_\phi^2-\beta_1\nu\rho C_yC_\phi\right)\right\}-2\zeta_2\left\{1+\gamma\left(\left(\frac{\beta_2^2}{2}-\beta_2\right)\nu^2C_\phi^2+\beta_2\nu\rho C_yC_\phi\right)\right\}\end{array}\right].$$

The above MSE expression can further be written as

$$MSE(t_{bk})=\bar{Y}^2\left(1+\zeta_1^2U_1+\zeta_2^2U_2+2\zeta_1\zeta_2U_3-2\zeta_1U_4-2\zeta_2U_5\right), \tag{4}$$

where

$U_1=1+\gamma\left\{C_y^2+\left(2\beta_1^2\nu^2+\beta_1\nu\right)C_\phi^2-4\beta_1\nu\rho C_yC_\phi\right\}$, $U_2=1+\gamma\left\{C_y^2+\left(2\beta_2^2-2\beta_2\right)\nu^2C_\phi^2+4\beta_2\nu\rho C_yC_\phi\right\}$,

$U_3=1+\gamma\left\{C_y^2+\left(\left(\beta_2^2/2\right)-\beta_2-\beta_1\beta_2+\left(\beta_1\left(\beta_1+1\right)/2\right)\right)\nu^2C_\phi^2-2\left(\beta_1-\beta_2\right)\nu\rho C_yC_\phi\right\}$,

$U_4=1+\gamma\left\{\left(\beta_1\left(\beta_1+1\right)/2\right)\nu^2C_\phi^2-\beta_1\nu\rho C_yC_\phi\right\}$ and

$U_5=1+\gamma\left\{\left(\left(\beta_2^2/2\right)-\beta_2\right)\nu^2C_\phi^2+\beta_2\nu\rho C_yC_\phi\right\}$.

Minimizing (4) with respect to the scalars ζ_1 and ζ_2, we get the optimum values of scalars ζ_1 and ζ_2 as

$$\zeta_{1(opt)}=\frac{\left(U_2U_4-U_3U_5\right)}{\left(U_1U_2-U_3^2\right)},\ \zeta_{2(opt)}=\frac{\left(U_1U_5-U_3U_4\right)}{\left(U_1U_2-U_3^2\right)}.$$

The minimum MSE at optimum values of ζ_1 and ζ_2 is given as

$$minMSE\left(t_{bk}\right)=\bar{Y}^2\left[1-\frac{\left(U_1U_5^2+U_2U_4^2-2U_3U_4U_5\right)}{\left(U_1U_2-U_3^2\right)}\right]. \tag{5}$$

The expression of minimum MSE obtained in (5) is important in order to derive the efficiency conditions of next section.

3. Efficiency Conditions

On comparing the minimum MSE of the proposed estimators t_{bk} with the minimum MSE of the existing estimators, we get the following conditions.

i. From (5) and (A.1), $MSE\left(t_r\right)>MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2+U_2U_5^2-2U_3U_4U_5\right)}{\left(U_1U_2-U_3^2\right)}>1-\gamma\left[C_y^2+C_\phi^2-2\rho C_yC_\phi\right]. \tag{6}$$

ii. From (5) and (A.4), $MSE\left(t^*\right) > MSE\left(t_{bk}\right)$ where $t^* = t_{lr}, t_{sr}, t_g$ and $t_{ai}, i = 2,\ldots,5$ when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > 1 - \gamma C_y^2\left(1-\rho^2\right). \tag{7}$$

iii. From (5) and (A.2), $MSE\left(t_{s_i}\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > 1 - \gamma\left[R_i^{\,2}C_\phi^2 + C_y^2\left(1-\rho^2\right)\right]. \tag{8}$$

iv. From (5) and (A.5), $MSE\left(t_{gk}\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > 1 - \frac{MSE\left(t_{lr}\right)}{\bar{Y}^2 + MSE\left(t_{lr}\right)} + \frac{\gamma C_\phi^2\left[MSE\left(t_{lr}\right) + \dfrac{\gamma \bar{Y}^2 C_\phi^2}{16}\right]}{4\bar{Y}^2\left[1 + \dfrac{MSE\left(t_{lr}\right)}{\bar{Y}^2}\right]}. \tag{9}$$

i. From (5) and (A.6), $MSE\left(t_{ai}\right) > MSE\left(t_{bk}\right), i = 6, 7, ..., 10,$ when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > 1 - \gamma\left[C_y^2 + R_i^2C_\phi^2 - 2R_i\rho C_yC_\phi\right]. \tag{10}$$

ii. From (5) and (A.7), $MSE\left(t_k\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > \frac{\left(A_1E_1^2 + B_1D_1^2 - C_1D_1E_1\right)}{\left(4\mathrm{A}_1\mathrm{B}_1 - \mathrm{C}_1^2\right)}. \tag{11}$$

iii. From (5) and (A.8), $MSE\left(t_{ss}\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > \frac{\left(A_2E_2^2 + B_2D_2^2 - C_2D_2E_2\right)}{\left(4\mathrm{A}_2\mathrm{B}_2 - \mathrm{C}_2^2\right)}. \tag{12}$$

iv. From (5) and (A.9), $MSE\left(t_{hs}\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > 1 - \frac{\left(B_3^2C_3 + A_3D_3B_3D_3E_3\right)}{\bar{Y}^2\left(E_3^2 - 4A_3C_3\right)}. \tag{13}$$

v. From (5) and (A.3), $MSE\left(t_{zc}\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > 1 - \gamma\left[\lambda^2C_\phi^2 + C_y^2 - 2\lambda\rho C_yC_\phi\right]. \tag{14}$$

vi. From (5) and (A.10), $MSE\left(t_{bg}\right) > MSE\left(t_{bk}\right)$, when

$$\frac{\left(U_1U_5^2 + U_2U_4^2 - 2U_3U_4U_5\right)}{\left(U_1U_2 - U_3^2\right)} > \frac{\left(A_4G_4^2 + B_4D_4^2 - D_4F_4G_4\right)}{4A_4B_4 - F_4^2}. \tag{15}$$

Under the conditions (6) to (15), the proposed class of estimators t_{bk} repress the usual mean estimator, classical ratio and regression estimators, Srivastava (1967) type estimator, log type estimator defined on the lines of Bhushan and Gupta (2019), Singh et al. (2008) estimators, Abd-Elfattah et al. (2010) estimators, Grover and Kaur (2011) estimator, Singh and Solanki (2012) estimator, Haq and Shabbir (2014) estimator, Zaman and Kadilar (2019) estimator and Bhushan and Gupta (2020) estimator. Further, these conditions are supported with a numerical study and a simulation study using different real and artificially generated populations.

4. Numerical study

To exemplify the properties of the proposed estimators, we consider a numerical study over three real populations. The description of these populations is given below.

Population 1: (Source: Sukhatme and Sukhatme 1970, pp. 256)
y: Number of villages in the circles, ϕ: A circle consisting of more than five villages, $N=89,\ n=23,\ \bar{Y}=3.36,\ P=0.124,\ C_y=0.601,\ C_\phi=2.678,\ \rho=0.766$ and $\beta_2(\phi)=6.162.$

Population 2: (Source: Sukhatme and Sukhatme 1970, pp.256)
y: Area (in acres) under the wheat crop within the circles, ϕ: A circle consisting of more than five villages, $N=89,\ n=23,\ \bar{Y}=1102,\ P=0.124,\ C_y=0.65,\ C_\phi=2.678,\ \rho=0.624$ and $\beta_2(\phi)=6.162.$

Population 3: (Source: Zaman et al. 2014)
y: The number of teachers, ϕ. The number of teachers is more than 60, $N=111,\ n=30,\ \bar{Y}=29.279,\ P=0.117,\ C_y=0.872,\ C_\phi=2.758,\ \rho=0.797$ and $\beta_2(\phi)=3.898.$

We have computed the percent relative efficiency (PRE) of proposed estimator t_{bk} regarding the existing estimators for the above three populations using the expressions given below.

$$E_1=\frac{V(\bar{y})}{MSE(t_{bk})}\times 100,\ E_2=\frac{MSE(t_r)}{MSE(t_{bk})}\times 100,\ E_3=\frac{MSE(t^*)}{MSE(t_{bk})}\times 100,\ E_4=\frac{MSE(t_{s_1})}{MSE(t_{bk})}\times 100,$$

$$E_5=\frac{MSE(t_{s_2})}{MSE(t_{bk})}\times 100,\ E_6=\frac{MSE(t_{s_3})}{MSE(t_{bk})}\times 100,\ E_7=\frac{MSE(t_{s_4})}{MSE(t_{bk})}\times 100,\ E_8=\frac{MSE(t_{s_5})}{MSE(t_{bk})}\times 100,$$

$$E_9=\frac{MSE(t_{s_6})}{MSE(t_{bk})}\times 100,\ E_{10}=\frac{MSE(t_{s_7})}{MSE(t_{bk})}\times 100,\ E_{11}=\frac{MSE(t_{s_8})}{MSE(t_{bk})}\times 100,\ E_{12}=\frac{MSE(t_{s_9})}{MSE(t_{bk})}\times 100,$$

$$E_{13}=\frac{MSE(t_{s_{10}})}{MSE(t_{bk})}\times 100,\ E_{14}=\frac{MSE(t_{a_2})}{MSE(t_{bk})}\times 100,\ E_{15}=\frac{MSE(t_{a_3})}{MSE(t_{bk})}\times 100,\ E_{16}=\frac{MSE(t_{a_4})}{MSE(t_{bk})}\times 100,$$

$$E_{17}=\frac{MSE(t_{a_5})}{MSE(t_{bk})}\times 100,\ E_{18}=\frac{MSE(t_{a_6})}{MSE(t_{bk})}\times 100,\ E_{19}=\frac{MSE(t_{gk})}{MSE(t_{bk})}\times 100,\ E_{20}=\frac{MSE(t_{ss})}{MSE(t_{bk})}\times 100,$$

$$E_{21}=\frac{MSE(t_{k(1)})}{MSE(t_{bk})}\times 100,\ E_{22}=\frac{MSE(t_{k(2)})}{MSE(t_{bk})}\times 100,\ E_{23}=\frac{MSE(t_{k(3)})}{MSE(t_{bk})}\times 100,$$

$$E_{24}=\frac{MSE\left(t_{k(4)}\right)}{MSE\left(t_{bk}\right)}\times 100,\ E_{25}=\frac{MSE\left(t_{hs(1)}\right)}{MSE\left(t_{bk}\right)}\times 100,\ E_{26}=\frac{MSE\left(t_{hs(2)}\right)}{MSE\left(t_{bk}\right)}\times 100,$$

$$E_{27}=\frac{MSE\left(t_{hs(3)}\right)}{MSE\left(t_{bk}\right)}\times 100,\ E_{28}=\frac{MSE\left(t_{hs(4)}\right)}{MSE\left(t_{bk}\right)}\times 100,\ E_{29}=\frac{MSE\left(t_{zc}\right)}{MSE\left(t_{bk}\right)}\times 100,$$

$$E_{30}=\frac{MSE\left(t_{bg}\right)}{MSE\left(t_{bk}\right)}\times 100.$$

The results of the numerical study for the above populations are summarized in Table 2 in terms of PRE.

Table 2 PRE of proposed estimators with respect to existing estimators using real populations

PRE	Population 1	Population 2	Population 3	PRE	Population 1	Population 2	Population 3
E_1	373.2731	204.9612	403.5888	E_{16}	172.8386	142.4597	196.1586
E_2	5236.5200	2630.1930	2406.2000	E_{17}	262.3543	160.1693	271.6260
E_3	154.2529	125.1542	147.2256	E_{18}	162.1201	125.8676	209.2832
E_4	7565.6290	3604.2480	4184.559	E_{19}	117.9907	104.4803	127.8807
E_5	157.1369	126.5080	150.6540	E_{20}	104.2286	101.0027	108.3730
E_6	168.7675	131.9678	153.9119	E_{21}	133.5548	124.6365	153.7964
E_7	519.4621	296.5929	228.5162	E_{22}	111.7166	104.6689	143.5028
E_8	173.6318	134.2512	170.8243	E_{23}	134.4995	125.4805	159.6628
E_9	298.1202	220.7649	213.3823	E_{24}	137.9081	128.5135	161.5949
E_{10}	832.0532	544.8643	482.5478	E_{25}	133.4931	124.5758	153.6853
E_{11}	162.9486	127.8978	151.5439	E_{26}	109.9066	103.1453	143.0841
E_{12}	2002.4690	1179.3440	682.0388	E_{27}	134.4535	125.4353	159.6322
E_{13}	155.9608	125.6893	149.4293	E_{28}	137.8992	128.5046	161.5793
E_{14}	325.8917	185.5262	347.7239	E_{29}	952.0523	547.8039	395.5610
E_{15}	275.0225	165.1371	327.4708	E_{30}	140.9190	114.0550	132.8709

From the findings of Table 2, it can be observed that the proposed class of estimator is found to be superior in terms of the PRE than the contemporary estimators discussed in this study for each population.

5. Simulation Study

To generalize the results of numerical study, we accomplish a simulation study over few hypothetically generated populations which are discussed below.

1. A normal population of size N = 500 is generated using R software with mean = 30 and variance = 25.

2. An exponential population of size N = 500 is generated using R software with rate parameter = 5.
3. A population is generated using Gamma distribution of size N = 500, shape parameter = 5 and scale parameter = 2.

We have classified the above populations according to their corresponding median and consider all those values which are greater than median as attribute. We have drawn samples of sizes 80, 70 and 100 respectively from the above populations using simple random sampling without replacement. With 15,000 iterations, we have calculated the PRE of the proposed estimator regarding the existing estimators using the following expression,

$$PRE = \frac{\frac{1}{15,000}\sum_{i=1}^{15,000}\left(T-\bar{Y}\right)^2}{\frac{1}{15,000}\sum_{i=1}^{15,000}\left(t_{bk}-\bar{Y}\right)^2},$$

where T denotes the existing estimators discussed in this study. The results of simulation study are displayed in Table 3.

Table 3 PRE of proposed estimators with respect to existing estimators using artificially generated populations

PRE	Population 1	Population 2	Population 3	PRE	Population 1	Population 2	Population 3
E_1	281.2866	279.0826	202.6693	E_{16}	244.0311	249.2277	156.3866
E_2	2965.2720	1818.2250	5394.0390	E_{17}	124.1395	111.3315	182.6533
E_3	123.0629	100.4697	108.9550	E_{18}	161.8478	180.6542	109.7309
E_4	4464.6970	3104.6530	6895.2870	E_{19}	109.5848	187.2094	201.6826
E_5	146.8853	111.3260	165.0404	E_{20}	102.9930	199.6416	126.4265
E_6	125.5598	101.6128	116.7590	E_{21}	144.5867	163.5922	124.6023
E_7	125.5598	101.8339	116.7590	E_{22}	187.7590	188.5348	146.3597
E_8	308.4670	201.8524	442.5793	E_{23}	142.1927	160.9458	121.4100
E_9	163.3974	119.9181	220.4989	E_{24}	145.3284	123.2940	125.2533
E_{10}	411.5566	266.9437	700.5629	E_{25}	144.3518	163.5822	124.6018
E_{11}	124.4844	101.2070	112.6432	E_{26}	179.4259	168.4521	145.7933
E_{12}	163.3974	123.4153	220.4989	E_{27}	142.1851	160.9390	121.4006
E_{13}	136.9734	107.5879	136.4663	E_{28}	145.3228	121.3273	125.3254
E_{14}	182.3203	201.8691	113.7581	E_{29}	537.8710	136.3054	1101.7710
E_{15}	244.0311	251.6482	156.3866	E_{30}	114.3417	321.0630	104.8773

6. Discussion of Results

The following discussion may be read out from the results of numerical study and simulation study reported respectively in Table 2 and Table 3.

1. From Table 2 based on results of numerical study, it is observed from the results of population 1 that the PRE of the proposed estimator with respect to the existing estimators such as: usual mean estimator remain 373.2731; classical ratio estimator remain 5236.5200; classical regression estimator remain 154.2529; Singh et al. (2008) estimators $(t_{s_i}, i=1,2,...,10)$ remain 155.9608 to 7565.6290; Abd-Elfattah et al. (2010) estimators $(t_{a_i}, i=2,3,...,6)$ remain 162.1201 to 325.8917; Grover and Kaur (2011) estimator remain 117.9907; Singh and Solanki (2012) estimator remain 104.2286; members of Koyuncu (2012) estimator $(t_{k(i)}, i=1,2,3,4)$ remain 111.7166 to 137.9081; Haq and Shabbir (2014) estimator $(t_{hs(i)}, i=1,2,3,4)$ remain 109.9066 to 137.8992; Zaman and Kadilar (2019) estimator remain 952.0523 and Bhushan and Gupta (2020) remain 140.9190.
2. From Table 2, proceeding in the same manner, it has been noticed that the PRE of the proposed estimator regarding the existing estimators is greater than 100 for population 2 and population 3 which shows the dominance of the proposed estimator over the existing estimators.
3. From Table 3 based on the results of simulation study, it is observed from the results of population 1 that the PRE of the proposed estimator with respect to the existing estimators such as: usual mean estimator remain 281.2866; classical ratio estimator remain 2965.272; classical regression estimator remain 123.0629; Singh et al. (2008) estimators $(t_{s_i}, i=1,2,...,10)$ remain 124.4844 to 4464.697; Abd-Elfattah et al. (2010) estimators $(t_{a_i}, i=2,3,...,6)$ remain 124.1395 to 244.0311; Grover and Kaur (2011) estimator remain 109.5848; Singh and Solanki (2012) estimator remain 102.9930; members of Koyuncu (2012) estimator $(t_{k(i)}, i=1,2,3,4)$ remain 142.1927 to 187.7590; Haq and Shabbir (2014) estimator $(t_{hs(i)}, i=1,2,3,4)$ remain 142.1851 to 179.4259; Zaman and Kadilar (2019) estimator remain 537.8710 and Bhushan and Gupta (2020) remain 114.3417.
4. The similar conclusion can be drawn from the simulation results of population 2 and population 3 reported in Table 3.
5. Further, to have another clear view of the results reported in Tables 2 and 3 by PRE, we displayed two multiple bar diagrams given in Figures 1 and 2 respectively for real and artificially generated populations.

7. Conclusion

In this paper, we have suggested an improved class of estimators for the estimation of population mean using information on an auxiliary attribute along with its properties. The proposed class of estimator includes the usual mean estimator, classical ratio estimator and Abd-Elfattah et al. (2010) estimators for suitably chosen values of scalars. The theoretical conditions have been derived and supported by a numerical study conducted over three real populations and a simulation study over three artificially generated populations. The findings of the numerical and simulation study demonstrate that the proposed estimator performs better than the usual mean estimator, classical ratio and regression estimators, Singh et al. (2008) estimators, Abd-Elfattah et al. (2010) estimators, Grover and Kaur (2011) estimator, Singh and Solanki (2012) estimator, Koyuncu (2012) estimator, Haq and Shabbir (2014) estimator, Zaman and Kadilar (2019) estimator and Bhushan and Gupta (2020) estimator. Thus, the survey practitioners are recommended to consider the proposed class of estimator for the estimation of population mean provided the information is available on an auxiliary attribute.

Moreover, the performance of the proposed estimator can be examined for the estimation of population variance utilizing the knowledge of an auxiliary attribute and it is authors future step of research.

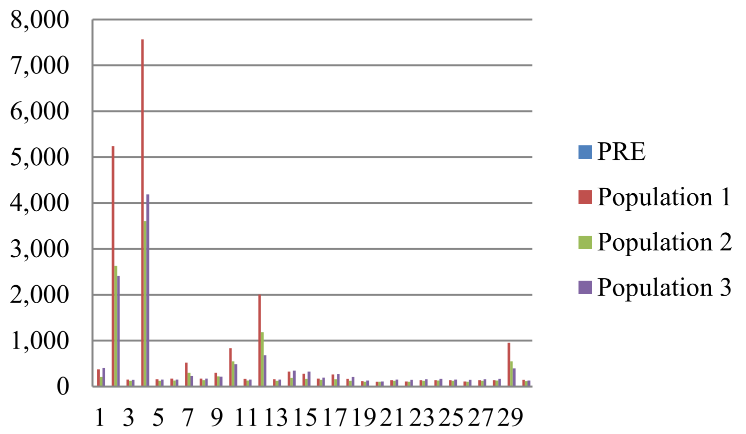

Figure 1 PRE of different estimators using real populations

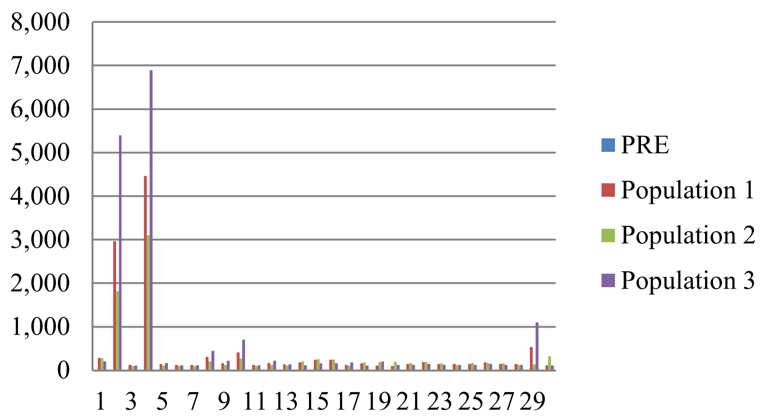

Figure 2 PRE of different estimators using artificially generated populations

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Appendix A

To the first order of approximation, the MSE of the estimators t_r, t_{lr}, t_{sr}, t_g, t_s, t_{gk}, t_{a_i} ,$i=2,\ldots,10$, t_k, t_{ss}, t_{hs_1}, t_{hs_2}, t_{zc} and t_{bg} are respectively given below.

$$MSE(t_r)=\gamma\bar{y}^2\left[C_y^2+C_\phi^2-2\rho C_yC_\phi\right], \tag{A.1}$$

$$MSE(t_{lr})=\bar{Y}^2\left[C_y^2+\beta_\phi^2C_\phi^2-2\beta_\phi\rho C_yC_\phi\right],$$

$$MSE(t_{sr})=\bar{Y}^2\left[C_y^2+\tau^2C_\phi^2-2\tau\rho C_yC_\phi\right],$$

$$MSE(t_g)=\bar{Y}^2\left[C_y^2+k^2C_\phi^2+2k\rho C_yC_\phi\right],$$

$$MSE(t_{si})=\gamma\bar{Y}^2\left[R_i^2C_\phi^2+C_y^2\left(1-\rho^2\right)\right],\ i=1,2,\ldots,10\ , \tag{A.2}$$

$$MSE(t_{gk})=\bar{Y}^2\left\{(\alpha-1)^2+\alpha^2\gamma M_1-\frac{\alpha\gamma}{2}\left(M_2+\frac{C_\phi^2}{2}-\rho C_yC_\phi\right)\right\}$$
$$+\beta^2P^2\gamma C_\phi^2+2\beta P\bar{Y}\gamma\left(\alpha M_2-\frac{C_\phi^2}{2}\right),$$

$$MSE(t_{ai})=\gamma\bar{Y}^2\left[C_y^2+R_i^2C_\phi^2-2R_i\rho C_yC_\phi\right],\ i=2,3,4,5,6\ ,$$

$$MSE(t_k)=\left[\bar{Y}^2\lambda_1^2A_1+\bar{Y}^2B_1+\bar{Y}\lambda_1\lambda_2C_1+\bar{Y}^2\lambda_1D_1+\bar{Y}\lambda_2E_1\right],$$

$$MSE(t_{ss})=\bar{Y}^2\left[1+d_1^2A_2+d_2^2B_2+2d_1d_2C_2-2d_1D_2-2d_2E_2\right],$$

$$MSE(t_{hs})=\left[\bar{Y}^2+h_2^2A_3+h_1^2C_3+h_1D_3+h_2B_3+h_1h_2E_3\right],$$

$$MSE(t_{zc})=\bar{Y}^2\left[\lambda^2C_\phi^2+C_y^2-2\lambda\rho C_yC_\phi\right], \tag{A.3}$$

$$MSE(t_{bg})=\bar{Y}^2w_1^2A_4+w_2^2B_4+\bar{Y}^2w_1D_4+\bar{Y}w_2G_4+\bar{Y}w_1w_2F_4+\bar{Y}^2,$$

where $\gamma=(N-n)/Nn,\ C_y=S_y/\bar{Y},\ C_\phi=S_\phi/P,\ S_y^2=(N-1)^{-1}\Sigma_{i=1}^N\left(y_i-\bar{Y}\right)^2$,

$S_\phi^2=(N-1)^{-1}\Sigma_{i=1}^N(\phi_i-P)^2$, $\nu=\eta P/(\eta P+\lambda)$, $R_1=1$, $R_2=P/\left(P+\beta_2(\phi)\right)$,

$R_3=P/\left(P+C_\phi\right)$, $R_4=P\beta_2(\phi)/\left(P\beta_2(\phi)+C_\phi\right)$, $R_4=PC_\phi/(PC_\phi+\beta_2(\phi))$, $R_6=P/(P+\rho)$,

$R_7=PC_\phi/\left(PC_\phi+\rho\right)$, $R_8=P\rho/\left(P\rho+C_\phi\right)$, $R_9=P\beta_2(\phi)/\left(P\beta_2(\phi)+\rho\right)$,

$R_{10}=P\rho/\left(P\rho+\beta_2(\phi)\right)$, $M_1=\left(C_y^2+C_\phi^2-2\rho C_yC_\phi\right)$, $M_2=\left(C_\phi^2-\rho C_yC_\phi\right)$,

$A_1=1+\gamma\left(C_y^2+\nu^2C_\phi^2-2\nu\rho C_yC_\phi\right)$, $B_1=1+\left(\delta^2+\nu^2+\delta(\delta-1)-2\delta\nu\right)\gamma C_\phi^2$,

$C_1=2+2(\delta-\nu)\gamma\rho C_yC_\phi+\left(2\nu^2+\delta(\delta-1)-2\delta\nu\right)\gamma C_\phi^2$, $D_1=\gamma\nu\rho C_yC_\phi-2-(3/4)\nu^2\gamma C_\phi^2$,

$E_1=\delta\nu-(3/4)\nu^2-\delta(\delta-1)\gamma C_\phi^2-2$, $A_2=1+\gamma\left\{C_y^2+g(2g+1)\nu^2C_\phi^2-4\rho C_yC_\phi\right\}$, $B_2=\gamma P^2C_\phi^2$,

$C_2 = 2\gamma P\left(\gamma\nu C_\phi^2 - \rho C_y C_\phi\right),\ D_2 = 1+\gamma\left\{\left(g\left(g+1\right)/2\right)\nu^2 C_\phi^2 - \nu\rho C_y C_\phi\right\},$

$E_2 = \gamma g\nu P C_\phi^2 - \gamma\rho P C_y C_\phi,\ A_3 = 1-\delta\gamma C_\phi^2 + 2\delta^2\gamma C_\phi^2 - 2\delta\nu\gamma C_\phi^2 + 3\nu^2\gamma C_\phi^2,$

$B_3 = -2\bar{Y} + \bar{Y}\delta\gamma C_\phi^2 - \bar{Y}\delta^2\gamma C_\phi^2 + \bar{Y}\delta\nu\gamma C_\phi^2 - \left(11/4\right)\bar{Y}\nu^2\gamma C_\phi^2,$

$C_3 = \bar{Y}^2 + 3\bar{Y}^2\nu^2\gamma C_\phi^2 + \bar{Y}^2\gamma C_y^2 - 2\bar{Y}^2\nu\rho C_y C_\phi,\ D_3 = -2\bar{Y}^2 - \left(11/4\right)\bar{Y}^2\nu^2\gamma C_\phi^2 + \bar{Y}^2\nu\rho C_y C_\phi,$

$E_3 = 2\bar{Y} - \bar{Y}\delta\gamma C_\phi^2 + \bar{Y}\delta^2\gamma C_\phi^2 - 2\bar{Y}\delta\nu\gamma C_\phi^2 + 6\bar{Y}\nu^2\gamma C_\phi^2 + 2\bar{Y}\delta\rho C_y C_\phi - 2\bar{Y}\nu\rho C_y C_\phi,$

$A_4 = 1+\gamma\left(C_y^2 + \alpha^2\nu^2 C_\phi^2 + 4\alpha\nu\rho C_y C_\phi - \alpha\nu C_\phi^2\right),\ B_4 = 1+\gamma\left(C_\phi^2 + \alpha^2\nu^2 C_\phi^2 - \alpha\nu^2 C_\phi^2 + 4\alpha\nu C_\phi^2\right),$

$D_4 = \gamma\left(\alpha\nu^2 C_\phi^2 - 2\alpha\nu\rho C_y C_\phi\right) - 2,\ G_4 = \gamma\left(\alpha\nu^2 C_\phi^2 - 2\alpha\nu C_\phi^2\right) - 2,$

$F_4 = 2 + 2\gamma\left(2\alpha\nu C_\phi^2 + 2\alpha\nu\rho C_y C_\phi + \rho C_y C_\phi - \alpha\nu^2 C_\phi^2 + \alpha^2\nu^2 C_\phi^2\right).$

The MSE of estimators $t_{lr}, t_{sr}, t_g, t_{gk}, t_k, t_{ss}, t_{hs1}, t_{hs2}$ and t_{bg} are respectively minimized for

$$\beta_{\phi(opt)} = \frac{\rho C_y}{C_\phi},\ \ \beta_{1(opt)} = \frac{\rho C_y}{C_\phi},\ \ \beta_{2(opt)} = -\frac{\rho C_y}{C_\phi},\ \ \alpha_{(opt)} = \frac{-C_\phi^2\left[2 - \frac{\gamma M_2}{2} + \frac{\gamma}{2}\left(\frac{C_\phi^2}{2} - \rho C_y C_\phi\right)\right]}{2\left[\gamma M_2^2 - C_\phi^2\left(1+\gamma M_1\right)\right]},$$

$$\beta_{(opt)} = \frac{\bar{Y}\left[M_2\left\{2 + \frac{\gamma}{2}M_2 + \frac{\gamma}{2}\left(\frac{C_\phi^2}{2} - \rho C_y C_\phi\right)\right\} - \left(1+\gamma M_1\right)C_\phi^2\right]}{2P\left[\gamma M_2^2 - C_\phi^2\left(1+\gamma M_1\right)\right]},\ \ \lambda_{1(opt)} = \frac{C_1E_1 - 2B_1D_1}{4A_1B_1 - C_1^2},$$

$$\lambda_{2(opt)} = \frac{\bar{Y}\left(C_1D_1 - 2A_1E_1\right)}{4A_1B_1 - C_1^2},\ \ d_{1(opt)} = \frac{B_2D_2 - C_2E_2}{A_2B_2 - C_2^2},\ \ d_{2(opt)} = \frac{A_2E_2 - C_2D_2}{A_2B_2 - C_2^2},$$

$$h_{1(opt)} = \frac{B_3E_3 + 2A_3D_3}{E_3^2 - 4A_3C_3},\ \ h_{2(opt)} = \frac{2B_3C_3 - D_3E_3}{E_3^2 - 4A_3C_3},\ \ w_{1(opt)} = \frac{G_4F_4 - 2B_4D_4}{4A_4B_4 - F_4^2}\ \text{and}$$

$$w_{2(opt)} = \frac{\bar{Y}\left(D_4F_4 - 2G_4A_4\right)}{4A_4B_4 - F_4^2}.$$

The minimum MSE of estimators $t_{lr}, t_{gk}, t_k, t_{ss}, t_{hs}$ and t_{bg} at the above optimum values is respectively given as

$$\min MSE\left(t^*\right) = \bar{Y}^2\gamma C_y^2\left(1-\rho^2\right)\ \text{where}\ t^* = t_{lr},\ t_{sr}\ \text{and}\ t_g. \tag{A.4}$$

$$\min MSE\left(t_{gk}\right) = \frac{\bar{Y}^2 MSE\left(t_{lr}\right)}{\bar{Y}^2 + MSE\left(t_{lr}\right)} - \frac{\gamma C_\phi^2\left[MSE\left(t_{lr}\right) + \frac{\gamma\bar{Y}^2C_\phi^2}{16}\right]}{\frac{4\left[1+MSE\left(t_{lr}\right)\right]}{\bar{Y}^2}}. \tag{A.5}$$

$$\min MSE\left(t_{ai}\right) = \gamma\bar{Y}^2\left[C_y^2\left(1-\rho^2\right)\right] = min MSE\left(t_{lr}\right), i = 7,8,9,10. \tag{A.6}$$

$$\min MSE\left(t_k\right) = \bar{Y}^2\left[1 - \frac{\left(A_1E_1^2 + B_1D_1^2 - 2C_1D_1E_1\right)}{4A_1B_1 - C_1^2}\right]. \tag{A.7}$$

$$\min MSE\left(t_{ss}\right)=\bar{Y}^2\left[1-\frac{\left(A_2E_2^2+B_2D_2^2-2C_2D_2E_2\right)}{A_2B_2-C_2^2}\right]. \tag{A.8}$$

$$\min MSE\left(t_{hs}\right)=\bar{Y}^2+\frac{\left(B_3^2C_3+A_3D_3^2-2B_3D_3E_3\right)}{(E_3^2-4A_3C_3)}\ . \tag{A.9}$$

$$\min MSE\left(t_{bg}\right)=\bar{Y}^2\left[1-\frac{\left(A_4G_4^2+B_4D_4^2-2D_4F_4G_4\right)}{\left(4A_4B_4-F_4^2\right)}\right]. \tag{A.10}$$