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Instant Exchange Analytics: A Compelling Mathematical Model for Real-Time Debt Instrument and Cash Conversion Rates

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Abstract

This research investigates a model for determining instantaneous debt instrument prices for converting debt securities into cash over time. The study employs three interest rate models: Vasicek, Cox-Ingersoll-Ross, and Hull-White, using historical daily treasury bill interest rates from January 2004 to October 2020. The findings suggest that the Hull-White model outperforms the Vasicek and Cox-Ingersoll-Ross models, simulating interest rates for determining debt-to-cash exchange rates. Additionally, a real-time continuous debt-to-cash conversion prototype is developed using zero-coupon government bonds with maturities of no more than 10 years, traded in the Thai secondary debt market as a case study.

Keywords: Zero-coupon bond pricing, maximum likelihood estimation, stochastic interest rate models.

1. Introduction

The debt securities market plays a vital role in the economy, acting as a source of capital and investment for both businesses and investors. The returns on debt securities are crucial for determining and implementing interest rate policies. Debt securities are popular assets for managing risk in investment portfolios of individual investors and funds, offering low-risk investments with higher returns than bank deposits. Additionally, the debt securities market is an excellent temporary investment option for domestic and foreign investors waiting for suitable investment opportunities in other projects.

The Thai debt securities market has become increasingly popular in recent years due to its low risk, steady returns, and accessible trading options for investors. Investors can directly invest in the primary market by purchasing securities from issuers or trading in the secondary market. Alternatively, inexperienced, small-scale investors can indirectly invest in debt securities through debt funds.

Direct investment in the primary market involves investing in newly issued debt securities. Private company bonds have two offering types: private placement (PP) for institutional or large investors with specific financial qualifications and public offering (PO) for the public. Investors can buy bonds directly from financial institutions, authorized dealers, or through online channels and mobile banking applications. Investment minimums, such as 100,000 THB or 100 units at 1,000 THB per unit, are often set. General investors can purchase government bonds, like savings bonds, with a minimum of 1,000 THB at designated banks. Other government bonds, such as treasury bills and bonds issued by the Bank of Thailand, are available for institutional and large investors with specific financial qualifications through competitive or non-competitive bidding.

Investors can buy, sell, or exchange debt securities in the secondary market to increase liquidity or manage their finances, trading with commercial banks or through the Bond Electronic Exchange (BEX) system.

Individual investors can invest in debt securities through debt mutual funds. However, there are limitations, such as the time it takes to convert debt securities into cash, fees, and transaction costs. For short-term debt securities, investors can receive cash on the next business day (T+1), while for medium-term debt securities, some funds may take (T+2) or have other restrictions specified by the fund itself.

This has led researchers to study debt security pricing to establish an exchange rate between debt securities and cash and develop a model for individual investors who want to use money from selling debt securities at a particular time but cannot receive the cash immediately. This is important because if it takes a long time to sell the securities, investors may miss out on other investment opportunities or need the cash for operating expenses or immediate use, which affects their liquidity. Furthermore, debt securities can only be traded on business days by authorized dealers, such as commercial banks and securities companies, which are not open for trading every day or all the time and require a large amount of money for each debt security transaction (Thai Bond Market Association 2018).

In summary, individual investors can invest in debt securities through debt mutual funds, but there are limitations concerning the time to convert securities into cash, fees, and transaction costs. Researchers are studying debt security pricing to create a model for individual investors to improve the process of converting securities into cash and enhance their liquidity. Trading of debt securities is limited to authorized dealers' business days and requires a significant amount of money for each transaction.

2. Literature Review

In interest rate modeling for determining debt security prices, researchers have primarily focused on single-factor interest rate models in both equilibrium and arbitrage-free markets. These models, including the Vasicek (1977), Cox-Ingersoll-Ross (CIR), and Hull-White models, are compared using Root Mean Square Error to select the most suitable model for pricing debt securities. Relevant studies in this area include:

Mongkolkiatchai (2006) used cointegration regression analysis to examine the relationship between interest rates derived from Vasicek and CIR models and short-term interest rates in the Thai market. The study found that both models were related to the short-term interest rates examined, with the CIR model providing a better explanation of the relationship than the Vasicek model.

Dagistan (2010) assessed the measurement and management of interest rate risk in government bonds issued by the United States, Germany, and Canada by simulating future interest rates using stochastic interest rate models. The study aimed to assess interest rate risk in government bonds using stochastic interest rate models to evaluate the effectiveness of Value at Risk (VaR) as a risk measurement tool and analyze the sensitivity of risk measurement to changes in parameters within the stochastic interest rate model. The findings revealed that single-factor stochastic interest rate models in equilibrium markets and arbitrage-free markets produced similar results when measuring risk values. The most suitable model for simulating risk varied by country, necessitating different models for each nation.

Kaewcharoenkij and Panpocha (2018) investigated pricing callable bonds, which allow early redemption rights, using the CIR and Hull-White interest rate models. The interest rates obtained from these models were employed in Monte Carlo simulations to calculate the value of callable bonds. The study concluded that interest rates from the Hull-White model more closely followed the trend of yield curves than those from the CIR model.

Orlando et al. (2020) explored a novel method for forecasting interest rates in the Vasicek and CIR models by dividing real data into sets to examine the distribution of probability using a copula-based approach. The data sets were then utilized to estimate parameters and evaluate the model's performance with the Root Mean Square Error. The research demonstrated that the CIR model offered superior forecasting performance compared to the Vasicek model when data sets were divided according to the distribution of the models employed for forecasting. However, the study encountered a limitation due to its focused dataset, which concentrated on the period surrounding the 2008 financial crisis, spanning approximately five years, and relied on monthly data. This specific temporal and data granularity focus could have implications for the generalizability and applicability of the findings across different economic contexts or timeframes.

The research's use of daily yield data from the Thai Bond Market Association, encompassing a broad range of securities including 1-month, 3-month, 6-month, and 1-year Treasury Bills, as well as 2-year to 10-year zero-coupon government bonds, for the period from January 2004 to October 2019, provides a rich and comprehensive dataset. This dataset is particularly valuable for analyzing trends and patterns over a significant time frame, allowing for a robust examination of the behavior of various debt instruments in different market conditions.

The focus on developing a model for determining instantaneous and continuous cash-equivalent exchange rates for debt securities is a critical advancement in the field. This approach goes beyond traditional methods that often rely on more static or periodic assessments. By applying interest rate simulation models, researchers aim to calculate exchange rates that are not only accurate at a given moment but also adaptable to the continuous fluctuations inherent in financial markets.

Such a model would be invaluable for market participants, providing them with tools to better assess risk, price debt securities more accurately, and make more informed investment decisions. It also holds the potential to enhance the understanding of the dynamic nature of interest rates and their impact on various forms of debt. This research, therefore, represents a significant step towards more responsive and sophisticated financial modeling, reflecting the real-time complexities of the bond market.

3. Basic Knowledge and Related Theories

3.1. Examination of the change points in the variance of time series data

3.1.1 Likelihood ratio test

Let $X_1, X_2, ..., X_n$ be independent random variables with the same distribution and values of the random variables $x_1, x_2, ..., x_n$ having the probability density functions as follows:

$$L(x_1, x_2, ..., x_n; \theta) = f_{x_1, x_2, ..., x_n}(x_1, x_2, ..., x_n; \theta), \;\; LRT = \frac{L(x_1, x_2, ..., x_n; \theta_0)}{L(x_1, x_2, ..., x_n; \theta_1)}.$$

3.1.2 Testing for a single change point in the average value of time series data (Single Changepoint)

Let Y_t represent the data of the time series at any time t, where $t \geq 0$, θ_t represents the parameter of the average value of the time series data at any time t, where $t \geq 0$. To test for changes in the average value, assume that $\theta_t = \theta$ is constant, and the time series data follows the given conditions as follows:

$$Y_t \mid \theta_t \sim N(\theta_t, 1).$$

To determine a single change point in the average value of time series data, the Likelihood Ratio Test can be performed as follows:

$$LR = \max_{\tau} \left\{ l(y_{1:\tau}) + l(y_{\tau+1:n}) - l(y_{1:n}) \right\}.$$

The point where the average value changes is when $LR > \lambda$ when λ is the penalty value. To find the position of the point where the average value of the time series data changes, it can be found as follows:

$$\tau = \arg\max \left\{ l(y_{1:\tau}) + l(y_{\tau+1:n}) - l(y_{1:n}) \right\}.$$

3.1.3 Testing for multiple change points in the average value or variance of time series data (Multiple Changepoint)

Let Y_t represent the data of the time series at any time t, k represents the number of change points in the average value or variance of time series data, τ represents the change points in the average value or variance of time series data when $\tau = (\tau_0, \tau_1, \tau_2, ..., \tau_{k+1})$, $\tau_0 = 0$ and $\tau_{k+1} = n$. The likelihood ratio test can be defined for cases where the number of change points in the average value or variance is equal to 0 or 1, as follows:

$$\min_{k \in \{0,1\}, \tau} \left\{ \sum_{j=1}^{k+1} \left[-l(y_{\tau_{j-1}}) \right] + \lambda k \right\},$$

where k represents the number of change points, a new Likelihood Ratio can be defined as follows:

$$\min_{k,\tau} \left\{ \sum_{j=1}^{k+1} \left[-l\left(y_{\tau_{j-1}:\tau_j} \right) \right] + \lambda k \right\}, \tag{1}$$

Equation (1) is called the penalized likelihood. For the general form of the penalized likelihood, it can be written as follows:

$$\min_{k,\tau} \left\{ \sum_{j=1}^{k+1} \left[-l(y_{\tau_{j-1}:\tau_j}) \right] + \lambda f(k) \right\}.$$

3.2. Interest rate models

3.2.1 Vasicek interest rate model (Brigo and Fabio 2013, pp.58)

$$dr(t) = \alpha\left(\beta - r(t)\right)dt + \sigma_r dW(t).$$

3.2.2 CIR interest rate model (Brigo and Fabio 2013, pp.64-69)

$$dr(t) = \alpha\left(\beta - r(t)\right)dt + \sigma_r \sqrt{r(t)}dW(t).$$

3.2.3 Hull-White interest rate model (Brigo and Fabio 2013, pp.161)

$$dr(t) = \left[\theta(t) - \beta(t)r\right]dt + \sigma_r dW(t),$$

and $\theta(t)$ can be found from $\theta(t) = \dfrac{\partial f(0,t)}{\partial T} + \beta f(0,t) + \dfrac{\sigma_r^2}{2\beta}\left(1 - e^{-2\beta t}\right).$

3.3. Parameter estimation in models using maximum likelihood estimation

3.3.1 Vasicek and Hull-White interest rate models

$$a=\hat{\alpha}=\frac{1}{dt}\ln\left(\frac{n\sum_{j=1}^{n} r_{t_j} r_{t_{j-1}}-\sum_{j=1}^{n} r_{t_j}\sum_{j=1}^{n} r_{t_{j-1}}}{n\sum_{j=1}^{n} r_{t_{j-1}}^2-\left(\sum_{j=1}^{n} r_{t_{j-1}}\right)^2}\right),$$

$$b=\hat{\beta}=\frac{1}{n\left(1-e^{-adt}\right)}\left(\sum_{j=1}^{n} r_{t_j}-e^{-adt}\sum_{j=1}^{n} r_{t_{j-1}}\right),$$

$$\hat{\sigma}_r^2=\frac{2a}{n\left(1-e^{-2adt}\right)}\sum_{j=1}^{n}\left(r_{tj}-r_{t_{j-1}}e^{-adt}-b\left(1-e^{-adt}\right)\right)^2.$$

3.3.2 CIR interest rate model

$$c=\frac{4ax}{\hat{\sigma}_r^2\left(1-e^{-a(t-s)}\right)},$$

$$\ln\left(L(\alpha,\beta,\sigma;x)\right)=(n-1)\ln(c)+\sum_{j=2}^{n}\ln\left(p_{\chi^2_{(d,\lambda)}}\left(cx_j\mid x_{j-1}\right)\right),$$

where $x=r_1,r_2,...,r_n$.

4. Research Methodology

Step 1. Data collection

Collect daily interest rate data for debt securities with maturities of up to 10 years. Organize the data into two sets: Set 1 consists of interest rates for treasury bills with maturities of 1 month, 3 months, 6 months, and 1 year, and zero-coupon government bonds with maturities ranging from 2 to 10 years, totaling 4,037 data points. Set 2 is similar to Set 1, but includes the most recent 68 data points. The data is collected from the debt securities information between the years 2004-2021(Bank of Thailand).

Step 2. Detecting variance change points

The interest rate time series data generally exhibits an empirical distribution, which is a combination of several probability distributions, and multiple variance changepoints can be found in real data. Therefore, detecting variance changepoints is conducted to find the range of points where one level of variance transitions to another.

To select an appropriate penalty value for the time series data, the method by Assistant Professor Rebecca Killick of Lancaster University in England is employed, using the CROPS (Change points for a range of penalties) as the penalty value in the cpt.var function. The pen.value is then set as a range between the minimum and maximum values to find the appropriate number of variances changepoints.

Step 3. Parameter estimation in models

Estimating the model parameters using the maximum likelihood estimation method for 1-month treasury bill interest rates, the parameter estimation is divided into two cases: one using a single parameter estimate and the other using the parameter estimate from the variance change point closest to the current value, following the procedure outlined in Section 2.2.

Step 4. Interest rate simulation

In this research, the exact simulation method is used to simulate interest rates, with each interest rate model having different equations for simulating the rates, as follows:

4.1) Vasicek interest rate model

$$r_{t_{k+1}} = e^{-a(t_{k+1}-t_k)} r_{t_k} + b\left(1-e^{-a(t_{k+1}-t_k)}\right) + \hat{\sigma}_r \sqrt{\frac{1}{2a}\left(1-e^{-2a(t_{k+1}-t_k)}\right)} Z_{k+1},$$

where Z_{k+1} has a normal distribution with a mean of 0 and a variance of 1.

4.2) CIR interest rate model

$$r_{t_{k+1}} = \frac{\hat{\sigma}_r^2\left(1-e^{-a(t-s)}\right)}{4a} \chi_d^2 \left(\frac{4ae^{-a(t-s)}}{\hat{\sigma}_r^2\left(1-e^{-a(t-s)}\right)} r_{t_k} \right).$$

4.3) Hull-White interest rate model

$$r_{t_{k+1}} = e^{-a(t_{k+1}-t_k)} r_{t_k} + \hat{\alpha}_{t_{k+1}} - \widehat{\alpha}_{t_k} e^{-a(t_{k+1}-t_k)} + \sqrt{\frac{\hat{\sigma}_r^2}{2a}\left(1-e^{-2a(t_{k+1}-t_k)}\right)} Z_{k+1},$$

where Z_{k+1} has a normal distribution with a mean of 0 and a variance of 1.

In the case of the Hull-White model, forward rates are also considered, and an instantaneous forward rate curve is constructed using the following method:

Instantaneous forward rates can be obtained by using debt security prices and maturity dates with the following equation:

$$P(t,T) = \exp\left(-\int_t^m f(t,s)ds \right), t < m < T.$$

For the given maturity dates of 1 month, 3 months, 6 months, and 1 year up to 10 years, fixed forward rates can generally be calculated over time periods $0-\frac{1}{12}$, $\frac{1}{12}-\frac{3}{12}$, $\frac{3}{12}-\frac{6}{12}$, $\frac{6}{12}-1$, $1-2$ years, up to 9-10 years, totaling 13 time periods. However, this research aims to determine the instantaneous forward rates at any time t using linear interpolation between the intervals.

Assume that the forward interest rate curve is a straight line with the following equation:

$$f_k(t) = m_k t + n_k,$$

where k represents the intervals of forward interest rates, totaling 13 intervals. In period $0-\frac{1}{12}$, we know $n_1(t_1) = 0.004379$, which is the data of the spot rate interest. The general form for programming is

$$m_1 = -\frac{2}{t_1^2} \ln P(0,t_1) + n_1(t_1),$$

$$f(y) = \ln \frac{P(0,t_{k+1})}{P(0,t_k)} + \frac{y}{2}\left(t_k^2 - t_{k-1}^2\right) + \left(t_{k-1} \times (m_{k-1}) - y + n_{k-1}\right) \times \left(t_k - t_{k-1}\right),$$

$$n_k = t_{k-1} \times \left(m_{k-1} - m_k\right) + n_{k-1}, \tag{2}$$

where $k = 2,3,\ldots,n,$ with n representing the bond's maturity date.

To find the values from m_2 onwards, the uniroot command in R programming is used for its ease of implementation. Equation (2) is used to write the function for calculating the instantaneous forward interest rates.

Step 5. Determining debt security prices

The pricing of debt securities for each simulation model is based on various maturity terms. The total price is calculated on July 17, 2020, using dataset 1 and dataset 2 as unseen data.

To find the present value of zero-coupon bonds, all three models use the same equation for pricing, which is

$$P(r,t,T) = A(t,T)\exp\left(-B(t,T)\right)r(t),$$

and the basic equation for determining the present value of a single payment installment, assuming the interest rate is a continuously compounded rate, is

$$P(r,t,T) = \exp\left(-r(t)\times(T-t)\right).$$

5. Research Results

5.1 Identifying the change points of variance

To detect points of variance change in time series data, a penalty value adjustment method is employed using the pen.value.full() command. This facilitates the identification of appropriate points of variance change within the time series data.

```
  [1]   5.000000   5.024446   5.064259   5.080639   5.139735   5.144217   5.184113   5.223307   5.253170   5.272673
 [11]   5.289940   5.294247   5.314790   5.334898   5.417536   5.454724   5.461267   5.472587   5.496023   5.507363
 [21]   5.542615   5.558972   5.585501   5.609403   5.610490   5.677053   5.679646   5.725983   5.873282   5.874163
 [31]   5.876700   5.892188   5.915709   5.944090   6.075919   6.163473   6.228336   6.248174   6.252951   6.255329
 [41]   6.279804   6.317013   6.321068   6.335345   6.336895   6.361586   6.389109   6.406338   6.453424   6.460895
 [51]   6.472579   6.483018   6.530014   6.530746   6.611425   6.625576   6.636985   6.676542   6.687520   6.730904
 [61]   6.736785   6.753086   6.784655   6.798884   6.888332   6.890794   6.913748   6.923323   6.928709   6.977241
 [71]   7.010377   7.015745   7.018443   7.021259   7.033732   7.056849   7.077457   7.205606   7.250468   7.317726
 [81]   7.341794   7.362258   7.387747   7.483237   7.493376   7.495399   7.536865   7.576532   7.882935   7.951179
 [91]   7.964774   7.966620   7.975752   7.981770   8.131007   8.140605   8.210832   8.258492   8.267634   8.328277
[101]   8.362430   8.381228   8.490679   8.589566   8.602772   8.670094   8.677692   8.688097   8.718847   8.894013
[111]   8.978919   8.997039   9.018221   9.095214   9.105223   9.113893   9.202880   9.203601   9.271565   9.289187
[121]   9.291982   9.336085   9.429698   9.434492   9.475213   9.494351   9.569036   9.595653   9.643210   9.654852
[131]   9.687359   9.751551   9.895454  10.372889  10.373849  10.428029  10.470617  10.597850  10.599956  10.657177
[141]  10.668317  10.833070  10.872205  11.234193  11.549767  11.638725  12.114715  12.193036  12.341323  12.395189
[151]  12.462117  12.474763  12.620975  12.927494  12.942428  13.574895  13.961631  14.004696  14.156268  14.248468
[161]  14.489133  14.502517  14.623793  14.645551  14.957261  15.048660  15.207538  15.391019  15.473202  16.063520
[171]  16.205002  16.357355  16.562045  16.618745  16.868672  17.652134  17.795206  18.932037  18.953151  19.109581
[181]  19.165299  19.372541  19.382030  19.384661  19.422195  19.891444  19.920493  19.999129  20.213197  20.582485
[191]  20.687735  20.829046  21.565227  22.485083  23.091679  23.849483  24.087156  24.601383  24.947806  24.974334
[201]  25.003478  25.155338  25.521268  27.214456  28.016897  28.621755  29.078946  30.419330  31.708618  31.777709
[211]  32.464939  38.496867  39.067966  40.964880  42.559699  42.690184  43.115471  50.047927  53.149819  53.462787
[221]  54.756305  57.946098  65.760224  66.093463  67.412382  78.428832  79.159469  88.752832 102.395358 107.660663
[231] 111.496613 122.891199 131.216239 170.847473 172.597610 174.316945 183.608223 199.253089 353.684967
```

Figure 1 Results of using the command pen.value.full()

From the results of the pen.value.full() command used to determine the appropriate change points for the variance in the time series data, it is found that the values at positions 237 and 238 in Figure 1 do not differ significantly. However, when considering the values at positions 238 and 239, a significant difference is found. To clarify the appropriate penalty value, the command plot (detect.var, diagnostic=TRUE) is used, yielding the following results.

From Figure 2, it is observed that selecting the number of points with changes in variance within the range not exceeding 100, the graph is quite steep (the penalty value is still changing) which may not be suitable for selecting as the penalty. After that, the slope of the graph gradually decreases until around 200 onwards, the graph becomes straighter. This range is therefore suitable for selecting the penalty value. Once an appropriate penalty value is selected, it is used to determine the points with data variance changes as Figure 3.

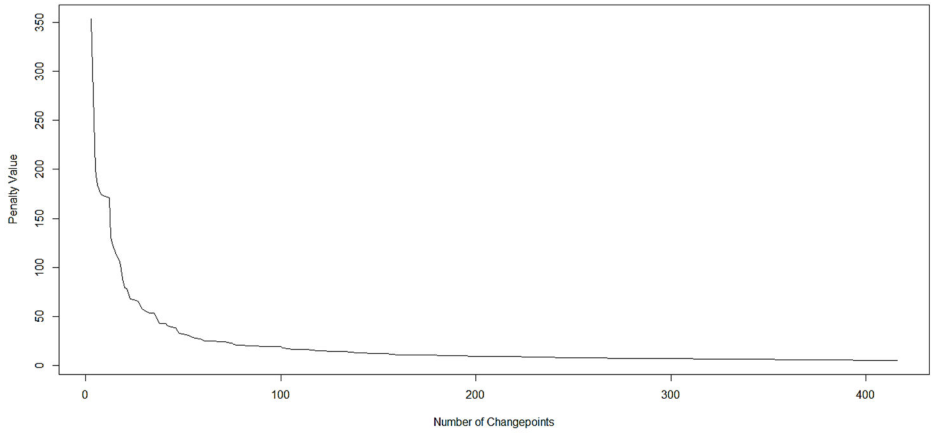

Figure 2 Results of using the command plot(detect.var, diagnostic=TRUE)

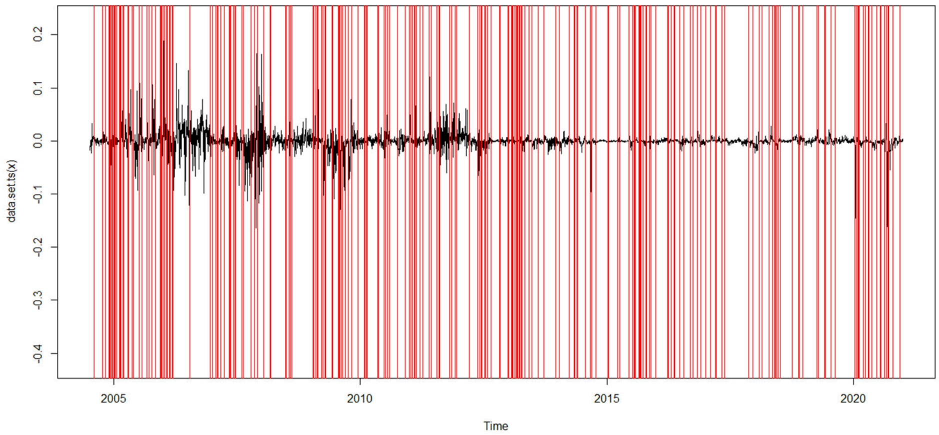

Figure 3 Results from determining the points with changes in variance for the data

5.2. Estimating parameters in the models

5.2.1 Estimating parameters in the Vasicek model and the Hull-White model

Case 1: $r_1, r_2, \ldots, r_{4037}$, there are different parameter values as follows:

$$a = 0.01310077,\ b = -0.006323367,\ \text{and}\ \hat{\sigma}_r = 0.00362324.$$

Case 2: $r_{4022}, r_{4023}, \ldots, r_{4037}$, there are different parameter values as follows:

$$a = -0.08744766,\ b = -0.002443262,\ \text{and}\ \hat{\sigma}_r = 0.0004306986.$$

5.2.2 Estimating parameters in the Cox-Ingersoll-Ross (CIR) model.

Case 1: $r_1, r_2, \ldots, r_{4037}$, there are different parameter values as follows:

$$a = 0.04052855,\ b = 0.01615882,\ \text{and}\ \hat{\sigma}_r = 0.02410338.$$

Case 2: $r_{4022}, r_{4023}, \ldots, r_{4037}$, there are different parameter values as follows:

$$a = 0.029320551,\ b = 0.025916455,\ \text{and}\ \hat{\sigma}_r = 0.006897393.$$

5.3. Pricing debt securities

The process of pricing debt securities from each simulation model is a step taken to find the yield curve of debt securities according to each maturity. All debt security prices are calculated as of July 17, 2023. However, an example of debt security prices obtained from the Hull-White model in Case 2 will be presented in Table 1.

Table 1 Results of pricing using actual data and simulating interest rates from the Hull-White model with accurate estimation using the estimated parameters obtained from Step 2.1 in Case 2

Maturity	General Pricing Formula	Accurate Simulation Method	Absolute Difference	95% Confidence Interval
1 month	0.9996324	0.9996349	0.0000025	(0.9996345, 0.9996353)
3 months	0.9988543	0.9988642	0.0000099	(0.9988623, 0.9988662)
6 months	0.9975546	0.9975745	0.0000199	(0.9975690, 0.9975799)
1 year	0.9950493	0.9950492	0.0000001	(0.9950333, 0.9950652)
2 years	0.9904664	0.9907997	0.0003333	(0.9907552, 0.9908442)
3 years	0.9826690	0.9831008	0.0004318	(0.9830114, 0.9831902)
4 years	0.9715949	0.9714632	0.0001317	(0.9713300, 0.9715964)
5 years	0.9586743	0.9583879	0.0002864	(0.9581978, 0.9585780)
6 years	0.9447097	0.9443797	0.0003300	(0.9441170, 0.9446424)
7 years	0.9287378	0.9283813	0.0003565	(0.9280415, 0.9287210)
8 years	0.9106572	0.9079308	0.0027264	(0.9075333, 0.9083283)
9 years	0.8951920	0.8926911	0.0025009	(0.8921933, 0.8931890)
10 years	0.8769335	0.8768146	0.0001189	(0.8762231, 0.8774062)

The results of using new (Unseen Data) for approximately 2 months, or 68 data points, are intended to compare the actual calculated values from the market yield data with the calculated values from the interest rate simulation model. This is done by creating a yield curve for each debt security's maturity. An example will be taken from the Hull-White model, Case 2, are shown in Figures 4-16.

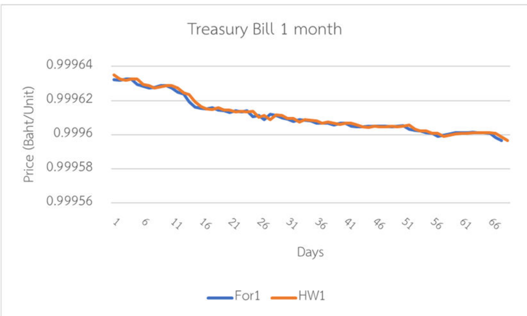

Figure 4 1-month Treasury bills

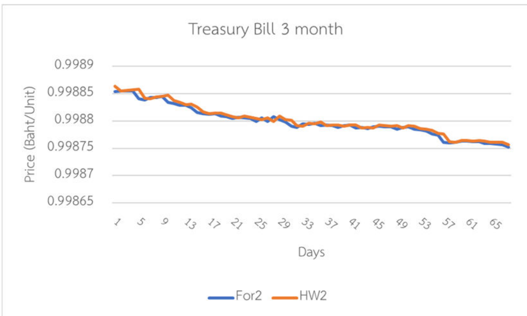

Figure 5 3-month Treasury bills

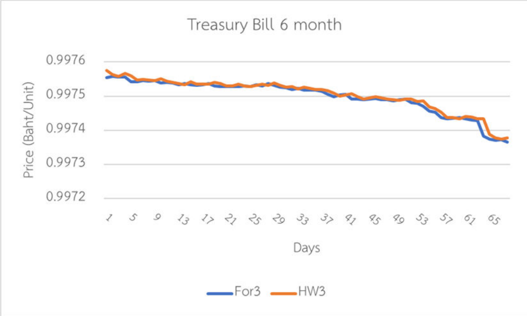

Figure 6 6-month Treasury bills

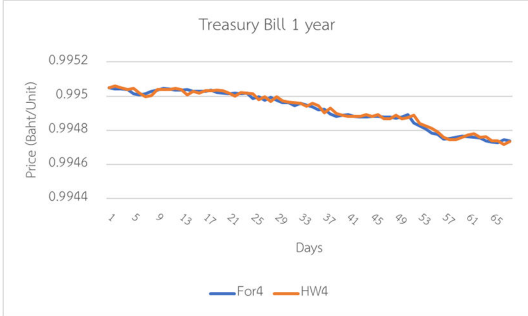

Figure 7 1-year Treasury bills

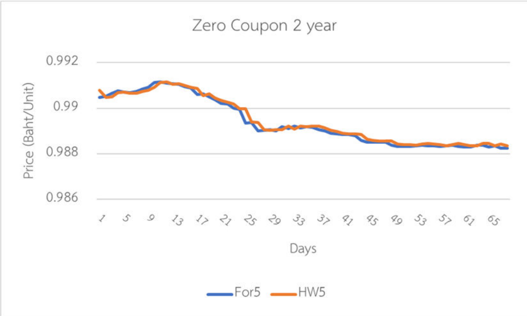

Figure 8 2-year zero-coupon bonds

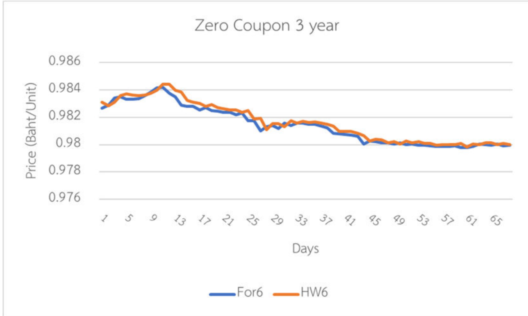

Figure 9 3-year zero-coupon bonds

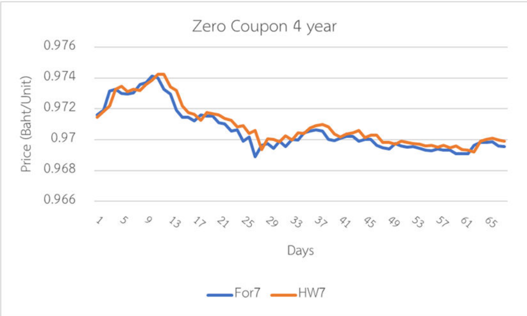

Figure 10 4-year zero-coupon bonds

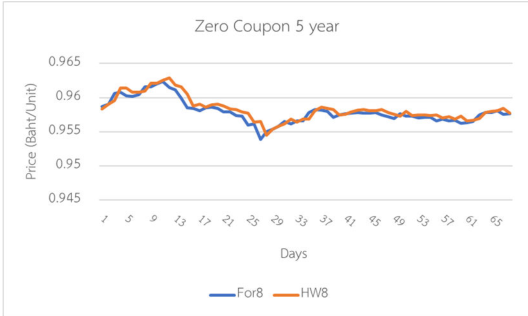

Figure 11 5-year zero-coupon bonds

Figure 12 6-year zero-coupon bonds

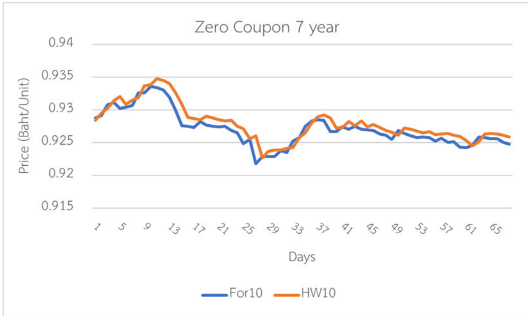

Figure 13 7-year zero-coupon bonds

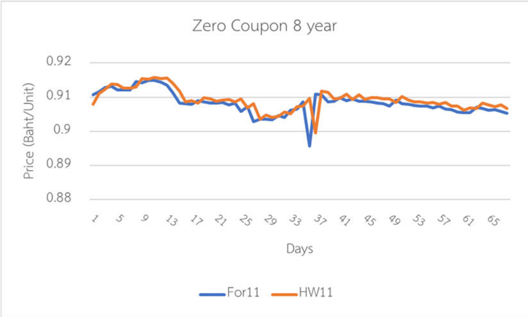

Figure 14 8-year zero-coupon bonds

Figure 15 9-year zero-coupon bonds

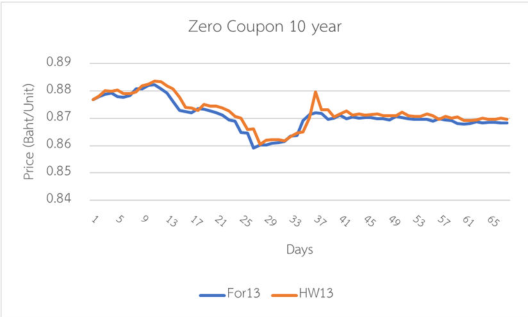

Figure 16 10-year zero-coupon bonds

To compare the effectiveness of different models in pricing, the performance is measured using the square root of the mean squared error (RMSE). The results are shown in Table 2.

From Table 2, it can be concluded that the Hull-White model has a lower RMSE value in Case 2 compared to Case 1 and has a lower RMSE value than both the Vasicek and CIR models in both cases.

Therefore, the Hull-White model will be used to simulate interest rates for pricing debt securities. Following the results in Table 2, which measure the RMSE values based on prices and considering two separate cases, the parameter estimates in the model may vary. Consequently, we will re-estimate the parameters by using a sample of 4,037 level data points of 1-month treasury bills to perform a Likelihood Ratio Test. The procedure will be like the one conducted in Step 2, and the results are as Figure 17.

Table 2 RMSE values for each interest rate simulation model

Debt Security Type	Interest Rate Simulation Model Type	RMSE	
		Case 1	Case 2
TB1M	Vasicek	0.000002309	0.000002015
	CIR	0.000001610	0.000001986
	Hull-White	0.000001605	0.000001402
TB3M	Vasicek	0.000043047	0.000019445
	CIR	0.000023444	0.000019276
	Hull-White	0.000009042	0.000005113
TB6M	Vasicek	0.000195267	0.000102849
	CIR	0.000122633	0.000104352
	Hull-White	0.000023278	0.000009701
TB1Y	Vasicek	0.000512660	0.000131733
	CIR	0.000213441	0.000138929
	Hull-White	0.000053217	0.000016188
ZCB2Y	Vasicek	0.001836975	0.000783460
	CIR	0.000923507	0.000817519
	Hull-White	0.000236396	0.000140217
ZCB3Y	Vasicek	0.005434602	0.001949944
	CIR	0.002940100	0.002256900
	Hull-White	0.000474132	0.000312154
ZCB4Y	Vasicek	0.012191304	0.005493881
	CIR	0.007529609	0.006295104
	Hull-White	0.000551634	0.000480179
ZCB5Y	Vasicek	0.020995814	0.010320526
	CIR	0.013797453	0.011869457
	Hull-White	0.001045534	0.000743168
ZCB6Y	Vasicek	0.032043987	0.016013214
	CIR	0.021534694	0.018741798
	Hull-White	0.001260885	0.000962595
ZCB7Y	Vasicek	0.044685087	0.022839046
	CIR	0.030960477	0.027233846
	Hull-White	0.001536147	0.001224416
ZCB8Y	Vasicek	0.060824978	0.031016332
	CIR	0.042626969	0.037639392
	Hull-White	0.003299271	0.002680550
ZCB9Y	Vasicek	0.075999663	0.037676502
	CIR	0.053257557	0.047036301
	Hull-White	0.002688173	0.001879333
ZCB10Y	Vasicek	0.092017012	0.044133732
	CIR	0.007673414	0.057202510
	Hull-White	0.002732361	0.002241598

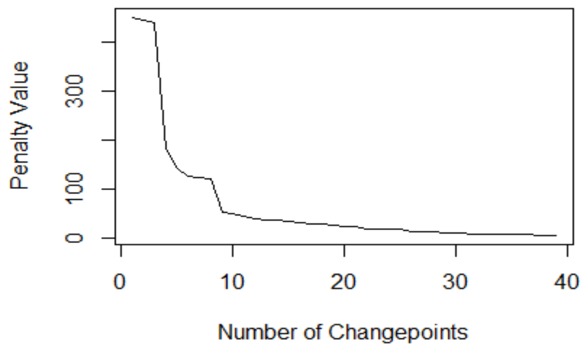

Figure 17 The number of change points used to determine the penalty value

From Figure 18, define the range of penalty values to be between 5 and 500, and select a penalty value of 24.900007, which is the range where the slope of the graph does not change. The purpose is simply to cover the points where there are sudden changes in interest rates and to test the effect of parameters on the modeling of interest rates for the model only.

```
 [1]   5.000000   5.728175   6.222025   7.873802   8.072259   8.219864   8.293070   9.603205  11.136052
[10]  12.044915  13.452201  16.443989  17.094430  17.307502  18.297980  21.735713  24.060487  24.900007
[19]  27.660082  28.242899  34.861040  36.334431  37.361621  39.412802  50.626276  52.036489 121.733800
[28] 124.156848 125.687150 141.391361 181.460447 440.705471 450.469739
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Figure 18 Penalty value in determining change points

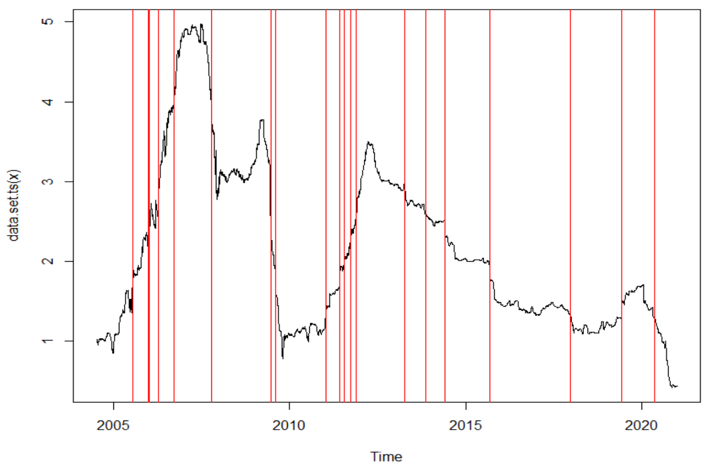


Figure 19 Optimal time for maximum log-likelihood of yield curve for treasury bill with 1-month maturity, using penalty value of 24.900007

After identifying the change points in Figure 19, we estimated the parameters for each model and simulated the corresponding interest rates. As an example, we present the results for the Hull-White model. In the Hull-White model, we consider only the dataset $r_1, r_2, ..., r_{251}$ with the following estimated parameters:

$$a = -1.211564, \; b = 0.005741036, \; \text{and} \; \hat{\sigma}_r = 0.003968953.$$

Next, we consider the results obtained by varying the parameters in the model in two cases: Case 1) where the value of a is positive and $\hat{\sigma}_r$ is constant, and Case 2) where the value of a is negative and $\hat{\sigma}_r$ is constant. The simulated yield curve results in the Hull-White model are then compared to the actual yield curve data in both cases.

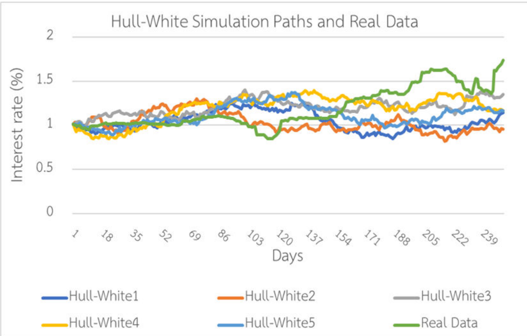


Figure 20 a is positive and $\hat{\sigma}_r$ is constant

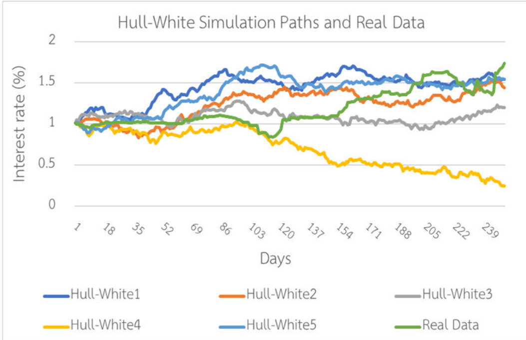


Figure 21 a is negative and $\hat{\sigma}_r$ is constant

From Figures 20 and 21, it can be observed that in the early stages or for a small number of days, the results obtained from the modeling are close to the actual market return. However, as the time horizon increases, the results diverge further away from the actual data.

6. Application

The use of the model for pricing debt securities can have multiple applications, including:

1) Investment: Investors can use the model to help make investment decisions in the debt securities market by using the model's results to evaluate risks and potential returns from different investment opportunities.

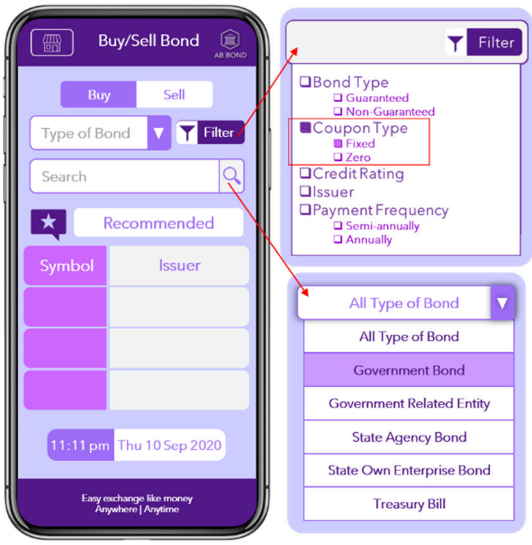
2) Market trend forecasting: Business operators or analysts can use the model to forecast market trends in the debt securities market, which can help with investment planning and risk management in the future.

3) Platform development for trading: Developers of trading platforms can use the model to determine the true market price of debt securities and build a standardized, systematized trading platform that enables users to trade conveniently and easily.

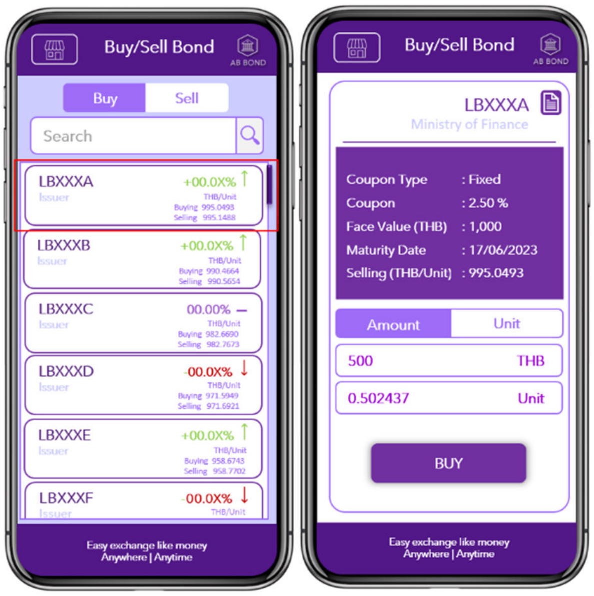
4) Research and analysis: Researchers can use the model to study and analyze the impact of parameter changes on the model's output or to test new investment strategies and concepts, thereby analyzing future debt securities market investments. Additionally, the model's results can also be used to support investment decision-making in banks and financial institutions.

Furthermore, the model can be used to study the impact of changes in various market conditions on debt security returns, such as changes in interest rates or economic conditions, which can help predict potential future returns in uncertain situations. All these applications can significantly improve investment planning and risk management efficiency for users of the model.

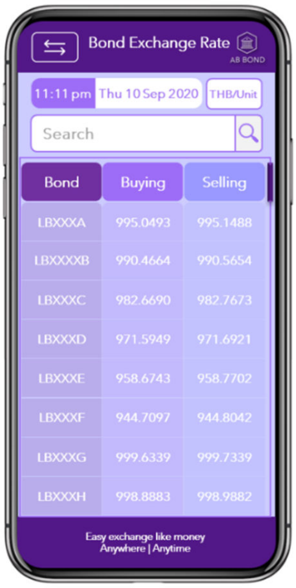
a) All Transactions Page



b) Buy/Sell Bond Function



c) Buy/Sell Bond function (continued)



d) Bond Exchange Rate function

Figure 22 an application platform or website that has functions for trading and exchanging between bonds and cash in real-time and continuously

This is an example of the benefits that can be further developed from this research include the creation of an application platform or website that has functions for trading and exchanging between bonds and cash in real-time and continuously. The bond prices, both bid and ask, are calculated from the bond pricing model developed in this research. The ask price will also include a 0.01% operation cost or handling fee of the bid price.

7. Summary of the Study

The mathematical models used to simulate interest rates, Vasicek model, CIR model, and Hull-White model, were studied to apply them in pricing bonds with no coupon payments. The goal was to determine the exchange rate of these bonds to cash quickly and reliably. A confidence interval of 95% was used to establish the bond prices for various maturities, including 1-month, 6-month, 1-year treasury bills, and coupon bonds ranging from 2 to 10 years. The interest rate for the 1-month treasury bill was used to set the benchmark rate for other bonds. The Hull-White model showed the best results, closely predicting the actual bond prices, with an RMSE value lower than the other models, for both short-term and long-term bonds. The RMSE value was used to measure the direct prediction error of the bond prices.

Regarding model verification, the estimated parameter values affected the simulated interest rates and, consequently, the bond prices. The researchers separated the parameter values into different ranges and found that considering the parameter values in these ranges resulted in more accurate bond price predictions. Additionally, the Hull-White model remained the best model when considering all the possible parameter values. However, it is crucial to examine the parameter values' characteristics to obtain better predictions.

8. Recommendations

1) The interest rate modeling should consider other economic factors in order to improve its accuracy. For example, interest rate policies that affect interest rates should be considered, and other models should be compared for additional insights.

2) A risk modeling should be created to measure the value of the risk that arises from the instant exchange of bonds into cash.

3) A variety of methods should be used to validate the model to assess the accuracy and causal relationship of the results.

Acknowledgements

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