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Stochastic Properties of Topp-Leone Generated Family of Distributions

Amit Kumar Misra, Ruby Chanchal* and Vaishali Gupta
Department of Statistics, School of Physical and Decision Sciences,
Babasaheb Bhimrao Ambedkar University, Lucknow, India.
*Corresponding author; e-mail: rubychanchal21@gmail.com

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Abstract

Topp-Leone generated family of distributions contains continuous distributions having bathtub shaped hazard rates. In this paper, we compare two random variables from this family of distributions using stochastic orderings. We also consider a special case of this family of distributions, namely, Topp-Leone exponential distribution and investigate few reliability indicators of this distribution such as hazard rate function, reversed hazard rate function, mean residual life function, and expected inactivity time. Renyi entropy measure for the Topp-Leone exponential distribution has also been discussed. Moreover, we define the Topp-Leone generated log-logistic distribution and the Topp-Leone generated Lomax distribution using the genesis of the Topp-Leone generated family of distributions. We also present real data applications to discuss the importance of this family of distributions.

Keywords: Hazard rate function, stochastic orderings, reversed hazard rate function, expected inactivity time, mean residual life function.

1. Introduction

The Topp-Leone (TL) distribution was first announced by Topp and Leone (1955) which is a simple bounded J-shaped distribution. Its probability density function (p.d.f.) and cumulative distribution function (c.d.f.) are given by

$$f_{\text{TL}}(u) = 2\vartheta(1-u)\left[u(2-u)\right]^{\vartheta-1}, \quad 0 \le u \le 1, 0 < \vartheta < 1,$$

and

$$F_{\text{TL}}(u) = \left[u(2-u)\right]^{\vartheta}, \quad 0 \le u \le 1, 0 < \vartheta < 1,$$

respectively. We can see that it is not very versatile due to having only one parameter and its domain is limited to $(0,1)$. Also, this distribution had not been given due consideration until Nadarajah and Kotz (2003) studied different aspects of this distribution. Let the continuous random variable U follows the TL distribution with two parameters ϑ and κ. Then, according to Nadarajah and Kotz (2003), the p.d.f. of U is written as

$$f(u)=\frac{2\vartheta}{\kappa}\left(\frac{u}{\kappa}\right)^{\vartheta-1}\left(1-\frac{u}{\kappa}\right)\left(2-\frac{u}{\kappa}\right)^{\vartheta-1}, 0<u<\kappa, 0<\vartheta<1, \kappa>0.$$

Clearly, the c.d.f. of U is

$$F(u)=\left(\frac{u}{\kappa}\right)^{\vartheta}\left(2-\frac{u}{\kappa}\right)^{\vartheta}, 0\le u<\kappa, 0<\vartheta<1, \kappa>0.$$

They identified that the hazard rate function of the TL distribution has a bathtub shape for all $\vartheta\in(0,1)$. Several authors then further studied the TL distribution. For example, Ghitany et al. (2005) investigated few reliability indicators of this distribution and their stochastic orderings. Dorp and Kotz (2006) represented income distributions with the help of the TL distribution. Zhou et al. (2006) examined the case of independent TL random variables X and Y and derived the exact distributions of $X+Y, XY$ and $X/\left(X+Y\right)$. A summary on kurtosis of the TL distribution was discussed by Kotz and Seier (2007). Also, Al-Zahrani (2012) and Genç (2012) discussed the goodness of fit tests and moments of order statistics for the TL distribution, respectively.

To make the TL distribution more applicable, some generalizations of it have also been considered in the literature. For example, Vicaria et al. (2008) presented two-sided generalized TL distribution. Authors talked about some properties of this family of distributions and defined a procedure for estimating the parameters using maximum likelihood estimation. Pourdarvish et al. (2015) proposed exponentiated TL distribution and studied its hazard rate function, moments, and order statistics. Recently, Al-Shomrani et al. (2016) introduced a generalization of the TL distribution using $Z(u)$ as the base-line c.d.f. in the TL distribution and derived the hazard rate function and its moments. Rezaei et al. (2017) also proposed a generalization of the TL distribution. They acknowledged $\left[Z(u)\right]^{\theta}$ as the base-line distribution and named it Topp-Leone generated (TL-G) family of distributions. The distributions of this family also have the hazard rate functions with bathtub shape for all $\vartheta\in(0,1)$, and maybe utilized for modeling lifetime events. Assume that the p.d.f. of the TL-G family of distributions having parameters ϑ and θ is

$$f(u;\vartheta,\theta,\varepsilon)=2\vartheta\theta z(u;\varepsilon)Z(u;\varepsilon)^{\theta\vartheta-1}\left(1-Z(u;\varepsilon)^{\theta}\right)\left(2-Z(u;\varepsilon)^{\theta}\right)^{\vartheta-1},\ u\in\mathbb{R},\theta,\vartheta>0 \tag{1}$$

and the corresponding c.d.f. is

$$F(u;\vartheta,\theta,\varepsilon)=\left(Z(u;\varepsilon)^{\theta}\left(2-Z(u;\varepsilon)^{\theta}\right)\right)^{\vartheta}, u\in\mathbb{R},\theta,\vartheta>0, \tag{2}$$

where $Z(u;\varepsilon)$ and $z(u;\varepsilon)$ represent the c.d.f. and the p.d.f. of the base-line distribution, respectively, and ε contains the parameters which specify the base-line distribution. For ease of notation, we write $U\sim$ TL-G$(\vartheta,\theta,\varepsilon)$ for a random variable U having p.d.f. written as (1). If we take the base-line distribution as $U(0,b)$ along with $\theta=1$, then the TL-G family of distributions reduces to the TL distribution. Recently, some authors introduced and studied new distributions by choosing different $Z(\cdot;\varepsilon)$, see, for example, Aryal et al. (2017), Brito et al. (2017), Sharma (2018), Korkmaz et al. (2019), and Shekhawat and Sharma (2020).

Motivated from the TL-G family of distributions suggested through Rezaei et al. (2017), our aim is to compare two random variables from this family of distributions in terms of stochastic orders. We also consider one of its special cases, namely, the TL-exponential distribution and study some well-known reliability indicators such as hazard rate function, mean residual life function, reversed hazard rate function, and expected inactivity time for this distribution. For the definitions of the reliability measures, we refer to Belzunce et al. (2016) to the reader. Moreover, we define two another special cases of this family of distributions, namely, the TL generated log-logistic distribution and the TL generated Lomax distribution.

Now, we first recall stochastic orderings, which are partial orderings between two random variables to judge their comparative behavior. Over the years several different stochastic orders and their properties have evolved considerably. For definitions and applications of various stochastic orders, one can cite Müller and Stoyan (2002), Shaked and Shanthikumar (2007), and Belzunce et al. (2016). In this paper, we mainly use the likelihood ratio order, the dispersive order, and the star-shaped order.

Consider two random variables named U and V having distributions along with $\mathbb{R}_+ = [0,\infty)$. Let $f_U(\cdot)$ and $f_V(\cdot)$ be the corresponding probability density functions, and $F_U(\cdot)$ and $F_V(\cdot)$ be the cumulative distribution functions of U and V, respectively.

Definition 1 *U is considered to be smaller than V in the*

1. *likelihood ratio order (indicated as $U \le_{\text{lr}} V$) if $f_V(u)/f_U(u)$ is non-decreasing in $u \in \mathbb{R}_+$;*
2. *dispersive order (indicated as $U \le_{\text{disp}} V$) if $F_U^{-1}(b) - F_U^{-1}(a) \le F_V^{-1}(b) - F_V^{-1}(a)$, for all $0 < a \le b < 1$, where, F^{-1} denotes the right continuous inverse of F;*
3. *star-shaped order (indicated as $U \le_* V$) if $F_U^{-1}(b)F_V^{-1}(a) \le F_U^{-1}(a)F_V^{-1}(b)$, for all $0 < a \le b < 1$, where F^{-1} denotes the right continuous inverse of F.*

The upcoming lemma is beneficial in obtaining the consequences of the next section.

Lemma 1. (Saunders and Moran, 1978) Let *$F_\nu, \nu \in \mathbb{R}$, denote a class of distribution functions such that F_ν is supported on some interval $(u_0, u_1) \subseteq (0,\infty)$ having a density function f_ν, which does not disappear on any sub-interval of (u_0, u_1). Then*

$$F_\nu \le_{\text{disp}} F_{\nu^*}; \quad \nu, \nu^* \in \mathbb{R}, \nu \le \nu^*, \tag{3}$$

if and only if $\dfrac{F_\nu'(u)}{f_\nu(u)}$ is non-increasing in u, where F_ν' is the derivative of F_ν with respect to ν.

And

$$F_\nu \le_* F_{\nu^*}; \; \nu, \nu^* \in \mathbb{R}, \nu \le \nu^*, \tag{4}$$

if and only if $\dfrac{F_\nu'(u)}{uf_\nu(u)}$ is non-increasing in u, where F_ν' is the derivative of F_ν with respect to ν.

Note that the inequalities in (3) and (4) reverse as the quantities $\dfrac{F_\nu'(u)}{f_\nu(u)}$ and $\dfrac{F_\nu'(u)}{uf_\nu(u)}$, respectively, non-decrease in u.

This paper is presented in the following scenario. In Section 2, we show that the reversed hazard rate function of the TL-G family of distributions is non-increasing (and hence, the expected inactivity time function is non-decreasing) if the reversed hazard rate function of the base-line distribution is non-increasing. Also, we make stochastic comparisons between two random variables from the TL-G family of distributions in terms of the dispersive order and the star-shaped order, and with the help of an example, we show that the likelihood ratio order may not be taken in some situations. In Section 3, we discuss a particular case of this family of distributions called the TL-exponential distribution, and examine few reliability indicators of this distribution, and derive some results. Also, we define the TL generated log-logistic distribution and the TL generated Lomax distribution using the genesis of the TL-G family of distributions. Section 4 provides real data applications to compare the fits of two models of the TL-G family of distributions and also, compare the fits of the TL-exponential distribution with the Lomax distribution and the Burr-XII distribution using two real data sets.

2. Main Results

In this section, we first discuss the condition under which the reversed hazard rate function (the expected inactivity time function) of the TL-G$(\vartheta,\theta,\varepsilon)$ distribution is non-increasing (non-decreasing). Then, we provide some comparison results based on the dispersive and the star-shaped orders.

2.1. The reversed hazard rate function

Let $\tilde{r}(u;\varepsilon), u>0,$ denotes the reversed hazard rate function of the base-line distribution of the TL-G$(\vartheta,\theta,\varepsilon)$ family of distributions, i.e.,

$$\tilde{r}(u;\varepsilon)=\frac{z(u;\varepsilon)}{Z(u;\varepsilon)},\quad u>0.$$

The theorem below shows that if $\tilde{r}(u;\varepsilon)$ is non-increasing in $u\in(0,\infty)$, therefore the reversed hazard rate function of the TL-G$(\vartheta,\theta,\varepsilon)$ distribution is non-increasing in $u\in(0,\infty)$.

Theorem 2 *Let $U\sim$ TL-G$(\vartheta,\theta,\varepsilon)$ and let $\tilde{r}(u;\varepsilon)$ be the reversed hazard rate function of the base-line distribution. Then, for fixed $\theta,\vartheta>0$ and for each fixed ε, the reversed hazard rate function of U is non-increasing in $u\in(0,\infty)$ if $\tilde{r}(u;\varepsilon)$ is non-increasing in $u\in(0,\infty)$.*

Proof Let $U\sim$ TL-G$(\vartheta,\theta,\varepsilon)$ with p.d.f and c.d.f given by Equations (1) and (2), respectively. Further, let $\tilde{r}_U(u;\vartheta,\theta,\varepsilon)$ denotes the reversed hazard rate function of U. Then,

$$\tilde{r}_U(u;\vartheta,\theta,\varepsilon)=\frac{f_U(u;\vartheta,\theta,\varepsilon)}{F_U(u;\vartheta,\theta,\varepsilon)}=\frac{\mathrm{d}}{\mathrm{d}u}\log\left(F(u;\vartheta,\theta,\varepsilon)\right).$$

Using Equation (2), we obtain that

$$\begin{aligned}\tilde{r}(u;\vartheta,\theta,\varepsilon)&=\frac{\mathrm{d}}{\mathrm{d}u}\left(\vartheta\theta\log\left(Z(u;\varepsilon)\right)+\vartheta\log\left(2-Z(u;\varepsilon)^{\theta}\right)\right)\\&=\vartheta\theta\frac{z(u;\varepsilon)}{Z(u;\varepsilon)}-\frac{\vartheta\theta z(u;\varepsilon)Z(u;\varepsilon)^{\theta-1}}{2-Z(u;\varepsilon)^{\theta}}\\&=\frac{2\vartheta\theta z(u;\varepsilon)\left(1-Z(u;\varepsilon)^{\theta}\right)}{Z(u;\varepsilon)\left(2-Z(u;\varepsilon)^{\theta}\right)}=2\vartheta\theta\tilde{r}(u;\varepsilon)\left(1-\frac{1}{2-Z(u;\varepsilon)^{\theta}}\right),\ u>0.\end{aligned}$$

Now, it is straightforward to look that $\tilde{r}_U(u;\vartheta,\theta,\varepsilon)$ is non-increasing in $u\in(0,\infty)$ whenever $\tilde{r}(u;\varepsilon)$ is non-increasing in $u\in(0,\infty)$.

Remark 1 This is popularly recognized that the non-increasing reversed hazard rate implies non-decreasing expected inactivity time (see Chandra and Roy 2001). Then, using Theorem 2, it follows that the expected inactivity time of the TL-G$(\vartheta,\theta,\varepsilon)$ random variable is non-decreasing in $u\in(0,\infty)$ if $\tilde{r}(u;\varepsilon)$ is non-increasing in $u\in(0,\infty)$.

2.2. Stochastic comparisons of the TL-G family of distributions

In this subsection, we compare two random variables from the TL-G family of distributions in terms of the dispersive order. Moreover, the star-shaped order. The following theorem provides the stochastic comparisons of the TL-G family of distributions when the parameter θ varies.

Theorem 3 *Let* $U\sim$ TL-G$(\vartheta,\theta_1,\varepsilon)$ *and* $V\sim$ TL-$G(\vartheta,\theta_2,\varepsilon)$ *be two random variables having base-line distribution* $Z(u;\varepsilon)$ *with support* $(0,\infty)$. *Then, for fixed* $\vartheta>0$ *and for each fixed* ε, $V\le_{disp}U$ $(V\le_* U)$ *whenever* $\theta_1\le\theta_2$ *and* $\dfrac{\log(Z(u;\varepsilon))}{\tilde{r}(u;\varepsilon)}\left(\dfrac{\log(Z(u;\varepsilon))}{u\tilde{r}(u;\varepsilon)}\right)$ *is non-decreasing in* $u\in(0,\infty)$, *where* $\tilde{r}(u;\varepsilon)$ *is the reversed hazard rate function of base-line distribution.*

Proof For a fixed $\vartheta>0$ and for any fixed ε, let the p.d.f. of U is written as

$$f_{\theta_1}(u)=2\vartheta\theta_1 z(u;\varepsilon)Z(u;\varepsilon)^{\theta_1\vartheta-1}\left(1-Z(u;\varepsilon)^{\theta_1}\right)^{\vartheta-1}\left(2-Z(u;\varepsilon)^{\theta_1}\right)^{\vartheta-1},\ u>0,\theta_1>0.$$

Clearly, the c.d.f. of U is $F_{\theta_1}(u)=\left(Z(u;\varepsilon)^{\theta_1}\left(2-Z(u;\varepsilon)^{\theta_1}\right)\right)^{\vartheta}$, $u>0,\theta_1>0$.

Then,

$$\begin{aligned}
F'_{\theta_1}(u)&\equiv\frac{\partial}{\partial\theta_1}F_{\theta_1}(u)\\
&=-\vartheta Z(u;\varepsilon)^{\theta_1\vartheta+\theta_1}\left(2-Z(u;\varepsilon)^{\theta_1}\right)^{\vartheta-1}\log\left(Z(u;\varepsilon)\right)+\vartheta\log\left(Z(u;\varepsilon)\right)Z(u;\varepsilon)^{\theta_1\vartheta}\left(2-Z(u;\varepsilon)^{\theta_1}\right)^{\vartheta}\\
&=\vartheta Z(u;\varepsilon)^{\theta_1\vartheta}\left(2-Z(u;\varepsilon)^{\theta_1}\right)^{\vartheta}\log\left(Z(u;\varepsilon)\right)\left(1-\frac{Z(u;\varepsilon)^{\theta_1}}{2-Z(u;\varepsilon)^{\theta_1}}\right)\\
&=2\vartheta Z(u;\varepsilon)^{\theta_1\vartheta}\left(2-Z(u;\varepsilon)^{\theta_1}\right)^{\vartheta}\log\left(Z(u;\varepsilon)\right)\left(\frac{1-Z(u;\varepsilon)^{\theta_1}}{2-Z(u;\varepsilon)^{\theta_1}}\right).
\end{aligned}$$

Therefore, $\dfrac{F'_{\theta_1}(u)}{f_{\theta_1}(u)}=\dfrac{1}{\theta_1}\dfrac{\log(Z(u;\varepsilon))}{\tilde{r}(u;\varepsilon)}$, which is non-decreasing in $u\in(0,\infty)$ as $\dfrac{\log\left(Z(u;\varepsilon)\right)}{\tilde{r}(u;\varepsilon)}$ is non-decreasing in $u\in(0,\infty)$. Hence, on adopting Lemma 1, we achieved that $V\le_{\text{disp}}U$ whenever $\theta_1\le\theta_2$.

Further, we have

$$\frac{F'_{\theta_1}(u)}{uf_{\theta_1}(u)}=\frac{1}{u\theta_1}\frac{\log\left(Z(u;\varepsilon)\right)}{\tilde{r}(u;\varepsilon)},$$

which is also non-decreasing in $u\in(0,\infty)$ as $\dfrac{\log\left(Z(u;\varepsilon)\right)}{u\tilde{r}(u;\varepsilon)}$ is non-decreasing in $u\in(0,\infty)$. Hence, on adopting Lemma 1, we have that $V\le_* U$ whenever $\theta_1\le\theta_2$. The following example supports the existence of the assumptions made in Theorem 3.

Example 1 Let $U \sim \text{TL-G}(\vartheta, \theta_1, \varepsilon)$ and $V \sim \text{TL-G}(\vartheta, \theta_2, \varepsilon)$ be two random variables with base-line distribution $G(u;\tau,\rho) = 1 - e^{-\tau u^{\rho}}$, $u > 0, \tau > 0, \rho > 1$. Here $\varepsilon = (\tau, \rho)$. Then, $g(u;\tau,\rho) = \tau\rho u^{\rho-1} e^{-\tau u^{\rho}}$, $u > 0, \tau > 0, \rho > 1$, and $\tilde{r}(u;\tau,\rho) = \dfrac{\tau\rho u^{\rho-1}}{e^{\tau u^{\rho}} - 1}$, $u > 0, \tau > 0, \rho > 1$. Now, let

$$\omega_1(u) = \frac{\log(G(u;\tau,\rho))}{\tilde{r}(u;\tau,\rho)} = \frac{\left(e^{\tau u^{\rho}} - 1\right)\log\left(1 - e^{-\tau u^{\rho}}\right)}{\tau\rho u^{\rho-1}}, \quad u > 0,$$

and

$$\omega_2(u) = \frac{\log(G(u;\tau,\rho))}{u\tilde{r}(u;\tau,\rho)} = \frac{\left(e^{\tau u^{\rho}} - 1\right)\log\left(1 - e^{-\tau u^{\rho}}\right)}{\tau\rho u^{\rho}}, \quad u > 0.$$

Now, for $\tau = 2$ and $\rho = 2.5$, with the help of R-software, we plot $\omega_1(u)$ and $\omega_2(u)$ as given in Figures 1 and 2, respectively . Clearly, $\omega_1(u)$ and $\omega_2(u)$ are non-decreasing in $u \in (0, \infty)$ and hence, on adopting Theorem 3, we achieved that $V \le_{\text{disp}} U (V \le_{*} U)$.

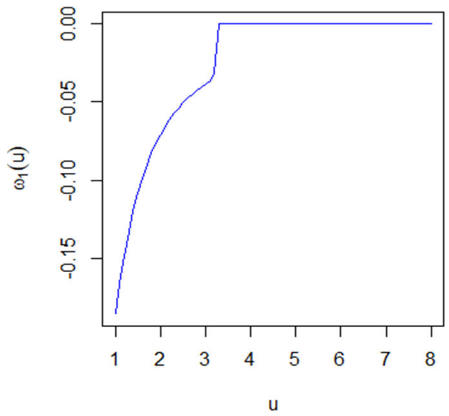

Figure 1 $\omega_1(u)$ shows $V \le_{\text{disp}} U$

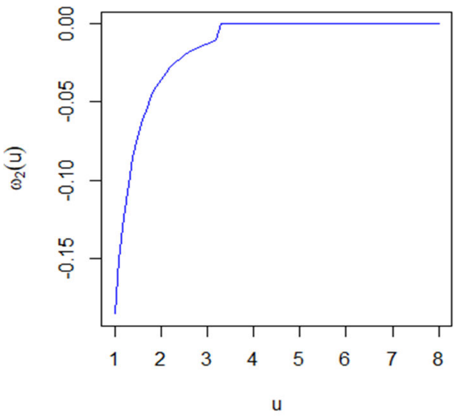

Figure 2 $\omega_2(u)$ shows $V \le_{*} U$

The following theorem directly follows from Theorem 5.1 of Sharma (2018).

Theorem 4 *Let* $U \sim \text{TL-G}(\vartheta_1, \theta_1, \varepsilon)$ *and* $V \sim \text{TL-G}(\vartheta_2, \theta_2, \varepsilon)$ *be two random variables. Then, for* $\theta_1 = \theta_2 \,(>0)$ *and for any fixed* ε, $U \le_{\text{lr}} V$ *whenever* $\vartheta_1 < \vartheta_2$.

One may be interested in comparing the random variables U and V in terms of the likelihood ratio order when $\theta_1 \ne \theta_2$. The upcoming counterexample demonstrates that the likelihood ratio order may not hold when $\theta_1 \ne \theta_2$.

Counterexample 1 Let $U \sim \text{TL-G}\left(\vartheta_1, \theta_1, (h_1, k_1)\right)$ and $V \sim \text{TL-G}\left(\vartheta_2, \theta_2, (h_2, k_2)\right)$ with base-line c.d.f. $Z(u;h,k) = 1 - e^{-(u/h)^k}$, $u > 0, h > 0, k > 0$, i.e., U and V follows Topp-Leone Generated Weibull (TLGW) distributions proposed by Aryal et al. (2017). Then, this can be easily get that

$$\frac{f_V(u)}{f_U(u)}=\frac{\vartheta_2\theta_2}{\vartheta_1\theta_1}\frac{k_2}{k_1}\left(\frac{h_1^{k_1}}{h_2^{k_2}}\right)u^{k_2-k_1}e^{-\left((u/h_2)^{k_2}-(u/h_1)^{k_1}\right)}\frac{\left(1-e^{-(u/h_2)^{k_2}}\right)^{\theta_2\vartheta_2-1}}{\left(1-e^{-(u/h_1)^{k_1}}\right)^{\theta_1\vartheta_1-1}}\frac{\left(1-\left(1-e^{-(u/h_2)^{k_2}}\right)^{\theta_2}\right)}{\left(1-\left(1-e^{-(u/h_1)^{k_1}}\right)^{\theta_1}\right)}\frac{\left(2-\left(1-e^{-(u/h_2)^{k_2}}\right)^{\theta_2}\right)^{\vartheta_2-1}}{\left(2-\left(1-e^{-(u/h_1)^{k_1}}\right)^{\theta_1}\right)^{\vartheta_1-1}},$$

$$u>0,\theta_i,\vartheta_i,h_i,k_i>0,\ i=1,2.$$

On taking $\vartheta_1=0.1,\ \vartheta_2=0.2,\ h_1=1,\ h_2=2,\ k_1=2,$ and $k_2=3$ in the above equation, we get

$$\frac{f_V(u)}{f_U(u)}=\frac{3}{8}\frac{\theta_2}{\theta_1}ue^{-\left((u/2)^3-u^2\right)}\frac{\left(1-e^{-(u/2)^3}\right)^{(\theta_2\times 0.2)-1}}{\left(1-e^{-u^2}\right)^{(\theta_1\times 0.1)-1}}\frac{\left(1-\left(1-e^{-(u/2)^3}\right)^{\theta_2}\right)}{\left(1-\left(1-e^{-u^2}\right)^{\theta_1}\right)}\frac{\left(2-\left(1-e^{-(u/2)^3}\right)^{\theta_2}\right)^{-0.8}}{\left(2-\left(1-e^{-u^2}\right)^{\theta_1}\right)^{-0.9}}=\varphi(u;\theta_1,\theta_2). \tag{5}$$

Now, we plot $\varphi(u;1,2)$ and $\varphi(u;2,1)$ (see Figures 3 and 4). Clearly, in both the cases, $\varphi(u)$ is not monotone, which means that neither $U \not\leq_{\text{lr}} V$ nor $V \not\leq_{\text{lr}} U$. This shows that the likelihood ratio order may no longer exists between U and V if $\theta_1 \neq \theta_2$.

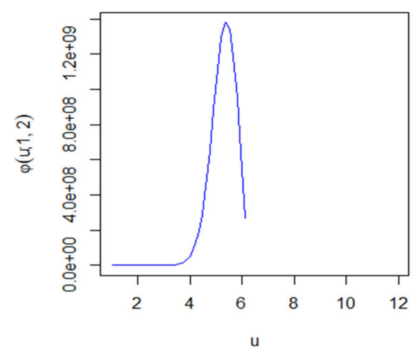

Figure 3 The ratios of the density functions when $\theta_1=1,\ \theta_2=2$

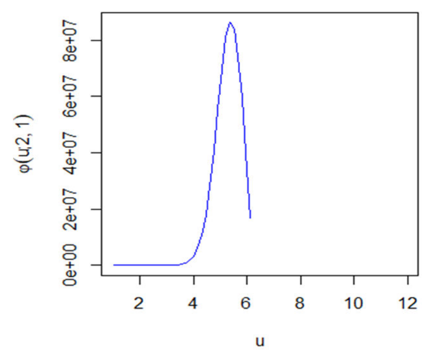

Figure 4 The ratios of the density functions when $\theta_1=2,\ \theta_2=1$

3. Some Special Cases of the TL-G Family of Distributions

This section of the article is devoted to some particular cases of the TL-G family of distributions. First, we consider the TL-exponential distribution by using the exponential distribution as the base-line distribution. In addition, we study some reliability indicators and the Renyi entropy measure for the TL-exponential distribution. Second, we define the Topp-Leone generated log-logistic distribution (TL-log logistic), and the Topp-Leone generated Lomax (TLGLo) distribution using the log-logistic distribution and the Lomax distribution as the base-line distributions, respectively. Also, we show some graphical representations of the hazard rate functions for both the distributions.

3.1. The TL-exponential distribution

In this subsection, we use the exponential distribution as the base-line distribution with $Z(u;\mu)=1-e^{-\mu u}$ and $\theta=1$, then the p.d.f. of TL-exponential distribution with parameters ϑ and μ is written as

$$f_U(u)=2\vartheta\mu e^{-2\mu u}\left(1-e^{-2\mu u}\right)^{\vartheta-1},\ u>0,\mu>0,\vartheta>0, \tag{6}$$

and the corresponding c.d.f. is

$$F_U(u)=\left(1-e^{-2\mu u}\right)^{\vartheta},\ \ u>0,\mu>0,\vartheta>0.$$

For a random variable U with p.d.f. written as (6), we write $U \sim \text{TL-Exp}(\vartheta, \mu)$. Figure 5 indicates the shapes of p.d.f. of TL-exponential distribution for $\vartheta > 1$ and for distinct values of μ.

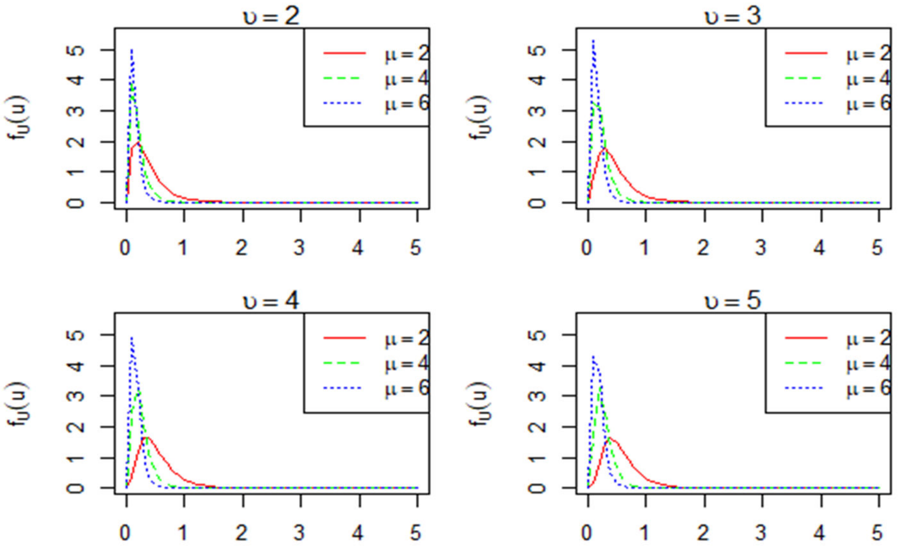

Figure 5 Shapes of the p.d.f. of TL-exponential distribution for $\vartheta > 1$ and for distinct values of μ

Few authors have studied the shapes of the hazard rate functions and the p.d.f. of TL-exponential distribution for distinct points of ϑ and μ (see, Al-Shomrani et al. (2016) and Sebastian et al. (2019)). Now, we discuss some reliability indicators of TL-exponential distribution.

3.1.1 Hazard rate and mean residual life functions of TL-exponential distribution

These two reliability indicators are essential vital features in survival analysis and reliability theory. Moreover, they defined earlier in terms of the time to failure of a device. But, the mean residual life function becomes more appropriate reliability measure than the hazard rate function because it compiles the whole remaining life, whereas the hazard rate function gives instantaneous failure at time $u > 0$. For the definitions and other details on the hazard rate and the mean residual life functions, we refer the readers to Ramos-Romero and Sordo-Díaz (2001), Belzunce et al. (2002), and Lai and Xie (2006). The hazard rate function for the TL-exponential distribution is written as

$$r_U(u) = \frac{f_U(u)}{1-F_U(u)} = \frac{2\vartheta}{\mu} \frac{e^{-2\mu u}\left(1-e^{-2\mu u}\right)^{\vartheta-1}}{1-\left(1-e^{-2\mu u}\right)^{\vartheta}}, \quad u > 0, \vartheta > 0, \mu > 0.$$

Al-Shomrani et al. (2016) and Sebastian et al. (2019) discussed the shape of the hazard rate function for various points of ϑ, and for $\mu = 3$ and $\mu = 0.3$, respectively. They graphically showed that the TL-exponential distribution has strictly non-decreasing hazard rate function for $\vartheta > 1$ and has strictly non-increasing hazard rate function for $\vartheta < 1$. Now, the following theorem provides the proof of the above-mentioned graphical observations for any $\mu > 0$, i.e.; we show that different form of the hazard rate function of the $\text{TL-Exp}(\vartheta, \mu)$ distribution is strictly non-decreasing for $\vartheta > 1$, strictly non-increasing for $\vartheta < 1$, and constant for $\vartheta = 1$ for any $\mu > 0$.

Theorem 5 *Let* $U \sim \text{TL-Exp}(\vartheta, \mu)$ *and let* $r_U(\cdot)$ *be the hazard rate function of* U. *Then,* $r_U(u)$ *is strictly non-decreasing (strictly non-increasing, constant) in* $u \in (0, \infty)$ *for* $\vartheta > 1$ $(\vartheta < 1, \vartheta = 1)$ *for any* $\mu > 0.$

Proof The p.d.f., $f_U(u)$, of the random variable U is written as (6). Define

$$\eta(u) = -\frac{\frac{\mathrm{d}}{\mathrm{d}u} f_U(u)}{f_U(u)}, \quad u > 0.$$

Obviously,
$$\eta(u) = \frac{2\mu\left(1 - \vartheta e^{-2\mu u}\right)}{1 - e^{-2\mu u}}, \; u > 0,$$

as well as
$$\frac{\mathrm{d}}{\mathrm{d}u}\eta(u) = \frac{4\mu^2\left(\vartheta - 1\right)e^{-2\mu u}}{\left(1 - e^{-2\mu u}\right)^2}, \; u > 0.$$

Clearly, for any fixed $\mu > 0$, we have $\frac{\mathrm{d}}{\mathrm{d}u}\eta(u) > 0(< 0, = 0)$, for every $u > 0$, if $\vartheta > 1(\vartheta < 1, \vartheta = 1)$, which means that $\eta(u)$ is strictly non-decreasing (strictly non-increasing, constant) in $u \in (0, \infty)$ for $\vartheta > 1(\vartheta < 1, \vartheta = 1)$. Now, on adopting Theorem 2.1 of Lai and Xie (2006), we achieved that the hazard rate function, $r_U(u)$, is strictly non-decreasing (strictly non-increasing, constant) in $u \in (0, \infty)$ for $\vartheta > 1(\vartheta < 1, \vartheta = 1)$ for any $\mu > 0$.

Now, the mean residual life function for the TL-exponential distribution is written as

$$m_U(u) = \frac{\int_u^\infty \left(1 - F_U(t)\right)\mathrm{d}t}{1 - F_U(u)} = \frac{\int_u^\infty \left(1 - (1 - e^{-2\mu t})^\vartheta\right)\mathrm{d}t}{1 - \left(1 - e^{-2\mu u}\right)^\vartheta}, \quad u > 0, \vartheta > 0, \mu > 0.$$

On using binomial expansion (any index), we obtain the expression for the mean residual life function of TL-exponential distribution as follows.

$$\begin{aligned} m_U(u) &= \frac{1}{1 - \left(1 - e^{-2\mu u}\right)^\vartheta} \int_u^\infty \sum_{m=1}^\infty \binom{\vartheta}{m} (-1)^{m-1} \left(e^{-2\mu t}\right)^m \mathrm{d}t, \quad \text{where } \binom{\vartheta}{m} = \frac{\vartheta(\vartheta - 1)\cdots(\vartheta - m + 1)}{m!} \\ &= \frac{1}{1 - \left(1 - e^{-2\mu u}\right)^\vartheta} \sum_{m=1}^\infty \binom{\vartheta}{m} (-1)^{m-1} \int_u^\infty e^{-2\mu t m}\mathrm{d}t, \quad \text{by Fubini's theorem} \\ &= \frac{1}{1 - \left(1 - e^{-2\mu u}\right)^\vartheta} \sum_{m=1}^\infty \binom{\vartheta}{m} (-1)^{m-1} \frac{1}{2\mu m} e^{-2\mu u m}, \quad u > 0. \end{aligned}$$

3.2.2. Reversed hazard rate function and expected inactivity time of TL- exponential distribution

Despite being similar to the above functions, these two functions are frequently used in life testing problems. They play a significant role in censored data research. Also, they are relevant in fields like Forensic Sciences (see, for example, Kalbeisch and Lawless (1989), Chandra and Roy (2001), and Kayid and Ahmad (2004), and references cited therein).

For the $\text{TL-Exp}(\vartheta, \mu)$ distribution, the reversed hazard rate function and the expected inactivity time are written as

$$\tilde{r}_U(u)=\frac{f_U(u)}{F_U(u)}=\frac{2\vartheta}{\mu}\frac{1}{\left(e^{2\mu u}-1\right)},\quad u>0,\vartheta>0,\mu>0,$$

and

$$\tilde{m}_U(u)=\frac{\int_0^u F_U(t)\mathrm{d}t}{F_U(u)}=\frac{1}{\left(1-e^{-2\mu u}\right)^{\vartheta}}\int_0^u\left(1-e^{-2\mu t}\right)^{\vartheta}\mathrm{d}t,\quad u>0,\vartheta>0,\mu>0,$$

respectively. The expected inactivity time measures the actual time of a device that has already failed in the interval $[0,u]$. On adopting binomial expansion (any index), we obtain that

$$\begin{aligned}\tilde{m}_U(u)&=\frac{1}{\left(1-e^{-2\mu u}\right)^{\vartheta}}\int_0^u\sum_{m=0}^{\infty}\binom{\vartheta}{m}(-1)^m\left(e^{-2\mu t}\right)^m\mathrm{d}t,\quad \text{where}\binom{\vartheta}{0}=1 \text{ and } \binom{\vartheta}{m}=\frac{\vartheta(\vartheta-1)\cdots(\vartheta-m+1)}{m!},m\geq 1\\ &=\frac{1}{\left(1-e^{-2\mu u}\right)^{\vartheta}}\sum_{m=0}^{\infty}\binom{\vartheta}{m}(-1)^m\int_0^u e^{-2\mu tm}\mathrm{d}t,\quad \text{by Fubini's theorem}\\ &=\frac{1}{\left(1-e^{-2\mu u}\right)^{\vartheta}}\sum_{m=0}^{\infty}\binom{\vartheta}{m}(-1)^m\left(1-\frac{1}{2\mu m}e^{-2\mu u m}\right),\quad u>0.\end{aligned}$$

Now, we state the following corollary which comes directly under the observation that $\tilde{r}_U(u)$ is non-increasing in $u\in(0,\infty)$ and Remark 1.

Corollary 1 *If* $U\sim$ TL-Exp(ϑ,μ), *then* U *has non-increasing reversed hazard rate function and non-decreasing expected inactivity time.*

3.1.3. Renyi entropy

The degree of unpredictability of a random variable U is defined in terms of entropy, and the one which is well-suited in this case is the Renyi entropy (Renyi (1961)). For a continuous random variable U having p.d.f. $f_U(\cdot)$, the Renyi entropy is written as

$$E_R(\delta)=\frac{1}{1-\delta}\log\left(\int f_U^{\delta}(u)\mathrm{d}u\right),$$

where $\delta>0$ and $\delta\neq 1$. For the TL-Exp(ϑ,μ), the expression for the Renyi entropy is written as

$$E_R(\delta)=\frac{1}{1-\delta}\log\left(\int_0^{\infty}(2\vartheta\mu)^{\delta}e^{-2\mu\delta u}\left(1-e^{-2\mu u}\right)^{\delta(\vartheta-1)}\mathrm{d}u\right).$$

Now, on using binomial expansion (any index), we obtain that

$$E_R(\delta)=\frac{1}{1-\delta}\log\left(\int_0^{\infty}(2\vartheta\mu)^{\delta}e^{-2\mu\delta u}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^m e^{-2\mu m u}\mathrm{d}u\right),$$

$$\text{where}\binom{\delta(\vartheta-1)}{m}=\frac{\delta(\vartheta-1)(\delta(\vartheta-1)-1)\cdots(\delta(\vartheta-1)-m+1)}{m!},\ m\geq 1$$

$$\begin{aligned}&=\frac{1}{1-\delta}\log\left(\int_0^{\infty}(2\vartheta\mu)^{\delta}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^m e^{-2\mu u(\delta+m)}\mathrm{d}u\right)\\ &=\frac{1}{1-\delta}\log\left((2\vartheta\mu)^{\delta}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^m\int_0^{\infty}e^{-2\mu u(\delta+m)}\mathrm{d}u\right),\quad \text{by Fubini's theorem}\end{aligned}$$

$$=\frac{1}{1-\delta}\log\left((2\vartheta\mu)^{\delta}\sum_{m=0}^{\infty}\binom{\delta(\vartheta-1)}{m}(-1)^{m}\frac{1}{2\mu(\delta+m)}\right).$$

3.2. The TL-log logistic distribution and the TLGLo distribution

In this subsection, we take the log-logistic and the Lomax distributions (see, Lomax (1954)) as the base-line distributions with $Z(u;\beta,\eta)=\frac{(\eta u)^{\beta}}{1+(\eta u)^{\beta}}$ and $Z^{*}(u;\beta,\eta)=1-(1+\eta u)^{-\beta}$, respectively. Then, the p.d.f. of the TL-log logistic distribution and the TLGLo distribution are given by

$$f_U(u)=2\vartheta\theta\beta\eta^{\beta\theta\vartheta}\frac{u^{\beta\theta\vartheta-1}}{\left[1+(\eta u)^{\beta}\right]^{1+\theta\vartheta}}\left[1-\left(\frac{(\eta u)^{\beta}}{1+(\eta u)^{\beta}}\right)^{\theta}\right]\left[2-\left(\frac{(\eta u)^{\beta}}{1+(\eta u)^{\beta}}\right)^{\theta}\right]^{\vartheta-1},$$
$$u>0,\vartheta>0,\theta>0,\beta>0,\eta>0, \tag{7}$$

and

$$f_U^{*}(u)=2\vartheta\theta\beta\eta\left(1+\eta u\right)^{-(\beta+1)}\left[1-(1+\eta u)^{-\beta}\right]^{\theta\vartheta-1}\left[1-\left(1-(1+\eta u)^{-\beta}\right)^{\theta}\right]\left[2-\left(1-(1+\eta u)^{-\beta}\right)^{\theta}\right]^{\vartheta-1},$$
$$u>0,\vartheta>0,\theta>0,\beta>0,\eta>0, \tag{8}$$

respectively, and their corresponding c.d.f. are

$$F_U(u)=\left[\frac{(\eta u)^{\beta}}{1+(\eta u)^{\beta}}\right]^{\theta\vartheta}\left[2-\left(\frac{(\eta u)^{\beta}}{1+(\eta u)^{\beta}}\right)^{\theta}\right]^{\vartheta},\ u>0,\vartheta>0,\theta>0,\beta>0,\eta>0,$$

and

$$F_U^{*}(u)=\left[1-(1+\eta u)^{-\beta}\right]^{\theta\vartheta}\left[2-\left(1-(1+\eta u)^{-\beta}\right)^{\theta}\right]^{\vartheta},\ u>0,\vartheta>0,\theta>0,\beta>0,\eta>0.$$

Figures 6 and 7 represent the possible shapes of the p.d.f. of the TL-log logistic distribution and the TLGLo distribution, respectively. Figures 8 and 9 illustrate the possible shapes of the hazard rate functions of the TL-log logistic distribution and the TLGLo distribution, respectively.

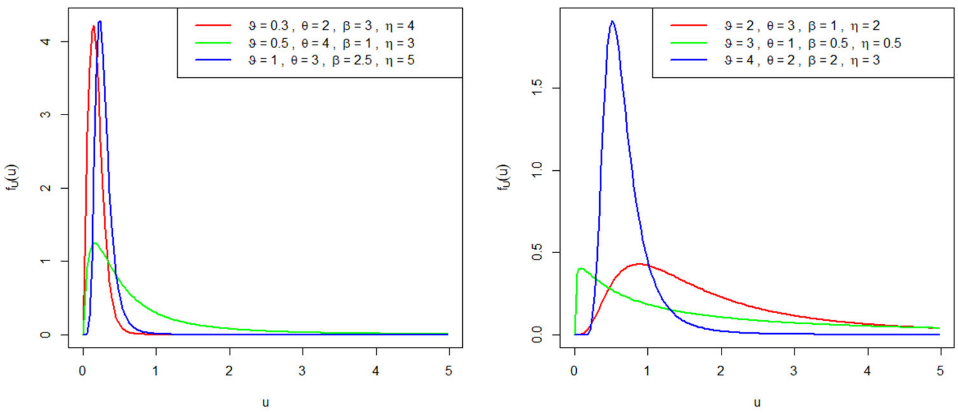

Figure 6 Shapes of the p.d.f. of the TL-log logistic distribution

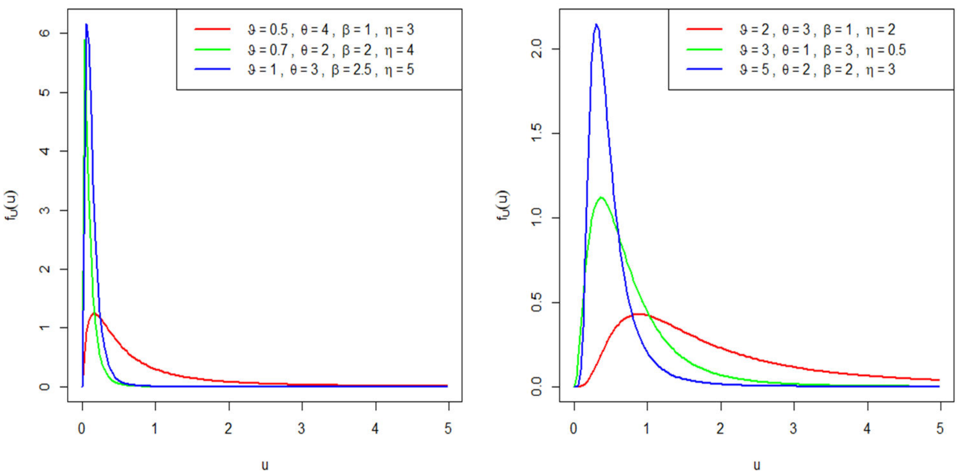

Figure 7 Shapes of the p.d.f. of the TLGLo distribution

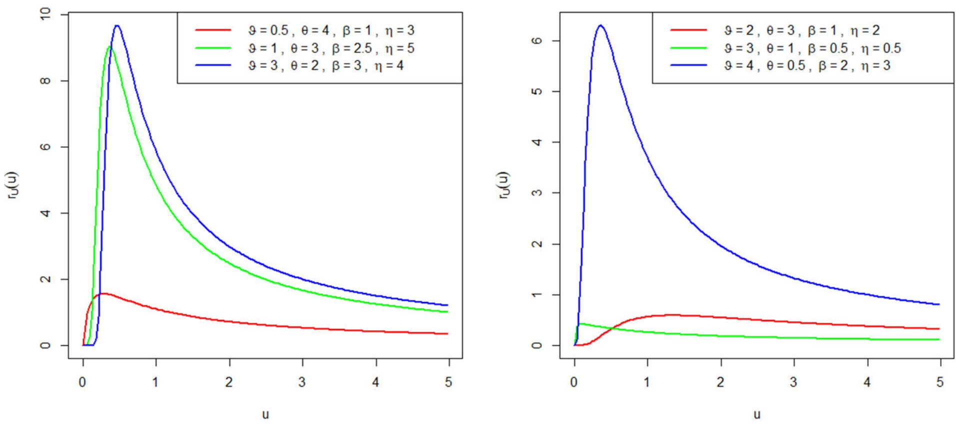

Figure 8 Shapes of the hazard rate function of the TL-log logistic distribution

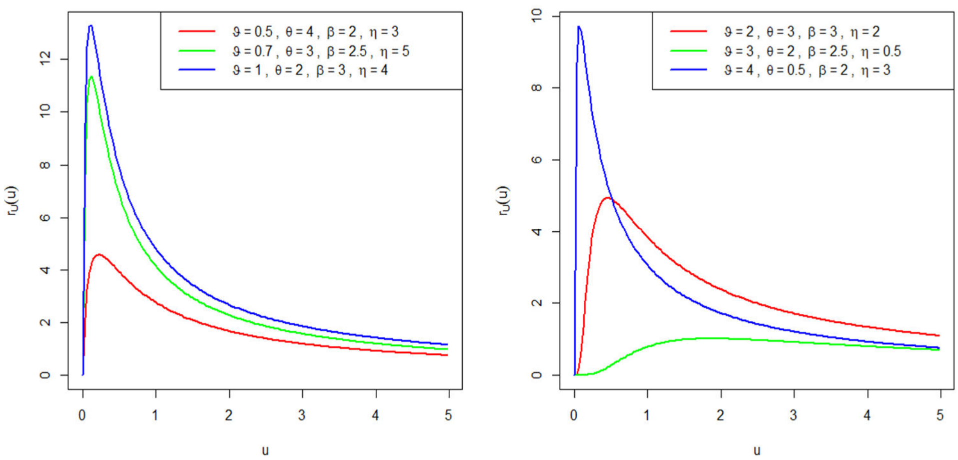

Figure 9. Shapes of the hazard rate function of the TLGLo distribution

4. Real Data Applications

In this section, first we analyze a real data set to compare the fits of two models of the TL-G family of distributions, named TLGW distribution and TL-log logistic distribution (defined in Subsection 3.2), and then we provide the applications of the TL-exponential distribution (defined in Subsection 3.1) to two real data sets, and compare the fits of this distribution with the Lomax distribution and the Burr-XII distribution. The criteria used for choosing the best fitted distribution are: Akaike information criterion (AIC), Akaike information criterion corrected (AICC), Bayesian information criterion (BIC), Kolmogorov-Smirnov (KS) statistic, and its p-value. All the calculations were conducted using the R software.

4.1. Comparative study on two models of the TL-G family of distributions

In this subsection, we consider two models of the TL-G family of distributions, named TLGW distribution and TL-log logistic distribution (defined in Subsection 3.2), respectively, and compare the performances of both models with the help of a real data set. The density function of the four parameter TLGW is given by

$$f_U(u) = 2\vartheta\theta\beta\eta^{\beta}u^{\beta-1}e^{-(\eta u)^{\beta}}\left(1-e^{-(\eta u)^{\beta}}\right)^{\theta\vartheta-1}\left[1-\left(1-e^{-(\eta u)^{\beta}}\right)^{\theta}\right]\left[2-\left(1-e^{-(\eta u)^{\beta}}\right)^{\theta}\right]^{\vartheta-1},$$

$$u>0, \vartheta>0, \theta>0, \beta>0, \eta>0.$$

respectively. We consider a data set includes 63 observations of the strengths of 1.5 cm of glass fibers collected by UK National Physical Laboratory and also applied by Smith and Naylor (1987). The data is given as:

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49,1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61,1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

First, we provide some descriptive statistics of the data set in the following Table 1.

Table 1 Descriptive statistics for the strength data of glass fibers

Descriptive statistics	Data set
Minimum	0.550
Median	1.590
Mean	1.507
Maximum	2.240
Standard deviation	0.3241257

Further, we plot the probability-probability plots (PP-plots) in Figure 10 for the TLGW and TL-log logistic models which provides a suitable fits to the data.

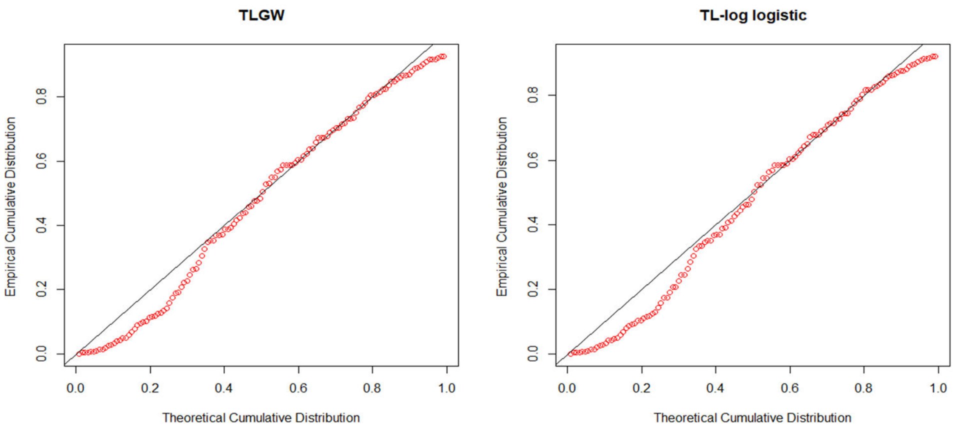

Figure 10 The PP-plots of the strength data of glass fibers fitted with the TLGW and TL-log logistic models.

Now, we compute the estimates for the unknown parameters of each model using the method of maximum likelihood (ML) estimation and then we compare the results through statistics: AIC, AICC, BIC, KS statistic, and its p-value, and $-\log l$ indicates the log-likelihood function evaluated at the ML estimates for the both data sets (see, Table 2). The best model fits the smallest AIC, AICC, BIC, KS statistic, and the largest p-value.

Table 2 ML estimates and log-likelihood functions for the TLGW and TL-log logistic models, and the statistics AIC, AICC, BIC, and KS statistics with its p-values for the strength data of glass fibers

Models	ML estimates	-log l	AIC	AICC	BIC	KS statistic	p-value
TL-log logistic	$\hat{\vartheta}=0.1964573$ $\hat{\theta}=1.6199747$ $\hat{\beta}=14.5917323$ $\hat{\eta}=0.5459271$	12.6589	33.3178	34.00746	41.89034	0.12698	0.69
TLGW	$\hat{\vartheta}=0.4127334$ $\hat{\theta}=1.8399951$ $\hat{\beta}=6.4448543$ $\hat{\eta}=0.5491863$	14.56609	37.13218	37.82184	45.70472	0.14286	0.5412

Since, the results from Table 2 show that the TL-log logistic model has the smallest values of AIC, AICC, BIC, KS statistic, and the largest p-value, so, this could be considered as the best model as compared to the TLGW model.

4.2. Comparative study on the TL-exponential with the Lomax distribution and the Burr-XII distribution

In this subsection, we present the applications of the TL-exponential distribution to two real data sets and compare the fits of this distribution with the two parameter Lomax distribution (see, Lomax (1954)) and Burr-XII distribution (see, Burr (1942)). The density functions of the two parameter Lomax and Burr-XII distributions are given by

$$g(u)=\vartheta\mu(1+\mu u)^{-(\vartheta+1)},\ u>0,\vartheta>0,\mu>0,$$

and

$$h(u)=\vartheta\mu u^{\mu-1}(1+u^{\mu})^{-(\vartheta+1)},\ u>0,\vartheta>0,\mu>0,$$

respectively.

Consider, the first data set consists of 20 observations which describes the relaxation times (in minutes) of 20 patients who receiving an analgesic as recorded by Gross and Clark (1975). The data is given as:

1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3.0, 1.7, 2.3, 1.6, 2.0.

The second data set consists of 31 observations which is the strength data of glass of the aircraft window recorded by Fuller et al. (1994). The data is given as:

18.83, 20.80, 21.657, 23.03, 23.23, 24.05, 24.321, 25.50, 25.52, 25.80, 26.69, 26.77, 26.78, 27.05, 27.67, 29.90, 31.11, 33.20, 33.73, 33.76, 33.89, 34.76, 35.75, 35.91, 36.98, 37.08, 37.09, 39.58, 44.045, 45.29, 45.381.

We show some overview of the data sets 1 and 2 in the following Table 3.

Table 3 Descriptive statistics for data sets 1 and 2

Descriptive statistics	Data set 1	Data set 2
Minimum	1.1	18.83
Median	1.7	29.9
Mean	1.9	30.81
Maximum	4.1	45.38
Standard deviation	0.704123	7.253381

The fitting of these distributions to the both data sets have shown in Figures 11 and 12 by using PP-plots, which indicate that the TL-exponential distribution can also be a good option to fit the data as well as other distributions.

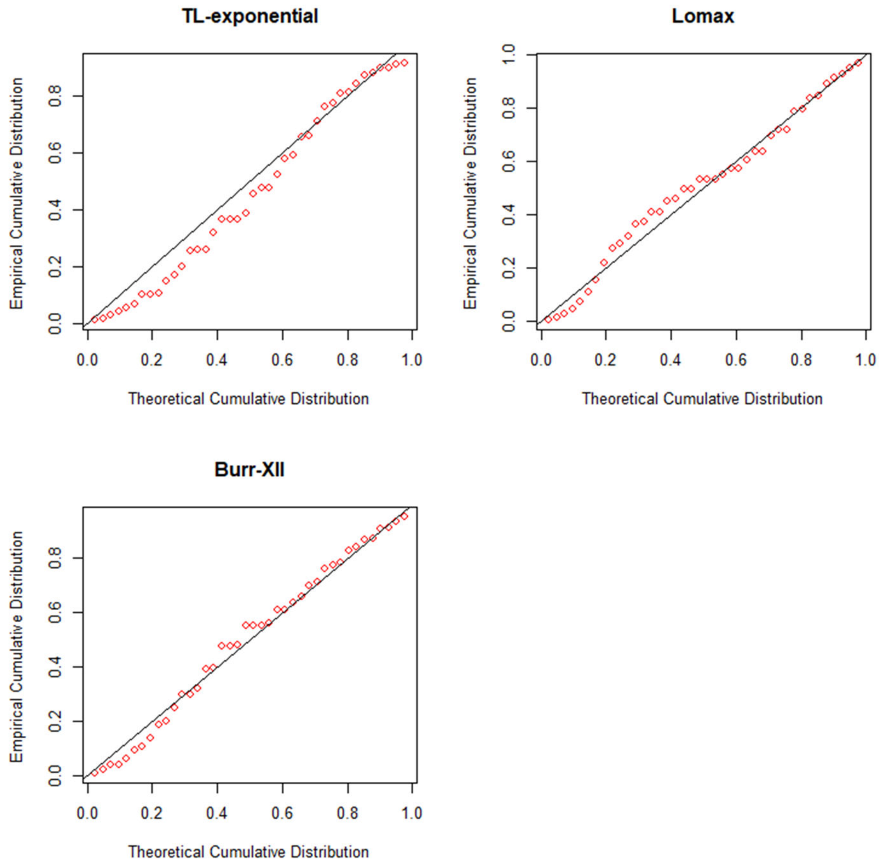

Figure 11 The PP-plots of relaxation times of patient's data fitted with the TL-exponential, Lomax, and Burr-XII distributions

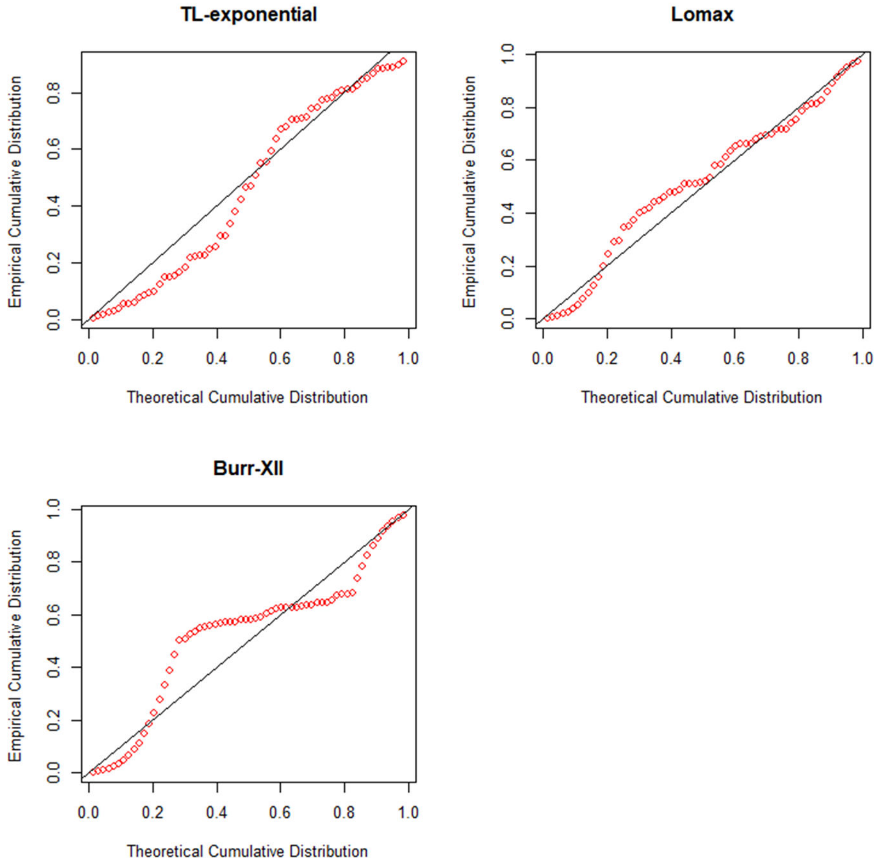

Figure 12 The PP-plots of the strength data of glass of the aircraft window fitted with the TL-exponential, Lomax, and Burr-XII distributions

Now, we compute the ML estimates for the unknown parameters of each distribution and compare the results based on AIC, AICC, BIC, KS statistics, and its p-values for the both data sets (see, Table 4). The values of $-\log l$ for each distributions are also calculated for the both data sets.

Table 4 ML-estimates and log-likelihood functions for the TL-exponential, Lomax, and Burr-XII distributions, and the statistics AIC, AICC, BIC, and KS statistics with its p-values for data sets 1 and 2

	Models	ML estimates	$-\log l$	AIC	AICC	BIC	KS statistic	p-value
Data set 1	TL-exponential	$\hat{\vartheta}=36.682456$ $\hat{\mu}=1.117614$	16.26061	36.5212	37.2271	38.51268	0.15	0.978
	Lomax	$\hat{\vartheta}=1.374595e+03$ $\hat{\mu}=3.821833e-04$	32.84344	69.6868	70.3927	71.67834	0.40	0.08152
	Burr-XII	$\hat{\vartheta}=0.01277514$ $\hat{\mu}=132.81326644$	21.84344	46.4143	47.1201	48.40576	0.25	0.5596
Data set 2	TL-exponential	$\hat{\vartheta}=93.75757060$ $\hat{\mu}=0.08300174$	104.1314	212.2686	212.697	215.1366	0.12903	0.9634
	Lomax	$\hat{\vartheta}=4.634453e+02$ $\hat{\mu}=6.918740e-05$	137.2984	278.5968	279.025	281.4648	0.45161	0.003178
	Burr-XII	$\hat{\vartheta}=0.05783577$ $\hat{\mu}=5.08354219$	174.3869	352.7738	353.202	355.6418	0.54839	0.000125

Since, from Table 4, it can be easily seen that TL-exponential distribution has the lowest AIC, AICC, BIC, KS statistic, and the largest p-values as compared to the Lomax and Burr-XII distributions, and thus, TL-exponential distribution considered as the best distribution among the Lomax and Burr-XII distributions.

5. Discussion

The comparisons of two random variables from the TL-G family of distributions with respect to the dispersive and the star-shaped orders have been discussed. Also, a special case of this family of distributions, called TL-exponential distribution has been considered. Some reliability indicators of this distribution have been studied. We have defined two members of TL-G family of distributions, named TL-log logistic and the TLGLo distributions using the base-line distributions as the log-logistic and Lomax distributions, respectively. Also, the real data applications have been provided to compare the fits of distributions using AIC, AICC, BIC, KS statistic. We have also computed the ML-estimates for the unknown parameters of each distribution. We have presented the fits of the TLGW and the TLGLo distributions using a real data set, and provided the applications of the TL-exponential distribution with two real data sets and compared the fit of this distribution with the Lomax and Burr-XII distributions.

6. Conclusions

With the help of a real data set, we show that the model TL-log logistic has the smallest values of AIC, AICC, BIC, KS statistic, and the largest p-value in Table 2, which describe that TL-log logistic model is the best model as compared to the TLGW model. Also, with the help of two other real data sets, we show that the TL-exponential distribution has the lowest values of AIC, AICC, BIC, KS statistic, and the largest p-value as compared to the Lomax and Burr-XII distributions (see, Table 4), and hence, the TL-exponential distribution considered as the best model among the Lomax and Burr-

XII distributions. The study presents the importance of the TL-G family of distributions and suggests for further research on different aspects of the TL-log logistic and the TLGLo models which are not discussed in this paper.

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