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A New and Generalized Class of Log-logistic Modified Weibull Power Series Distributions with Applications

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Abstract

A new generalized class of distributions called the log-logistic modified Weibull power series (LLoGMWPS) distribution is developed and presented. The LLoGMWPS class of distributions generalizes several distributions including the log-logistic exponential power series, log-logistic Weibull power series, log-logistic Rayleigh power series, log-logistic power series class of distributions and a host of other distributions including log-logistic modified Weibull, log-logistic Weibull, and log-logistic distributions. The special case of the log-logistic modified Weibull Poisson (LLoGMWP) and log-logistic modified Weibull Logarithmic (LLoGMWL) distributions are studied in detail. We apply the method of maximum likelihood to estimate the parameters of this new distribution. Finally, real data examples are presented to illustrate the usefulness and applicability of both LLoGMWP and LLoGMWL distributions.

Keywords: Generalized distribution, power series distribution, modified Weibull distribution, Log-logistic modified Weibull distribution, maximum likelihood estimation.

1. Introduction

Several useful ways of generating new probability distributions from classic ones to relative new distributions are given in the literature on statistical distributions and modeling. Murthy et al. (2004) stated that, distributions with bathtub-shaped failure rate are sufficiently complex and, therefore, difficult to model. The distribution proposed by Hjorth (1980) is such an example. Rajarshi and Rajarshi (1988) presented a revision of these distributions, and Haupt and Schabe (1992) developed a new lifetime model with bathtub-shaped failure rates. Unfortunately, these models are not sufficient to address various practical situations, so new classes of distributions were presented based on modifications of Weibull distribution to satisfy non-monotonic failure rate. For a comprehensive review of these models, please refer to Mudholkar and Srivastava (1993), and Pham and Lai (2007), where the authors summarized some generalizations of the Weibull distribution.

The development and applications of generalized distributions have led to important and useful contributions to the statistical literature in recent times. Oluyede and Yang (2015) on the beta generalized Lindley distribution, Oluyede et al. (2014) on the gamma-Dagum distribution, as well general family of univariate distributions generated from Weibull distribution that was introduced by Gurvich et al. (1997) where the authors developed a new statistical distribution for characterizing the random strength of brittle materials. There are other generalizations that include the exponentiated

Weibull (EW) by Gupta and Kundu (2001), the modified Weibull (MW) by Lai et al. (2003) and the beta exponential (BE) presented by Nadarajah and Kotz (2006). Some more recent extensions are the generalized modified Weibull (GMW) studied by Carrasco et al. (2008), the beta modified Weibull (BMW) reported by Silva et al. (2010), Nadarajah et al. (2011), the Weibull-G family reported by Bourguignon et al. (2014) and the gamma-exponentiated Weibull distributions (GEW) by Pinho et al. (2012). The generalized class of compound distributions have applications in various fields of study such as economics, engineering, public health, industrial reliability and medicine.

There are several new distributions that have been developed by compounding well known continuous distributions such as the exponential, Weibull, Burr XII and exponentiated exponential distributions with the power series distribution that includes the Poisson, logarithmic, geometric and binomial distributions as particular cases. Compound distributions are important due to their flexibility in modelling distributions with both monotonic and non-monotonic hazard rate functions which are encountered in real life. These compound distribution include the class of Weibull-power series (WPS) distributions byMorais and Barreto-Souza (2011). Silva et al. (2013) studied the extended Weibull power series family, which includes as special models the exponential power series and Weibull power series distributions. Silva and Cordeiro (2015) introduced a new family of Burr XII power series models, and Oluyede et al. (2019) further proposed the Burr-Weibull power series class of distributions. Oluyede et al. (2016) recently proposed a log-logistic Weibull Poisson distribution which has applications in several areas including lifetime data analysis, reliability and economics.

Our primary motivations include the advantages of generalized distributions with respect to having hazard rate functions that exhibits increasing, decreasing, up-side-down and bathtub shapes, as well as the versatility and flexibility of the log-logistic and modified Weibull distributions in modelling lifetime data. In this context, we propose and study the new class of distributions called the log-logistic modified Weibull power series class of distributions. This new class of distributions inherits these desirable properties and has quite flexible variety of shapes. There is an added advantage to this model, in that it also has added dispersion parameter, depending on the overall form that accounts for the scale of the underlying random variable. The distribution also has exponential dumping in the upper tail making the distribution suitable for modelling samples that display power behaviour for intermediate observations and decrease in tail probability for large observations or beyond a certain threshold or specified value. The proposed new class of distributions has quite a large number of sub-models and also generalizes the log-logistic, Weibull and modified Weibull distributions.

This paper is organized as follows. In Section 2, we present the generalized class of distributions called the log-logistic modified Weibull power series (LLoGMWPS) class of distributions. The hazard rate function, quantile function and various sub-classes are presented. Also, some additional structural properties of the LLoGMWPS class of distributions including moments, conditional moments, mean deviations, order statistics, entropy and estimates of model parameters are discussed in Section 2. The special cases of the log-logistic modified Weibull Poisson and log-logistic modified Weibull logarithmic distributions are presented and discussed in Section 3 followed by applications and comparisons with other models. Concluding remarks are given in Section 4.

2. The Log-Logistic Modified Weibull Power Series Class of Distributions

In this section, the log-logistic modified Weibull power series (LLoGMWPS) class of distributions and some of its statistical properties are presented. We first of all present the log-logistic and modified Weibull distributions. The cumulative distribution function (cdf) and probability density function (pdf) of log-logistic distribution are given by $F_{LLoG}(x) = 1-(1+x^c)^{-1}$ and $f_{LLoG}(x) = cx^{c-1}(1+x^c)^{-2}$, for $c>0$ and $x \geq 0$. The modified Weibull (MW) distribution BY Lai et al. (2003) is given by

$$F_{MW}(x;\alpha,\beta,\lambda) = 1-\exp(-\alpha x^{\beta} e^{\lambda x}), \quad x \geq 0, \alpha > 0, \beta > 0, \lambda \geq 0. \tag{1}$$

The corresponding pdf is given by

$$f_{_{MW}}(x;\alpha,\beta,\lambda)=\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)\exp(-\alpha x^{\beta}e^{\lambda x}), \tag{2}$$

for $x\geq 0$, $\alpha>0$, $\beta>0$, and $\lambda\geq 0$. Note that the parameter α control the scale of the distribution, β controls the shape, whereas λ can be considered to be an accelerating factor in the imperfection time and a factor of fragility in the survival of the individual as time increases. When $\lambda=0$, we obtain Weibull distribution. Weibull distribution is well known and has been extensively used for modeling data in several areas including reliability. Weibull distribution is particularly useful for modeling monotone hazard rates.

Consider a sequence of N independent and identically distributed random variables, say X_i, $i=1,\ldots,N$, from the log-logistic modified Weibull distribution (Oluyede et al., 2018). That is, the cdf of X is given by

$$F_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)=1-(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x}), \tag{3}$$

and the corresponding pdf is given by

$$f_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)=e^{-\alpha x^{\beta}e^{\lambda x}}\Big(1+x^{c}\Big)^{-1}\left\{\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)+\frac{cx^{c-1}}{(1+x^{c})}\right\}, \tag{4}$$

for c, α, $\beta>0$, $\lambda\geq 0$ and $x\geq 0$. The survival and hazard rate functions are given by

$$S_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)=1-F_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)=(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x})$$
$$\text{and}\quad h_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)=\frac{f_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)}{S_{_{LLoGMW}}(x;c,\alpha,\beta,\lambda)}=\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)+\frac{cx^{c-1}}{(1+x^{c})},$$

respectively.

Now, let N be a discrete random variable following a power series distribution assumed to be truncated at zero, whose probability mass function (pmf) is given by

$$P(N=n)=\frac{a_n\theta^n}{C(\theta)},\quad n=1,2,\ldots,$$

where $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$ is finite, $\theta>0$, and $\{a_n\}_{n\geq 1}$ a sequence of positive real numbers. Considering $X_{(1)}=\min(X_1,\ldots,X_N)$ and conditioning upon $N=n$, the conditional distribution of $X_{(1)}$ given $N=n$ is obtained as

$$G_{X_{(1)}|N=n}(x)=1-\prod_{i=1}^{n}(1-F_{_{LLoGMW}}(x))=1-S^{n}_{_{LLoGMW}}(x)=1-(1+x^{c})^{-n}\exp(-n\alpha x^{\beta}e^{\lambda x}).$$

Proposition 1 The cdf of $X_{(1)}$, say F_θ, is given by

$$F_\theta(x)=1-\frac{C\Big(\theta(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x})\Big)}{C(\theta)}.$$

Proof: Note that

$$\begin{aligned} F_\theta(x) &= \sum_{n=1}^{\infty} G_{X_{(1)}|N=n}(x)P(N=n) \\ &= \sum_{n=1}^{\infty}\left[1-(1+x^c)^{-n}\exp(-n\alpha x^\beta e^{\lambda x})\right]\frac{a_n\theta^n}{C(\theta)} \\ &= \sum_{n=1}^{\infty}\frac{a_n\theta^n}{C(\theta)} - \sum_{n=1}^{\infty}\frac{a_n\left[\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})\right]^n}{C(\theta)} \\ &= 1-\frac{C\Big(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})\Big)}{C(\theta)}. \end{aligned}$$

Proposition 2 $F_{_{LLoGMW}}(x) = \lim_{\theta\to 0^+} F_\theta(x)$.

Proof: Recall that $C(\theta)=\sum_{n=1}^{\infty}a_n\theta^n$. Then,

$$F_\theta(x) = 1-\frac{\sum_{n=1}^{\infty}a_n\left[\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})\right]^n}{\sum_{n=1}^{\infty}a_n\theta^n} = 1-\frac{\sum_{n=1}^{\infty}a_n\theta^n S^n(x)}{\sum_{n=1}^{\infty}a_n\theta^n}.$$

Letting $\theta\to 0^+$, the second term to the right of this equation is undetermined. Thus, applying l'Hopital's rule, we have

$$\begin{aligned} \lim_{\theta\to 0^+} F_\theta(x) &= 1-\lim_{\theta\to 0^+}\frac{\sum_{n=1}^{\infty}na_n\theta^{n-1}S^n(x)}{\sum_{n=1}^{\infty}na_n\theta^{n-1}} \\ &= 1-\lim_{\theta\to 0^+}\frac{a_1S(x)+\sum_{n=2}^{\infty}na_n\theta^{n-1}S^n(x)}{a_1+\sum_{n=2}^{\infty}na_n\theta^{n-1}} \\ &= 1-S_{_{LLoGMMW}}(x) \\ &= F_{_{LLoGMW}}(x). \end{aligned}$$

Remark 1 Let $C'(\theta)$ be the derivative of $C(\theta)$, that is, $C'(\theta)=\sum_{n=1}^{\infty}na_n\theta^{n-1}$. Then the density of F_θ, say f_θ, is given by

$$f_\theta(x) = \frac{dF_\theta(x)}{dx} = \frac{\theta f(x)C'(\theta S(x))}{C(\theta)}.$$

2.1. Some generalized sub-classes of distributions

In this section, we present several new and known sub-classes of distributions including the log-logistic power series (LLoGPS), log-logistic modified exponential power series (LLoGMEPS), log-logistic Rayleigh power series (LLoGRPS), and other sub-classes of distributions.

- If $\lambda=0$, we obtain the log-logistic Weibull power series (LLoGWPS) class of distributions.
- If $\lambda=0$, and $\beta=1$, we obtain the log-logistic exponential power series (LLoGEPS) class of distributions.
- When $\alpha\to 0^+$, we have the log-logistic power series (LLoGPS) class of distributions.
- If $\lambda=0$ and $\beta=2$, we have the log-logistic Rayleigh power series (LLoGRPS) class of distributions.
- If $\beta=1$, we obtain the log-logistic modified exponential power series (LLoGMEPS) class of distributions.

- If $\beta = 2$, we have the log-logistic modified Rayleigh (LLoGMRPS) class of distributions.
- Several new classes of distributions can be readily obtained by setting c=1; c=1, β=1; c=1, β=2; $c = 1, \beta$=1, λ=0; c=1, λ=0; $\alpha \to 0^+$, c=1 and c=α=1, respectively.

2.2. Hazard rate and reverse hazard functions

We present the hazard rate and reverse hazard functions of the proposed LLoGMWPS class of distributions in this section. The hazard rate function (hrf) is given by $h_\theta(x) = f_\theta(x)/S_\theta(x)$, where $S_\theta(x) = 1 - F_\theta(x)$. Explicitly,

$$h_\theta(x) = \theta f(x) \frac{C'(\theta S(x))}{C(\theta S(x))},$$

where $f_{_{LLoGMW}}(x) = f(x)$ is the log-logistic modified Weibull pdf. On the other hand, the reverse hazard function, $\tau_\theta(x) = f_\theta(x)/F_\theta(x)$, is given by

$$\tau_\theta(x) = \theta f(x) \frac{C'(\theta S(x))}{C(\theta) - C(\theta S(x))}.$$

2.3. Log-logistic modified Weibull poisson, geometric, logarithmic and binomial distributions

The cdfs, pdfs and hrfs of the log-logistic modified Weibull Poisson (LLoGMWP), log-logistic modified Weibull geometric (LLoGMWG), log-logistic modified Weibull logarithmic (LLoGMWL) and log-logistic modified Weibull binomial (LLoGMWB) distributions are presented in this section.

2.3.1 Log-logistic modified Weibull poisson distribution

The log-logistic modified Weibull Poisson (LLoGMWP) distribution is a special case of the LLoGMWPS class of distributions with $C(\theta) = e^\theta - 1$ and $a_n = \frac{1}{n!}$. The cdf is given by

$$F_{LLoGMWP}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{e^{(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))} - 1}{e^\theta - 1}, \tag{5}$$

for $c, \alpha, \beta, \theta > 0$, $\lambda \geq 0$. The pdf is given by

$$\begin{aligned} f_{LLoGMWP}(x; c, \alpha, \beta, \lambda, \theta) &= \frac{\theta e^{-\alpha x^\beta e^{\lambda x}}(1+x^c)^{-2} e^{\theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}}}{e^\theta - 1} \\ &\times \left(c x^{c-1} + \alpha x^{\beta-1} e^{\lambda x}(\beta + \lambda x)(1+x^c)\right). \end{aligned} \tag{6}$$

The hrf is

$$\begin{aligned} h_{LLoGMWP}(x) &= \frac{\theta e^{-\alpha x^\beta e^{\lambda x}}(1+x^c)^{-2} e^{\theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}}}{e^{\theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}} - 1} \\ &\times \left(c x^{c-1} + \alpha \beta x^{\beta-1} e^{\lambda x}(\beta + \lambda x)(1+x^c)\right). \end{aligned}$$

2.3.2 Log-logistic modified Weibull geometric distribution

The log-logistic modified Weibull geometric (LLoGMWG) distribution is obtained from the LLoGMWPS class of distributions with $C(\theta) = \theta(1-\theta)^{-1}$ and $a_n = 1$. The cdf of the LLoGMWG distribution is given by

$$F_{LLoGMWG}(x; c, \alpha, \beta, \lambda, \theta) = 1 - \frac{(1-\theta)(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}}{1 - \theta(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}} \text{ for } c, \alpha, \beta, \theta > 0, \lambda \geq 0.$$

The corresponding pdf is given by

$$\begin{aligned} f_{LLoGMWG}(x;c,\alpha,\beta,\lambda,\theta) &= \frac{(1-\theta)(e^{-\alpha x^{\beta}e^{\lambda x}}(1+x^c)^{-2})}{(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})^2} \\ &\quad \times \left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)\right), \end{aligned}$$

and the hrf is given by

$$h_{LLoGMWG}(x) = \frac{(1+x^c)^{-1}\left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)\right)}{1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}}.$$

2.3.3 Log-logistic modified Weibull logarithmic distribution

The log-logistic modified Weibull logarithmic (LLoGMWL) distribution is obtained from the the LLoGMWPS class of distributions with $C(\theta) = -\log(1-\theta)$ and $a_n = \frac{1}{n}$. The cdf is given by

$$F_{LLoGMWL}(x;c,\alpha,\beta,\lambda,\theta) = 1 - \frac{\log(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})}{\log(1-\theta)}, \tag{7}$$

for $c,\alpha,\beta,\theta > 0$ and $\lambda \geq 0$. The corresponding pdf is given by

$$f_{LLoGMWL}(x;c,\alpha,\beta,\lambda,\theta) = \frac{\theta e^{-\alpha x^{\beta}e^{\lambda x}}(1+x^c)^{-2}\left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)\right)}{-(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})\log(1-\theta)}, \tag{8}$$

and the hrf is

$$h_{LLoGMWL}(x;c,\alpha,\beta,\lambda,\theta) = \frac{\theta e^{-\alpha x^{\beta}e^{\lambda x}}(1+x^c)^{-2}\left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)\right)}{-(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})\log(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})}. \tag{9}$$

2.3.4 Log-logistic modified Weibull binomial distribution

The log-logistic modified Weibull binomial distribution (LLoGMWB) is a special case of the LLoGMWPS class of distributions with $C(\theta) = (1+\theta)^m - 1$, $(n \leq m)$, and $a_n = \binom{m}{n}$. The cdf is given by

$$F_{LLoGMWB}(x;c,\alpha,\beta,\lambda,\theta) = 1 - \frac{(1+\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})^m - 1}{(1+\theta)^m - 1}, \tag{10}$$

for $c,\alpha,\beta,\theta > 0$ and $\lambda \geq 0$. The corresponding pdf and hrf are given by

$$\begin{aligned} f_{LLoGMWB}(x;c,\alpha,\beta,\lambda,\theta) &= e^{-\alpha x^{\beta}}(1+x^c)^{-2}\left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)\right) \\ &\quad \times \frac{m(1+\theta e^{-\alpha x^{\beta}}(1+x^c)^{-k})^{m-1}}{(1+\theta)^m - 1}, \end{aligned}$$

and

$$\begin{aligned} h_{LLoGMWB}(x;c,\alpha,\beta,\lambda,\theta) &= e^{-\alpha x^{\beta}}(1+x^c)^{-2}\left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)\right) \\ &\quad \times \frac{m(1+\theta e^{-\alpha x^{\beta}e^{\lambda x}}(1+x^c)^{-1})^{m-1}}{(1+\theta e^{-\alpha x^{\beta}e^{\lambda x}}(1+x^c)^{-1})^m - 1}, \end{aligned}$$

respectively.

2.4. Quantile function

The quantile function of the proposed LLoGMWPS class of distributions is obtained by inverting $F_{\theta}(x) = u, 0 \leq u \leq 1$, and $F_{\theta}(x)) = 1 - C(\theta S(x))/C(\theta)$. This is equivalent to solving the equation

$$\ln(C(\theta S(x))) + \ln(C(\theta)) + \ln(1-u) = 0, \tag{11}$$

which can be done using numerical methods. Consequently, random number can be generated based on Eqn. (11).

2.5. Moments, mean deviations, order statistics and entropy

In this section, moments, order statistics and Rényi entropy from LLoGMWPS class of distributions are presented. Mean deviation from the mean and the mean deviation from the median are also derived in this section. We note that using the result of a power series raised to a positive integer s (Gradshetyn and Ryzhik, 2000), that is,

$$\left(\sum_{j=0}^{\infty} a_j y^j\right)^s = \sum_{j=0}^{\infty} b_{s,j} y^j,$$

where the coefficients $b_{s,j}$ for $(j = 1, 2, ...)$ are determined by the recurrence equations $b_{s,j} = (ja_0)^{-1}\sum_{m=1}^{j}[m(s+1)-j]a_m b_{s,j-m}$, and $b_{s,0} = a_0^s$, we have

$$\begin{aligned}\left(C(\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}})\right)^s &= \left(\sum_{j=0}^{\infty} a_j[\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^j\right)^s \\ &= \sum_{j=0}^{\infty} b_{s,j}[\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^j. \qquad (12)\end{aligned}$$

2.5.1 Moments

Moments of the LLoGMWPS class of distribution are presented in this subsection. The r^{th} moment of the LLoGMWPS class of distributions is given by

$$E(X^r) = \int_0^\infty x^r \frac{\theta f(x) C'(\theta S(x))}{C(\theta)} dx.$$

Now, using the transformation $y = (1+x^c)^{-1}$, we have

$$\begin{aligned}E(X^r) &= \frac{\theta\alpha}{C(\theta)} \sum_{j,k,p=0}^{\infty} b_{s,j}\theta^j \frac{(-1)^k[\alpha(j+1)]^k[\lambda(k+1)]^p}{k!p!} \\ &\times\left[\beta B\left(j+1-\frac{r+k\beta+\beta+p}{c}, \frac{r+k\beta+\beta+p}{c}\right)\right. \\ &\left.+\lambda B\left(j+1-\frac{r+k\beta+\beta+p+1}{c}, \frac{r+k\beta+\beta+p+1}{c}\right)\right] \\ &+\frac{\theta c}{C(\theta)} \sum_{j,k,p=0}^{\infty} b_{s,j}\theta^j \frac{(-1)^k[\alpha(j+1)]^k[k\lambda]^p}{k!p!} \\ &\times B\left(j+2-\frac{r+c+k\beta+p}{c}, \frac{r+c+k\beta+p}{c}\right), \qquad (13)\end{aligned}$$

where $B(a,b) = \int_0^1 t^{a-1}(1-t)^{b-1}dt$ is the complete beta function.

2.5.2 Conditional moments

The r^{th} conditional moment of the LLoGMWPS class of distributions is given by

$$E(X^r \mid X > t) = \frac{1}{\overline{F}_\theta(t)} \int_t^\infty x^r \frac{\theta f(x) C'(\theta S(x))}{C(\theta)} dx$$

$$
\begin{aligned}
E(X^r \mid X > t) = \quad & \frac{1}{\overline{F}_\theta(t)} \frac{\theta\alpha}{C(\theta)} \sum_{j,k,p=0}^{\infty} b_{s,j}\theta^j \frac{(-1)^k[\alpha(j+1)]^k[\lambda(k+1)]^p}{k!p!} \\
& \times\left[\beta B_{(1+t^c)^{-1}}\left(j+1-\frac{r+k\beta+\beta+p}{c}, \frac{r+k\beta+\beta+p}{c}\right)\right. \\
& \left.+\lambda B_{(1+t^c)^{-1}}\left(j+1-\frac{r+k\beta+\beta+p+1}{c}, \frac{r+k\beta+\beta+p+1}{c}\right)\right] \\
& +\frac{1}{\overline{F}_\theta(t)} \frac{\theta c}{C(\theta)} \sum_{j,k,p=0}^{\infty} b_{s,j}\theta^j \frac{(-1)^k[\alpha(j+1)]^k[k\lambda]^p}{k!p!} \\
& +B_{(1+t^c)^{-1}}\left(j+2-\frac{r+c+k\beta+p}{c}, \frac{r+c+k\beta+p}{c}\right), \qquad (14)
\end{aligned}
$$

where $B_{(1+t^c)^{-1}}(a,b)$ is the incomplete beta function.

2.5.3 Mean and median deviations

The amount of scatter in a population is measured to some extent by the totality of deviations from the mean and median. These are known as the mean deviation about the mean and the mean deviation about the median, and are defined by

$$\delta_1(x) = \int_0^\infty |x-\mu| f_\theta(x)dx \quad \text{and} \quad \delta_2(x) = \int_0^\infty |x-M| f_\theta(x)dx, \qquad (15)$$

respectively, where $\mu = E(X)$ and $M = Median(X)$ denotes the median. The measures $\delta_1(x)$ and $\delta_2(x)$ can be calculated using the relationships

$$\delta_1(x) = 2\mu F_\theta(\mu) - 2\mu + 2\int_\mu^\infty x f_\theta(x)dx, \qquad (16)$$

$$\delta_2(x) = -\mu + 2\int_M^\infty x f_\theta(x)dx, \qquad (17)$$

respectively. When $r = 1$, we get the mean $\mu = E(X)$ from Eqn. (13). Note that $T(\mu) = \int_\mu^\infty x f_\theta(x)dx$ is given by

$$
\begin{aligned}
T(\mu) &= \int_\mu^\infty x f_\theta(x)dx \\
&= \frac{\theta\alpha}{C(\theta)} \sum_{j,k,p=0}^{\infty} b_{s,j}\theta^j \frac{(-1)^k[\alpha(j+1)]^k[\lambda(k+1)]^p}{k!p!} \\
&\quad \times\left[\beta B_{(1+\mu^c)^{-1}}\left(j+1-\frac{1+k\beta+\beta+p}{c}, \frac{1+k\beta+\beta+p}{c}\right)\right. \\
&\quad \left.+\lambda B_{(1+\mu^c)^{-1}}\left(j+1-\frac{1+k\beta+\beta+p+1}{c}, \frac{1+k\beta+\beta+p+1}{c}\right)\right] \\
&\quad +\frac{\theta c}{C(\theta)} \sum_{j,k,p=0}^{\infty} b_{s,j}\theta^j \frac{(-1)^k[\alpha(j+1)]^k[k\lambda]^p}{k!p!} \\
&\quad \times c B_{(1+\mu^c)^{-1}}\left(j+2-\frac{1+c+k\beta+p}{c}, \frac{1+c+k\beta+p}{c}\right).
\end{aligned}
$$

Consequently, the mean deviation about the mean and the mean deviation about the median are

$$\delta_1(x) = 2\mu F_\theta(\mu) - 2\mu + 2T(\mu) \quad \text{and} \quad \delta_2(x) = -\mu + 2T(M),$$

respectively.

2.5.4 Order statistics

The pdf of the i^{th} order statistic from the LLoGMWPS class of distributions is given by

$$\begin{aligned} f_{i:n}(x) &= \frac{1}{B(i,n-i+1)} f_\theta(x) F_\theta^{i-1}(x)(1-F_\theta(x))^{n-i} \\ &= \frac{1}{B(i,n-i+1)} f_\theta(x) \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j F_\theta^{j+i-1}(x) \\ &= \frac{f_\theta(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} \binom{n-i}{j} (-1)^j \\ &\quad \times \left(1 - \frac{C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))}{C(\theta)}\right)^{j+i-1}. \end{aligned}$$

Using binomial expansion

$$\begin{aligned} \left(1 - \frac{C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))}{C(\theta)}\right)^{j+i-1} &= \sum_{s=0}^{j+i-1} (-1)^s \binom{j+i-1}{s} \\ &\quad \times \left(\frac{C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))}{C(\theta)}\right)^s. \end{aligned}$$

Now, the pdf of the i^{th} order statistics can be written as

$$\begin{aligned} f_{i:n}(x) &= \frac{f_\theta(x)}{B(i,n-i+1)} \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \binom{n-i}{j} \binom{j+i-1}{s} (-1)^{j+s} \\ &\quad \times \left(\frac{C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x}))}{C(\theta)}\right)^s. \end{aligned}$$

Note that,

$$\begin{aligned} (C(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})))^s &= \left(\sum_{w=0}^{\infty} a_w \theta (1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})\right)^w \\ &= \sum_{w=0}^{\infty} e_{s,w} \left(\theta(1+x^c)^{-1}\exp(-\alpha x^\beta e^{\lambda x})\right)^w, \end{aligned}$$

where if $e_{s,w} = (wa_0)^{-1}\sum_{m=1}^{w}[m(s+1)-w]a_m e_{s,w-m}$, and $e_{s,0} = a_0^s$, so that the pdf of the i^{th} order statistic is

$$\begin{aligned} f_{i:n}(x) &= \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{w=0}^{\infty} \binom{n-i}{j} \binom{j+i-1}{s} \frac{(-1)^{j+s} e_{s,w} \theta^w}{(w+1)B(i,n-i+1)(C(\theta))^s} \\ &\quad \times (w+1)[(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}]^w f_\theta(x) \\ &= \sum_{j=0}^{n-i} \sum_{s=0}^{j+i-1} \sum_{w=0}^{\infty} \binom{n-i}{j} \binom{j+i-1}{s} \frac{(-1)^{j+s} e_{s,w} \theta^w}{(w+1)B(i,n-i+1)(C(\theta))^s} f^*(x), \end{aligned}$$

where

$$\begin{aligned} f^*(x) &= (w+1)[(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}]^w f_\theta(x) \\ &= (w+1)[(1+x^c)^{-1} e^{-\alpha x^\beta e^{\lambda x}}]^w \\ &\quad \times [e^{-\alpha x^\beta e^{\lambda x}} (1+x^c)^{-2}(cx^{c-1} + (1+x^c)\alpha\beta x^{\beta-1} e^{\lambda x}(\beta+\lambda x))]. \end{aligned}$$

The corresponding cdf is given by

$$\begin{aligned}
F_{i:n}(x) &= \sum_{k=i}^{n}\binom{n}{k}F_{\theta}^{k}(x)(1-F_{\theta}(x))^{n-k} \\
&= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\binom{n-k}{j}\binom{n}{k}(-1)^{j}[F_{\theta}(x)]^{j+k} \\
&= \sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{m=0}^{j+k}\binom{j+k}{m}\binom{n-k}{j}\binom{n}{k}(-1)^{j+m} \\
&\quad \times\left(\frac{C(\theta(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x}))}{C(\theta)}\right)^{m},
\end{aligned}$$

where $F_{\theta}(x) = 1 - \frac{C(\theta(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x}))}{C(\theta)}$, and

$$C(\theta(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x})) = \sum_{j=0}^{\infty}a_{j}\left[\theta(1+x^{c})^{-1}\exp(-\alpha x^{\beta}e^{\lambda x})\right]^{j}.$$

2.5.5 Rényi entropy

The most widely used information measures are Rényi entropy (Rényi, 1961) and Shannon entropy (Shannon, 1974). Statistically, Rényi entropy is defined as an extension of Shannon entropy and is given as

$$I_{R}(v) = \frac{1}{1-v}\log\left(\int_{0}^{\infty}[f(x;c,\alpha,\beta,\lambda,\theta)]^{v}dx\right), v \neq 1, v > 0. \tag{18}$$

Rényi entropy tends to Shannon entropy as $v \to 1$. With

$$\int_{0}^{\infty}f^{v}(x;c,\alpha,\beta,\lambda,\theta)dx = \int_{0}^{\infty}f_{\theta}^{v}(x)dx$$

written as $$\int_{0}^{\infty}f_{\theta}^{v}(x)dx = \left(\frac{\theta}{C(\theta)}\right)^{v}\int_{0}^{\infty}(f(x)C'(\theta S(x)))^{v}dx,$$

and
$$\begin{aligned}
C'(\theta(1+x^{c})^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}) &= \sum_{n=1}^{\infty}na_{n}[\theta(1+x^{c})^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}]^{n-1} \\
&= \sum_{n=0}^{\infty}(n+1)a_{n+1}[\theta(1+x^{c})^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}]^{n} \\
&= \sum_{n=0}^{\infty}b_{n}[\theta(1+x^{c})^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}]^{n}, \qquad (19)
\end{aligned}$$

where $b_{n} = (n+1)a_{n+1}$, and $\left(\sum_{n=0}^{\infty}b_{n}y^{n}\right)^{v} = \sum_{n=0}^{\infty}d_{v,n}y^{n}$, where $y = \theta(1+x^{c})^{-1}e^{-\alpha x^{\beta}}$, $d_{v,n} = b_{0}^{n}$, and $d_{v,m} = \frac{1}{mb_{0}}\sum_{j=0}^{m}(jv-m+j)b_{j}d_{v,m-j}$, we have

$$\begin{aligned}\int_0^\infty f_\theta^v(x)dx &= \left(\frac{\theta}{C(\theta)}\right)^v\sum_{n=0}^\infty d_{v,n}\int_0^\infty [\theta(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}]^n \\ &\quad\times\left[(1+x^c)^{-1}e^{-\alpha x^\beta e^{\lambda x}}\left(\alpha\beta x^{\beta-1}e^{\lambda x}(\beta+\lambda x)+\frac{cx^{c-1}}{1+x^c}\right)\right]^v dx \\ &= \left(\frac{\theta}{C(\theta)}\right)^v\sum_{n,p,k,l,s=0}^\infty \frac{(-1)^p[\alpha(n+1)]^p}{p!}\frac{[\lambda(p+v-k)]^l}{l!}\binom{v}{k}\binom{v-k}{s} \\ &\quad\times\theta^n d_{v,n}c^k\alpha^{v-k}\lambda^s\beta^{v-k-s}\int_0^\infty x^{ck+v\beta-k\beta-v+s+l}(1+x^c)^{-n+v-k-2v}dx.\end{aligned}$$

Now, let $y=(1+x^c)^{-1}$, and set $\delta^*=ck+v\beta-k\beta-v+s+l+1$, then Rényi entropy for LLoGMWPS class of distributions is given by

$$\begin{aligned}I_R(v) &= \frac{1}{1-v}\log\left(\left(\frac{\theta}{C(\theta)}\right)^v\sum_{n,p,k,l,s=0}^\infty\frac{(-1)^p[\alpha(n+1)]^p}{p!}\frac{[\lambda(p+v-k)]^l}{l!}\right. \\ &\quad\left.\times\binom{v}{k}\binom{v-k}{s}\theta^n d_n c^k\alpha^{v-k}\lambda^s\beta^{v-k-s}B\left(n+v+k-\frac{\delta^*+1}{c},\frac{\delta^*+1}{c}\right)\right),\end{aligned}$$

for $v\neq 1$, and $v>0$.

2.6. Estimation

Let $X\sim LLoGMWPS(c,\alpha,\beta,\lambda,\theta)$ and $\boldsymbol{\Delta}=(c,\alpha,\beta,\lambda,\theta)^T$ be the parameter vector. The log-likelihood function $\ell=\ell(\boldsymbol{\Delta})$ based on a random sample of size n is given by

$$\begin{aligned}\ell=\ell_n(\Delta) &= n\ln(\theta)-n\ln(C(\theta))+\sum_{i=1}^n\ln(f(x_i))+\sum_{i=1}^n\ln(C'(\theta S(x_i))) \\ &= n\ln(\theta)-n\ln(C(\theta))+\sum_{i=1}^n(-\alpha x_i^\beta e^{\lambda x_i})-2\sum_{i=1}^n\ln(1+x_i^c) \\ &\quad+\sum_{i=1}^n\ln[cx_i^{c-1}+(1+x_i^c)\alpha x_i^{\beta-1}e^{\lambda x_i}(\beta+\lambda x_i)] \\ &\quad+\sum_{i=1}^n\ln(C'(\theta e^{-\alpha x_i^\beta e^{\lambda x_i}}(1+x_i^c)^{-1})).\end{aligned}$$

The equations obtained by setting the partial derivatives to zero are not in closed form and the values of the parameters c,α,β,λ and θ must be found via iterative methods. The maximum likelihood estimates of the parameters, denoted by $\hat{\boldsymbol{\Delta}}$ is obtained by solving the nonlinear equation $(\frac{\partial\ell}{\partial c},\frac{\partial\ell}{\partial\alpha},\frac{\partial\ell}{\partial\beta},\frac{\partial\ell}{\partial\lambda},\frac{\partial\ell}{\partial\theta})^T=\mathbf{0}$, using a numerical method such as Newton-Raphson procedure. The Fisher information matrix given by $\mathbf{I}(\boldsymbol{\Delta})=[\mathbf{I}_{\theta_i,\theta_j}]_{5X5}=E(-\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j})$, $i,j=1,2,3,4,5$, can be numerically obtained by MATLAB or NLMIXED in SAS or R software. The total Fisher information matrix $n\mathbf{I}(\boldsymbol{\Delta})$ can be approximated by

$$\mathbf{J}_n(\hat{\boldsymbol{\Delta}})\approx\left[-\left.\frac{\partial^2\ell}{\partial\theta_i\partial\theta_j}\right|_{\boldsymbol{\Delta}=\hat{\boldsymbol{\Delta}}}\right]_{5X5},\quad i,j=1,2,3,4,5. \tag{20}$$

For a given set of observations, the matrix given in equation [29] is obtained after the convergence of the Newton-Raphson procedure via NLMIXED in SAS or R software. Note that the expectations in the Fisher Information Matrix (FIM) can be obtained numerically. Let $\hat{\Delta}=(\hat{c},\hat{\alpha},\hat{\beta},\hat{\lambda},\hat{\theta})$

be the maximum likelihood estimate of $\mathbf{\Delta} = (c, \alpha, \beta, \lambda, \theta)$. Under the usual regularity conditions and that the parameters are in the interior of the parameter space, but not on the boundary, we have: $\sqrt{n}(\hat{\mathbf{\Delta}} - \mathbf{\Delta}) \xrightarrow{d} N_5(\underline{0}, I(\mathbf{\Delta})^{-1})$, where $I(\mathbf{\Delta})$ is the expected Fisher information matrix. The asymptotic behavior is still valid if $I(\mathbf{\Delta})$ is replaced by the observed information matrix evaluated at $\hat{\mathbf{\Delta}}$, that is $J(\hat{\mathbf{\Delta}})$. The multivariate normal distribution $N_5(\underline{0}, J^{-1}(\hat{\mathbf{\Delta}}))$, where the mean vector $\underline{\mathbf{0}} = (0,0,0,0,0)^T$, can be used to construct confidence intervals and confidence regions for the individual model parameters and for the survival and hazard rate functions. That is, the approximate $100(1-\eta)\%$ two-sided confidence intervals for $c, \lambda, \alpha, \beta$ and θ are given by:

$$\widehat{c} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\alpha} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\beta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}})}, \quad \widehat{\lambda} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})},$$

$$\text{and} \quad \widehat{\theta} \pm Z_{\frac{\eta}{2}}\sqrt{\mathbf{I}_{\theta\theta}^{-1}(\widehat{\mathbf{\Delta}})},$$

respectively, where $\mathbf{I}_{cc}^{-1}(\widehat{\mathbf{\Delta}}), \mathbf{I}_{\alpha\alpha}^{-1}(\widehat{\mathbf{\Delta}}), \mathbf{I}_{\beta\beta}^{-1}(\widehat{\mathbf{\Delta}}), \mathbf{I}_{\lambda\lambda}^{-1}(\widehat{\mathbf{\Delta}})$ and $\mathbf{I}_{\theta\theta}^{-1}(\widehat{\mathbf{\Delta}})$ are the diagonal elements of $\mathbf{I}_n^{-1}(\hat{\mathbf{\Delta}}) = (n\mathbf{I}(\hat{\mathbf{\Delta}}))^{-1}$ and $Z_{\frac{\eta}{2}}$ is the upper $\frac{\eta}{2}^{th}$ percentile of a standard normal distribution.

3. Some Special Cases of the LLoGMWPS Class of Distributions

Two special cases of the log-logistic modified Weibull power series (LLoGMWPS) class of distributions, namely the log-logistic modified Weibull Poisson and log-logistic modified Weibull Logarithmic distributions are presented including pdfs, cdfs, hazard rate and reverse hazard functions, moments, conditional moments, distribution of order statistics and Rényi entropy. Plots of the pdf and hrf for selected values of the model parameters are given. The method of maximum likelihood estimation is used for estimating parameters and simulation is conducted to illustrate the performance of the two special cases of the LLoGMWPS class of distributions. Real life data sets are applied to demonstrate the flexibility and applicability of the two special cases.

3.1. Log-logistic modified Weibull poisson distribution

The cdf and pdf of the LLoGMWP distribution are given by Eqns. (5) and (6). Plots of the LLoGMWP pdf for selected values of the model parameters are given in Figure 1.

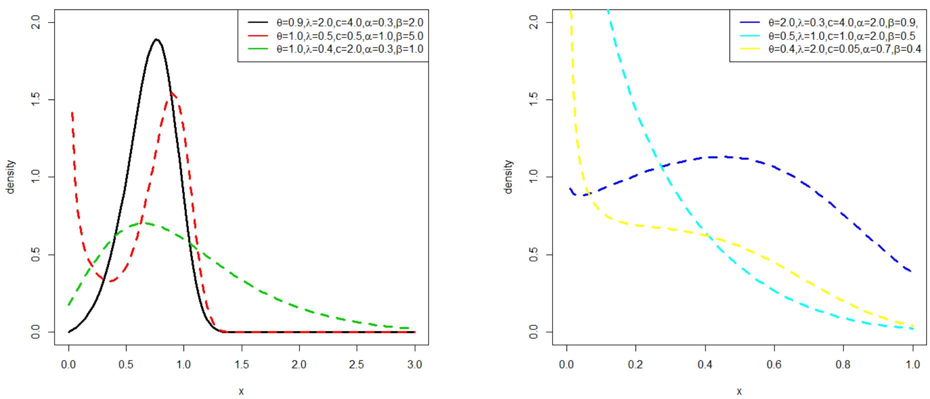

Figure 1 Plots of the pdf of the LLoGMWP distribution for the selected values of the model parameters

Figure 1 shows that the plots of the pdf can be right-skewed, reverse-J and decreasing among many other shapes among many potential shapes. The density can exhibit different shapes depending on the values of the parameters, as shown in these plots.

3.1.1 Hazard rate function

In this section, the hrf of the LLoGMWP distribution is presented. Plots of the hrf for selected values of the model parameters are also given. The hrf of the LLoGMWP distribution is given by

$$\begin{aligned} h_{LLoGMWP}(x) &= \frac{\theta e^{-\alpha x^{\beta} e^{\lambda x}} (1+x^c)^{-2} e^{\theta (1+x^c)^{-1} e^{-\alpha x^{\beta} e^{\lambda x}}}}{e^{\theta (1+x^c)^{-1} e^{-\alpha x^{\beta} e^{\lambda x}}} - 1} \\ &\times \left(c x^{c-1} + \alpha \beta x^{\beta - 1} e^{\lambda x} (\beta + \lambda x)(1 + x^c) \right) \end{aligned}$$

for $x \geq 0$, $c, \alpha, \beta > 0$, $\lambda \geq 0, \theta > 0$.

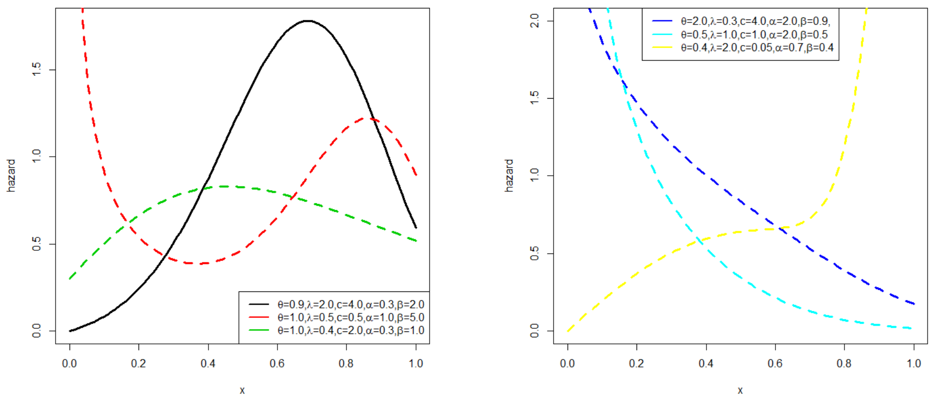

Figure 2 Plots of the hrf LLoGMWP distribution for the selected values of the model parameters

The graphs in Figure 2 exhibit decreasing, increasing, uni-modal and bathtub followed by upside down bathtub shapes for the selected values of the model parameters. This very attractive flexibility makes the LLoGMWP hrf useful and suitable for non-monotonic empirical hazard behaviours which are more likely to be encountered in practice or real life situations.

3.1.2 Quantile function

The LLoGMWP quantile function can be obtained by inverting $F_{LLoGMWP}(x) = u$, $0 \leq u \leq 1$, where

$$F_{LLoGMWP}(x) = 1 - \frac{1 - e^{\theta((1+x^c)^{-1} e^{-\alpha x^{\beta} e^{\lambda x}})}}{1 - e^{\theta}}. \tag{21}$$

The quantile function of the LLoGMWP distribution is obtained by the solving non-linear equation

$$\alpha x^{\beta} e^{\lambda x} + \log(1 + x^c) + \log\left(\left(\frac{1}{\theta} \log\left(1 - (1-u)(1-e^{\theta})\right)\right)\right) = 0, \tag{22}$$

using numerical methods. Consequently, random number can be generated based on (22). Table of quantiles from the LLoGMWP distribution for selected values of model parameters are given below.

Table 1 LLoGMWP quantile for selected values

	$(c,\lambda,\theta,\alpha,\beta)$				
u	(0.1,0.2,2.8,0.4,0.5)	(2.8,3.5,4.0,3.0,1.0)	(2.0,1.0,1.0,1.0,0.3)	(1.0,0.9,1.0,1.0,1.0)	(1.0,1.0,1.0,0.6,1.0)
0.1	0.0732	0.0964	0.1241	0.0846	0.1067
0.2	0.2539	0.1417	0.0642	0.1677	0.2123
0.3	0.4259	0.1740	0.1390	0.2505	0.3174
0.4	0.5736	0.2007	0.2258	0.3344	0.4231
0.5	0.7138	0.2249	0.3222	0.4215	0.5310
0.6	0.8611	0.2484	0.4296	0.5145	0.6439
0.7	1.0327	0.2730	0.5532	0.6181	0.7667
0.8	1.2619	0.3011	0.7045	0.7418	0.9093
0.9	1.6561	0.3391	0.9191	0.9137	1.1009

Table 2 LLoGMWP moments for selected values

	$(c,\theta,\lambda,\alpha,\beta)$				
$Moments$	(1.0,3.0,2.0,3.0,1.0)	(1.0,3.0,1.0,1.5,2.0)	(1.0,3.0,0.9,1.5,1.0)	(0.9,2,0.8,0.5,2.0)	(0.4,1.0,1.0,3.0,1.0)
μ_1'	4.3759	1.5883	1.4355	1.0340	0.5515
μ_2'	6.1566	2.8842	2.1994	0.4162	0.4804
μ_3'	2.7520	6.1940	3.6307	0.5738	0.4916
μ_4'	3.0311	6.4748	6.5323	0.9791	0.5564
μ_5'	6.4121	7.0603	12.9794	1.9405	0.6783
SD	6.5129	0.6011	0.3723	0.2541	0.8774
CV	1.4883	0.3784	0.2593	0.1212	0.4197
CS	7.6433	2.1399	1.4564	1.3556	0.7610
CK	1.4500	12.9121	7.2475	6.4151	0.2858

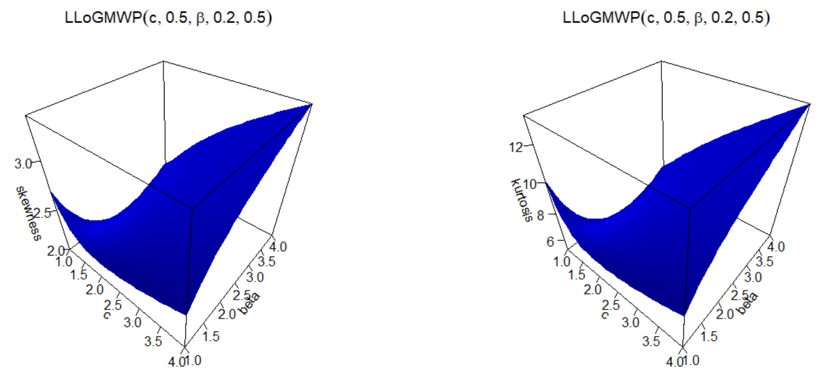

Figure 3 Plots of skewness and kurtosis of the LLoGMWP distribution

The first five moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected values of the parameters of the LLoGMWP distribution are listed in Table 2.

The 3D plots of skewness and kurtosis for the LLoGMWP distribution are given in Figures 3 and 4. We observe that

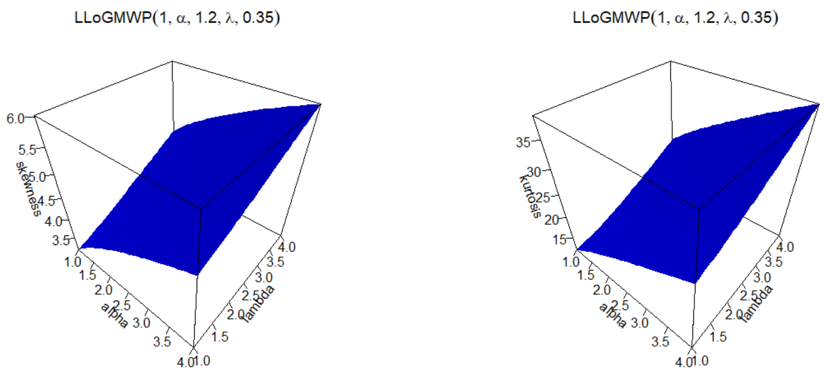

Figure 4 Plots of skewness and kurtosis of the LLoGMWP distribution

- When we fix the parameters α, λ and θ, the skewness and kurtosis of LLoGMWP increase as c and β increase.
- When we fix the parameters c, β and θ, the skewness and kurtosis of LLoGMWP increase as α and λ increase.

3.1.3 Estimation

Let $X \sim LLoGMWP(c, \alpha, \beta, \lambda, \theta)$ and $\mathbf{\Delta} = (c, \alpha, \beta, \lambda, \theta)^T$ be the parameter vector. The log-likelihood function $\ell = \ell(\mathbf{\Delta})$ based on a random sample of size n is given by

$$\begin{aligned} \ell = \ell_n(\Delta) &= n\ln(\theta) - n\ln(C(\theta)) + \sum_{i=1}^{n}\ln(g(x_i)) + \sum_{i=1}^{n}\ln(C'(\theta S(x_i)) \\ &= n\ln(\theta) - n\ln(e^{\theta}-1) + \sum_{i=1}^{n}(-\alpha x_i^{\beta}e^{\lambda x_i}) - 2\sum_{i=1}^{n}\ln(1+x_i^c) \\ &\quad + \sum_{i=1}^{n}\ln[cx_i^{c-1} + (1+x_i^c)\alpha\beta x_i^{\beta-1}e^{\lambda x_i}(\beta+\lambda x_i)] \\ &\quad + \theta\sum_{i=1}^{n}[1-e^{-\alpha x_i^{\beta}e^{\lambda x_i}}(1+x_i^c)^{-1}]. \end{aligned} \tag{23}$$

We obtain the maximum likelihood estimates of parameters denoted by $\hat{\mathbf{\Delta}}$ by solving the non-linear equation $(\frac{\partial\ell}{\partial c}, \frac{\partial\ell}{\partial\alpha}, \frac{\partial\ell}{\partial\beta}, \frac{\partial\ell}{\partial\lambda}, \frac{\partial\ell}{\partial\theta})^T = \mathbf{0}$.

3.1.4 Simulation

In this section, the performance of the maximum likelihood estimates is examined by conducting simulation studies for different sample sizes. We examine the performance of the LLoGMWP distribution by conducting various simulations for different sizes (n=25, 50, 100, 200, 400, 800, 1600) via the R package. We simulate $N = 1000$ samples and the true parameters values $I: \theta = 0.5, c = 0.5, \alpha = 0.1, \beta = 0.5, \lambda = 1.2$, $II: \theta = 1.0, c = 1.0, \alpha = 0.1, \beta = 0.3, \lambda = 1.0$, The mean, average bias and root mean square error (RMSE) were computed. The mean for the estimated parameter

$\hat{\theta}$, average bias and RMSE are given by

$$Mean = \frac{\sum_{i=1}^{N}\hat{\theta}}{N}, ABias(\hat{\theta}) = \frac{\sum_{i=1}^{N}\hat{\theta}_i}{N} - \theta \quad \text{and} \quad RMSE(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{N}(\hat{\theta}_i - \theta)^2}{N}}.$$

The table lists the mean MLEs of the parameters along with the respective average bias and root mean squared errors (RMSEs).

Table 3 Monte Carlo simulation results: Mean, Average Bias and RMSE

		I			II		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
θ	25	0.9575	1.5802	0.4523	3.2679	20.4624	2.2679
	50	0.7539	0.5933	0.2539	1.2195	1.6281	0.2195
	100	0.6183	0.4247	0.1183	1.0887	0.8129	0.0887
	200	0.5425	0.3424	0.0425	1.1013	0.6287	0.1013
	400	0.4845	0.2698	-0.0154	1.0549	0.4887	0.0549
	800	0.4284	0.2144	-0.0715	1.0621	0.3540	0.0621
	1600	0.4983	0.0528	-0.0016	1.0954	0.2774	0.0954
c	25	0.5686	0.2964	0.0686	2.0350	1.5543	1.0350
	50	0.5410	0.1724	0.0410	1.9783	1.3260	0.9783
	100	0.5288	0.0994	0.0288	1.9590	1.2525	0.9590
	200	0.5211	0.0627	0.0211	1.9371	1.1921	0.9371
	400	0.5208	0.0467	0.0208	1.9510	1.1390	0.9510
	800	0.5292	0.0371	0.0292	0.9928	0.4008	-0.0072
	1600	0.5167	0.0168	0.0167	0.9556	0.0825	-0.0443
α	25	0.1792	0.2468	0.0792	0.2089	0.5388	0.1089
	50	0.1642	0.1543	0.0642	0.1533	0.1906	0.0533
	100	0.1338	0.0955	0.0338	0.1369	0.1325	0.0369
	200	0.1187	0.0634	0.0187	0.1332	0.1076	0.0332
	400	0.1050	0.0420	0.0050	0.1199	0.0757	0.0199
	800	0.1044	0.0262	0.0044	0.1188	0.0535	0.0188
	1600	0.1037	0.0041	0.0037	0.1218	0.0440	0.0218
β	25	1.2602	1.2992	0.7602	1.2374	1.4298	0.9374
	50	1.1215	1.1941	0.6215	1.0798	1.2891	0.7798
	100	0.9277	1.0101	0.4277	0.8783	1.1181	0.5783
	200	0.7493	0.7094	0.2493	0.7078	0.8834	0.4078
	400	0.7333	0.7232	0.2333	0.5520	0.6036	0.2520
	800	0.5392	0.0613	0.0392	0.4253	0.2894	0.1253
	1600	0.5132	0.0521	0.0132	0.3965	0.1078	0.0965
λ	25	1.0804	0.8024	-0.1195	0.8538	0.7104	-0.1461
	50	0.9752	0.6738	-0.2247	0.8051	0.5963	-0.1948
	100	1.0338	0.5376	-0.1661	0.8353	0.4925	-0.1646
	200	1.0936	0.3791	-0.1063	0.8566	0.3821	-0.1433
	400	1.1255	0.3376	-0.0744	0.9089	0.2751	-0.0910
	800	1.2210	0.0983	0.0210	0.9320	0.1745	-0.0679
	1600	1.2005	0.0140	0.0005	0.9923	0.1562	-0.0077

The results in Table 3 show that the mean MLEs converge to the true values and the average bias decreases for all parametric values as the sample size n increases, and the RMSEs decay toward zero.

3.1.5 Applications

In this section, applications of LLoGMWP distribution to real data are presented. The maximum likelihood estimates (MLEs) of the LLoGMWP parameters $\Delta = (c, \alpha, \beta, \lambda, \theta)$ are computed by maximizing the objective function via the subroutine NLMIXED in SAS as well as the function nlm in R (R Development Core Team (2011)). The estimated values of the parameters (standard error in parenthesis), -2log-likelihood statistic $(-2\ln(L))$, Akaike Information Criterion $(AIC = 2p -$

$2\ln(L)$), Bayesian Information Criterion ($BIC = p\ln(n) - 2\ln(L)$) and $AICC$=$AIC + 2\frac{p(p+1)}{n-p-1}$, where $L = L(\hat{\Delta})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters are presented in Tables 4 and 5. The Cramér-von Mises (W^*) and Anderson-Darling (A^*) goodness-of-fit statistics, described by Chen and Balakrishnan (1995) are also presented in the tables. These statistics can be used to verify which distribution fits better to the data. In general, the smaller the values of W^* and A^*, the better the fit. The Sum of Squares (SS) from the probability plots is given by

$$SS = \sum_{j=1}^{n} \left[F(x_{(j)}) - \left(\frac{j - 0.375}{n + 0.25} \right) \right]^2,$$

and was also computed for the fitted models, where $j = 1, 2....., n$ and $x_{(j)}$ are the ordered values of the observed data.

We can use the likelihood ratio (LR) test to compare the fit of the LLoGMWP distribution with its sub-models for a given data set. For example, to test $\beta = 1$, the LR statistic is $\omega = 2[\ln(L(\hat{c}, \hat{\alpha}, \hat{\beta}, \hat{\lambda}, \hat{\theta})) - \ln(L(\tilde{c}, \tilde{\alpha}, 1, \tilde{\lambda}, \tilde{\theta}))]$, where $\hat{c}$, $\hat{\alpha}$, $\hat{\beta}$, $\hat{\lambda}$, and $\hat{\theta}$ are the unrestricted estimates, and $\tilde{c}$, $\tilde{\alpha}$, $\tilde{\lambda}$, and $\tilde{\theta}$ are the restricted estimates. The LR test rejects the null hypothesis if $\omega > \chi^2_\epsilon$, where χ^2_ϵ denote the upper $100\epsilon\%$ point of the χ^2 distribution with 1 degrees of freedom.

The data sets in this section are used to illustrate the flexibility of the LLoGMWP distribution and its sub-models (LLoGP, LLoGEP) for data modeling. We compare the LLoGMWP distribution with the gamma-Dagum (GD) (Oluyede et al., 2014), Exponentiated Kumaraswamy Dagum (EKD) (Huang and Oluyede, 2014), Beta Weibull log-logistic BWLLoG (Makubate et al., 2018), Exponentiated Kumaraswamy Weibull (EKW) (Eissa and Abdulaziz, 2014), LoG-Logistic Weibull Poisson (LLoGWP) (Oluyede et al., 2016), and Burr XII Weibull Logarithmic (BrWLn) (Oluyede et al., 2018) distributions. The pdfs of the GD, EKD, BWLLoG, EKW, LLoGWP, and BrWLn distributions are given by

$$\begin{aligned}
f_{GD}(x) &= \frac{\lambda\beta\delta x^{-\delta-1}}{\Gamma(\alpha)\theta^{\alpha}}(1+\lambda x^{-\delta})^{-\beta-1}\bigg(-\log[1-(1+\lambda x^{-\delta})^{-\beta}]\bigg)^{\alpha-1} \\
&\quad \times[1-(1+\lambda x^{-\delta})^{-\beta}]^{(1/\theta)-1}, \text{ for } \alpha,\lambda,\delta,\beta,\theta>0, x>0. \qquad (24)\\
f_{EKD}(x) &= \alpha\lambda\delta\phi\theta x^{-\delta-1}(1+\lambda x^{-\delta})^{-\alpha-1}\left[1-(1+\lambda x^{-\delta})^{-\alpha}\right]^{\phi-1} \\
&\quad \times\left[1-\left[1-(1+\lambda x^{-\delta})^{-\alpha}\right]^{\theta-1}\right], \text{ for } \alpha,\lambda,\delta,\phi,\theta>0, x>0. \qquad (25)\\
f_{BWLLoG}(x) &= \frac{\alpha\beta c}{B(a,b)}x^{c-1}(1+x^c)^{-2}\frac{\left[1-(1+x^c)^{-1}\right]^{\beta-1}}{\left[(1+x^c)^{-1}\right]^{\beta+1}} \\
&\quad \times exp\left\{-\alpha b\left[(1+x^c)-1\right]^{\beta}\right\} \\
&\quad \times\left[1-exp\left\{-\alpha\left[(1+x^c)-1\right]^{\beta}\right\}\right]^{a-1}, \text{ for } a,b,c,\alpha,\beta>0, x>0. \qquad (26)\\
f_{EKW}(x) &= \theta abc\lambda^c x^{c-1}exp-(\lambda x)^c[1-exp-(\lambda x)^c]^{a-1} \\
&\quad \times\left[1-[1-exp-(\lambda x)^c]^a\right]^{b-1}\left[1-[1-[1-exp-(\lambda x)^c]^a]^b\right]^{\theta-1}, \qquad (27)
\end{aligned}$$

for $a, b, c, \lambda, \theta > 0$, $x > 0$,

$$\begin{aligned}
f_{LLoGWP}(x) &= \frac{\theta e^{\theta\left(1-\left(1+\left(\frac{x}{s}\right)^c\right)^{-1}e^{-\alpha x^{\beta}}\right)}}{e^{\theta-1}}\left(1+\left(1+\left(\frac{x}{s}\right)^c\right)^{-1}\right)e^{-\alpha x^{\beta}} \\
&\quad \times\left[\left(1+\left(\frac{x}{s}\right)^c\right)^c\frac{c}{s}\left(\frac{x}{s}\right)^{c-1}+\alpha\beta x^{\beta-1}\right], \text{ for } s,c,\alpha,\beta,\theta>0, x>0. \qquad (28)
\end{aligned}$$

$$f_{BrWLn}(x) = \frac{\theta e^{-\alpha x^{\beta}} \left(1+x^{c}\right)^{-k-1} \left(kcx^{c-1} + \alpha\beta x^{\beta-1}\left(1+x^{c}\right)\right)}{-\left(1-\theta\left(1+x^{c}\right)^{-k} e^{-\alpha x^{\beta}}\right) \log\left(1-\theta\right)}, \tag{29}$$

for $k, c, \alpha, \beta, \theta > 0$, $x > 0$.

3.1.5.1 Failure time of 50 components data

The first set of data represent failure times of 50 components from Murthy et al. (2004). Estimates of the parameters of LLoGMWP distribution and its related sub-models (standard error in parentheses), AIC, AICC, BIC, W*, A*, KS, P-value and SS are give in Table 4. The asymptotic variance-covariance matrix of the MLEs for the LLoGMWP distribution is given by:

$$\begin{pmatrix} 0.0739 & 0.0134 & 0.0004 & -0.4383 & -0.0170 \\ 0.0134 & 0.0157 & -0.0011 & -0.0156 & -0.0144 \\ 0.0004 & -0.0011 & 0.0007 & -0.0418 & 0.0054 \\ -0.4383 & -0.0156 & -0.0418 & 4.7653 & -0.1960 \\ -0.0170 & -0.0144 & 0.00542 & -0.1960 & 0.2903 \end{pmatrix} \tag{30}$$

and the approximate 95% two-sided confidence intervals for $\lambda, c, \alpha, \beta$ and θ are given by 0.2478 ± 0.5329, 0.7776 ± 0.2459, 0.0065 ± 0.0523, 0.7740 ± 4.2786, and 0.3708 ± 1.0560, respectively.

Table 4 Estimates of model for failure times of 50 components data

	Estimates					Statistics								
Model	λ	c	α	β	θ	$-2\log Ł$	AIC	$AICC$	BIC	W^*	A^*	SS	KS	$P-VALUE$
LLoGMWP	0.2478	0.7776	0.0066	0.7740	0.3708	197.6134	207.6134	208.9771	217.1735	0.0941	0.6137	0.0853	0.0994	0.6696
	(0.2719)	(0.1246)	(0.0267)	(2.1832)	(0.5388)									
LLoGP	-	0.9098	-	-	0.9558	210.9849	214.9849	215.2402	218.8089	0.2140	1.3296	0.5448	0.1372	0.2774
	-	(0.1029)	-	-	(0.4911)									
LLoGEP	-	0.7372	0.7800	-	3.6898×10^{-08}	206.2124	212.2124	212.7342	217.9485	0.1523	0.9451	0.3066	0.1839	0.0592
	-	(0.1238)	(0.3954)	-	(0.2131)									
	α	λ	δ	ϕ	θ									
EKD	1.0279	1.846	1486.00	0.7485	9250.1000	204.7135	214.7135	216.0771	224.2736	0.1499	0.9460	3.4806	0.1331	0.3101
	(0.3081)	(0.0000014)	(0.7378)	(0.000002)	(1.5469)									
	λ	β	δ	α	θ									
GD	0.0004	0.5934	1.2483	12.5790	0.6682	208.2542	218.2543	219.6179	227.8144	0.2148	1.3219	0.2162	0.1606	0.1359
	(0.0018)	(0.2332)	(0.6123)	(0.0391)	(0.3247)									
	c	a	b	θ	λ									
EKW	0.5110	10.0989	7.1749	0.1173	0.2359	202.8997	212.8998	214.2634	222.4599	0.1318	0.8531	0.1353	0.1363	0.2845
	(0.3118)	(9.7929)	(11.3697)	(0.1166)	(0.2870)									
	c	a	b	α	β									
BWLLoG	0.4256	1.4584	13.3910	0.0709	1.2299	204.6975	214.6975	216.0612	224.2577	0.1549	0.9658	0.1616	0.1187	0.4475
	(2.4626)	(3.2680)	(56.1346)	(0.2913)	(7.1157)									
	s	c	α	β	θ									
LLoGWP	1.6389	0.6939	0.0268	1.5685	5.5051×10^{-06}	200.4967	210.4967	211.8603	220.0568	0.1123	0.7305	0.0970	0.1075	0.5723
	(0.8462)	(0.1230)	(0.0457)	(0.6521)	(0.2441)									
	k	c	α	β	θ									
BrWLn	1.051×10^{-10}	1.2728	0.2679	0.6168	5.3097×10^{-09}	228.4620	238.4625	239.8261	248.0226	0.1524	0.9498	1.8593	0.2750	0.0007
	(1.0310×10^{-06})	(0.6154)	(0.04024)	(0.0721)	(0.0107)									

The LR for the following hypothesis testing H_0 : LLoGP against H_a : LLoGMWP and H_0 : LLoGEP against H_a : LLoGMWP are 13.3715 (p-value=0.0038) and 8.599 (p-value= 0.0136). We conclude that there is significant difference between the non-nested models and LLoGMWP distribution. The goodness-of-fit statistics W^*, A^*, KS and its p-value show that the LLoGMWP distribution is far better than the sub-models, and the non-nested models. Also, the values of AIC, AICC and BIC shows that the LLoGMWP distribution fits better than the non-nested distributions.

Plots of the fitted densities and probability plots are given in Figure 5. For probability plots figure, the plot with the smallest SS value corresponds to the model with points that are closer to the diagonal line. See Table 4 for the SS-values.

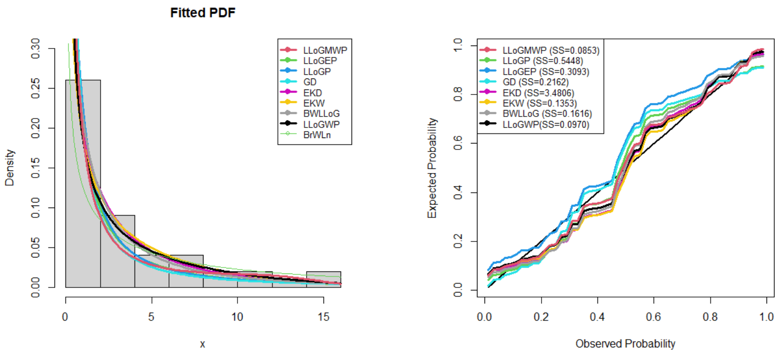

Figure 5 Fitted PDF and probability plots for failure times of 50 components data

3.1.5.2 Aircraft windshield failure data

The dataset analyzed in Table 5 represents the data sets for a particular aircraft windshield reported by Murthy et al. (2004). The dataset is divided into two different sets of data being the 88 observations that are classified as failed windshields and the 65 observations which are service times of windshields that had not failed at the time of observation. The measurement unit is 1000 hours.

Table 5 Estimates of model for aircraft windshield failure data

	Estimates					Statistics								
Model	λ	c	α	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-VALUE$	SS
LLoGMWP	0.3222	0.1024	0.1776	0.06971	25.0918	253.6165	263.6165	264.3857	275.7706	0.0469	0.5376	0.0642	0.9689	0.0728
	(0.1233)	(0.4118)	(0.9597)	(0.2213)	(30.4056)									
LLoGP	-	2.3891	-	-	5.6505	286.4005	290.4005	290.5487	295.5622	0.3113	2.2440	0.1304	0.1147	0.2087
	-	(0.1737)	-	-	(0.76491)									
LLoGEP	-	0.4780	0.2590	-	$1.2934X10^{-08}$	419.3331	425.3331	425.6333	432.6257	0.1512	1.3009	0.1785	0.5743	0.3222
	-	(0.5083)	(0.1490)	-	(0.0090)									
	λ	β	δ	α	θ									
GD	5.0919	6.6830	0.7827	0.4698	0.0010	260.4438	270.4440	271.2132	282.5981	0.0658	0.6310	0.0775	0.6934	0.5995
	(0.0101)	(0.0038)	(0.2214)	(0.1492)	(0.0016)									
	α	λ	δ	ϕ	θ									
EKD	2.8909	16.1328	1.9388	10.5093	0.4057	261.9840	271.9840	272.7532	284.1381	0.0649	0.6581	0.0756	0.7224	0.0881
	(1.4745)	(12.3388)	(0.4457)	(5.4744)	(0.2462)									
	c	a	b	θ	λ									
EKW	1.8858	3.6003	1.53512	0.3442	0.34900	261.1730	271.1730	271.9422	283.3271	0.0643	0.6435	0.0753	0.7278	0.0843
	(0.5653)	(2.2276)	(1.1677)	(0.2446)	(0.0721)									
	c	a	b	α	β									
BWLLoG	5.8439	0.2885	5.3215	0.0005	2.5840	255.4797	265.4797	266.2490	277.6338	0.1185	0.7838	0.0680	0.9421	7.8727
	(0.0018)	(0.0307)	$(2.0250x10^{-6})$	$(1.1249x10^{-6})$	(0.0113)									
	s	c	α	β	θ									
LLoGWP	2.5559	4.3028	0.0714	0.8998	0.3062	258.8826	268.8826	269.6519	281.0367	0.0770	0.5822	0.0770	0.5822	0.0877
	(0.6394)	(0.5457)	(0.1189)	(0.6295)	(2.8061)									
	k	c	α	β	θ									
BrWLn	1.7919×10^{-06}	1.9719	0.5418	0.4345	1.8402×10^{-09}	450.7534	460.7547	461.5239	472.9088	0.3257	2.4118	0.3257	2.591×10^{-10}	1.8593
	(1.0310×10^{-06})	(2.3539×10^{-05})	(0.0250)	(0.0459)	(0.0052)									

The estimated variance-covariance matrix for the LLoGMWP distribution is given by:

$$\begin{pmatrix} 0.0152 & 0.03261 & -0.1009 & -0.0025 & -2.9859 \\ 0.03261 & 0.1696 & -0.1149 & -0.06028 & -2.8573 \\ -0.1009 & -0.1149 & 0.9210 & -0.0891 & 28.9171 \\ -0.0025 & -0.0603 & -0.0891 & 0.0489 & -3.282 \\ -2.9858 & -2.8573 & 28.9171 & -3.2824 & 924.5021 \end{pmatrix}$$

and the approximate 95% two-sided confidence intervals for $\lambda, c, \alpha, \beta$ and θ are given by $0.3222 \pm$

0.2416, 0.1024 ± 0.8072, 1.1776 ± 1.8810, 0.0691 ± 0.4337, and 25.0918 ± 59.5950, respectively.

The following hypothesis were tested using the LR test. The values for LR test statistic for testing H_0 : LLoGP against H_a : LLoGMWP and H_0 : LLoGEP against H_a: LLoGMWP are 32.7840 (p-value<0.00001) and 165.7166 (p-value<0.00001). We conclude that there is significant difference between LLoGP, LLoGEP and LLoGMWP distributions. However, the goodness-of-fit statistics W^*, A^* and KS and its p-value show that the LLoGMWP distribution fits better than the sub-models and the non-nested models. Lastly, the values of AIC, AICC and BIC shows that the LLoGMWP distribution fits better than the non-nested distributions. The probability plots and fitted densities are given in Figure 6.

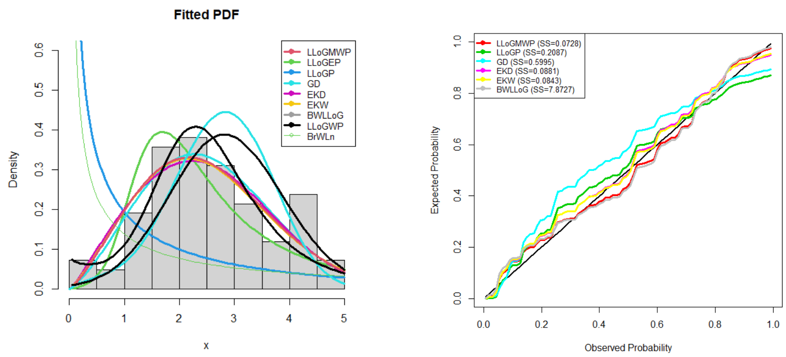

Figure 6 Fitted densities and probability plots for aircraft windshield failure data

3.2. Log-Logistic modified Weibull logarithmic distribution

The cdf and pdf of the LLoGMWL distribution are given by Eqns. (7) and (8), respectively. Plots of the pdf of the LLoGMWL distribution are given in the Figure 5.

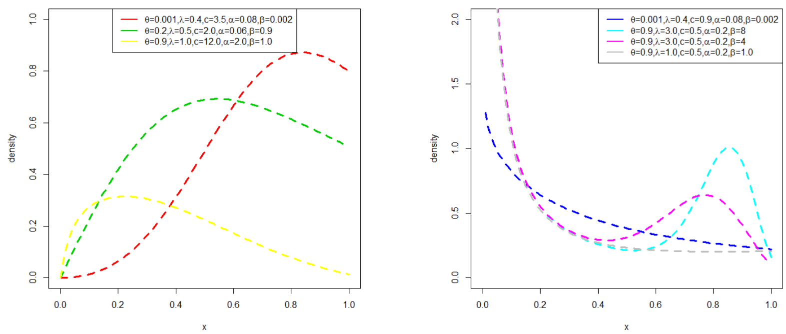

Figure 7 Plots of the LLoGMWL density function

Figure 7 show that plots of the the LLoGMWL pdf can be decreasing, increasing-decreasing, right-skewed, uni-modal and reverse-J shapes for different selected values of the parameters.

3.2.1 Hazard rate function

The hrf of the LLoGMWL distribution is given in Eqn. (9). The graphs below show the hrf of LLoGMWL distribution in different shapes.

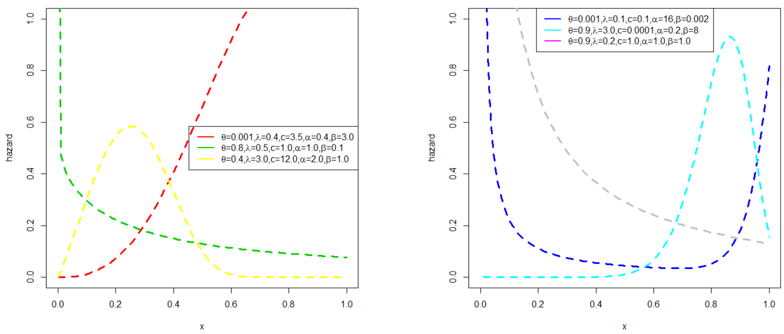

Figure 8 Plots of the LLoGMWL hazard function

The graphs in Figure 8 exhibit increasing, decreasing, bathtub and unimodal shapes for the selected values of the model parameters.

3.2.2 Quantile function

The quantile function of the LLoGMWL distribution can be obtained by inverting $F_{LLoGMWL}(x) = u$, $0 \leq u \leq 1$, where

$$F_{LLoGMWL}(x) \quad = \quad 1 - \frac{\log(1 - \theta(1 + x^c)e^{-\alpha x^\beta e^{\lambda y}})}{\log(1 - \theta)}.$$

The quantile function of the LLoGMWL distribution is obtained by solving the non-linear equation

$$\alpha x^\beta e^{\lambda x} + \log(1 + x^c) + \log\left(\frac{1}{\theta}\left(1 - \frac{1 - \theta}{(1 - \theta)^u}\right)\right) = 0. \tag{31}$$

Quantiles of the LLoGMWL distribution are presented in Table 6 for selected values of the model parameters.

Table 6 LLoGMWL quantiles for selected parameter values

	$(c, \lambda, \theta, \alpha, \beta)$				
u	(0.1,0.2,2.8,0.4,0.5)	(0.1,3.5,4.0,3.0,0.8)	(0.2,1.0,1.0,1.0,0.3)	(0.1,0.9,1.0,1.0,0.5)	(0.9,1.0,1.0,0.6,1.0)
0.1	0.0607	0.0134	0.0003	0.0083	0.0182
0.2	0.2236	0.0318	0.0043	0.0312.	0.0418
0.3	0.3921	0.0524	0.0182	0.0674	0.0727
0.4	0.5388	0.0748	0.0476	0.1167	0.1135
0.5	0.6777	0.0989	0.0967	0.1810	0.1681
0.6	0.8230	0.1254	0.1691	0.2636	0.2423
0.7	0.9918	0.1555	0.2716	0.3715	0.3457
0.8	1.2166	0.1917	0.4194	0.5191	0.4959
0.9	1.6030	0.2422	0.6565	0.7484	0.7369

The first six moments, standard deviation (SD), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for selected values of the parameters of the LLoGMWL distribution are listed in Table 7.

The 3D plots of skewness and kurtosis for the LLoGMWL distribution are given in Figures 9 and 10. We observe that

Table 7 LLoGMWL moments for selected parameter values

	$(c,\theta,\lambda,\alpha,\beta)$				
$Moments$	(1.0,1.0,1.0,1.0,1.0)	(0.2,0.0,0.1,1.0,0.2)	(1.0,1.0,1.0,0.9,1.0)	(0.6,0.6,0.5,0.5,0.05)	(0.3,1.0,1.6,1.0,0.1)
μ_1'	0.0802	0.0530	0.0288	0.0319	0.0996
μ_2'	0.0567	0.0360	0.0194	0.0213	0.0707
μ_3'	0.0443	0.0275	0.0147	0.0160	0.0560
μ_4'	0.0365	0.0222	0.0120	0.0128	0.0451
μ_5'	0.0310	0.0188	0.0102	0.0107	0.0382
μ_6'	0.0271	0.0162	0.0088	0.0092	0.0332
SD	0.2243	0.1821	0.1363	0.1425	0.2465
CV	2.7982	3.4367	4.7233	4.4695	2.4747
CS	2.8062	3.6496	5.2023	4.8617	2.3936
CK	9.6071	15.5148	30.1957	25.5557	7.3354

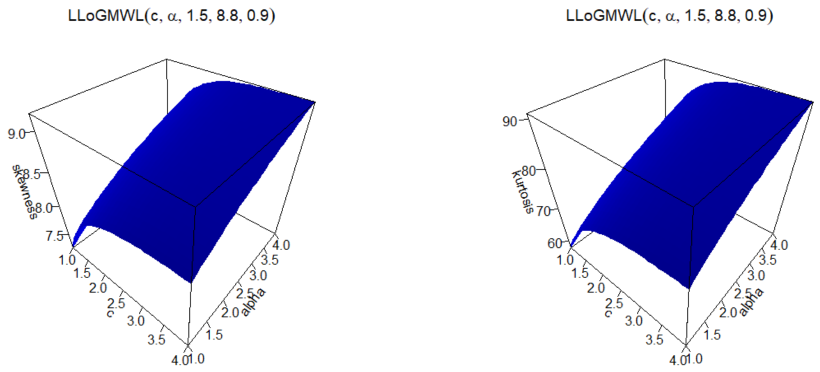

Figure 9 Plots of skewness and kurtosis of the LLoGMWL distribution.

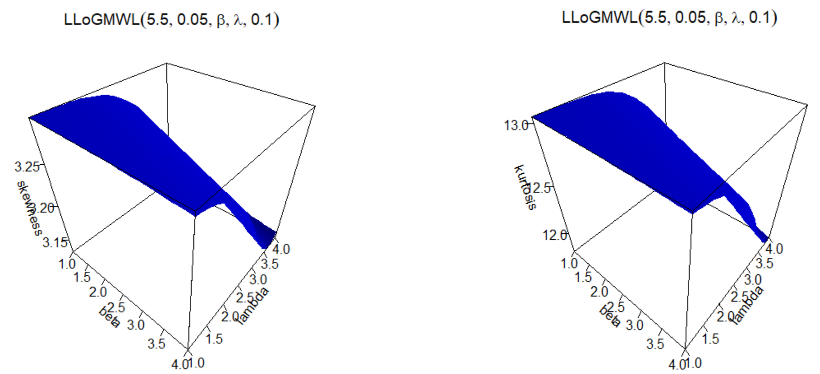

Figure 10 Plots of skewness and kurtosis of the LLoGMWl distribution.

- When we fix the parameters β, λ and θ, the skewness and kurtosis of LLoGMWL increase as c and α increase.
- When we fix the parameters c, α and θ, the skewness and kurtosis of LLoGMWL decrease as β and λ increase.

3.2.3 Estimation

Let $X \sim LLoGMWL(c,\alpha,\beta,\lambda,\theta)$ and $\mathbf{\Delta} = (c,\alpha,\beta,\lambda,\theta)^T$ be the parameter vector. The log-likelihood function $\ell = \ell(\mathbf{\Delta})$ for a single observation x of X is given by

$$\begin{aligned}\ell = \log L &= \ln(\theta) - 2\ln(1+x^c) - \alpha x^{\beta} e^{\lambda x} + \ln\left(cx^{c-1} + \alpha x^{\beta-1} e^{\lambda x}(\beta+\lambda x)\right) \\ &- \ln\left(-\left(1-\theta(1+x^c)^{-1} e^{-\alpha x^{\beta} e^{\lambda x}}\right)\right) - \ln(\ln(1-\theta)).\end{aligned}$$

Elements of the score vector are given in the web-appendix.

3.2.4 Simulation study

A simulation study was conducted to check the performance of the maximum likelihood estimates. The simulation study was repeated $N = 1000$ times each with sample size $n = 35, 65, 90, 200, 400, 800$ with the true parameters values $I : \theta = 0.6, \lambda = 0.5, c = 0.8, \alpha = 4.0, \beta = 1.8$, and $II : \theta = 0.5, \lambda = 0.7, c = 1.0, \alpha = 4.0, \beta = 8$. The table below shows the mean MLEs with respective their average bias and Root Mean Square Error (RMSE). The results in Tables 8 show that the mean MLEs converge to the true value and the RMSEs decay toward zero as the sample size n increases. Also, the average bias decreases as the sample size n increases for all parametric values.

Table 8 Monte Carlo simulation results: Mean, Average Bias and RMSE

		I			II		
Parameter	n	Mean	Average Bias	RMSE	Mean	Average Bias	RMSE
θ	25	0.6977	0.0977	0.3026	0.8623	0.3623	0.2491
	50	0.6474	0.0474	0.2564	0.8415	0.3415	0.2297
	100	0.6337	0.0337	0.2398	0.8067	0.2067	0.1985
	200	0.6242	0.0244	0.2333	0.7794	0.2794	0.1757
	400	0.6127	0.0127	0.2285	0.6875	0.1875	0.1335
	800	0.6079	0.0079	0.2252	0.4896	0.0104	0.0993
λ	25	1.2407	0.7407	2.5277	1.2228	0.5228	2.7739
	50	0.9818	0.4818	1.7046	1.1944	0.4940	2.5901
	100	0.7942	0.2942	0.8581	1.0297	0.3297	2.2550
	200	0.7169	0.2169	0.8580	0.8585	0.1585	1.8065
	400	0.6039	0.1039	0.6147	0.7842	0.0842	1.3980
	800	0.4540	-0.0454	0.4992	0.7401	0.0410	1.3820
c	25	0.9122	0.1122	0.4366	1.515	0.5152	0.3310
	50	0.8915	0.0915	0.2559	1.4356	0.4356	0.2295
	100	0.8635	0.0635	0.1838	1.3101	0.0310	0.1510
	200	0.8630	0.0630	0.1410	1.1474	0.1474	0.1112
	400	0.8382	0.0382	0.1032	1.0624	0.0624	0.0781
	800	0.8309	0.0309	0.0882	1.0011	0.0011	0.0558
α	25	7.3775	3.3776	13.5132	11.6146	7.6146	20.6478
	50	5.3212	1.3212	4.8617	8.4929	4.4929	9.7755
	100	4.6752	0.6752	3.0319	7.2175	3.2175	5.6443
	200	4.3856	0.3856	2.4560	6.5282	2.5286	4.4378
	400	4.1856	0.1856	2.1026	6.0693	2.0693	3.9642
	800	4.0351	0.0350	1.7837	4.3504	0.3504	0.7296
β	25	2.0542	0.9054	3.2785	8.9300	0.9300	5.4126
	50	1.9266	0.1266	0.6707	8.2174	0.2174	3.4122
	100	1.9138	0.1138	0.5261	8.6611	0.6611	2.6054
	200	1.9115	0.1115	0.4689	8.4289	0.4288	1.9195
	400	1.9225	0.1225	0.4215	8.1695	0.1695	1.3855
	800	1.8273	0.0270	0.3901	7.8895	-0.1104	1.2641

3.2.5 Applications

In this subsection, some datasets are used to illustrate the usefulness and applicability of the LLoGMWL distribution. The LLoGMWL distribution is compared with its sub-models: log-logistic Weibull (LLoGW), log-logistic Rayleigh (LLoGR) and log-logistic (LLoG) distributions. We also compare the LLoGMWL distribution with Gamma Dagum (GD) (Oluyede et al., 2014), Exponentiated Kumaraswamy Dagum (EKD) (Huang and Oluyede, 2014), Exponentiated Kumaraswamy Weibull (EKW) (Eissa and Abdulaziz, 2014), LoG-Logistic Weibull Poisson (LLoGWP) (Oluyede et al., 2016), and Burr XII Weibull Logarithmic (BrWLn) (Oluyede et al., 2018) distributions.

3.2.5.1 Caterpillar body mass data

The first dataset comes from R software package "Stat2Data" Caterpillars data, consisting of 267 observations corresponding to body mass (in grams) of Manduca Sexta caterpillars. The estimated variance-covariance matrix of the LLoGMWL distribution is given by

$$\begin{pmatrix} 0.01584 & 0.00108 & -0.00016 & 0.00014 & 0.00096 \\ 0.00108 & 0.00140 & -0.00001 & 0.00009 & 0.00106 \\ -0.00016 & -0.00001 & 0.000001 & -0.00001 & -0.00001 \\ 0.00145 & 0.00009 & -0.00001 & 0.00013 & 0.00008 \\ 0.00096 & 0.00106 & -0.00001 & 0.00008 & 0.00169 \end{pmatrix}$$

and the approximate 95% two-sided confidence intervals for $\lambda, c, \alpha, \beta$ and θ are given by 0.5371 ± 0.2467, 0.5722 ± 0.0735, 0.0008 ± 0.0025, 0.5926 ± 0.0227, and 0.8651 ± 0.0806, respectively.

Table 9 Estimates of model for caterpillar body mass data

	Estimates					Statistics								
Model	λ	c	α	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-VALUE$	SS
LLoGMWL	0.5371	0.5723	0.0008	0.5926	0.8651	281.2231	291.2231	291.4530	309.1594	0.3005	1.9010	0.0606	0.2500	0.2747
	(0.1259)	(0.0375)	(0.0013)	(0.0116)	(0.0412)									
LLoGW	-	0.4153	0.4903	0.5498	-	343.5370	349.5370	349.6282	360.2987	0.5982	4.0694	0.6935	0.0074	0.5975
	-	(0.0364)	(0.0925)	(0.0805)	-									
LLoGR	-	0.4268	0.0149	-	-	348.5562	433.9442	433.9896	441.1187	0.5364	3.6470	0.2085	1.644×10^{-10}	4.1926
	-	(0.0251)	(0.0025)	-	-									
LLoG	-	0.5139	-	-	-	448.3505	450.3505	450.3656	453.9377	0.6294	4.2539	0.2318	6.969×10^{-13}	7.0787
	-	(0.0259)	-	-	-									
	λ	β	δ	α	θ									
GD	3.3889	3.0748	0.0612	3.0667	0.0015	341.9914	351.9913	352.2211	369.9275	0.5309	3.6239	0.1064	0.0046	0.5396
	(0.000009)	(0.00005)	(0.0027)	(0.00001)	(0.00001)									
	α	λ	δ	ϕ	θ									
EKD	0.0050	312.5400	2.3520	2.0050	22.2950	324.4124	334.4124	334.3423	352.3486	0.4682	3.2383	0.0912	0.0235	0.5282
	(0.3081)	(0.0000014)	(0.7378)	(0.000002)	(1.5469)									
	c	a	b	θ	λ									
EKW	0.8727	3.7969	0.9770	0.0891	0.2629	357.6407	447.8232	448.1462	469.3467	1.1149	7.2879	0.1594	2.574×10^{-06}	0.9779
	(0.7512)	(3.4702)	(0.8459)	(0.0807)	(0.2582)									
	s	c	α	β	θ									
LLoGWP	0.2276	0.4769	0.0975	1.0561	1.7582×10^{-09}	322.8534	332.8395	333.0694	350.7757	0.4386	3.0787	0.0732	0.1136	0.2969
	(0.0553)	(0.3135)	(0.0385)	(0.1771)	(0.0020)									
	k	c	α	β	θ									
BrWLn	1.6843	0.4602	1.8759×10^{-05}	4.6779	6.4784×10^{-03}	292.0598	302.0598	302.2897	319.9961	0.3701	2.4162	0.0814	0.0577	0.3097
	(0.0553)	(0.3135)	(0.0385)	(0.1771)	(0.0020)									

To test if the LLoGMWL distribution is significantly different from LLoGW, LLoGR and LLoG distributions, the LR test is used and the following hypothesis are tested. H_0 : LLoGW against H_a : LLoGMWL, H_0 : LLoGR against H_a : LLoGMWL and H_0 : LLoG against H_a : LLoGMWL. The values of LR test statistic and corresponding p-values are: 62.3139 (p-value<0.00001), 67.3331 (p-value<0.00001) and 164.1274 (p-value<0.00001), respectively. We conclude that there are significant differences between each one of LLoGW, LLoGR, LLoG and LLoGMWL distributions. The

LLoGMWL distribution is performing far better when compared to the nested and non-nested models when looking at the W^*, A^* and SS values. Also, the values of AIC, AICC and BIC shows that the LLoGMWL distribution fits better than the non-nested distributions. Plots of the fitted densities and probability plots are given in Figure 11.

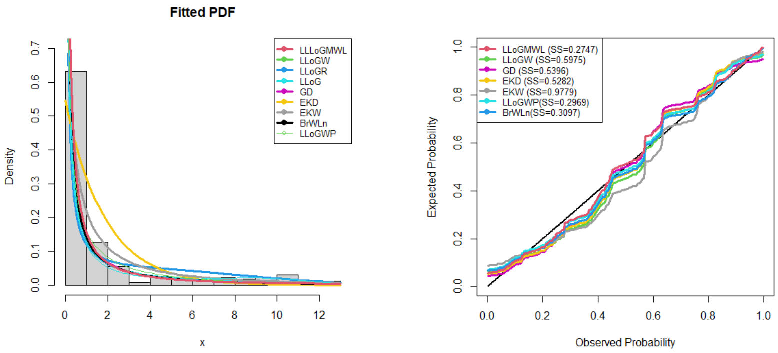

Figure 11 Fitted pdf and probability plots for caterpillar body mass data

3.2.5.2 Kevlar 49/Epoxy strands data

Cooray and Ananda (2008) analyzed this data. The data set consist of 101 data points that represent the stress-rupture of kevlar 49/epoxy strands life which are subjected to constant sustained pressure at the 90 percent stress level until all have failed, so that the complete data set with the exact times of failures is recorded. Andrews and Herzberg (1985) and Barlow (1984) presented this failure times in hours.The estimated variance-covariance matrix of the LLoGMWL distribution is given by

$$\begin{pmatrix} 0.0053 & -0.0021 & 7.648\times10^{-6} & -0.0064 & -0.0057 \\ -0.0021 & 0.0618 & -1.1325\times10^{-5} & 0.0064 & 0.0012 \\ 7.648\times10^{-6} & -1.1325\times10^{-5} & 0.0001 & -4.2426\times10^{-5} & 3.9738\times10^{-06} \\ -0.0064 & 0.0064 & -4.2426\times10^{-5} & 0.0154 & 0.0063 \\ -0.0057 & 0.0012 & 3.9738\times10^{-6} & 0.0063 & 0.0129 \end{pmatrix}$$

and the approximate 95% two-sided confidence intervals for $\lambda, c, \alpha, \beta$ and θ are given by 0.0103 ± 0.1434, 2.2224 ± 0.4875, $1.0 \times 10^{-8} \pm 0.2433$, 0.6007 ± 0.2225, and 0.6984 ± 0.0274, respectively.

The LR test statistics of the hypothesis; H_0 : LLoGW against H_a : LLoGMWL, H_0 : LLoGR against H_a : LLoGMWL, H_0 : LLoG against H_a : LLoGMWL are 82.3328 (p-value<0.00001), 88.6469 (p-value<0.00001)and 112.8602 (p-value<0.00001), respectively. We conclude that there is significant difference between each one of the nested LLoGW, LLoGR, LLoG distributions and the LLoGMWL distribution. When using goodness-of-fit statistics W^*, A^*, KS and its p-value and SS values, it is clear that the LLoGMWL distribution is performing far much better when compared to both the nested and non-nested models. Also, the smallest values of the AIC, AICC and BIC shows that the LLoGMWL distribution fits better than the non-nested models. Plots of the fitted densities and probability plots are given in Figure 12, respectively.

Table 10 Estimates of model for failure time to kevlar 49/epoxy strands data

	Estimates					Statistics								
Model	λ	c	α	β	θ	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	KS	$P-VALUE$	SS
LLoGMWL	0.0103	2.2224	1.0×10^{-8}	0.6007	0.6984	124.6988	134.6988	135.3304	147.7744	0.0094	0.0291	0.0444	0.9744	0.0274
	(0.0732)	(0.2487)	(0.0139)	(0.1242)	(0.1136)									
LLoGW	-	0.8734	0.3246	1.4600	-	207.0316	213.0316	213.2791	220.8770	0.1850	1.0609	0.1030	0.2341	0.2214
	-	(0.1197)	(0.1002)	(0.2149)	-									
LLoGR	-	0.9114	0.1423	-	-	213.3457	229.1762	229.2986	234.4064	0.4110	2.2007	0.1755	0.0039	0.2475
	-	(0.1119)	(0.0376)	-	-									
LLoG	-	1.2265	-	-	-	237.5890	239.5390	239.6294	242.2041	0.4759	2.5805	0.2168	0.0002	1.3146
	-	(0.1057)	-	-	-									
	λ	β	δ	α	θ									
GD	245.8243	0.3347	12.969	0.1518	9.7490	196.2327	206.2327	206.8642	219.3083	0.1349	0.8790	0.0587	0.8771	0.0352
	(0.0567)	(0.0.2353)	(2.8917)	(0.0757)	(4.5375)									
	α	λ	δ	ϕ	θ									
EKD	0.0179	6.7074	3.7473	0.8577	8.4926	196.3005	206.3005	206.9321	219.3761	0.1363	0.8853	0.0597	0.8641	0.0580
	(0.0116)	(6.5662)	(0.9029)	(0.2266)	(0.3646)									
	c	a	b	θ	λ									
EKW	0.4812	5.1187	11.0960	0.3773	0.4858	205.0968	215.0974	215.7289	228.1730	0.1666	0.9584	0.0859	0.4440	0.1682
	(1.1202)	(14.8182)	(66.4748)	(0.3696)	(1.2823)									
	s	c	α	β	θ									
LLoGWP	1.5153	2.7714	0.6232	0.6516	1.1597×10^{-07}	199.0136	209.0136	209.6452	222.0892	0.0396	0.3579	0.0549	0.9206	0.0386
	(0.2438)	(0.5053)	(0.1321)	(0.1053)	(0.0272)									
	k	c	α	β	θ									
BrWLn	0.2334	5.8336	0.4259	0.7695	0.7290	196.1191	206.1191	206.7507	219.1947	0.0428	0.3107	0.0516	0.9505	4.1609
	(0.1268)	(2.0869)	(0.3960)	(0.1738)	(0.5966)									

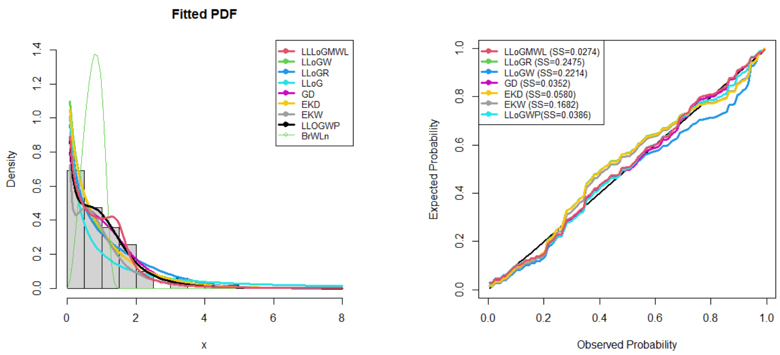

Figure 12 Fitted pdf and probability plots for failure time to kevlar 49/epoxy strands data

4. Concluding Remarks

We have presented a new class of generalized distributions called the LLoGMWPS distribution that is suitable for applications in various areas including reliability, survival analysis, just to mention a few areas. This general class of distributions and some of its structural properties including hazard rate function, quantile function, moments, conditional moments, mean deviations, Rényi entropy, distribution of order statistics, maximum likelihood estimates, asymptotic confidence intervals are presented. Two special cases of the LLoGMWPS class of distributions, namely, LLoGMWP and LLoGMWL distributions are considered in details. Applications of these two special cases to real data sets are given in order to illustrate the applicability and usefulness of the proposed class of distributions.

References

Andrews DF, Herzberg AM. Data: A Collection of Problems from Many Fields for the Student and Research Worker. Springer Ser. Statist. New York; 1985.

Barlow RE, Toland RH, Freeman T. A bayesian analysis of stress-rupture life of kevlar 49/epoxy spherical pressure vessels. In Proceeding of Canadian Conference in Applied Statistics. New York: Marcel Dekker; 1984.

Bourguignon M, Silva RB, Cordeiro GM. The Weibull-G family of probability distributions. J Data Sci. 2014; 12(1): 53-68.

Carrasco M, Ortega EMM, Cordeiro GM. A generalized Weibull distribution for lifetime modeling. Comput Stat Data An. 2008; 53(2), 450-462.

Chen G, Balakrishnan N. A general purpose approximate goodness-of-fit test. J Qual Technol. 1995; 27(2): 154-161.

Cooray K, Ananda, MM. A generalization of the half-normal distribution with application to lifetime data. Commun Stat Theor M. 2008; 37(9), 1323-1337.

Eissa F, Abdulaziz, RK. The exponentiated Kumaraswamy Weibull distribution with application to real data. Int J Probab Stat. 2014; 6(6): 167-182.

Gradshetyn IS, Ryzhik IM. Tables of Integrals, Series and Products. Sixth Edition. AP: San Diego.; 2000.

Gupta RD, Kundu D. Exponentiated exponential family: an alternative to gamma and Weibull distributions. Biometrical J. 2001; 43(1): 117-130.

Gurvich MR, DiBenedetto AT, Ranade, SV. A new statistical distribution for characterizing the random strength of brittle materials. J Mater Sci. 1997; 32: 2559-2564.

Haupt E, Schabe H. A new model for a lifetime distribution with bathtub shaped failure rate. Microelectron Reliab. 1992; 32(5): 633-639.

Hjorth U. A reliability distribution with increasing, decreasing, constant and bathtub failure rates. Technometrics. 1980; 22(1): 99-107.

Huang S, Oluyede BO. Exponentiated Kumaraswamy-Dagum distribution with application to income and lifetime data. J Stat Distrib Appl. 2014; 1(8):1-20.

Lai CD, Xie M, Murthy DNP. A modified Weibull distribution. IEEE Trans Reliab. 2003; 52(1): 33-37.

Makubate B, Oluyede BO, Motobetso G, Huang S, Fagbamigbe A. The beta Weibull-G family of distributions: model, properties and application. Int J Probab Stat. 2018; 12(2):12-32.

Morais AL, Barreto-Souza W. A compound class of Weibull and power series distributions. Comput Stat Data An. 2011; 5(3): 1410-1425.

Mudholkar GS, Srivastava, DK. Exponentiated Weibull family for analyzing bathtub failure-rate data. IEEE Trans Reliab. 1993; 42(2): 299-302.

Murthy, DNP, Xie M, Jiang, R. Weibull Models. John Wiley & Sons; 2004.

Nadarajah S, Cordeiro GM, Ortega EMM. General results for the beta-modified Weibull distribution. J Stat Comput Sim. 2011; 81(10): 1211-1232.

Nadarajah S, Kotz S. The beta exponential distribution. Reliab Eng Syst Safe. 2006; 91(6): 689–697.

Oluyede BO, Bindele F, Makubate B, Huang, S. A new class of generalized log-logistic and modified Weibull distribution with applications. Int J Probab Stat. 2018; 7(3): 72-93.

Oluyede BO, Huang S, Pararai, M. A new class of generalized dagum distribution with applications to income and lifetime data. J Stat Econ Methods. 2014; 3(2): 125-151.

Oluyede, BO, Makubate B, Fagbamigbe AF, Mdlongwa P. A new burr XII-Weibull-logarithmic distribution for survival and lifetime data analysis: model, theory and applications. Stats. 2018; 1(1): 77-91.

Oluyede BO., Mdlongwa P, Makubate B, Huang S. The burr Weibull power series class of distributions. Austrian J Stat. 2019; 48(1): 1-13.

Oluyede BO, Warahena-Liyanage G, Pararai M. A new compound class of log-logistic Weibull-Poisson distribution: model, properties and applications. J Stat Comput Sim. 2016; 86(7): 1363-1391.

Oluyede BO, Yang T. A new class of generalized Lindley distributions with applications. J Stat Comput Sim. 2015; 85(10): 2072-2100.

Pham H, Lai CD. On recent generalizations of the Weibull distribution. IEEE Trans Reliab. 2007; 56(3): 454-458.

Pinho GB, Cordeiro GM, Nobre JS. The gamma-exponentiated Weibull distribution. J Stat Theory Appl. 2012; 11(4): 379-395.

Rajarshi S, Rajarshi, MB. Bathtub distributions: A review. Commun Stat Theor M. 1988; 17(8): 2521-2597.

R Development Core Team. A Language and Environment for Statistical Computing. R Foundation for Stat Comput. Vienna: Austria.; 2011.

Renyi A. On measures of entropy and information. In Proceedings of the Fourth Berkeley Symposium on Math. Stat. Probab. University of California Press. 1961; 4(1): 547561.

Shannon, EA. A mathematical theory of communication. Bell Syst Tech J. 1974; 27(10): 461-464.

Silva, GO, Ortega, EMM, Cordeiro, GM. The beta modified Weibull distribution. Lifetime Data Anal. 2010; 16: 409-430.

Silva RB, Cordeiro GM. The Burr XII power series distributions: A new compounding family. Braz J Probab Stat. 2015; 29(3): 565-589.

Silva RB, Bourguignon M, Dais CRB, Cordeiro GM. The compound family of extended Weibull power series distributions. Comput Stat Data An. 2013; 58: 352-367.

Appendix

A. The r^{th} Moment of the LLoGMWPS Class of distribution is given by

$$
\begin{aligned}
E(X^r) &= \int_0^\infty x^r \frac{\theta f(x) C'(\theta S(x))}{C(\theta)} dx \\
&= \frac{\theta}{C(\theta)} \sum_{j=0}^{\infty} b_{s,j} \theta^j \int_0^\infty x^r (1+x^c)^{-j} e^{-j\alpha x^\beta e^{\lambda x}} \\
&\quad \times [(1+x^c)^{-2} e^{-\alpha x^\beta e^{\lambda x}} (cx^{c-1} + (1+x^c)(\beta+\lambda x) e^{\lambda x} \alpha x^{\beta-1})] dx \\
&= \frac{\theta}{C(\theta)} \sum_{j=0}^{\infty} b_{s,j} \theta^j \int_0^\infty x^r (1+x^c)^{-j-2} e^{-(j+1)\alpha x^\beta e^{\lambda x}} \\
&\quad \times [(cx^{c-1} + (1+x^c)(\beta+\lambda x) e^{\lambda x} \alpha x^{\beta-1})] dx \\
&= \frac{\theta}{C(\theta)} \sum_{j=0}^{\infty} b_{s,j} \theta^j \Big[\int_0^\infty x^{r+k\beta} (1+x^c)^{-j-2} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha(j+1) e^{\lambda x})^k}{k!} \\
&\quad \times (1+x^c) \alpha x^{\beta-1} e^{\lambda x} (\beta+\lambda x) dx \\
&\quad + \int_0^\infty x^{r+k\beta} c x^{c-1} (1+x^c)^{-j-2} \sum_{k=0}^{\infty} \frac{(-1)^k (\alpha(j+1) e^{\lambda x})^k}{k!} dx \Big] \\
&= \frac{\theta\alpha}{C(\theta)} \sum_{j=0}^{\infty} b_{s,j} \theta^j \Big[\sum_{k,p=0}^{\infty} \frac{(-1)^k [\alpha(j+1)]^k [\lambda(k+1)]^p}{k! p!} \\
&\quad \times \int_0^\infty x^{r+k\beta+p+\beta-1} (1+x^c)^{-j-2+1} (\beta+\lambda x) dx \\
&\quad + \sum_{k,p=0}^{\infty} \frac{(-1)^k [\alpha(j+1)]^k [k\lambda]^p c}{k! p!} \int_0^\infty x^{r+k\beta+p+c-1} (1+x^c)^{-j-2} dx \Big].
\end{aligned}
$$

B. The elements of the score vector for LLoGMWPS Class of distribution $U(\boldsymbol{\Delta}) = \left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial c}\right)$ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial \theta} =& \frac{n}{\theta} - \frac{C'(\theta)}{C(\theta)} + \sum_{i=1}^{n} \frac{C''(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}}{C'(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))},\\
\frac{\partial \ell}{\partial \alpha} =& -\sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} + \sum_{i=1}^{n} \frac{(1+x_i^c)(\beta+\lambda x_i)x_i^{\beta-1}e^{\lambda x_i}}{c x_i^{c-1} + (1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}}\\
& - \sum_{i=1}^{n} \frac{C''(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))\theta e^{-\alpha x_i^{\beta} e^{\lambda x_i}} x_i^{\beta} e^{\lambda x_i}}{C'(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))},\\
\frac{\partial \ell}{\partial \beta} =& -\alpha \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} \ln(x_i) + \sum_{i=1}^{n} \frac{\alpha(1+x_i^c)^{-1}[x_i^{\beta-1}e^{\lambda x_i}(1+(\beta+\lambda x_i)\ln(x_i))]}{c x_i^{c-1} + (1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}}\\
& - \sum_{i=1}^{n} \frac{C''(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))\theta(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta} e^{\lambda x_i}} \alpha e^{\lambda x_i} x_i^{\beta}\ln(x_i)}{C'(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))},\\
\frac{\partial \ell}{\partial \lambda} =& -\alpha \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i} x_i + \sum_{i=1}^{n} \frac{(1+x_i^c)\alpha x_i^{\beta} e^{\lambda x_i}(1+\beta+\lambda x_i)}{c x_i^{c-1} + (1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}}\\
& - \sum_{i=1}^{n} \frac{C''(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))\theta(1+x_i^c)^{-1}e^{-\alpha x_i^{\beta} e^{\lambda x_i}} \alpha e^{\lambda x_i} x_i^{\beta+1}}{C'(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))}\\
\text{and} \quad \frac{\partial \ell}{\partial c} =& -2\sum_{i=1}^{n} \frac{x_i^c \ln(x_i)}{1+x_i^c} + \sum_{i=1}^{n} \frac{x_i^{c-1}\ln(x_i)[c+\alpha x_i^{\beta-1}e^{\lambda x_i}(\beta+\lambda x_i)] + x_i^{c-1}}{c x_i^{c-1} + (1+x_i^c)\alpha x_i^{\beta-1}e^{\lambda x_i}(\beta+\lambda x_i)}\\
& - \sum_{i=1}^{n} \frac{C''(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))\theta e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-2}x_i^c\ln(x_i)}{C'(\theta(e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}))}.
\end{aligned}$$

C. The r^{th} Moment of the LLoGMWP distribution is given by

$$\begin{aligned}
E(X^r) =& \int_0^{\infty} x^r \frac{\theta f(x) C'(\theta S(x))}{C(\theta)} dx\\
=& \frac{\theta}{e^{\theta}-1} \sum_{n=0}^{\infty} b_n \theta^n \int_0^{\infty} x^r (1+x^c)^{-n} e^{-n\alpha x^{\beta} e^{\lambda x}}\\
& \times [(1+x^c)^{-2} e^{-\alpha x^{\beta} e^{\lambda x}} (c x^{c-1} + (1+x^c)(\beta+\lambda x)e^{\lambda x}\alpha x^{\beta-1})]dx\\
=& \frac{\theta}{e^{\theta}-1} \sum_{n=0}^{\infty} b_n \theta^n \int_0^{\infty} x^r (1+x^c)^{-n-2} e^{-(n+1)\alpha x^{\beta} e^{\lambda x}}\\
& \times [(c x^{c-1} + (1+x^c)(\beta+\lambda x)e^{\lambda x}\alpha x^{\beta-1})]dx\\
=& \frac{\theta}{e^{\theta}-1} \sum_{n=0}^{\infty} b_n \theta^n \Bigg[\int_0^{\infty} x^{r+k\beta}(1+x^c)^{-n-2} \sum_{k=0}^{\infty} \frac{(-1)^k(\alpha(n+1)e^{\lambda x})^k}{k!}\\
& \times (1+x^c)\alpha x^{\beta-1} e^{\lambda x}(\beta+\lambda x)dx\\
& + c\int_0^{\infty} x^{r+k\beta+c-1}(1+x^c)^{-n-2} \sum_{k=0}^{\infty} \frac{(-1)^k(\alpha(n+1)e^{\lambda x})^k}{k!} dx\Bigg]\\
=& \frac{\theta\alpha}{e^{\theta}-1} \sum_{n=0}^{\infty} b_n \theta^n \Bigg[\sum_{k,p=0}^{\infty} \frac{(-1)^k[\alpha(n+1)]^k[\lambda(k+1)]^p}{k!p!}\\
& \times \int_0^{\infty} x^{r+k\beta+p+\beta-1}(1+x^c)^{-n-1}(\beta+\lambda x)dx\\
& + \sum_{k,p=0}^{\infty} \frac{(-1)^k[\alpha(n+1)]^k[k\lambda]^p c}{k!p!} \int_0^{\infty} x^{r+k\beta+p+c-1}(1+x^c)^{-n-2}dx\Bigg].
\end{aligned}$$

D. The elements of the score vector for LLoGMWP distribution is given as $U(\mathbf{\Delta}) = \left(\frac{\partial \ell}{\partial \theta}, \frac{\partial \ell}{\partial \alpha}, \frac{\partial \ell}{\partial \beta}, \frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial c}\right)$ are given by

$$\begin{aligned}
\frac{\partial \ell}{\partial \theta} &= \frac{n}{\theta} - \frac{ne^{\theta}}{e^{\theta}-1} + \sum_{i=1}^{n}[1-e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}],\\
\frac{\partial \ell}{\partial \alpha} &= \theta \sum_{i=1}^{n}[1-e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}]x_i^{\beta} - \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i}\\
&\quad + \sum_{i=1}^{n} \frac{(1+x_i^c)(\beta+\lambda x_i)x_i^{\beta-1}e^{\lambda x_i}}{cx_i^{c-1}+(1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}},\\
\frac{\partial \ell}{\partial \beta} &= -\alpha \sum_{i=1}^{n} x_i^{\beta} e^{\lambda x_i}\ln(x_i) + \theta \sum_{i=1}^{n} e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-1}\alpha e^{\lambda x_i} x_i^{\beta}\ln(x_i)\\
&\quad + \sum_{i=1}^{n} \frac{\alpha e^{\lambda(1+x_i^c)^{-1}[x_i^{\beta-1}x_i}(1+(\beta+\lambda x_i)\ln(x_i))]}{cx_i^{c-1}+(1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}},\\
\frac{\partial \ell}{\partial \lambda} &= \frac{\sum_{i=1}^{n}(1+x_i^c)\alpha x_i^{\beta-1}e^{\lambda x_i}(\beta+\lambda x_i)ln(x_i)}{cx_i^{c-1}+(1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}} + \alpha \sum_{i=1}^{n} x_i ln(x_i)\\
\frac{\partial \ell}{\partial c} &= -2\sum_{i=1}^{n} \frac{x_i^c \ln(x_i)}{1+x_i^c} + \sum_{i=1}^{n} e^{-\alpha x_i^{\beta} e^{\lambda x_i}}(1+x_i^c)^{-2}x_i \ln(x_i)\\
&\quad - \sum_{i=1}^{n} \frac{[xi^c + cxi^{c-1}\ln(x_i)] + \alpha x_i^{\beta-1}e^{\lambda x_i}(\beta+\lambda x_i)x_i \ln x_i}{cx_i^{c-1}+(1+x_i^c)\alpha x_i^{\beta-1}(\beta+\lambda x_i)e^{\lambda x_i}}.
\end{aligned}$$

E. The elements of the score vector for LLoGMWL distribution are given as

$$\begin{aligned}
\frac{\partial \ell}{\partial \theta} &= \frac{1}{\theta} - \frac{(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}}{1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}} - \frac{1}{(1-\theta)\ln(1-\theta)},\\
\frac{\partial \ell}{\partial c} &= \frac{-2x^c\ln(x)}{(1+x^c)} + \frac{x^{c-1}\left(1+c\ln(x)+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)x^c\ln(x)\right)}{(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c))}\\
&\quad + \frac{\theta x^c e^{-\alpha x^{\beta}e^{\lambda x}}\ln(x)(1+x^c)^{-1}}{(1-\theta)(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}},\\
\frac{\partial \ell}{\partial \alpha} &= -x^{\beta}e^{\lambda x} + \frac{(1+x^c)x^{\beta-1}e^{\lambda x}(\beta+\lambda x)}{\left(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c)^{-1}\right)} + \frac{\theta x^{\beta}e^{-x^{\beta}e^{\lambda x}}(\alpha+\lambda x)}{(1-\theta)(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}},\\
\frac{\partial \ell}{\partial \beta} &= \alpha x^{\beta-1}\ln(x)e^{\lambda x} + \frac{\alpha e^{\lambda x}(1+x^c)\left(x^{\beta-1}+(\beta+\lambda x)x^{\beta-1}\ln(x)\right)}{(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c))}\\
&\quad + \frac{\theta\alpha x^{\beta-1}\ln(x)e^{\lambda x}e^{-\alpha x^{\beta}e^{\lambda x}}}{(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})},\\
\frac{\partial \ell}{\partial \lambda} &= -\alpha x^{\beta}e^{\lambda x} + \frac{(1+x^c)\alpha x^{\beta-1}\left(xe^{\lambda x}(\beta+\lambda x)+xe^{\lambda x}\right)}{(cx^{c-1}+\alpha x^{\beta-1}e^{\lambda x}(\beta+\lambda x)(1+x^c))}\\
&\quad + \frac{\theta\alpha x^{\beta+1}\ln(x)e^{\lambda x}(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}}}{(1-\theta(1+x^c)^{-1}e^{-\alpha x^{\beta}e^{\lambda x}})}.
\end{aligned}$$

F. LLoGMWP DISTRIBUTION R ALGORITHMS

```
#### define LLoGMWL cdf
LLoGMWL_cdf=function(x,c,alpha ,beta,lambda ,theta){
  (1-log((1-exp(theta*(1-(1+x^c)^(-1)*exp(-alpha*(x^beta)
  *exp(lambda*x)))))/(1 -exp(theta ))))/log(1-theta)
}

#### define LLoGMWL_hazard
```

```
LLoGMWL_hazard=function(x,theta,alpha,lambda,beta,c){
  LLoGMWL_pdf(x,c,alpha,beta,lambda,theta)/
(1-LLoGMWL_cdf(x,c,alpha,beta,lambda,theta))
}

#Define the quantile of  LLLoGMWL
quantile =function(theta,lambda,c,alpha,beta,u){
  f=function(x){log(1+x^c)+(alpha*x^beta*exp(lambda*x))
    +log ((1/ theta)*(1-(1-theta)^(1-u)))
  }
  rc<-uniroot(f, lower=0,upper=1000,tol=1e-9)
  result=rc$root
}
LLLoGMWL_LL<-function(par){
  bb=(par[1]*(exp(-(par[4])*(x^(par[5]))
  *exp(par[2]*x) )*((1+x^par[3])^(-2))))
  cc=(par[4]*(x^(par[5]))*(exp(par[2]*x))
  *(par[5]+par[2]*x)*(1+x^(par[3])) + par[3]*(x^(par[3]-1)))
  dd=(-(1-par[1]*(exp(-(par[4])*(x^par[5])*exp(par[2]*x))
  *((1+x^(par[3]))^(-1))))*(log(1-par[1])))
  -sum(log(bb*cc/dd))
}
theta=0.8
lambda=0.6
c=0.5
alpha=3.0
beta=1.8

n1<-c(25)

# To check one sample at a time, use n1<-c(sample size)

for (i in 1: length(n1)){
  n=n1[i]
  N=1000
  mle_theta<-c(rep(0,N))
  mle_lambda<-c(rep(0,N))
  mle_c<-c(rep(0,N))
  mle_alpha<-c(rep(0,N))
  mle_beta<-c(rep(0,N))

  LC_theta<-c(rep(0,N))
  UC_theta<-c(rep(0,N))
  LC_lambda<-c(rep(0,N))
  UC_lambda<-c(rep(0,N))
  LC_c<-c(rep(0,N))
  UC_c<-c(rep(0,N))
  LC_alpha<-c(rep(0,N))
  UC_alpha<-c(rep(0,N))
  LC_beta<-c(rep(0,N))
  UC_beta<-c(rep(0,N))

  count_theta=0
  count_lambda=0
  count_c=0
  count_alpha=0
  count_beta=0

  temp=1

  HH1<-matrix(c(rep(5,25)), nrow=5, ncol=5)
  HH2<-matrix(c(rep(5,25)), nrow=5, ncol=5)
  for (i in 1:N)
  {
```

```
#print (i)
#flush.console()
repeat{
  x<-c(rep(0,n))

  #Generate a random variable from uniform distribution
  u<-0
  u<-runif(n,min=0,max=1)

  for (k in 1:n){
    x[k]<-quantile(theta,lambda,c,alpha,beta,u[k])
  }

  #Maximum likelihood estimation
  mle.result<-nlminb(c(theta,lambda,c,alpha,beta),
  LLLoGMWL_LL, lower = 0, upper = Inf)

  temp=mle.result$convergence
  if (temp==0){
    temp_theta<-mle.result$par[1]
    temp_lambda<-mle.result$par[2]
    temp_c<-mle.result$par[3]
    temp_alpha<-mle.result$par[4]
    temp_beta<-mle.result$par[5]

    HH1<-hessian(LLLoGMWL_LL, c(temp_theta,
  temp_lambda, temp_c, temp_alpha,temp_beta))
    if ( sum(is.nan(HH1))==0 & (diag(HH1)[1]>0)
& (diag(HH1)[2]>0) & (diag(HH1)[3]>0) & (diag(HH1)[4]>0)
& (diag(HH1)[5]>0)){
      HH2<-solve(HH1)
      #print(det(hh1))
    }
    else{
      temp=1}
  }

  if ((temp==0) & (diag(HH2)[1]>0) & (diag(HH2)[2]>0)
      & (diag(HH2)[3]>0) & (diag(HH2)[4]>0)&
      (diag(HH2)[5]>0) & (sum( is.nan(HH2))==0)){
    break
  }
}
#print(temp)
temp=1

mle_theta[i]<-mle.result$par[1]
mle_lambda[i]<-mle.result$par[2]
mle_c[i]<-mle.result$par[3]
mle_alpha[i]<-mle.result$par[4]
mle_beta[i]<-mle.result$par[5]

HH<-hessian(LLLoGMWL_LL,c(mle_theta[i],mle_lambda[i],
mle_c[i],mle_alpha[i],mle_beta[i]))
H<-solve(HH)
LC_theta[i]<-mle_theta[i]-1.96*sqrt(diag(H)[1])
UC_theta[i]<-mle_theta[i]+1.96*sqrt(diag(H)[1])
if ((LC_theta[i]<=theta) & (theta<=UC_theta[i])){
  count_theta=count_theta+1
}

LC_lambda[i]<-mle_lambda[i]-1.96*sqrt(diag(H)[2])
UC_lambda[i]<-mle_lambda[i]+1.96*sqrt(diag(H)[2])
if ((LC_lambda[i]<=lambda) & (lambda<=UC_lambda[i])){
```

```
    count_lambda=count_lambda+1
  }

  LC_c[i]<-mle_c[i]-1.96*sqrt(diag(H)[3])
  UC_c[i]<-mle_c[i]+1.96*sqrt(diag(H)[3])
  if ((LC_c[i]<=c) & (c<=UC_lambda[i])){
    count_c=count_c+1
  }

  LC_alpha[i]<-mle_alpha[i]-1.96*sqrt(diag(H)[4])
  UC_alpha[i]<-mle_alpha[i]+1.96*sqrt(diag(H)[4])
  if ((LC_alpha[i]<=alpha) & (alpha<=UC_alpha[i])){
    count_alpha=count_alpha+1
  }

  LC_beta[i]<-mle_beta[i]-1.96*sqrt(diag(H)[5])
  UC_beta[i]<-mle_beta[i]+1.96*sqrt(diag(H)[5])
  if ((LC_beta[i]<=beta) & (beta<=UC_beta[i])){
    count_beta=count_beta+1
  }

  #Calculate Mean
  Mean_theta<-sum(mle_theta)/N
  Mean_lambda<-sum(mle_lambda)/N
  Mean_c<-sum(mle_c)/N
  Mean_alpha<-sum(mle_alpha)/N
  Mean_beta<-sum(mle_beta)/N

  print(cbind(Mean_theta, Mean_lambda,Mean_c,
  Mean_alpha, Mean_beta ))

  #Calculate Average Bias
  Bias_theta<-sum(mle_theta-theta)/N
  Bias_lambda<-sum(mle_lambda-lambda)/N
  Bias_c<-sum(mle_c-c)/N
  Bias_alpha<-sum(mle_alpha-alpha)/N
  Bias_beta<-sum(mle_beta-beta)/N

  print(cbind(Bias_theta, Bias_lambda,Bias_c,
 Bias_alpha, Bias_beta ))

  #Calculate RMSE
  RMSE_theta<-sqrt(sum((theta-mle_theta)^2)/N)
  RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
  RMSE_c<-sqrt(sum((c-mle_c)^2)/N)
  RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/N)
  RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)

  print(cbind(RMSE_theta, RMSE_lambda,RMSE_c,
RMSE_alpha, RMSE_beta ))

  #Converge Probability
  CP_theta<-count_theta/N
  CP_lambda<-count_lambda/N
  CP_c<-count_c/N
  CP_alpha<-count_alpha/N
  CP_beta<-count_beta/N

  print(cbind(CP_theta, CP_lambda,CP_c,CP_alpha, CP_beta ))

  # Average Width
  AW_theta<-sum(abs(UC_theta-LC_theta))/N
  AW_lambda<-sum(abs(UC_lambda-LC_lambda))/N
  AW_c<-sum(abs(UC_c-LC_c))/N
```

```
    AW_alpha<-sum(abs(UC_alpha-LC_alpha))/N
    AW_beta<-sum(abs(UC_beta-LC_beta))/N

    print(cbind(AW_theta, AW_lambda,AW_c,AW_alpha, AW_beta))
  }
}
```

G.LLoGMWL DISTRIBUTION R ALGORITHMS

```
#### define LLoGMWP cdf
LLoGMWP_cdf=function(y,c,alpha ,beta,lambda ,theta){
  (1-exp(theta*(1-(1+x^c)^(-1)*(exp(-alpha*(x^beta)*
  exp(lambda*x))))))/(1-exp(theta))
}

#### define LLoGMWP_hazard
LLoGMWP_hazard=function(x,theta,alpha,lambda,beta,c){
  LLoGMWP_pdf(x,c,alpha,beta,lambda,theta)/
(1-LLoGMWP_cdf(x,c,alpha,beta,lambda,theta))
}
LLoMWP_pdf <- function(lambda,c,alpha,beta,theta) {
  aa=(theta*(exp(theta*(1+x^c)^(-1)*exp(-alpha*(x^beta)
  *exp(lambda*x)))))/(exp(theta)-1)
  bb=((1+x^c)^(-2)*(exp(-alpha*(x^beta)*exp(lambda*x))))
  cc=(alpha*(x^(beta-1))*exp(lambda*x)*(beta+lambda*x)*
(1+x^c) + c*(x^(c-1)))
  -sum(log(aa*bb*cc))
}

#Define the quantile of  LLoGMWP
quantile =function(theta,lambda,c,alpha,beta,u){
  f=function(x){
    log(1+x^c)+alpha*x^(beta)*exp(lambda*x)+
      log((log(1-(1-u)*(1-exp(theta)))/ theta))}
  rc<-uniroot(f, lower=0,upper=1000,tol=1e-9)
  result=rc$root
}
LLoGMWP_LL<-function(par){
  aa=(par[1]*exp(par[1]*((1+x^par[3])^(-1))*
(exp(-par[4]*(x^par[5])*(exp(par[2]*x)))))/(exp(par[1])-1))
  bb=(((1+x^par[3])^(-2))*(exp(-par[4]*(x^par[5])*
(exp(par[2]*x)) )))
  cc=(par[3]*(x^(par[3]-1))+(1+x^par[3])*par[4]*
(x^(par[5]-1))*(exp(par[2]*x))*(par[5]+(par[2]*x)) )
  -sum(log(aa*bb*cc))
}

theta=0.95
lambda=0.5
c=0.5
alpha=6.
beta=3.75
n1<-c(25)

# To check one sample at a time, use n1<-c(sample size)

for (j in 1: length(n1)){
  n=n1[j]
  N=1000
  mle_theta<-c(rep(0,N))
  mle_lambda<-c(rep(0,N))
  mle_c<-c(rep(0,N))
  mle_alpha<-c(rep(0,N))
  mle_beta<-c(rep(0,N))
```

```
LC_theta<-c(rep(0,N))
UC_theta<-c(rep(0,N))
LC_lambda<-c(rep(0,N))
UC_lambda<-c(rep(0,N))
LC_c<-c(rep(0,N))
UC_c<-c(rep(0,N))
LC_alpha<-c(rep(0,N))
UC_alpha<-c(rep(0,N))
LC_beta<-c(rep(0,N))
UC_beta<-c(rep(0,N))
count_theta=0
count_lambda=0
count_c=0
count_alpha=0
count_beta=0

temp=1
HH1<-matrix(c(rep(2,25)), nrow=5, ncol=5)
HH2<-matrix(c(rep(2,25)), nrow=5, ncol=5)
for (i in 1:N)
{
  print (i)
  flush.console()
  repeat{
    x<-c(rep(0,n))

    #Generate a random variable from uniform distribution
    u<-0
    u<-runif(n,min=0,max=1)

    for (k in 1:n){
      x[k]<-quantile(theta,lambda,c,alpha,beta,u[k])
    }

    #Maximum likelihood estimation
    mle.result<-nlminb(c(theta,lambda,c,alpha,beta),
              LLoGMWP_LL, lower = 0, upper = Inf)

    temp=mle.result$convergence
    if (temp==0){
      temp_theta<-mle.result$par[1]
      temp_lambda<-mle.result$par[2]
      temp_c<-mle.result$par[3]
      temp_alpha<-mle.result$par[4]
      temp_beta<-mle.result$par[5]

      HH1<-hessian(LLoGMWP_LL, c(temp_theta, temp_lambda,
temp_c, temp_alpha,temp_beta))
      if (( rcond (HH1)> 1e-9)&sum(is.nan(HH1))==0 & (diag(HH1)[1]>0) & (diag(HH1)[2]>0)
  & (diag(HH1)[3]>0) & (diag(HH1)[4]>0)& (diag(HH1)[5]>0)){
        HH2<-solve(HH1)
        #print(det(HH1))
      }
      else{
        temp=1}
    }

    if ((temp==0) & (diag(HH2)[1]>0) & (diag(HH2)[2]>0)
        & (diag(HH2)[3]>0) & (diag(HH2)[4]>0)& (diag(HH2)[5]>0)
        & (sum( is.nan(HH2))==0)){
      break
    }
    else{
      temp=1}
```

```
}
temp=1

mle_theta[i]<-mle.result$par[1]
mle_lambda[i]<-mle.result$par[2]
mle_c[i]<-mle.result$par[3]
mle_alpha[i]<-mle.result$par[4]
mle_beta[i]<-mle.result$par[5]


HH<-hessian(LLoGMWP_LL,c(mle_theta[i],mle_lambda[i],
mle_c[i],mle_alpha[i],mle_beta[i]))
H<-solve(HH)
LC_theta[i]<-mle_theta[i]-qnorm(0.975)*sqrt(diag(H)[1])
UC_theta[i]<-mle_theta[i]+qnorm(0.975)*sqrt(diag(H)[1])
if ((LC_theta[i]<=theta) & (theta<=UC_theta[i])){
  count_theta=count_theta+1
}

LC_lambda[i]<-mle_lambda[i]-qnorm(0.975)*sqrt(diag(H)[2])
UC_lambda[i]<-mle_lambda[i]+qnorm(0.975)*sqrt(diag(H)[2])
if ((LC_lambda[i]<=lambda) & (lambda<=UC_lambda[i])){
  count_lambda=count_lambda+1
}

LC_c[i]<-mle_c[i]-qnorm(0.975)*sqrt(diag(H)[3])
UC_c[i]<-mle_c[i]+qnorm(0.975)*sqrt(diag(H)[3])
if ((LC_c[i]<=c) & (c<=UC_c[i])){
  count_c=count_c+1
}

LC_alpha[i]<-mle_alpha[i]-qnorm(0.975)*sqrt(diag(H)[4])
UC_alpha[i]<-mle_alpha[i]+qnorm(0.975)*sqrt(diag(H)[4])
if ((LC_alpha[i]<=alpha) & (alpha<=UC_alpha[i])){
  count_alpha=count_alpha+1
}

LC_beta[i]<-mle_beta[i]-qnorm(0.975)*sqrt(diag(H)[5])
UC_beta[i]<-mle_beta[i]+qnorm(0.975)*sqrt(diag(H)[5])
if ((LC_beta[i]<=beta) & (beta<=UC_beta[i])){
  count_beta=count_beta+1
}

#Calculate Mean
Mean_theta<-sum(mle_theta)/N
Mean_lambda<-sum(mle_lambda)/N
Mean_c<-sum(mle_c)/N
Mean_alpha<-sum(mle_alpha)/N
Mean_beta<-sum(mle_beta)/N

print(cbind(Mean_theta, Mean_lambda,Mean_c,Mean_alpha, Mean_beta ))

#Calculate Average Bias
Bias_theta<-sum(mle_theta-theta)/N
Bias_lambda<-sum(mle_lambda-lambda)/N
Bias_c<-sum(mle_c-c)/N
Bias_alpha<-sum(mle_alpha-alpha)/N
Bias_beta<-sum(mle_beta-beta)/N

print(cbind(Bias_theta, Bias_lambda,Bias_c, Bias_alpha, Bias_beta ))

#Calculate RMSE
RMSE_theta<-sqrt(sum((theta-mle_theta)^2)/N)
RMSE_lambda<-sqrt(sum((lambda-mle_lambda)^2)/N)
```

```
    RMSE_c<-sqrt(sum((c-mle_c)^2)/N)
    RMSE_alpha<-sqrt(sum((alpha-mle_alpha)^2)/N)
    RMSE_beta<-sqrt(sum((beta-mle_beta)^2)/N)

print(cbind(RMSE_theta, RMSE_lambda,RMSE_c,RMSE_alpha,RMSE_beta ))

    #Converge Probability
    CP_theta<-count_theta/N
    CP_lambda<-count_lambda/N
    CP_c<-count_c/N
    CP_alpha<-count_alpha/N
    CP_beta<-count_beta/N

    print(cbind(CP_theta, CP_lambda,CP_c,CP_alpha, CP_beta ))

    # Average Width
    AW_theta<-sum(abs(UC_theta-LC_theta))/N
    AW_lambda<-sum(abs(UC_lambda-LC_lambda))/N
    AW_c<-sum(abs(UC_c-LC_c))/N
    AW_alpha<-sum(abs(UC_alpha-LC_alpha))/N
    AW_beta<-sum(abs(UC_beta-LC_beta))/N

    print(cbind(AW_theta, AW_lambda,AW_c,AW_alpha, AW_beta))
  }
}
```