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Exponentiated Approach on Intervened Exponential Model with Real-Life Study

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Abstract

In this article, the exponentiated approach has been utilized to develop the new extension of Intervened Exponential distribution named as an exponentiated intervened exponential distribution. The various statistical properties of the proposed distribution have been derived, whereas the reliability characterization involves some special functions such as the reliability function, hazard rate, aging intensity, and the mean residual life function. In addition, the other major results presented in the article include order statistics, stress strength reliability, and stochastic ordering. Furthermore, the entropy measures that are Shannon and Rényi are also derived. The method of the maximum likelihood approach has been used for parameter estimation. Monte Carlo simulation study is recommended followed by the acceptance-rejection algorithm for data generation and in this study, the behavior of the estimated parameters is discussed based on the calculated values of bias and the mean square error. Lastly, the real-life data set is analyzed to ensure the applicability of the newly developed model.

Keywords: Bias, entropy, intervened distribution, mean square error, Monte Carlo simulation

1. Introduction

It is visible in the statistical literature, the influence of exponential distribution in certain areas such as Poisson process analysis, reliability theory, and the survival analysis is drastically increasing in one way or the other, due to its extensive applicability on the data analysis. Though the constant hazard rate and lack of memory are among its most impressive properties. But it is difficult, that a statistical distribution may be suitable to analyze every type of data set with greater flexibility. To tackle this obstacle, the statisticians and the researchers have developed methods to improve flexibility by creating new distributions. Hence, it has been seen the classical distributions such as the Exponential, Normal, Gamma, etc. have been modified and generalized with a new class of distributions to match the flexibility over the data sets having a great degree of skewness and kurtosis.

Despite all the existing methods that are used for creating new distributions, a new strategic concept called intervention has been brought into existence by Shanmugam (1985) along with the development of a new discrete model called intervened Poisson model. In another attempt by Shanmugam et al. (2002), similar concepts were utilized to develop the first continuous model called intervened exponential distribution (I_vED), which has fabulous applications in the medical background. Further, the I_vED was introduced as a lifetime distribution by Bhat and Pundir (2022). The cumulative density function (cdf) and the probability density function (pdf) of I_vED are given as,

$$F_{I_vED}(x;\omega) = \begin{cases} 1 - \frac{\rho e^{-(x-\phi)/\rho\lambda} - e^{-(x-\phi)/\lambda}}{(\rho-1)} & \rho \neq 1 \\ 1 - \left(1 + \frac{x-\phi}{\lambda}\right) e^{-(x-\phi)/\lambda} & \rho = 1 \end{cases}$$

and

$$f_{I_vED}(x;\omega) = \begin{cases} \frac{e^{-(x-\phi)/\rho\lambda} - e^{-(x-\phi)/\lambda}}{(\rho-1)\lambda} & \rho \neq 1 \\ \frac{(x-\phi)}{\lambda^2} e^{-(x-\phi)/\lambda} & \rho = 1 \end{cases}$$

where, $x > \phi > 0$. The diverse applicability of the I_vED model, motivated us to make new a development with the exponentiated approach. The new distribution is proposed and named as exponentiated intervened exponential distribution (EI_vD), for existing extensions of I_vED refer to Bhat and Pundir (2023a,b). The transformation technique used for bringing this new development is $z = exp(-x)$, where the random variable $(r.v)$ X follows the I_vED. Thus, for the proposed EI_vD, the cdf denoted by $F_{EI_vD}(.)$ and the pdf denoted by $f_{EI_vD}(.)$ are given as follows:

$$F_{EI_vD}(z;\omega) = \begin{cases} \frac{\rho e^{\phi/\rho\lambda} z^{1/\rho\lambda} - e^{\phi/\lambda} z^{1/\lambda}}{\rho-1} & \rho \neq 1 \\ \left[1 - \frac{(\phi+\log z)}{\lambda}\right] e^{\phi/\lambda} z^{1/\lambda} & \rho = 1 \end{cases} \tag{1}$$

and

$$f_{EI_vD}(z;\omega) = \begin{cases} \frac{e^{\phi/\rho\lambda} z^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z^{(1/\lambda)-1}}{(\rho-1)\lambda} & \rho \neq 1 \\ \frac{e^{\phi/\lambda}}{\lambda^2} \left(\log\left(1/z\right) - \phi\right) z^{(1/\lambda)-1} & \rho = 1 \end{cases} \tag{2}$$

where $0 < z < e^{-\phi}$. Also, the notation ω denotes the parametric space and is given by $\omega = \{\ (\phi, \lambda, \rho) : \phi > 0, \lambda > 0, \rho > 0\ \}$, where λ and ρ are rate and the intervention parameters. Furthermore the derived essential properties include the mean $(\mu_z) = e^{\phi}\left\{(1+\lambda)(1+\rho\lambda)\right\}^{-1}$, variance $\left(\sigma_z^2\right) = \frac{\lambda^2 e^{-2\phi}\left\{1+\rho^2+\rho^2\lambda^2+2\rho^2\lambda+2\rho\lambda\ \right\}}{(1+2\lambda)(1+2\rho\lambda)(1+\lambda)^2(1+\rho\lambda)^2}$, and the mode $(M_o) = \left\{\rho\left(1-\lambda\right)\left(1-\rho\lambda\right)^{-1}\right\}^{\rho\lambda/(1-\rho)}$ respectively. Also, it is important to address this issue that the quantile function is in unexplicit form, which plays the key role in simulation technique. However, other suitable algorithms could be preferred according to their suitability to overcome this challenge. Note, that $\Gamma(*, \star)$ is the lower incomplete gamma function.

The graphical demonstration of Eqn. (2) which is a pdf of the proposed model is shown in Figure 1 and its corresponding subplots in Figure 2 over the different set of values of parameters. This is visible from the graphical plot a variety of pdf shapes are exhibited for different values of parameters.

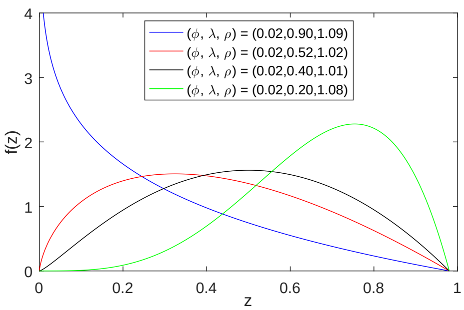

Figure 1 Pdf plot

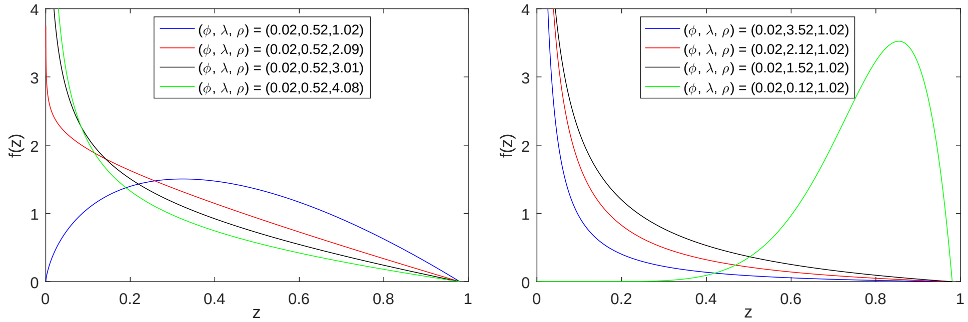

Figure 2 Pdf subplots

2. Statistical Properties of the New Distribution

In this section, a few fundamental statistical properties are discussed, which would include the moments (central and non-central), generating function for moments, skewness, kurtosis, etc. in the form of theorems.

Theorem 1 *The moments (non-central and central) for a r.v.* $Z \sim EI_vD(\phi, \lambda, \rho)$ *are given by*

$$(i) \quad \mu_r' = e^{-r\phi}\left\{(1+r\lambda)(1+r\rho\lambda)\right\}^{-1}; \;\; r = 1,2,...,n.$$

$$(ii) \quad \mu_r = \sum_{n=0}^{r} {}^rC_n\,(-\mu)^{r-n}\left[e^{-n\phi}\left\{(1+n\lambda)(1+n\rho\lambda)\right\}^{-1}\right]; \;\; r = 1,2,...,n.$$

Proof:

(i) Since $Z \sim EI_vD(\phi, \lambda, \rho)$, then the expression for r^{th} non-central moment is,

$$\begin{aligned}\mu_r' &= E(z^r) = \frac{1}{(\rho-1)\lambda}\int_0^{e^{-\phi}} z^r\left\{e^{\phi/\rho\lambda}z^{(1/\rho\lambda)-1} - e^{\phi/\lambda}z^{(1/\lambda)-1}\right\}dz \\ &= e^{-r\phi}\left\{(1+r\lambda)(1+r\rho\lambda)\right\}^{-1}\end{aligned}$$

where $r = 0,1,...,n.$

(ii) To obtain the r^{th} central moment for a random variable $Z \sim EI_vD(\phi, \lambda, \rho)$, the procedure is given as,

$$\begin{aligned}\mu_r &= E(z-\mu_z)^r = \frac{1}{(\rho-1)\lambda}\int_0^{e^{-\phi}} (z-\mu_z)^r\left\{e^{\phi/\rho\lambda}z^{(1/\rho\lambda)-1} - e^{\phi/\lambda}z^{(1/\lambda)-1}\right\}dz \\ &= \sum_{n=0}^{r} {}^rC_n\,(-\mu_z)^{r-n}\left[e^{-n\phi}\left\{(1+n\lambda)(1+n\rho\lambda)\right\}^{-1}\right]\end{aligned}$$

where $r = 0,1,...,n.$

Theorem 2 *The moment generating function* $M_z(t)$ *for a r.v.* Z *possessing* $EI_vD(\phi, \lambda, \rho)$ *is given by*

$$M_z(t) = \sum_{r=0}^{\infty}(t)^r\left\{r!\,(1+r\lambda)(1+r\rho\lambda)\,e^{r\phi}\right\}^{-1}.$$

Proof: Since $r.v.\ Z \sim EI_vD(\phi, \lambda, \rho)$, then $M_z(t)$ is obtained as

$$\begin{aligned} M_z(t) &= E(e^{zt}) = \int_0^{e^{-\phi}} e^{zt} f_{EI_vD}(z,\omega)dz \\ &= \sum_{r=0}^{\infty} (t)^r \left\{ r!\,(1+r\lambda)\,(1+r\rho\lambda)\,e^{r\phi} \right\}^{-1}. \end{aligned}$$

Remark 1 Since by definition, $\phi_z(t) = E(e^{\iota tz})$ is the characteristic function. So, in Theorem 2, it is obvious, if t is replaced by ιt, then it would define the characteristic function and to prove the result, the derivation of $\phi_z(t)$ would repeat the same procedure as it requires for moment generating function and the final result obtained is given by

$$\phi_z(t) = \sum_{r=0}^{\infty} (\iota t)^r \left\{ r!\,(1+r\lambda)\,(1+r\rho\lambda)\,e^{r\phi} \right\}^{-1}.$$

Remark 2 The expression of moment generating function $(M_z(t))$ is already proved, and is given in Theorem 2. Now, to obtain the cumulant generating function $(K_z(t))$. We know, by definition, $K_z(t) = \log\{M_z(t)\}$. So, after substituting $M_z(t)$ from Theorem 2 the required function obtained is as

$$K_z(t) = \log\left\{ \sum_{r=0}^{\infty} (t)^r \left\{ r!\,(1+r\lambda)\,(1+r\rho\lambda)\,e^{r\phi} \right\}^{-1} \right\}.$$

3. Background on Reliability Characterization

In reliability theory, the foremost step for any distribution is to derive its reliability function. Suppose Z be a non-negative $r.v.$, then the reliability function in mathematical form is described as:

$$R_{EI_vD}(z;\omega) = P(Z > z) = 1 - P(Z \leq z).$$

Therefore, for a $r.v.\ Z \sim EI_vD(\phi, \lambda, \rho)$ the derived reliability function is given by

$$R_{EI_vD}(z;\omega) = \begin{cases} 1 - \frac{\rho e^{\phi/\rho\lambda} z^{1/\rho\lambda} - e^{\phi/\lambda} z^{1/\lambda}}{\rho - 1} & \rho \neq 1 \\ 1 - \left[1 - \frac{(\phi + \log z)}{\lambda}\right] e^{\phi/\lambda} z^{1/\lambda} & \rho = 1 \end{cases}. \tag{3}$$

3.1. Hazard rate function

The remarkable lifetime characterization measures called hazard rate defined by $h_{EI_vD}(z;\omega) = f_{EI_vD}(z;\omega) \,/\, [1 - F_{EI_vD}(z;\omega)]$ (The ratio of pdf and the reliability function), and the reverse hazard rate function defined by $h^r_{EI_vD}(z;\omega) = f_{EI_vD}(z;\omega) \,/\, [F_{EI_vD}(z;\omega)]$ (The ratio of *pdf* and *cdf*) are derived for I_vED. Thus both the functions are given by:

$$h_{EI_vD}(z;\Theta) = \begin{cases} \frac{e^{\phi/\rho\lambda} z^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z^{(1/\lambda)-1}}{\lambda\left[(\rho-1) - \left\{\rho e^{\phi/\rho\lambda} z^{1/\rho\lambda} - e^{\phi/\lambda} z^{1/\lambda}\right\}\right]} & \rho \neq 1 \\ \frac{e^{\phi/\lambda}(\phi + \log z) z^{(1/\lambda)-1}}{\lambda\left\{(\lambda - \phi - \log z) e^{\phi/\lambda} z^{1/\lambda} - \lambda\right\}} & \rho = 1 \end{cases} \tag{4}$$

and

$$h^r_{EI_vD}(z;\Theta) = \begin{cases} \frac{e^{\phi/\rho\lambda} z^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z^{(1/\lambda)-1}}{\lambda\left\{\rho e^{\phi/\rho\lambda} z^{1/\rho\lambda} - e^{\phi/\lambda} z^{1/\lambda}\right\}} & \rho \neq 1 \\ \frac{(\phi + \log z)}{(\phi - \lambda + \log z)\lambda z} & \rho = 1 \end{cases}. \tag{5}$$

The graphical demonstration of the hazard rate function for the proposed model is shown in Figure 3 and its subplots in Figure 4 over the different sets of values of parameters. The bathtub shapes and the exponentially increasing curves for the function are seen.

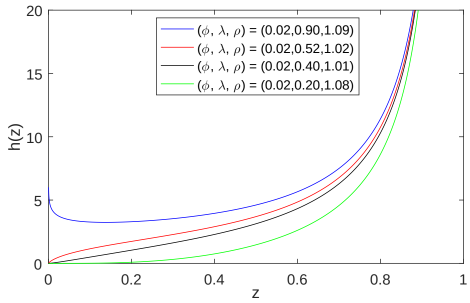

Figure 3 Hazard rate plot

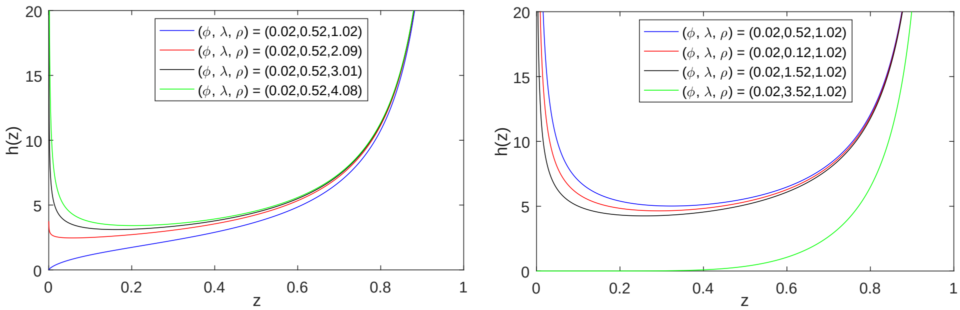

Figure 4 Hazard rate subplots

3.2. Mean residual life function

The mean residual life (MRL) function is a well-known statistical function, which finds massive applications in reliability theory. Let us suppose a system works up to time $t \geq 0$ without failure, then the total working hours for the system after the time t is considered as residual life, and this quantity is determined by the conditional $r.v.$ $Z - z|Z > z$ refer Finkelstein (2008). Thus to derive mathematically, the MRL function for a non-negative $r.v.$ $Z \sim EI_vD(\phi, \lambda, \rho)$, when $\rho \neq 1$, the steps required is given by

$$\begin{aligned} m_{EI_vD}(z;\omega) &= E\left[Z - t|_{Z>t}\right] = \frac{1}{R_{EI_vD}(z;\omega)} \int_z^{e^{-\phi}} R_{EI_vD}(z;\omega)dz \\ &= \frac{1}{(\rho-1)\,R_{EI_vD}(z;\omega)} \int_z^{e^{-\phi}} \left\{(\rho-1) - \rho e^{\phi/\rho\lambda} z^{1/\rho\lambda} + e^{\phi/\lambda} z^{1/\lambda}\right\} dz \\ &= \frac{a_1 e^{-\phi} - a_1 a_2 a_3 z + a_2 \lambda \rho^2 e^{\phi/\rho\lambda} z^{(1/\rho\lambda)+1} - a_3 \lambda e^{\phi/\lambda} z^{(1/\lambda)+1}}{a_1 a_2 a_3 R_{EI_vD}(z;\omega)} \end{aligned}$$

where $a_1 = (\rho - 1)$, $a_2 = (1 + \lambda)$, and $a_3 = (1 + \rho\lambda)$.

4. Stress-Strength Reliability

In this section, stress-strength modeling is being discussed, which is considered to have considerable importance in reliability theory and finds impressive applications in the field of engineering. Let us consider, Z_1 the strength of the system, which is undergoing some stress Z_2. Thus if $Z_2 > Z_1$ then the system breaks down (fails), and works normally if $Z_1 > Z_2$. Thus by using the formulae $R = P(Z_1 > Z_2)$ stress-strength reliability is obtained.

Let the random variables $Z_1 \sim EI_vD(\phi, \lambda_1, \rho_1)$ and $Z_2 \sim EI_vD(\phi, \lambda_2, \rho_2)$ with the same parameter ϕ. Then to obtain the stress-strength reliability R by using the given *cdf* of Z_2 and *pdf* of

Z_1 as given by

$$F_{EI_vD}(z;\omega_2)=\frac{\rho_2 e^{\phi/\rho_2\lambda_2}z^{1/\rho_2\lambda_2}-e^{\phi/\lambda_2}z^{1/\lambda_2}}{\rho_2-1}\qquad \rho_2\neq 1 \tag{6}$$

and

$$f_{EI_vD}(z;\omega_1)=\frac{e^{\phi/\rho_1\lambda_1}z^{(1/\rho_1\lambda_1)-1}-e^{\phi/\lambda_1}z^{(1/\lambda_1)-1}}{(\rho_1-1)\lambda_1}\qquad \rho_1\neq 1. \tag{7}$$

The stress-strength reliability estimate for newly developed EI_vD, when both $(\rho_1,\rho_2)\neq 1$ is derived as follows:

$$\begin{aligned} R &= \int_0^{e^{-\phi}}\left\{\int_0^{z} f_{Z_2}(z)dz\right\} f_{Z_1}(z)dz=\int_0^{e^{-\phi}} F_{Z_2}(z)f_{Z_1}(z)dz \\ &= \int_0^{e^{-\phi}}\left\{\frac{\rho_2 e^{\phi/\rho_2\lambda_2}z^{1/\rho_2\lambda_2}-e^{\phi/\lambda_2}z^{1/\lambda_2}}{\rho_2-1}\right\}\left\{\frac{e^{\phi/\rho_1\lambda_1}z^{(1/\rho_1\lambda_1)-1}-e^{\phi/\lambda_1}z^{(1/\lambda_1)-1}}{(\rho_1-1)\lambda_1}\right\}dz \\ &= \frac{\lambda_2^2}{(\rho_2-1)}\left\{\frac{\rho_2^3(\lambda_1+\lambda_2)(\rho_1\lambda_1+\lambda_2)-(\lambda_1+\rho_2\lambda_2)(\rho_1\lambda_1+\rho_2\lambda_2)}{(\lambda_1+\lambda_2)(\rho_1\lambda_1+\lambda_2)(\lambda_1+\rho_2\lambda_2)(\rho_1\lambda_1+\rho_2\lambda_2)}\right\}. \end{aligned}$$

5. Entropy Measures and Order Statistics

To measure the level of variations present among the distributions of the random variables different entropy measures have been given and the most prominent among them being the Rényi entropy by Rnyi (1961), and the Shannon entropy by Shannon (1948) respectively.

5.1. Rényi entropy

For a non-negative $r.v.\ Z\sim EI_vD(\phi,\lambda,\rho)$, when $\rho\neq 1$, the expression of Reńyi entropy of order δ is defined as

$$\hbar_R(\delta)=\frac{1}{1-\delta}\log\left[\int_0^{e^{-\phi}}\{f_{EI_vD}(z;\omega)\}^{\delta}dz\right];\quad \delta\geq 0,\delta\neq 1.$$

Now, in this integral, use the function $f_{EI_vD}(z;\omega)$, when $\rho\neq 1$, which is given in Eqn. (2), and on solving it, we get

$$\begin{aligned} \hbar_R(\delta) &= \frac{1}{1-\delta}\log\left[\int_0^{e^{-\phi}}\left\{\frac{e^{\phi/\rho\lambda}z^{(1/\rho\lambda)-1}-e^{\phi/\lambda}z^{(1/\lambda)-1}}{(\rho-1)\lambda}\right\}^{\delta}dz\right];\quad \delta\geq 0,\delta\neq 1 \\ &= \frac{1}{1-\delta}\log\left[\frac{1}{(\rho-1)\lambda}\sum_{n=0}^{\delta}\frac{(-1)^n\rho\lambda e^{\delta\phi}}{(\rho\lambda+\delta+n\rho-n-\rho\delta\lambda)}\right]. \end{aligned}$$

5.2. Shannon entropy

Shannon entropy for a $r.v.\ Z\sim EI_vD(\phi,\lambda,\rho)$ when $\rho\neq 1$ is obtained by using the following mathematical relation.

$$\tau(z)=-\int_0^{e^{-\phi}} f_{EI_vD}(z;\omega)\log\{f_{EI_vD}(z;\omega)\}dz.$$

Now, in this integral use the function $f_{EI_vD}(z;\omega)$, when $\rho\neq 1$, which is given in Eqn. (2), and after solving it we get

$$\tau(z)=-\int_0^{e^{-\phi}}\left[\frac{e^{\phi/\rho\lambda}z^{(1/\rho\lambda)-1}-e^{\phi/\lambda}z^{(1/\lambda)-1}}{(\rho-1)\lambda}\right]\log\left\{\frac{e^{\phi/\rho\lambda}z^{(1/\rho\lambda)-1}-e^{\phi/\lambda}z^{(1/\lambda)-1}}{(\rho-1)\lambda}\right\}dz$$

$$= \rho \sum_{r=1}^{\infty} \frac{(-1)^{2r}}{r a_4 a_5} - \frac{\phi}{\rho\lambda} + \frac{(\rho\lambda - 1)\left\{e^{\phi/\lambda}\Gamma\left(2, \phi/\lambda\right) - \rho^2 e^{\phi/\rho\lambda}\left(2, \phi/\rho\lambda\right)\right\}}{\rho\left(\rho - 1\right)}$$

where $a_4 = [1 + r(\rho - 1)]$ and $a_5 = [\rho + r(\rho - 1)]$.

5.3. Order statistics

The order statistics are of vital importance in distribution theory particularly in the field of reliability and survival analysis. Let $z = (z_1, z_2, ... ,z_n)$ be a random sample drawn from $EI_vD(\phi, \lambda, \rho)$ with desired sample size n. Then the ordered sample $z_{1:n} \le z_{2:n} \le ... \le z_{n:n}$ are denoted by $z_{i:n}$; $(i = 1, 2, ..., n)$, and the i^{th} order density function is given by

$$f_{i:n}(z; \omega) = N_1[F_{EI_vD}(z)]^{i-1}[1 - F_{EI_vD}(z)]^{n-i} f_{EI_vD}(z) \tag{8}$$

and the $(i, j)^{th}$ order density function for $(z_{(i:n)}, z_{(j:n)})$; $1 \le i \le j \le n$ is given by

$$\begin{aligned} f_{i:j:n}(z_i, z_j) =& N_2[F_{EI_vD}(z_i)]^{i-1}[F_{EI_vD}(z_j) - F_{EI_vD}(z_i)]^{j-i-1}[1 - F_{EI_vD}(z_j)]^{n-j} \\ & \cdot f_{EI_vD}(z_i) f_{EI_vD}(z_j) \end{aligned} \tag{9}$$

where $F(.)$ is *cdf* and $f(.)$ is the *pdf*, where both are given correspondingly in Eqns. (1) and (2) respectively. The constants N_1 and N_2 are defined as,

$$N_1 = \frac{n!}{(i-1)!(n-i)!} \text{ and } N_2 = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

Now to obtain the density functions of the first and the last (or n^{th}) order statistic, which are usually the $minimum$ and the $maximum$ values of $\{\ z_1, z_2, ... ,z_n\ \}$, the substitutions are to be made in Eqn. (8) for $i = 1$ and $i = n$, as given by:

$$\begin{aligned} f_{1:n}(z) &= n\left[1 - F_{EI_vD}(z_{(1)})\right]^{n-1} f_{EI_vD}(z_{(1)}) \\ f_{1:n}(z) &= \frac{n\left[(\rho-1) - \left\{\rho e^{\phi/\rho\lambda} z_{(1)}^{1/\rho\lambda} - e^{\phi/\lambda} z_{(1)}^{1/\lambda}\right\}\right]^{n-1}\left[e^{\phi/\rho\lambda} z_{(1)}^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z_{(1)}^{(1/\lambda)-1}\right]}{\lambda(\rho-1)^n} \end{aligned}$$

and

$$\begin{aligned} f_{n:n}(z) &= n\left[F_{EI_vD}(z_{(n)})\right]^{n-1} f_{EI_vD}(z_{(n)}). \\ f_{n:n}(z) &= \frac{n\left[\rho e^{\phi/\rho\lambda} z_{(n)}^{1/\rho\lambda} - e^{\phi/\lambda} z_{(n)}^{1/\lambda}\right]^{n-1}\left[e^{\phi/\rho\lambda} z_{(n)}^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z_{(n)}^{(1/\lambda)-1}\right]}{\lambda(\rho-1)^n}. \end{aligned}$$

Similarly, the joint order density function for i^{th} and the j^{th} statistic is obtained by substituting i^{th} and the j^{th} order cumulative density and the probability density functions in Eqn. (9) as derived by:

$$\begin{aligned} f_{i:j:n}(z_{(i)}, z_{(j)}) = \quad & \frac{N_2}{\lambda^2(\rho-1)^n}\left[\rho e^{\phi/\rho\lambda} z_{(i)}^{1/\rho\lambda} - e^{\phi/\lambda} z_{(i)}^{1/\lambda}\right]^{i-1} \\ . \quad & \left[\left\{\rho e^{\phi/\rho\lambda} z_{(i)}^{1/\rho\lambda} - e^{\phi/\lambda} z_{(i)}^{1/\lambda}\right\} - \left\{\rho e^{\phi/\rho\lambda} z_{(j)}^{1/\rho\lambda} - e^{\phi/\lambda} z_{(j)}^{1/\lambda}\right\}\right]^{j-i-1} \\ . \quad & \left[(\rho-1) - \left\{\rho e^{\phi/\rho\lambda} z_{(j)}^{1/\rho\lambda} - e^{\phi/\lambda} z_{(j)}^{1/\lambda}\right\}\right]^{n-j}\left[e^{\phi/\rho\lambda} z_{(i)}^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z_{(i)}^{(1/\lambda)-1}\right] \\ . \quad & \left[e^{\phi/\rho\lambda} z_{(j)}^{(1/\rho\lambda)-1} - e^{\phi/\lambda} z_{(j)}^{(1/\lambda)-1}\right]. \end{aligned}$$

6. Stochastic Ordering and Parameter Estimation

This section presents a detailed discussion on stochastic orderings and the estimation of parameters for the newly developed lifetime EI_vD in the following subsequent sections:

6.1. Stochastic ordering

The important tool counted in distribution theory for the lifetime distribution is considered as stochastic ordering, more detailed discussion on it given by Shaked and Shantikumar (2007). Let Z_1 and Z_2 be random variables both following $EI_vD(\phi,\lambda,\rho)$ having $f_{Z_1}(x)$, $f_{Z_2}(x)$ as $pdf's$ whereas $F_{Z_1}(x)$ and $F_{Z_2}(x)$ being its $cdf's$ respectively. The random variable Z_1 is smaller than Z_2 if the following measures of the stochastic ordering hold:

$[a]$ Stochastic order $(Z_1 \leq_{st} Z_2)$, if $F_{Z_1}(z) \geq F_{Z_2}(z)$ for all z.
$[b]$ Hazard rate order $(Z_1 \leq_{hr} Z_2)$, if $H_{Z_1}(z) \geq H_{Z_2}(z)$ for all z.
$[c]$ Mean residual life order $(Z_1 \leq_{MRL} Z_2)$, if $m_{Z_1}(z) \geq m_{Z_2}(z)$ for all z.
$[d]$ Likelihood ratio order $(Z_1 \leq_{LR} Z_2)$, if $\frac{f_1(z)}{f_2(z)}$ decreasing in z.

So according to above orderings following possible relationship is revealed.
$Z_1 \leq_{LR} Z_2 \Rightarrow Z_1 \leq_{hr} Z_2 \Rightarrow Z_1 \leq_{MRL} Z_2$ and $Z_1 \leq_{hr} Z_2 \Rightarrow Z_1 \leq_{st} Z_2$.
The compact illustration for the strongest likelihood ratio ordering for the $EI_vD(\phi,\lambda,\rho)$ is given by the following theorem.

Theorem 3 *Let $Z_1 \sim EI_vD(\phi_1,\lambda_1,\rho_1)$, and $Z_2 \sim EI_vD(\phi_2,\lambda_2,\rho_2)$. If $\phi_1=\phi_2=\phi$, $(\rho_1>\rho_2)>1$, and $(\lambda_1>\lambda_2)$ then $(Z_1 \leq_{lr} Z_2)$, $(Z_1 \leq_{st} Z_2)$, $(Z_1 \leq_{hr} Z_2)$, and $(Z_1 \leq_{MRL} Z_2)$.*

Proof: To prove the result, we have defined the ratio as

$$\frac{f_{Z_1}(z;\omega_1)}{f_{Z_2}(z;\omega_2)} = \frac{(\rho_2-1)\lambda_2}{(\rho_1-1)\lambda_1}\frac{\left\{e^{\phi_1/\rho_1\lambda_1}z^{(1/\rho_1\lambda_1)-1} - e^{\phi_1/\lambda_1}z^{(1/\lambda_1)-1}\right\}}{\left\{e^{\phi_2/\rho_2\lambda_2}z^{(1/\rho_2\lambda_2)-1} - e^{\phi_2/\lambda_2}z^{(1/\lambda_2)-1}\right\}}.$$

Then,

$$\frac{d}{dz}\log\left\{\frac{f_{Z_1}(z;\omega_1)}{f_{Z_2}(z;\omega_2)}\right\} = \frac{[\{B\rho_2\lambda_2\left(A_1-\rho_1A_2\right)\} - \{A\rho_1\lambda_1\left(B_1-\rho_2B_2\right)\}]}{AB\rho_1\lambda_1\rho_2\lambda_2 z}$$

where $A=\left\{e^{\phi_1/\rho_1\lambda_1}z^{(1/\rho_1\lambda_1)} - e^{\phi_1/\lambda_1}z^{(1/\lambda_1)}\right\}$, $B=\left\{e^{\phi_2/\rho_2\lambda_2}z^{(1/\rho_2\lambda_2)} - e^{\phi_2/\lambda_2}z^{(1/\lambda_2)}\right\}$,
$A_1 = e^{\phi_1/\rho_1\lambda_1}z^{(1/\rho_1\lambda_1)}$, $A_2 = e^{\phi_1/\lambda_1}z^{(1/\lambda_1)}$, $B_1 = e^{\phi_2/\rho_2\lambda_2}z^{(1/\rho_2\lambda_2)}$, and $B_2 = e^{\phi_2/\lambda_2}z^{(1/\lambda_2)}$.
Hence, If $\phi_1=\phi_2=\phi$, $(\rho_1>\rho_2)$, and $(\lambda_1>\lambda_2)$ then $\frac{d}{dz}\log\left\{\frac{f_{Z_1}(z;\omega_1)}{f_{Z_2}(z;\omega_2)}\right\} \leq 0$, which implies that $(Z_1 \leq_{lr} Z_2)$, $(Z_1 \leq_{st} Z_2)$, $(Z_1 \leq_{hr} Z_2)$, and $(Z_1 \leq_{MRL} Z_2)$.

6.2. Parameter estimation of the lifetime distribution EI_vD

In distribution theory, this is evident that for any probability model having an unknown parametric space, the foremost step is to estimate the parameters by employing any estimation procedure to study the model more deeply. Since there exist many estimation techniques but, the most often used method is the maximum likelihood estimation technique which has been used in this study. Let the n observations z_1, z_2, z_3, ... ,z_n be a randomly selected sample from $EI_vD(z,\omega)$ with a defined parametric space ω. Thus for the proposed model when $\rho \neq 1$ the complete data log-likelihood are described as follows

$$\log L = \sum_{i=1}^{n}\log\left\{e^{\phi/\rho\lambda}z_i^{1/\rho\lambda} - e^{\phi/\lambda}z_i^{1/\lambda}\right\} - \sum_{i=1}^{n}\log z_i - n\log(\rho-1) - n\log\lambda. \tag{10}$$

Take $M_1 = e^{\phi/\rho\lambda}z_i^{1/\rho\lambda}$, and $M_2 = e^{\phi/\lambda}z_i^{1/\lambda}$, then for the above equation it becomes

$$\log L = \sum_{i=1}^{n}\log\{M_1 - M_2\} - \sum_{i=1}^{n}\log z_i - n\log(\rho-1) - n\log\lambda. \tag{11}$$

Now, take the partial derivatives of Eqn. (11) with respect to $(w.r.t)$ desired parameters ϕ, λ, and ρ. we get the equations as:

$$\frac{\partial \log L}{\partial \phi} = \sum_{i=1}^{n} \frac{M_1 - \rho M_2}{\rho\lambda\left(M_1 - M_2\right)} \tag{12}$$

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n} \frac{\left(\phi + \log z_i\right)\left(M_1 - \rho M_2\right)}{\rho\lambda^2\left(M_2 - M_1\right)} - \frac{n}{\lambda} \tag{13}$$

$$\frac{\partial \log L}{\partial \rho} = \sum_{i=1}^{n} \frac{\left(\phi + \log z_i\right) M_1}{\rho^2\lambda\left(M_2 - M_1\right)} - \frac{n}{(\rho - 1)}. \tag{14}$$

To equate the above partial derivatives of parameters ϕ, λ, and ρ to zero i,e $\frac{\partial \log L}{\partial \phi} = 0$, $\frac{\partial \log L}{\partial \lambda} = 0$, and $\frac{\partial \log L}{\partial \rho} = 0$, we get the implicit normal likelihood equations. Thus the maximum likelihood estimates (MLE_s) are obtained, by employing an alternative iteration method approach such as the Newton-Rampson algorithm. It's worth noting that with the R programming language, one could use well-known functions like nlm or $optim$ to maximize the log-likelihood equations. For the proposed model $EI_vD(\phi, \lambda, \rho)$ the continuous partial differentials of the log-likelihood function $w.r.t$ the parameters are given in Eqns. (12), (13), and (14), which implies the second-order partial derivative also exist for the log-likelihood function. Let, the MLE_s of the parametric space $\omega = \{\ (\phi, \lambda, \rho) : \phi > 0,\ \lambda > 0\ ,\ \rho > 0\ \}$ are denoted by $\hat{\omega} = \{\ (\hat{\phi}\ ,\ \hat{\lambda}\ ,\ \hat{\rho})$: $\hat{\phi} > 0,\ \hat{\lambda} > 0\ ,\ \hat{\rho} > 0\ \}$, then the Fisher information matrix is defined as

$$I\left(\omega\right) = -E\begin{bmatrix} \frac{\partial^2 \log L}{\partial \phi^2} & \frac{\partial^2 \log L}{\partial \phi \partial \lambda} & \frac{\partial^2 \log L}{\partial \phi \partial \rho} \\ \frac{\partial^2 \log L}{\partial \lambda \partial \phi} & \frac{\partial^2 \log L}{\partial \lambda^2} & \frac{\partial^2 \log L}{\partial \lambda \partial \rho} \\ \frac{\partial^2 \log L}{\partial \rho \partial \phi} & \frac{\partial^2 \log L}{\partial \rho \partial \lambda} & \frac{\partial^2 \log L}{\partial \rho^2} \end{bmatrix}.$$

The second-order partial derivatives are the elements of the Fisher information matrix, which are listed below:

$$\frac{\partial^2 \log L}{\partial \phi^2} = \sum_{i=1}^{n} \frac{M_2\left\{\left(2\rho - 1\right)M_1 - \rho^2\right\}}{\left[\rho\lambda\left(M_1 - M_2\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \lambda^2} = \sum_{i=1}^{n} \frac{M_3\left\{\left(2\rho - \rho^2 - 1\right)M_1M_2M_3 + 2\rho\lambda\left(M_1 - M_2\right)\left(M_1 - \rho M_2\right)\right\}}{\rho\left[\rho\lambda^2\left(M_1 - M_2\right)\right]^2} + \frac{n}{\lambda^2}$$

$$\frac{\partial^2 \log L}{\partial \rho^2} = \sum_{i=1}^{n} \frac{M_1M_3\left\{2\rho\lambda\left(M_1 - M_2\right) - M_2M_3\right\}}{\left[\lambda\rho^2\left(M_1 - M_2\right)\right]^2} + \frac{n}{\left(\rho - 1\right)^2}$$

$$\frac{\partial^2 \log L}{\partial \lambda \partial \phi} = -\sum_{i=1}^{n} \frac{\left\{\left(2\rho - \rho^2 - 1\right)M_1M_2M_3 + \rho\lambda\left(M_1 - M_2\right)\left(M_1 - \rho M_2\right)\right\}}{\lambda\left[\rho\lambda\left(M_1 - M_2\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \rho \partial \phi} = -\sum_{i=1}^{n} \frac{\left\{\left(\rho - 1\right)M_1M_2M_3 + \rho\lambda\left(M_1 - \rho M_2\right)\left[M_1 - \left(\rho - 1\right)M_2\right]\right\}}{\rho\left[\rho\lambda\left(M_1 - M_2\right)\right]^2}$$

$$\frac{\partial^2 \log L}{\partial \lambda \partial \rho} = \sum_{i=1}^{n} \frac{M_3\left\{\left(\rho - 1\right)M_1M_2M_3 + \rho\lambda M_1\left(M_1 - M_2\right)\right\}}{\rho\lambda\left[\rho\lambda\left(M_1 - M_2\right)\right]^2}$$

Note that the term used in above expressions, $M_3 = (\phi + \log z_i)$. To compute the expectation of the above expressions is a tedious job. Therefore, the expectation operation E is dropped to get the observed Fisher information matrix (Cohen (1965)), which is defined as

$$I\left(\hat{\omega}\right) = -\begin{bmatrix} \frac{\partial^2 \log L}{\partial \phi^2} & \frac{\partial^2 \log L}{\partial \phi \partial \lambda} & \frac{\partial^2 \log L}{\partial \phi \partial \rho} \\ \frac{\partial^2 \log L}{\partial \lambda \partial \phi} & \frac{\partial^2 \log L}{\partial \lambda^2} & \frac{\partial^2 \log L}{\partial \lambda \partial \rho} \\ \frac{\partial^2 \log L}{\partial \rho \partial \phi} & \frac{\partial^2 \log L}{\partial \rho \partial \lambda} & \frac{\partial^2 \log L}{\partial \rho^2} \end{bmatrix}_{(\phi,\lambda,\rho)=(\hat{\phi},\hat{\lambda},\hat{\rho})}.$$

The asymptotic variance-covariance matrix of the parameters are obtained from the inversion matrix defined as $I\left(\hat{\omega}\right)^{-1}$. For all the parameters of the proposed model i, e ϕ , λ , and ρ the $100(1-\delta)\%$ confidence interval $(C.I)$ for ϕ , λ , and ρ are $\hat{\phi} \pm \chi_{\delta/2}\sqrt{V(\hat{\phi})}$, $\hat{\lambda} \pm \chi_{\delta/2}\sqrt{V(\hat{\lambda})}$, and $\hat{\rho} \pm \chi_{\delta/2}$ $\sqrt{V(\hat{\rho})}$ respectively. The variances $V(\hat{\phi})$, $V(\hat{\lambda})$, and $V(\hat{\rho})$ are diagonal elements of $I\left(\hat{\omega}\right)^{-1}$. Also $\chi_{\delta/2}$ is the upper $(\delta/2)th$ percentile of the standard normal distribution.

7. Simulation and Applications with Realistic Datasets

The following subsections provide a full overview of simulation study performance and real-life applications for the newly proposed EI_vD:

7.1. Simulation-based consistency justification

To judge the theoretical findings Monte Carlo simulation study of the proposed model (EI_vD) is carried out with the help of R programming language. Since the Acceptance rejection algorithm is used to generate the data as the inversion method does not exist due to the unexplicit form of quantile function. Different sample sizes $n =$ (30, 80, 130, 180, 250, 400) have been taken to check the performance of the parameters based on calculated absolute average bias $(Abias)$ and mean square error (MSE). The calculated results are presented in Table 1, as shown by:

Table 1 $Abias$ and MSE of parameters with different sample sizes

(ϕ, λ, ρ)	n	$Abias$			MSE		
		$\hat{\phi}$	$\hat{\lambda}$	$\hat{\rho}$	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\rho}$
$(0.28, 0.23, 1.02)$	30	0.03162	0.07956	1.9e+01	1.000e-03	0.00633	3.0e+02
	80	0.01272	0.05319	2.00800	1.617e-04	0.00283	4.03210
	130	0.00844	0.04371	0.90606	7.132e-05	0.00191	0.82094
	180	0.00711	0.04301	0.70744	5.056e-05	0.00185	0.50047
	250	0.00507	0.03634	0.50516	2.573e-05	0.00132	0.25519
	400	0.00345	0.03074	0.39361	1.188e-05	0.00095	0.15493
$(0.02, 0.45, 1.04)$	30	0.05848	0.14872	2.0e+01	3.420e-03	0.02212	4.0e+02
	80	0.02534	0.10487	2.02339	6.422e-04	0.01100	4.09409
	130	0.01912	0.08896	0.94288	3.656e-04	0.00791	0.88903
	180	0.01285	0.08896	0.66247	1.650e-04	0.00582	0.43886
	250	0.01063	0.07127	0.53397	1.130e-04	0.00508	0.28512
	400	0.00770	0.05931	0.39354	5.925e-05	0.00352	0.15487
$(1.25, 0.42, 1.07)$	30	0.05648	0.14187	1.9e+01	3.190e-03	0.02013	3.0e+01
	80	0.02411	0.08423	1.83875	5.814e-04	0.00710	3.38101
	130	0.01641	0.07526	0.96989	2.694e-04	0.00566	0.94068
	180	0.01219	0.06135	0.55295	1.485e-04	0.00376	0.30575
	250	0.01031	0.05904	0.49565	1.062e-04	0.00349	0.24567
	400	0.00698	0.04372	0.31237	4.870e-05	0.00191	0.09758

In Table 1 it is observed as the sample size becomes larger the decrease of $Abias$ and MSE is recorded, which declares the consistency of the parameters of the model.

7.2. Applications: analysis and discussion with realistic datasets

In this study, the superiority of $EI_vD(\phi, \lambda, \rho)$ have been demonstrated by analyzing real-life data sets, so that the most flexible model are selected among the competing models. The formulaes of different information criterias used to identify the best fitted model are given by: $AIC =$ - 2 $\log L(z;\omega)$ + $2k$, $BIC =$ - 2 $\log L(z;\omega)$ + $k\log(n)$, $HQIC =$ - $2\log L(z;\omega)$ + $2k\log[\log(n)]$, where k is the total number of parameters and n is the sample size. So, the distribution which is having the minimum values of $\{\ AIC,\ BIC,\ HQIC\ \}$ is considered the best fit. The EI_vD is compared with I_vED, exponentiated exponential distribution (EED), exponentiated generalized exponential distribution $(EGED)$, Weibull distribution (WD), and the two parameter Weibull distribution (WD^*) whose density functions are given in Eqns. (15) - (18) respectively.

$$EED = f(z;\omega) = \phi\lambda e^{-\lambda z}\left(1-e^{-\lambda z}\right)^{\phi-1} \tag{15}$$

$$EGED = f(z;\omega) = \phi\lambda\rho\left(e^{-\rho z}\right)^{\phi}\left(1-\left(e^{-\rho z}\right)^{\phi}\right)^{\lambda-1} \tag{16}$$

$$WD = f(z;\omega) = \frac{\phi}{\lambda}\left(\frac{z-\rho}{\lambda}\right)^{\phi-1} e^{-\left(\frac{z-\rho}{\lambda}\right)^{\phi}} \tag{17}$$

$$WD^* = f(z;\omega) = \frac{\phi}{\lambda}\left(\frac{z}{\lambda}\right)^{\phi-1} e^{-\left(\frac{z}{\lambda}\right)^{\phi}}. \tag{18}$$

The EI_vD is fitted to the real-life data set, that has been earlier analyzed and reported in the article by Ahmed (2021). The data set consisting of a sample of 50 devices are representing the lifetime performance in hours of the classical lamps. The data are given as:

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0.913, 0.786, 0.860, 0.904, 0.971, 0.616, 0.961, 0.789, 0.817, 0.722, 0.956, 0.835,
0.853, 0.692, 0.850, 0.677, 0.898, 0.965, 0.820, 0.964, 0.865, 0.947, 0.798, 0.746,
0.926, 0.709, 0.615, 0.747, 0.931, 0.913, 0.895, 0.745, 0.839, 0.766, 0.690, 0.531,
0.838, 0.846, 0.876, 0.817, 0.719, 0.907, 0.915, 0.879, 0.890, 0.865, 0.869, 0.772,
0.933, 0.875.
```

Table 2 Results of real data obtained from different measures

Models	Information Measures						
	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\rho}$	$\log L$	AIC	BIC	$HQIC$
EI_vD	0.01525	0.05921	2.00495	4.989e+01	-9.378e+01	-8.805e+01	-9.160e+01
WD^*	10.6877	0.87227	-	4.748e+01	-9.096e+01	-8.713e+01	-8.950e+01
EED	791.250	8.59530	-	3.464e+01	-6.528e+01	-6.146e+01	-6.382e+01
$EGED$	0.31255	840.285	27.7533	3.465e+01	-6.329e+01	-5.755e+01	-6.111e+01
I_vED	0.52424	0.15294	1.00044	2.285e+01	-3.970e+01	-3.396e+01	-3.751e+01
WD	0.78718	0.02669	0.53084	-1.855e+02	3.769e+02	3.827e+02	3.791e+02
Models	Goodness of Fit Tests						
	$\hat{\phi}$	$\hat{\lambda}$	$\hat{\rho}$	C_{VM}	AD	KS	$p\text{-}value$
EI_vD	0.01525	0.05921	2.00495	0.01619	0.14525	0.05496	0.9982
WD^*	10.6877	0.87227	-	0.04149	0.28300	0.06637	0.9803
EED	791.250	8.59530	-	0.34158	2.04089	0.17123	0.1016
$EGED$	0.31255	840.285	27.7533	0.34401	2.05493	0.16848	0.1170
I_vED	0.52424	0.15294	1.00044	0.45410	2.75362	0.24296	0.0055
WD	0.78718	0.02669	0.53084	0.62040	3.74017	0.91793	2.2e-16

Since the proposed model has acquired the minimum value of all the information measures and goodness of fit tests with a higher $p\text{-}value$ against the pre-existing models which are taken in comparison, and those models are EED, $EGED$, WD^*, I_vED, and WD. Hence confirms the model applicability to analyze real data in the field of reliability and survival analysis.

8. Conclusion

In this proposed study, a new intervened model has been developed which is $EI_vD\ (\ \phi,\ \lambda,\ \rho)$ that extends the I_vED introduced by Shanmugam et al. (2002). A variety of properties are presented to explore the distribution. The pdf and hazard rate plots that are drawn are having various shapes, the intrusting point for hazard rate function is that bath-tub shape and exponential increasing shapes are predicted. The parametric estimation is done by using the well-known maximum likelihood estimation technique. While carrying out the Monte Carlo simulation study the absolute bias and mean square error are noted and presented in Table 1, which both are decreasing with the increase of sample size. The real data set have been analyzed to show the flexibility of $EI_vD(\phi, \lambda, \rho)$ based on the calculated values of $\{\ AIC, BIC, HQIC\ \}$ and $\{\ C_{VM}, AD, KS, p\text{-}value\ \}$ which are furnished in Table 2. Since, it is noticed that the proposed distribution is having minimum values, of all the information measures, goodness of fit tests and higher $p\text{-}value$, which proves the superiority of the model among the competing distributions.

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