Thailand Statistician
April 2024; 22(2): 298-311
http://statassoc.or.th
Contributed paper

Comparison between the Two Haung-Kotz FGM Types by Fisher Information in Order Statistics and Their Concomitants

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Received: 6 January 2022
Revised: 11 April 2022
Accepted: 13 April 2022

Abstract

The Fisher information (FI) about the shape-parameter vectors of the two Huang-Kotz FGM types is investigated. We study analytically and numerically the FI matrix (FIM) related to an order statistic (OS) and its concomitant for each of these types. A comparison between the two types based on FI is carried out.

Keywords: Information measures, Haung-Kotz's bivariate distributions, correlation structure, concomitants of order statistics

1. Introduction

Farlie-Gumbel-Morgenstern (FGM) distribution was originally introduced by Morgenstern (1956) for Cauchy marginals. Gumbel (1960) investigated the same structure for exponential marginals. Farlie (1960), in a connection with his investigations of the correlation coefficient, suggested a new general form of a bivariate distribution for given arbitrary marginals inspired by the works of Morgenstern (1956) and Gumbel (1960). Later, Johnson and Kotz (1975, 1977) extended the suggested bivariate distribution to the multivariate case and coined the term "FGM"distribution function (DF). The FGM DF is defined by $F_{X,Y}(x,y)=F_X(x)F_Y(y)[1+\theta(1-F_X(x))(1-F_Y(y))],\ -1\leq\theta\leq 1$, where F_X and F_Y are the marginal DFs of some random variables (RVs) X and Y. While the classical FGM distribution is a flexible family and valuable in many applications, a well-known limitation of this distribution is the low dependence level it permits between RVs, where the maximal positive correlation coefficient is 0.33. Therefore, the applications of FGM distribution are confined to data that exhibits low correlation. Huang and Kotz (1984) used successive iterations in the FGM distribution to increase the correlation between the components, and showed that just one single iteration can result in tripling the covariance for certain marginals. Later, this model was extensively studied by Alawady et al. (2022), Barakat and Husseiny (2021), and Barakat et al. (2020, 2021). One of the most successful and well-known attempts to enhance the range of correlation and impart more flexibility of the FGM distribution is due to Huang and Kotz (1999). Huang and Kotz (1999) proposed two analogous extensions HK-FGM1(θ_1,p_1) and HK-FGM2(θ_2,p_2), defined respectively by

$$F^{(1)}_{X,Y}(x,y)=F_X(x)F_Y(y)\left[1+\theta_1(1-F^{p_1}_X(x))(1-F^{p_1}_Y(y))\right],p_1\geq 1, \tag{1}$$

with the probability density function (PDF)

$$f^{(1)}_{X,Y}(x,y)=f_X(x)f_Y(y)\left[1+\theta_1((1+p_1)F^{p_1}_X(x)-1)((1+p_1)F^{p_1}_Y(y)-1)\right],$$

$$\text{and} \quad F^{(2)}_{X,Y}(x,y) = F_X(x)F_Y(y)\left[1+\theta_2(1-F_X(x))^{p_2}(1-F_Y(y))^{p_2}\right], p_2 \geq 1, \tag{2}$$

with the PDF

$$f^{(2)}_{X,Y}(x,y) = f_X(x)f_Y(y)\big[1+\theta_2(1-F_X(x))^{p_2-1}(1-F_Y(y))^{p_2-1}$$
$$\times((1+p_2)F_X(x)-1)((1+p_2)F_Y(y)-1)\big].$$

The admissible range of the shape-parameter vectors (θ_1, p_1) and (θ_2, p_2) are

$$\Omega_1 = \{(\theta_1,p_1) : -p_1^{-2} \leq \theta_1 \leq p_1^{-1}, p_1 \geq 1\} \quad \text{and}$$
$$\Omega_2 = \left\{(\theta_2,p_2) : -1 \leq \theta_2 \leq \left(\frac{p_2+1}{p_2-1}\right)^{p_2-1}, p_2 > 1 \text{ or } -1 \leq \theta_2 \leq +1,\ p_2 = 1\right\}$$

respectively. The maximal positive correlations for the models (1) and (2) are given by 0.375 and 0.391, which are attained at $p_1 = 2$ and $p_2 = 1.1877$, respectively. There is a nuance between the two maximal positive correlations provided by the analogous models (1) and (2). Therefore, the trade off between the models (1) and (2) is not an easy task and in fact, no work can be found in literature about this problem. On the other hand, the most works about the extensions (1) and (2) are concerning to the family (1). Among those works are Abd Elgawad et al. (2020), Bairamov and Kotz (2002), Barakat et al. (2018, 2019), and Fisher and Klein (2007). One of the main aims of this paper is to compare the DFs (1) and (2) based on the FI in OSs and their concomitants.

Concomitants of OSs can emerge in sundry applications. In selection procedures, items or subjects may be chosen on the basis of their X characteristic, and an associated characteristic Y that is hard to measure or can be observed only later may be of interest. For a comprehensive review of different applications for the concomitants of OSs, see David and Nagaraja (1998, 2003). Nowadays, many recent works can be found in the literature for concomitants of OSs based on different generalizations of the FGM family, among them are Abd Elgawad (2021, 2022), Alawady et al. (2021a,b), Barakat et al. (2018, 2019, 2020, 2021, 2022), Eryilmaz (2016), Husseiny et al. (2022), and Philip and Thomas (2017).

Suppose $\{(X_i, Y_i), i = 1, 2, ..., n\}$ is a random sample from a bivariate DF $F_{X,Y}(x, y)$, with a PDF $f_{X,Y}(x, y)$. If we arranged the items of this sample by the $X-$variate to obtain the OSs $X_{1:n} \leq X_{2:n} \leq .... \leq X_{n:n}$, then the $Y-$variate associated with the rth OS $X_{r:n}$ would be called the concomitant of the rth OS, denoted by $Y_{[r:n]}$. Since the vectors $(X_1, Y_1), (X_2, Y_2), ..., (X_n, Y_n)$ are i.i.d, the conditional PDF of $Y_{[r:n]}$ given $X_{r:n} = x$ also equals the conditional DF of Y given X, i.e., $f_{Y_{[r:n]}|X_{r:n}}(y|x) = f_{Y|X}(y|x)$. Consequently, the joint PDF of $X_{r:n}$ and $Y_{[r:n]}$ can be expressed as (cf. David and Nagaraj, 2003)

$$f_{[r:n]}(x,y) = f_{Y|X}(y|x)f_{r:n}(x), \tag{3}$$

where $f_{r:n}(.)$ is the PDF of rth OS.

FI is an essential criterion in statistical inference especially in optimal and large sample studies in estimation theory. In the literature, there have been defined two forms of FI: one for the distribution parameters (which is our focus) and the other for the PDF (e.g, see Kharazmi and Asadi, 2018). The FI related to the distribution parameters tells us how much information about an unknown parameter we can get from a sample and FI is related to the sufficiency of a statistic and the efficiency of an estimator. The Cramér-Rao lower bound in its simplest form reflects the fact that the variance of any unbiased estimator is at least as high as the reciprocal of the FI value. Knowing FI that a sample involves an unknown parameter when the sample is large enough, helps finding bounds on the variance of a given estimator of that parameter and to approximate the sampling distribution of this estimator. Also, FI from censored samples that arises in a life-testing experiment is a useful tool for planning such experiments and for evaluating the performance of estimators and tests based on censored samples. Abo-Eleneen and Nagaraja (2002) investigated the properties of FI for FGM parent with dependence parameter θ, while Barakat et al. (2020) studied FI for the iterated FGM parent.

Consider a random vector (X,Y) of a PDF $f(x,y;\theta)$, where $\theta \in \Theta$ is an unknown parameter in a parameter space Θ. FI contained in the random vector (X,Y) about the parameter θ is (cf. Abo-Eleneen and Nagaraja, 2002 and Barakat et al., 2020)

$$I_\theta(X,Y) = \mathrm{E}\left(\frac{\partial \log f(x,y;\theta)}{\partial\theta}\right)^2 = -\mathrm{E}\left(\frac{\partial^2 \log f(x,y;\theta)}{\partial\theta^2}\right). \tag{4}$$

When the parameter θ is a vector $\underline{\theta} = (\theta_1,...,\theta_m)$, the FIM $I\!I(X,Y;\underline{\theta})$ is an $m\times m$ matrix, whose (i,j)th element is $I_{(\theta_i,\theta_j)}(X,Y) = -\mathrm{E}\left(\frac{\partial^2 \log f(X,Y;\underline{\theta})}{\partial\theta i\theta_j}\right)$.

The main aim of this paper is to evaluate the FIM related to the rth OS and its concomitant about the shape-parameter vectors $(\theta_i,p_i), i=1,2$, for HK-FGM1 and HK-FGM2, with a comparison. We investigate the FIMs

$$I\!I^{(i)}(X_{r:n},Y_{[r:n]}) = \begin{pmatrix} I_{\theta_i}(X_{r:n},Y_{[r:n]}) & I_{(\theta_i,p_i)}(X_{r:n},Y_{[r:n]}) \\ I_{(\theta_i,p_i)}(X_{r:n},Y_{[r:n]}) & I_{p_i}(X_{r:n},Y_{[r:n]}) \end{pmatrix},\ i=1,2. \tag{5}$$

Since our main aim is to study the FI in concomitants of OSs about only the unknown shape-parameter-vectors (θ_1,p_1) and (θ_2,p_2), our focus should be on the copulas related to the models (1) and (2). Copula is free of all unknown parameters except the shape parameters, and it can be obtained by letting the two RVs X and Y have uniform DFs.

The paper is organized as follows: In Section 2, a closed form for FIM for each of HK-FGM1 and HK-FGM2 is derived. Section 3, is devoted to numerical calculations, implemented by Mathematica 11.3, of the elements of the FIM in each of HK-FGM1 and HK-FGM2. From this numerical study, we will extract some important properties for the FIM related to the HK-FGM1 and HK-FGM2. Finally, at the end of Section 3, a comparison between HK-FGM1 and HK-FGM2 based on the FI is studied.

2. Closed Form for FIM for HK-FGM1 and HK-FGM2

Throughout this section and the paper, define the sequences $c(r,n) = r\binom{n}{r}$, $A_i^{(\ell)} = (-1)^i\theta_\ell^i$, $A_i = (-1)^i\binom{n-r}{i}$, $A_{ij}^{(\ell)} = (-1)^j(1+p_\ell)^j\binom{i+2}{j}$, $B_{ij}^{(\ell)} = (-1)^j(1+p_\ell)^j\binom{i}{j}$ and $C_{ij}^{(\ell)} = (-1)^j(1+p_\ell)^j\binom{i+1}{j}, \ell=1,2$, where the ranges of i and j will be separably defined in Theorems 1 and 2. Moreover, we use the symbols $\beta(a,b) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}$, $\Gamma(.)$, $H[.]$ and $\psi[.]$ to denote the beta function, gamma function, Harmonic number, and digamma function (the digamma function is the logarithmic derivative of the gamma function), respectively.

2.1. FIM for HK-FGM1

The copula density of (1) may be written in the form

$$f_{X,Y}^{(1)}(x,y) = [1+\theta_1 C_1(x,y;p_1)],\quad 0\le x,y\le 1, \tag{6}$$

where $C_1(x,y;p_1) = (1-(1+p_1)x^{p_1})(1-(1+p_1)y^{p_1})$. The FIM $I\!I^{(1)}(X_{r:n},Y_{[r:n]})$, defined in (5), for (6) is given in the next theorem.

Theorem 1 *Let $(\theta_1,p_1)\in\Omega_1^\star\cap\Omega_1$, where*

$$\Omega_1^\star = \{(\theta_1,p_1):\ \mid \theta_1 C_1(x,y,p_1)\mid < 1,\ \forall\, 0\le x,y\le 1\}. \tag{7}$$

Then, the elements $I_{\theta_1}(X_{r:n},Y_{[r:n]})$, $I_{(\theta_1,p_1)}(X_{r:n},Y_{[r:n]})$ and $I_{p_1}(X_{r:n},Y_{[r:n]})$ of the FIM $I\!I^{(1)}(X_{r:n},Y_{[r:n]})$ are

$$I_{\theta_1}(X_{r:n},Y_{[r:n]}) = c(r,n)\sum_{i=0}^{\infty}\sum_{j=0}^{i+2}\sum_{k=0}^{i+2}\frac{A_i^{(1)}A_{ij}^{(1)}A_{ik}^{(1)}}{jp_1+1}\beta(r+kp_1,n-r+1), \tag{8}$$

$$I_{(\theta_1,p_1)}(X_{r:n},Y_{[r:n]}) = c(r,n)\sum_{i=0}^{\infty}\sum_{j=0}^{i+1}\frac{A_i^{(1)}C_{ij}^{(1)}}{jp_1+1}\left[\sum_{k=0}^{i}B_{ik}^{(1)}\left(\beta(r+(k+1)p_1,n-r+1)\right.\right.$$
$$-\sum_{l=0}^{n-r}\frac{A_l(1+p_1)}{(r+(k+1)p_1+l)^2})$$
$$\left.+\sum_{l=0}^{i+1}\sum_{t=0}^{i}B_{it}^{(1)}C_{il}^{(1)}\beta(r+lp_1,n-r+1)(\frac{1}{(t+1)p_1+1}-\frac{1+p_1}{((t+1)p_1+1)^2})\right], \tag{9}$$

and

$$I_{p_1}(X_{r:n},Y_{[r:n]}) = c(r,n)\theta_1^2\sum_{i=0}^{\infty}A_i^{(1)}\left[\sum_{j=0}^{i+2}\sum_{k=0}^{i}\frac{A_{ij}^{(1)}B_{ik}^{(1)}}{jp_1+1}(\beta(r+(k+2)p_1,n-r+1)\right.$$
$$-\sum_{l=0}^{n-r}A_l(\frac{2(1+p_1)}{(r+(k+2)p_1+l)^2}-\frac{2(1+p_1)^2}{(r+(k+2)p_1+l)^3}))$$
$$+2\sum_{l=0}^{i+1}\sum_{t=0}^{i+1}C_{il}^{(1)}C_{it}^{(1)}\left(\frac{\beta(r+(t+1)p_1,n-r+1)}{(l+1)p_1+1}-\frac{(1+p_1)\beta(r+(t+1)p_1,n-r+1)}{((l+1)p_1+1)^2}\right.$$
$$\left.-\sum_{s=0}^{n-r}A_s\left(\frac{(1+p_1)}{((l+1)p_1+1)((t+1)p_1+r+s)^2}-\frac{(1+p_1)^2}{((l+1)p_1+1)^2((t+1)p_1+r+s)^2}\right)\right)$$
$$\left.+\sum_{l=0}^{i+2}\sum_{t=0}^{i}A_{il}^{(1)}B_{it}^{(1)}\beta(r+lp_1,n-r+1)\left(\frac{1}{(t+2)p_1+1}-\frac{2(1+p_1)}{((t+2)p_1+1)^2}+\frac{2(1+p_1)^2}{((t+2)p_1+1)^3}\right)\right]. \tag{10}$$

Proof: By combining (3), (4) and (6) we get

$$\frac{\partial^2\log f_{[r:n]}(x,y)}{\partial\theta_1^2} = \frac{\partial^2\log f_{X,Y}(x,y)}{\partial\theta_1^2} = -\frac{C_1^2(x,y;p_1)}{(1+\theta_1C_1(x,y;p_1))^2}. \tag{11}$$

The condition (7) allows us to expand $(1+\theta_1C_1(x,y;p_1))^{-1}$ by the binomial expansion. Thus, the relations (4) and (11) yield

$$\begin{aligned} I_{\theta_1}(X_{r:n},Y_{[r:n]}) &= c(r,n)\int_0^1\int_0^1\frac{C_1^2(x,y;p_1)}{(1+\theta_1C_1(x,y;p_1))}x^{r-1}(1-x)^{n-r}dxdy \\ &= c(r,n)\sum_{i=0}^{\infty}(-1)^i\theta_1^i\int_0^1\int_0^1 C_1^{i+2}(x,y;p_1)x^{r-1}(1-x)^{n-r}dxdy \end{aligned}$$
$$= c(r,n)\sum_{i=0}^{\infty}(-1)^i\theta_1^i\int_0^1\int_0^1(1-(1+p_1)x^{p_1})^{i+2}(1-(1+p_1)y^{p_1})^{i+2}x^{r-1}(1-x)^{n-r}dxdy.$$

Upon using the binomial expansion for $(1-(1+p_1)x^{p_1})^{i+2}$ and $(1-(1+p_1)y^{p_1})^{i+2}$, and after some simple algebra, we get (8). On the other hand, a combination of (4) and (6) yields

$$\frac{\partial^2\log f_{[r:n]}(x,y)}{\partial\theta_1\partial p_1} = \frac{\frac{\partial C_1(x,y;p_1)}{\partial p_1}}{(1+\theta_1C_1(x,y;p_1))^2}. \tag{12}$$

Thus, by using (4) and (12), we get

$$I_{(\theta_1,p_1)}(X_{r:n},Y_{[r:n]}) = c(r,n)\int_0^1\int_0^1\frac{\xi(x,y)}{(1+\theta_1C_1(x,y;p_1))}x^{r-1}(1-x)^{n-r}dxdy, \tag{13}$$

where

$$-\xi(x,y) = x^{p_1}(1-(1+p_1)y^{p_1})(1+(1+p_1)\log x) + y^{p_1}(1-(1+p_1)x^{p_1})(1+(1+p_1)\log y).$$

Again, the condition (7) allows us to expand $(1+\theta_1 C_1(x,y;p_1))^{-1}$ by the binomial expansion. Thus, (13) can be written as

$$I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]}) = c(r,n)\sum_{i=0}^{\infty} A_i^{(1)}(I_{1;i} + I_{2;i}), \tag{14}$$

where

$$I_{1;i} = \int_0^1\int_0^1 x^{r+p_1-1}(1-x)^{n-r}(1-(1+p_1)x^{p_1})^i(1-(1+p_1)y^{p_1})^{i+1}(1+(1+p_1)\log x)dxdy \tag{15}$$

and

$$I_{2;i} = \int_0^1\int_0^1 x^{r-1}y^{p_1}(1-x)^{n-r}(1-(1+p_1)y^{p_1})^i(1-(1+p_1)x^{p_1})^{i+1}(1+(1+p_1)\log y)dxdy. \tag{16}$$

By using the binomial expansion for the three middle terms of the integrand in $I_{1;i}$, the 3rd, 4th and 5th terms in $I_{2;i}$ and combining (14), (15) and (16) we get (9).

Finally, on the bases of (4), we have

$$I_{p_1}(X_{r:n}, Y_{[r:n]}) = \mathrm{E}(\frac{\partial \log f_{[r:n]}(x,y)}{\partial p_1})^2.$$

Therefore, (3) and (6) imply

$$I_{p_1}(X_{r:n}, Y_{[r:n]}) = c(r,n)\int_0^1\int_0^1 \frac{\theta_1^2\xi^2(x,y)}{(1+\theta_1 C_1(x,y;p_1))} x^{r-1}(1-x)^{n-r}dxdy, \tag{17}$$

where $\xi(x,y)$ is defined in (13). Once more, the condition (7) allows us to expand $(1+\theta_1 C_1(x,y;p_1))^{-1}$ by the binomial expansion. Thus, (17) after using the binomial expansion and some routine simplifications can be written as

$$I_{p_1}(X_{r:n}, Y_{[r:n]}) = c(r,n)\sum_{i=0}^{\infty} A_i^{(1)}\theta_1^2[J_{1;i} + J_{2;i} + J_{3;i}], \tag{18}$$

where

$$J_{1;i} = \sum_{j=0}^{i+2}\sum_{k=0}^{i} \frac{A_{ik}^{(1)}B_{ik}^{(1)}}{jp_1+1}$$

$$\times\left(\beta(r+(k+2)p_1, n-r+1) - \sum_{l=0}^{n-r} A_l(\frac{2(1+p_1)}{(r+(k+2)p_1+l)^2} - \frac{2(1+p_1)^2}{(r+(k+2)p_1+l)^3})\right), \tag{19}$$

$$J_{2;i} = 2\sum_{l=0}^{i+1}\sum_{t=0}^{i+1} C_{il}^{(1)}C_{it}^{(1)}\left(\frac{\beta(r+(l+1)p_1, n-r+1)}{(l+1)p_1+1} - \frac{(1+p_1)\beta(r+(l+1)p_1, n-r+1)}{((l+1)p_1+1)^2}\right.$$

$$\left. - \sum_{s=0}^{n-r} A_s\left(\frac{(1+p_1)}{((t+1)p_1+1)((t+1)p_1+r+s)^2} - \frac{(1+p_1)^2}{((t+1)p_1+1)^2((t+1)p_1+r+s)^2}\right)\right) \tag{20}$$

and

$$J_{3;i} = \sum_{l=0}^{i+2}\sum_{t=0}^{i} A_{il}^{(1)}B_{il}^{(1)}\beta(r+lp_1, n-r+1)\left(\frac{1}{(t+2)p_1+1} - \frac{2(1+p_1)}{((t+2)p_1+1)^2} + \frac{2(1+p_1)^2}{((t+2)p_1+1)^3}\right). \tag{21}$$

A combination of (18) with (19)-(21) proves the relation (10). This completes the proof of the theorem.

2.2. FIM for HK-FGM2

The copula density of (2) may be written in the form

$$f^{(2)}_{X,Y}(x,y) = [1+\theta_2 C_2(x,y;p_2)], \quad 0 \le x,y \le 1, \tag{22}$$

where $C_2(x,y;p_2) = (1-x)^{p_2-1}(1-(1+p_2)x)(1-y)^{p_2-1}(1-(1+p_2)y)$. The FIM $I\!I^{(2)}(X_{r:n},Y_{[r:n]})$ defined by (5), for (22) is given in the next theorem.

Theorem 2 *Let* $(\theta_2,p_2) \in \Omega_2^\star \cap \Omega_2$, *where*

$$\Omega_2^\star = \{(\theta_2,p_2): \ \mid \theta_2 C_2(x,y;p_2) \mid < 1, \ \forall\, 0 \le x,y \le 1\}. \tag{23}$$

Then, the elements $I_{\theta_2}(X_{r:n},Y_{[r:n]})$, $I_{(\theta_2,p_2)}(X_{r:n},Y_{[r:n]})$ *and* $I_{p_2}(X_{r:n},Y_{[r:n]})$ *of the FIM* $I\!I^{(2)}(X_{r:n},Y_{[r:n]})$ *are*

$$\begin{aligned} I_{\theta_2}(X_{r:n},Y_{[r:n]}) &= c(r,n)\sum_{i=0}^{\infty}\sum_{j=0}^{i+2}\sum_{k=0}^{i+2} A_i^{(2)}A_{ij}^{(2)}A_{ik}^{(2)} \\ \times &\quad \beta(j+1,(i+2)(p_2-1)+1)\beta(r+k,(i+2)(p_2-1)+n-r+1), \end{aligned} \tag{24}$$

$$\begin{aligned} I_{(\theta_2,p_2)}(X_{r:n},Y_{[r:n]}) &= c(r,n)\sum_{i=0}^{\infty} A_i^{(2)}\left[\sum_{j=0}^{i+1}\sum_{k=0}^{i+1} C_{ij}^{(2)}C_{ik}^{(2)}\beta(j+1,(i+1)(p_2-1)+1)\right. \\ \times\beta(r+k,(i+1)(p_2-1)+n-r+1)&(\psi[(i+1)(p_2-1)+n-r+1]-\psi[(i+1)(p_2-1)+n+k+1]) \\ &+\sum_{j=0}^{i+1}\sum_{k=0}^{i+1} C_{ij}^{(2)}C_{ik}^{(2)}\beta(r+j,(i+1)(p_2-1)+n-r+1)\beta(k+1,(i+1)p_2+1) \\ &\times(\psi[(i+1)(p_2-1)+1]-\psi[(i+1)(p_2-1)+k+2]) \\ &-\sum_{j=0}^{i+1}\sum_{k=0}^{i} C_{ij}^{(2)}C_{ik}^{(2)}\beta(r+k+1,(i+1)(p_2-1)+n-r+1)\beta(j+1,(i+1)(p_2-1)+1) \\ &\left.-\sum_{j=0}^{i+1}\sum_{k=0}^{i} C_{ij}^{(2)}C_{ik}^{(2)}\beta(k+2,(i+1)(p_2-1)+1)\beta(r+j,(i+1)(p_2-1)+n-r+1)\right] \end{aligned} \tag{25}$$

and

$$\begin{aligned} I_{p_2}(X_{r:n},Y_{[r:n]}) &= c(r,n)\theta_2^2\sum_{i=0}^{\infty} A_i^{(2)}\left[\sum_{j=0}^{i+2}\sum_{k=0}^{i+2} A_{ij}^{(2)}A_{ik}^{(2)}\beta(j+1,(i+2)(p_2-1)+1)\right. \\ &\times\frac{\Gamma(17+i)\Gamma(1+k)((H[16+i]-H[17+i+k])^2+\psi[1,17+i]-\psi[1,18+i+k])}{\Gamma(8+i+k)} \\ &+\sum_{j=0}^{i+2}\sum_{k=0}^{i+2} A_{ij}^{(2)}A_{ik}^{(2)}\beta(r+j,(i+2)(p_2-1)+n-r+1) \\ &\times\frac{\Gamma(3+i)\Gamma(1+k)((H[2+i]-H[3+i+k])^2+\psi[1,3+i]-\psi[1,4+i+k])}{\Gamma(4+i+k)} \\ &+\sum_{j=0}^{i+2}\sum_{k=0}^{i} A_{ij}^{(2)}B_{ik}^{(2)}\beta(j+1,(i+2)(p_2-1)+1)\beta(k+r+2,(i+2)(p_2-1)+n-r+1) \\ &+\sum_{j=0}^{i+2}\sum_{k=0}^{i} A_{ij}^{(2)}B_{ik}^{(2)}\beta(r+j,(i+2)(p_2-1)+n-r+1)\beta(k+3,(i+2)(p_2-1)+1) \\ +2\sum_{j=0}^{i+2} A_{ij}^{(2)}\beta(r+j,(i+2)(p_2-1)&+n-r+1)[\psi[(i+2)(p_2-1)+n-r+1]-\psi[(i+2)(p_2-1)+n+j+1]] \end{aligned}$$

$$\times\sum_{k=0}^{i+2}A_{ik}^{(2)}\beta(k+1,(i+2)(p_2-1)+1)[\psi[(i+2)(p_2-1)+1]-\psi[(i+2)(p_2-1)+k+2]]$$

$$-2\sum_{j=0}^{i}B_{ij}^{(2)}\beta(r+j+1,(i+2)(p_2-1)+n-r+1)[\psi[(i+2)(p_2-1)+n-r+1]$$

$$-\psi[(i+2)(p_2-1)+n+j+2]]\sum_{k=0}^{i+2}A_{ik}^{(2)}\beta(k+1,(i+2)(p_2-1)+1)-2\sum_{j=0}^{i}B_{ij}^{(2)}\beta(j+2,(i+2)(p_2-1)+1)$$

$$\times\sum_{k=0}^{i+2}A_{ik}^{(2)}\beta(r+k,(i+2)(p_2-1)+n-r+1)[\psi[(i+2)(p_2-1)+n-r+1]-\psi[(i+2)(p_2-1)+n+k+1]]$$

$$-2\sum_{j=0}^{i}B_{ij}^{(2)}\beta(r+j+1,(i+2)(p_2-1)+n-r+1)\sum_{k=0}^{i+2}A_{ik}^{(2)}\beta(k+1,(i+2)(p_2-1)+1)$$

$$\times[\psi[(i+2)(p_2-1)+1]-\psi[(i+2)(p_2-1)+k+2]]-2\sum_{j=0}^{i+2}A_{ij}^{(2)}\beta(r+j,(i+2)(p_2-1)+n-r+1)$$

$$\times\sum_{k=0}^{i}B_{ik}^{(2)}\beta(k+2,(i+2)(p_2-1)+1)[\psi[(i+2)(p_2-1)+1]-\psi[(i+2)(p_2-1)+k+3]]$$

$$\left.+\,2\sum_{j=0}^{i+1}\sum_{k=0}^{i+1}C_{ij}^{(2)}C_{ik}^{(2)}\beta(r+j+1,(i+2)(p_2-1)+n-r+1)\beta(k+2,(i+2)(p_2-1)+1)\right]. \quad (26)$$

Proof: By combining (3), (4) and (22), we get

$$\frac{\partial^2\log f_{[r:n]}(x,y)}{\partial\theta_2^2}=-\frac{C_2^2(x,y;p_2)}{(1+\theta_2C_2(x,y;p_2))^2}. \quad (27)$$

The condition (23) permits us to expand $(1+\theta_2C_2(x,y;p_2))^{-1}$ by the binomial expansion. Thus, the relations (3), (4) and (27) yield

$$I_{\theta_2}(X_{r:n},Y_{[r:n]})=c(r,n)\int_0^1\int_0^1\frac{C_2^2(x,y;p_2)}{(1+\theta_2C_2(x,y;p_2))}x^{r-1}(1-x)^{n-r}dxdy$$

$$=\quad c(r,n)\sum_{i=0}^{\infty}(-1)^i\theta_2^i\int_0^1\int_0^1C_2^{i+2}(x,y;p_2)x^{r-1}(1-x)^{n-r}dxdy=c(r,n)\sum_{i=0}^{\infty}(-1)^i\theta_2^i$$

$$\times\int_0^1\int_0^1(1-(1+p_2)x)^{i+2}(1-(1+p_2)y)^{i+2}(1-y)^{(i+2)(p_2-1)}x^{r-1}(1-x)^{n-r+(i+2)(p_2-1)}dxdy.$$

Upon using the binomial expansion for $(1-(1+p_2)x)^{i+2}$ and $(1-(1+p_2)y)^{i+2}$ and after using some simple algebra, we get (24). On the other hand, a combination of (3), (4) and (22) yields

$$\frac{\partial^2\log f_{[r:n]}(x,y)}{\partial\theta_2\partial p_2}=-\frac{\frac{\partial C_2(x,y;p_2)}{\partial p_2}}{(1+\theta_2C_2(x,y;p_2))^2}. \quad (28)$$

Therefore, the relations (4) and (28) yield

$$I_{(\theta_2,p_2)}(X_{r:n},Y_{[r:n]})=c(r,n)\int_0^1\int_0^1\frac{\zeta(x,y)}{(1+\theta_2C_2(x,y;p_2))}x^{r-1}(1-x)^{n-r}dxdy,$$

where

$$\zeta(x,y)=(1-x)^{p_2-1}(1-(1+p_2)x)(1-y)^{p_2-1}(1-(1+p_2)y)\log(1-x)$$

$$+(1-x)^{p_2-1}(1-(1+p_2)x)(1-y)^{p_2-1}(1-(1+p_2)y)\log(1-y)$$
$$-x(1-x)^{p_2-1}(1-y)^{p_2-1}(1-(1+p_2)y)-y(1-x)^{p_2-1}(1-y)^{p_2-1}(1-(1+p_2)x). \quad (29)$$

Again, the condition (23) allows us to expand $(1+\theta_2 C_2(x,y;p_2))^{-1}$ by the binomial expansion. Thus, (28) can be written as

$$I_{(\theta_2,p_2)}(X_{r:n},Y_{[r:n]}) = c(r,n)\sum_{i=0}^{\infty} A_i^{(2)}(I_{1;i}+I_{2;i}+I_{3;i}+I_{4;i}), \quad (30)$$

where
$$I_{1;i} = \int_0^1\int_0^1 x^{r-1}(1-x)^{n-r+(i+1)(p_2-1)}(1-(1+p_2)x)^{i+1}$$
$$\times(1-(1+p_2)y)^{i+1}(1-y)^{(i+1)(p_2-1)}\log(1-x)dxdy, \quad (31)$$

$$I_{2;i} = \int_0^1\int_0^1 x^{r-1}(1-x)^{n-r+(i+1)(p_2-1)}(1-(1+p_2)x)^{i+1} \quad (32)$$
$$\times(1-(1+p_2)y)^{i+1}(1-y)^{(i+1)(p_2-1)}\log(1-y)dxdy, \quad (33)$$

$$I_{3;i} = -\int_0^1\int_0^1 x^r(1-x)^{n-r+(i+1)(p_2-1)}(1-(1+p_2)x)^i(1-(1+p_2)y)^{i+1}(1-y)^{(i+1)(p_2-1)}dxdy, \quad (34)$$

and

$$I_{4;i} = -\int_0^1\int_0^1 yx^{r-1}(1-x)^{n-r+(i+1)(p_2-1)}(1-(1+p_2)x)^{i+1}(1-(1+p_2)y)^i(1-y)^{(i+1)(p_2-1)}dxdy. \quad (35)$$

Now, in each of the integrand of $I_{1;i}$ and $I_{2;i}$ expand $(1-(1+p_2)x)^{i+1}$ and $(1-(1+p_2)y)^{i+1}$ binomially. Moreover, use the binomial expansion for the 3rd and 4th terms in the integrand of $I_{3;i}$. Finally, use the binomial expansion for the 4th and 5th terms in the integrand of $I_{4;i}$. Thus, a combination of (29)-(35) yields (25), after using Gradshteyn and Ryzhik (2007) to solve some resulted integrals.

Finally, in view of (3), we have

$$\left(\frac{\partial \log f_{[r:n]}(x,y)}{\partial p_2}\right)^2 = \frac{(\frac{\partial C_2(x,y;p_2)}{\partial p_2})^2}{(1+\theta_2 C_2(x,y;p_2))^2}.$$

Therefore, on the bases of (4), (22) and (23) we get

$$I_{p_2}(X_{r:n},Y_{[r:n]}) = c(r,n)\int_0^1\int_0^1 \frac{(\theta_2\frac{\partial C_2(x,y;p_2)}{\partial p_2})^2}{(1+\theta_2 C_2(x,y;p_2))}x^{r-1}(1-x)^{n-r}dxdy.$$
$$= c(r,n)\sum_{i=0}^{\infty} A_i^{(2)}\theta_2^2\int_0^1\int_0^1 \zeta^2(x,y)C_2^i(x,y;p_2)x^{r-1}(1-x)^{n-r}dxdy, \quad (36)$$

where $\zeta(x,y)$ is defined by (29). Thus, upon expanding $\zeta^2(x,y)$ ($\zeta(x,y)$ consists of four terms), the double integral in (36) yield ten double integrals. Therefore, by the aiding of Mathematica 11.3, these double integrals yield explicitly the last required relation (26). This completes the proof of the theorem.

Remark 1 In Theorems 1 and 2, in order to check $(\theta_i',p_i')\in\Omega_i^\star$, for any pair $(\theta_i',p_i')\in\Omega_i, i=1,2$, we draw the function $\mathcal{F}(x,y;\theta_i',p_i') =\mid \theta_i' C_i(x,y;p_i')\mid$, with respect to $0\le x,y\le 1$, by using Mathematica 11.3. If the surface representing $\mathcal{F}$ falls entirely within the cuboid $\mathcal{K}=\{(x,y,z): 0\le x,y\le 1, -1<z<1\}$, then $(\theta_i',p_i')\in\Omega_i^\star\cap\Omega_i, i=1,2$, otherwise $(\theta_i',p_i')\notin\Omega_i^\star\cap\Omega_i, i=1,2$. It is worth mentioning that via numerous checking of diverse values in Ω_i, it was revealed that the case $(\theta_i',p_i')\notin\Omega_i^\star, i=1,2$, can only occur for the boundary values of (θ_i',p_i'), or close to them in the parmetric space Ω_i.

Remark 2 By putting $p_1 = p_2 = 1$ and $\theta_1 = \theta_2$, in Theorems 1 and 2, it easy to check that $I_{\theta_1}(X_{r:n}, Y_{[r:n]}) = I_{\theta_2}(X_{r:n}, Y_{[r:n]})$. Moreover, by putting $p_1 = p_2 = 1$ and $\theta_1 = \theta_2 = \theta$ starting in (11) and by noting that $\int_{-1}^{1} w^{i+2} dw = 0$, whenever i is odd, it easy we to show that

$$I_\theta(X_{r:n}, Y_{[r:n]}) = \frac{c(r,n)}{2^n} \sum_{i=0}^{\infty} \frac{\theta^{2i+3}}{2i+3} \int_{-1}^{1} w^{2i+2}(1-w)^{r-1}(1+w)^{n-r} dw,$$

which is the relation (3.6) in Abo-Eleneen and Nagaraja (2002).

3. Numerical Study

In Subsection 3.1, the elements of FIM $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$ (given by (8)-(10)) and $I\!I^{(2)}(X_{r:n}, Y_{[r:n]})$ (given by (24)-(26)) related to HK-FGM1 and HK-FGM2, respectively, are numerically studied by using Mathematica 11.3. Moreover, the percentages of FI conveyed by singly or multiply censored bivariate samples drawn from HK-FGM1 and HK-FGM2 are revealed. In Subsection 3.2, several comparisons between HK-FGM1 and HK-FGM2 based on FI are carried out.

3.1. Computing the FIMs $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$ and $I\!I^{(2)}(X_{r:n}, Y_{[r:n]})$

Tables 1 and 2 are devoted to the FIM $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$, while Tables 3 and 4 are allotted to study the FIM $I\!I^{(2)}(X_{r:n}, Y_{[r:n]})$. Moreover, Tables 1-4 provide values of each of the $I_{\theta_i}(X_{r:n}, Y_{[r:n]})$, $I_{(\theta_i,p_i)}(X_{r:n}, Y_{[r:n]})$ and $I_{p_i}(X_{r:n}, Y_{[r:n]}), i = 1, 2$, as a function of n, r ($r \le \frac{n+1}{2}$), θ_i and p_i, $i = 1, 2$, where $(\theta_i, p_i) \in \Omega_i^\star \cap \Omega_i, i = 1, 2$ (see, Remark 1). We compute the entries in $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$ and $I\!I^{(2)}(X_{r:n}, Y_{[r:n]})$ using (8)-(10) and (24)-(26), respectively, where we cut off the infinite series after 11 terms, which guarantees satisfactory accuracy. For Tables 1 and 3, we consider $n = 1, 2, 3, 5, 10, 15$, $\theta_1 = \theta_2 = 0.25, 0.50, 0.75, 0.99$, and $p_1 = p_2 = 1$. On the other hand, for Tables 2 and 4, we consider $n = 1, 2, 3, 5, 10, 15$, $\theta_1 = \theta_2 = -0.25, -0.15, 0.15, 0.25$, and $p_1 = p_2 = 2$. Finally, Tables 1-4 represent matrices: every entry $d = a, b, c$ consists of three numbers a, b and c separated by two commas, where $a = I_{\theta_i}(X_{r:n}, Y_{[r:n]})$, $b = I_{(\theta_i,p_i)}(X_{r:n}, Y_{[r:n]})$ and $c = I_{p_i}(X_{r:n}, Y_{[r:n]}), i = 1, 2$.

The first rows of Tables 1-4 represent $I\!I^{(1)}(X, Y)$ and $I\!I^{(2)}(X, Y)$ in a single pair (X, Y). Based on these tables and on the fact that the FIM $I\!I^{(i)}(X, Y)$ in a random sample of size n is $nI\!I^{(i)}(X, Y)$, $i = 1, 2$, we compute the proportion of the sample FIM $I\!I^{(i)}(X_{r:n}, Y_{[r:n]})$ contained in the single pair $(X_{r:n}, Y_{[r:n]}), i = 1, 2$ (see Tables 5-8). For example, when $n = 10$, $p_1 = 1$, and θ_1 ranges from 0.25 to 0.99, the FI $I_{\theta_1}(X_{1:10}, Y_{[1:10]})$ in the extreme pair $(X_{1:10}, Y_{[1:10]})$ ranges from 0.21 to 0.25 of the total FI, while the FI $I_{(\theta_1,p_1)}(X_{1:10}, Y_{[1:10]})$ in the extreme pair $(X_{1:10}, Y_{[1:10]})$ ranges from 0.16 to 0.21 of the total FI and the FI $I_{p_1}(X_{1:10}, Y_{[1:10]})$ in $(X_{1:10}, Y_{[1:10]})$ ranges from 0.14 to 0.15 of the total FI (see Table 5). Also, from Table 5 at $n = 15$, the FI $I_{\theta_1}(X_{1:15}, Y_{[1:15]})$, $I_{(\theta_1,p_1)}(X_{1:15}, Y_{[1:15]})$ and $I_{p_1}(X_{1:15}, Y_{[1:15]})$ in the extreme pair $(X_{1:15}, Y_{[1:15]})$ vary in the ranges of 0.16 to 0.20, 0.11 to 0.07 and 0.1 to 0.11 of the total information, respectively. In contrast, the FI in the central pair $(X_{5:15}, Y_{[5:15]})$ is no more than 0.01 of what is available in the complete sample for $I_{\theta_1}(X_{r:n}, Y_{[r:n]})$, $I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]})$ and $I_{p_1}(X_{r:n}, Y_{[r:n]})$. Similar results can be extracted from Tables 2 and 6, when $p_1 = 2$. Moreover, for $I\!I^{(2)}(X_{r:n}, Y_{[r:n]})$, Tables 3 and 7 and Tables 4 and 8 give parallel results for $p_2 = 1$ and $p_2 = 2$, respectively. Tables 5 and 6 show that the proportion of the sample FI in the FIM $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$ decreases with increasing p_1. On the other hand, Tables 7 and 8 show that the proportion of the sample FI $I_{\theta_2}(X_{r:n}, Y_{[r:n]})$, decreases with increasing p_2, while the proportion of the sample FI $I_{(\theta_2,p_2)}(X_{r:n}, Y_{[r:n]})$ and $I_{p_2}(X_{r:n}, Y_{[r:n]})$ increasing with p_2. Finally, Tables 1-4 almost show that $I_{\theta_i}(X_{r:n}, Y_{[r:n]})$ and $I_{p_i}(X_{r:n}, Y_{[r:n]})$ in the FIMs $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$ and $I\!I^{(2)}(X_{r:n}, Y_{[r:n]}), i = 1, 2$, increase with n and decrease with increasing r. In contrast, $I_{(\theta_i,p_i)}(X_{r:n}, Y_{[r:n]})$ has an erratic behaviour. Namely: In Table 1, the FI $I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]}) \ge 0$ increases with n and decreases with increasing r, while in Table 3, the FI

$I_{(\theta_2,p_2)}(X_{r:n}, Y_{[r:n]}) \leq 0$ decreases with n and increases with r. On the other hand, In Tables 2 and 4, the sign of $I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]})$ and $I_{(\theta_2,p_2)}(X_{r:n}, Y_{[r:n]})$ vary, with changing θ_1 and θ_2, respectively. Moreover, the values of them fluctuates disorderly with changing n or r.

3.1.1 Comparison between HK-FGM1 and HK-FGM2 based on the FI about the shape parameters

Tables 1-4 reveal the following similarity and divergence between HK-FGM1 and HK-FGM2 based on the FI about the shape parameters.

(1) $I_{\theta_1}(X_{r:n}, Y_{[r:n]})$ and $I_{\theta_2}(X_{r:n}, Y_{[r:n]})$ in Tables 1 and 3 (at $p_1 = p_2 = 1$) are equal when $\theta_1 = \theta_2$, respectively (see Remark 2).

(2) In Table 2, $I_{\theta_1}(X_{r:n}, Y_{[r:n]})$ is greater than $I_{\theta_2}(X_{r:n}, Y_{[r:n]})$ in Table 4, at $\theta_1 = \theta_2$ and $p_1 = p_2 = 2$.

(3) In Table 2, $I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]})$ is less than $I_{(\theta_2,p_2)}(X_{r:n}, Y_{[r:n]})$ in Table 4, at $\theta_1 = \theta_2 = -0.25, -0.15$ and $p_1 = p_2 = 2$. On the other hand, in Table 2, $I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]})$ is greater than $I_{(\theta_2,p_2)}(X_{r:n}, Y_{[r:n]})$ in Table 4, at $\theta_1 = \theta_2 = 0.15, 0.25$ and $p_1 = p_2 = 2$.

(4) In Table 1, $I_{(\theta_1,p_1)}(X_{r:n}, Y_{[r:n]}) \geq 0$ and $I_{(\theta_2,p_2)}(X_{r:n}, Y_{[r:n]}) \leq 0$ in Table 3, at $\theta_1 = \theta_2$ and $p_1 = p_2 = 1$.

(5) In Table 1, $I_{p_1}(X_{r:n}, Y_{[r:n]})$ is less than $I_{p_2}(X_{r:n}, Y_{[r:n]})$ in Table 3, at $\theta_1 = \theta_2$ and $p_1 = p_2 = 1$.

(6) In Table 2, $I_{p_1}(X_{r:n}, Y_{[r:n]})$ is almost greater than $I_{p_2}(X_{r:n}, Y_{[r:n]})$, in Table 4, at $\theta_1 = \theta_2$ and $p_1 = p_2 = 2$.

From the remarks (1)-(6), we get the following general conclusions, which are related to our numerical study.

- HK-FGM1 is better than HK-FGM2 relying on the FI about the shape parameter $\theta_1 = \theta_2$.
- HK-FGM2 is the best relying on the FI about the shape-parameter vector $(\theta_1, p_1) = (\theta_2, p_2)$, where $\theta_1 = \theta_2$ takes negative value, while if $\theta_1 = \theta_2$ takes positive value, HK-FGM1 will be the best.
- HK-FGM2 is the best relying on the FI about the shape parameter $p_1 = p_2 = 1$, where $\theta_1 = \theta_2$, while if $\theta_1 = \theta_2$ and $p_1 = p_2 = 2$, HK-FGM1 will be almost the best.

Table 1 $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$, at $p_1 = 1$

n	r	$\theta_1 = 0.25$	$\theta_1 = 0.5$	$\theta_1 = 0.75$	$\theta_1 = 0.99$
1	1	0.1137,0.0379,0.0208	0.1226,0.0813,0.0683	0.1434,0.1414,0.2258	0.1945,0.2484,0.5041
2	1	0.1137,0.0338,0.0176	0.1226,0.0739,0.0735	0.1434,0.1309,0.1839	0.1945,0.2338,0.3999
3	1	0.1367,0.0390,0.0199	0.1482,0.0863,0.0830	0.1755,0.1556,0.2070	0.2438,0.2864,0.4637
3	2	0.0678,0.0234,0.0136	0.0714,0.0492,0.0544	0.0793,0.0816,0.1336	0.0960,0.1288,0.2723
5	1	0.1793,0.0487,0.0244	0.1959,0.1091,0.1021	0.2354,0.2009,0.2602	0.3366,0.3832,0.5951
5	2	0.0808,0.0268,0.0143	0.0856,0.0575,0.0592	0.0960,0.0977,0.1443	0.1186,0.1597,0.2935
5	3	0.0482,0.0170,0.0096	0.0502,0.0352,0.0398	0.0543,0.0569,0.0956	0.0620,0.0853,0.1862
10	1	0.2393,0.0599,0.0293	0.2641,0.1365,0.1234	0.3248,0.2595,0.3206	0.4856,0.5236,0.7695
10	2	0.1550,0.0466,0.0237	0.1668,0.1025,0.0988	0.1936,0.1817,0.2471	0.2554,0.3197,0.5338
10	3	0.0925,0.0317,0.0169	0.0976,0.0679,0.0697	0.1085,0.1146,0.1693	0.1309,0.1884,0.3400
10	4	0.0511,0.0192,0.0108	0.0532,0.0403,0.0443	0.0572,0.0657,0.1051	0.0648,0.0986,0.2010
10	5	0.0306,0.0115,0.0068	0.0314,0.0238,0.0279	0.0331,0.0377,0.0653	0.0358,0.0541,0.1214
15	1	0.2681,0.0637,0.0309	0.2978,0.0997,0.1302	0.3715,0.2846,0.3421	0.5718,0.5955,0.8469
15	2	0.2012,0.0563,0.0279	0.2187,0.0741,0.1171	0.2594,0.2290,0.2981	0.3577,0.4244,0.6727
15	3	0.1451,0.0458,0.0234	0.1550,0.0513,0.0975	0.1767,0.1734,0.2415	0.2237,0.2933,0.5074
15	4	0.0996,0.0346,0.0184	0.1049,0.0332,0.0760	0.1159,0.1243,0.1841	0.1376,0.1964,0.3668
15	5	0.0645,0.0244,0.0135	0.0672,0.0513,0.0554	0.0724,0.0838,0.1319	0.0820,0.1261,0.2529
15	6	0.0396,0.0159,0.0093	0.0408,0.0332,0.0377	0.0432,0.0530,0.0888	0.0472,0.0768,0.1656
15	7	0.0247,0.0100,0.0061	0.0252,0.0206,0.0250	0.0263,0.0323,0.0579	0.0280,0.0456,0.1060
15	8	0.0197,0.0071,0.0045	0.0201,0.0145,0.0182	0.0208,0.0225,0.0422	0.0218,0.0311,0.0767

Table 2 $I\!I^{(1)}(X_{r:n}, Y_{[r:n]})$, at $p_1 = 2$

n	r	$\theta_1 = -0.25$	$\theta_1 = -0.15$	$\theta_1 = 0.15$	$\theta_1 = 0.25$
1	1	1.0184,-0.1786,0.0482	0.7332,-0.0.0703,0.0121	0.6461,0.0563,0.0105	0.6890,0.1006,0.0305
2	1	0.4987,-0.0600,0.0197	0.4741,-0.0335,0.0066	0.5219,0.0397,0.0066	0.5794,0.0712,0.0193
3	1	0.5036,-0.0517,0.0186	0.5031,-0.0317,0.0065	0.5840,0.0397,0.0067	0.6580,0.0765,0.0198
3	2	0.4889,-0.0767,0.0219	0.4160,-0.0372,0.0067	0.3977,0.0342,0.0063	0.4221,0.0607,0.0184
5	1	0.5935,-0.0546,0.0205	0.5971,-0.0342,0.0072	0.7047,0.0444,0.0074	0.8012,0.0871,0.0219
5	2	0.3838,-0.0469,0.0160	0.3841,-0.0287,0.0057	0.4319,0.0348,0.0059	0.4870,0.0655,0.0175
5	3	0.3231,-0.0486,0.0147	0.2956,-0.0261,0.0049	0.2949,0.0260,0.0049	0.3121,0.0462,0.0141
10	1	0.6930,-0.0556,0.0222	0.6966,-0.0349,0.0077	0.8284,0.0468,0.0078	0.9497,0.0934,0.0232
10	2	0.5829,-0.0578,0.0208	0.5870,-0.0361,0.0073	0.6905,0.0412,0.0076	0.7814,0.0903,0.0226
10	3	0.4453,-0.0536,0.0182	0.4493,-0.0333,0.0065	0.5212,0.0322,0.0068	0.5807,0.0782,0.0202
10	4	0.3080,-0.0437,0.0145	0.3107,-0.0269,0.0052	0.3538,0.0224,0.0055	0.3876,0.0596,0.0164
10	5	0.2080,-0.0325,0.0110	0.2072,-0.0197,0.0039	0.2265,0.0224,0.0042	0.2427,0.0404,0.0121
15	1	0.7234,-0.0542,0.0224	0.7268,-0.0342,0.0078	0.8667,0.0464,0.0078	0.9970,0.0934,0.0232
15	2	0.6636,-0.0581,0.0220	0.6675,-0.0364,0.0078	0.7908,0.0480,0.0079	0.9023,0.0950,0.0234
15	3	0.5812,-0.0595,0.0210	0.5855,-0.0371,0.0074	0.6876,0.0475,0.0077	0.7762,0.0922,0.0229
15	4	0.4832,-0.0574,0.0193	0.4876,-0.0356,0.0069	0.5666,0.0444,0.0072	0.6319,0.0844,0.0215
15	5	0.3785,-0.0516,0.0169	0.3824,-0.0319,0.0061	0.4391,0.0387,0.0065	0.4836,0.0722,0.0191
15	6	0.2775,-0.0428,0.0141	0.2805,-0.0264,0.0051	0.3177,0.0321,0.0054	0.3455,0.0572,0.0159
15	7	0.1930,-0.0327,0.0111	0.1947,-0.0200,0.0040	0.2162,0.0230,0.0043	0.2318,0.0416,0.0124
15	8	0.1411,-0.0239,0.0085	0.1402,-0.0144,0.0031	0.1490,0.0159,0.0032	0.1567,0.0282,0.0093

Table 3 $I\!I^{(2)}(X_{r:n}, Y_{[r:n]})$, at $p_2 = 1$

n	r	$\theta_2 = 0.25$	$\theta_2 = 0.5$	$\theta_2 = 0.75$	$\theta_2 = 0.99$
1	1	0.1137,-0.0476,0.1555	0.1226,-0.1047,0.6258	0.1434,-0.1918,1.4960	0.1945,-0.3751,3.0563
2	1	0.1137,-0.0354,0.0752	0.1226,-0.0823,0.3299	0.1434,-0.1560,0.8703	0.1945,-0.3287,2.0349
3	1	0.1367,-0.0415,0.0660	0.1482,-0.0986,0.3077	0.1755,-0.1967,0.8708	0.2438,-0.4206,2.2285
3	2	0.0678,-0.0232,0.0830	0.0714,-0.0497,0.3312	0.0793,-0.0853,0.7681	0.0960,-0.1448,1.4556
5	1	0.1793,-0.0528,0.0736	0.1959,-0.1279,0.3565	0.2354,-0.2623,1.0569	0.3366,-0.5847,2.8813
5	2	0.0808,-0.0277,0.0420	0.0856,-0.0624,0.1884	0.0960,-0.1133,0.4996	0.1186,-0.2047,1.1234
5	3	0.0482,-0.0152,0.0545	0.0502,-0.0321,0.2162	0.0543,-0.0531,0.4939	0.0620,-0.0836,0.9054
10	1	0.2393,-0.0661,0.0941	0.2641,-0.1645,0.4635	0.3248,-0.3530,1.4163	0.4856,-0.8413,4.0751
10	2	0.1550,-0.0491,0.0594	0.1668,-0.1152,0.2854	0.1936,-0.2225,0.8267	0.2554,-0.4433,2.0988
10	3	0.0925,-0.0336,0.0349	0.0976,-0.0753,0.1650	0.1085,-0.1348,0.4585	0.1309,-0.2347,1.0660
10	4	0.0511,-0.0205,0.0223	0.0532,-0.0444,0.1016	0.0572,-0.0752,0.2686	0.0648,-0.1193,0.5798
10	5	0.0306,-0.0110,0.0229	0.0314,-0.0231,0.0952	0.0331,-0.0375,0.2291	0.0358,-0.0558,0.4469
15	1	0.2681,-0.0715,0.1051	0.2978,-0.1810,0.5213	0.3715,-0.3993,1.6168	0.5718,-0.9920,4.7942
15	2	0.2012,-0.0594,0.0773	0.2187,-0.1430,0.3760	0.2594,-0.2883,1.1102	0.3577,-0.1650,2.9452
15	3	0.1451,-0.0477,0.05772	0.1550,-0.1102,0.2748	0.1767,-0.2067,0.7779	0.2237,-0.3888,1.8876
15	4	0.0996,-0.0367,0.0551	0.1049,-0.0819,0.2524	0.1159,-0.1452,0.6810	0.1376,-0.2475,1.5353
15	5	0.0645,-0.0265,0.0808	0.0672,-0.0577,0.3541	0.0724,-0.0980,0.9167	0.0820,-0.1553,1.9830
15	6	0.0396,-0.0176,0.1355	0.0408,-0.0376,0.5805	0.0432,-0.0618,1.4752	0.0472,-0.0930,3.1511
15	7	0.0247,-0.0105,0.1965	0.0252,-0.0219,0.8344	0.0263,-0.0351,2.1036	0.0280,-0.0509,4.4711
15	8	0.0197,-0.0055,0.2282	0.0201,-0.0113,0.9629	0.0208,-0.0176,2.4120	0.0218,-0.0248,5.0967

Table 4 $\mathbb{I}^{(2)}(X_{r:n}, Y_{[r:n]})$, at $p_2 = 2$

n	r	$\theta_2 = -0.25$	$\theta_2 = -0.15$	$\theta_2 = 0.15$	$\theta_2 = 0.25$
1	1	0.0188,0.0026,0.0042	0.0183,0.0015,0.0015	0.0174,-0.0015,0.0015	0.0171,-0.0025,0.0043
2	1	0.0288,0.0033,0.0042	0.0279,0.0020,0.0015	0.0257,-0.0019,0.0016	0.0253,-0.0031,0.0043
3	1	0.0375,0.0041,0.0041	0.0360,0.0024,0.0015	0.0329,-0.0023,0.0015	0.0322,-0.0038,0.0043
3	2	0.0116,0.0018,0.0036	0.0115,0.0011,0.0013	0.0139,-0.0011,0.0013	0.0114,-0.0018,0.0036
5	1	0.0525,0.0055,0.0043	0.0503,0.0032,0.0016	0.0455,-0.0031,0.0017	0.0444,-0.0051,0.0047
5	2	0.0168,0.0023,0.0024	0.0164,0.0013,0.0009	0.0155,-0.0013,0.0009	0.0153,-0.0021,0.0025
5	3	0.0095,0.0014,0.0032	0.0095,0.0008,0.0011	0.0096,-0.0008,0.0011	0.0097,-0.0014,0.0032
10	1	0.0786,0.0075,0.0056	0.0751,0.0044,0.0021	0.0675,-0.0042,0.0022	0.0657,-0.0069,0.0063
10	2	0.0369,0.0047,0.0023	0.0356,0.0028,0.0008	0.0327,-0.0027,0.0009	0.0319,-0.0044,0.0026
10	3	0.0161,0.0025,0.0012	0.0157,0.0015,0.0004	0.0148,-0.0014,0.0005	0.0145,-0.0023,0.0013
10	4	0.0083,0.0011,0.0019	0.0082,0.0007,0.0007	0.0080,-0.0007,0.0006	0.0080,-0.0011,0.0018
10	5	0.0074,0.0008,0.0033	0.0074,0.0005,0.0012	0.0075,-0.0005,0.0011	0.0076,-0.0008,0.0031
15	1	0.0944,0.0084,0.0067	0.0899,0.0049,0.0025	0.0805,-0.0047,0.0026	0.0784,-0.0077,0.0075
15	2	0.0553,0.0064,0.0034	0.0532,0.0038,0.0013	0.0485,-0.0036,0.0014	0.0473,-0.0059,0.0039
15	3	0.0303,0.0044,0.0026	0.0294,0.0026,0.0009	0.0272,-0.0025,0.0010	0.0266,-0.0041,0.0027
15	4	0.0157,0.0027,0.0065	0.0153,0.0016,0.0023	0.0143,-0.0015,0.0021	0.0141,-0.0025,0.0059
15	5	0.0083,0.0014,0.0181	0.0081,0.0008,0.0063	0.0078,-0.0008,0.0058	0.0077,-0.0013,0.0158
15	6	0.0057,0.0007,0.0370	0.0057,0.0004,0.0129	0.0057,-0.0004,0.0119	0.0057,-0.0006,0.0324
15	7	0.0061,0.0004,0.0563	0.0061,0.0003,0.0196	0.0063,-0.0003,0.0181	0.0063,-0.0004,0.0494
15	8	0.0078,0.0006,0.0653	0.0079,0.0004,0.0228	0.0082,-0.0004,0.0211	0.0083,-0.0007,0.0573

Table 5 Percentages of FI for $\mathbb{I}^{(1)}(X_{r:n}, Y_{[r:n]})$, at $p_1 = 1$

n	r	$\theta_1 = 0.25$	$\theta_1 = 0.5$	$\theta_1 = 0.75$	$\theta_1 = 0.99$
3	1	0.4,0.34,0.32	0.4,0.35,0.41	0.41,0.37,0.31	0.42,0.38,0.31
5	1	0.32,0.26,0.23	0.32,0.27,0.3	0.33,0.28,0.23	0.35,0.31,0.24
10	1	0.21,0.16,0.14	0.22,0.170.18	0.23,0.18,0.14	0.25,0.21,0.15
15	1	0.16,0.11,0.1	0.16,0.08,0.13	0.17,0.13,0.1	0.2,0.07,0.11

Table 6 Percentages of FI for $\mathbb{I}^{(1)}(X_{r:n}, Y_{[r:n]})$, at $p_1 = 2$

n	r	$\theta_1 = -0.25$	$\theta_1 = -0.15$	$\theta_1 = 0.15$	$\theta_1 = 0.25$
3	1	0.16,0.1,0.13	0.23,0.15,0.18	0.30,0.24,0.21	0.32,0.25,0.22
5	1	0.12,0.06,0.09	0.16,0.1,0.12	0.22,0.16,0.14	0.23,0.17,0.14
10	1	0.07,0.04,0.05	0.1,0.05,0.06	0.13,0.08,0.07	0.14,0.09,0.08
15	1	0.05,0.02,0.03	0.07,0.03,0.04	0.09,0.05,0.05	0.1,0.06,0.05

Table 7 Percentages of FI for $\mathbb{I}^{(2)}(X_{r:n}, Y_{[r:n]})$, at $p_2 = 1$

n	r	$\theta_2 = 0.25$	$\theta_2 = 0.5$	$\theta_2 = 0.75$	$\theta_2 = 0.99$
3	1	0.4,0.29,0.14	0.4,0.31,0.16	0.41,0.34,0.19	0.42,0.37,0.24
5	1	0.32,0.22,0.09	0.32,0.24,0.11	0.33,0.27,0.14	0.35,0.31,0.19
10	1	0.21,0.14,0.06	0.22,0.16,0.07	0.23,0.18,0.09	0.25,0.22,0.13
15	1	0.16,0.10,0.05	0.16,0.11,0.06	0.17,0.14,0.07	0.20,0.18,0.10

Table 8 Percentages of FI for $\mathbb{I}^{(2)}(X_{r:n}, Y_{[r:n]})$, at $p_2 = 2$

n	r	$\theta_2 = -0.25$	$\theta_2 = -0.15$	$\theta_2 = 0.15$	$\theta_2 = 0.25$
3	1	0.66,0.53,0.32	0.63,0.53,0.33	0.53,0.51,0.33	0.42,0.51,0.33
5	1	0.56,0.42,0.20	0.55,0.43,0.21	0.52,0.41,0.23	0.52,0.41,0.22
10	1	0.42,0.29,0.13	0.38,0.29,0.14	0.28,0.28,0.15	0.14,0.28,0.15
15	1	0.33,0.22,0.11	0.31,0.22,0.11	0.21,0.21,0.12	0.06,0.21,0.12

Acknowledgements

The authors would like to express their gratitude to anonymous referees for their constructive feedback on an earlier version of this paper, which had helped us to improve the manuscript significantly.

References

Abd Elgawad MA, Barakat HM, Alawady MA. Concomitants of generalized order statistics under the generalization of Farlie-Gumbel-Morgenstern type bivariate distributions. Bull Iran Math Soc. 2021; 47: 1045-1068.

Abd Elgawad MA, Barakat HM, Alawady MA. Concomitants of generalized order statistics from bivariate Cambanis family: Some information measures. Bull Iran Math Soc. 2022; 48: 563-585.

Abd Elgawad MA, Alawady MA, Barakat HM, Shengwu Xiong. Concomitants of generalized order statistics from Huang-Kotz Farlie-Gumbel-Morgenstern bivariate distribution: some information measures. Bull Malays Math Sci. Soc. 2020; 43: 2627-2645.

Abo-Eleneen ZA, Nagaraja HN. Fisher information in an order statistic and its concomitant. Ann Instit Stat Math. 2002; 54(3)2002; 667-680.

Alawady MA, Barakat HM, Abd Elgawad MA. Concomitants of generalized order statistics from bivariate Cambanis family of distributions under a general setting. Bull Malays Math Sci Soc. 2021a; 44: 3129-3159.

Alawady MA, Barakat HM, Xiong S, Abd Elgawad MA. On concomitants of dual generalized order statistics from Bairamov-Kotz-Becki-Farlie-Gumbel-Morgenstern bivariate distributions. Asian-European J Math. 2021b; 14 (10): 21501850.

Alawady MA, Abd Elgawad MA, Barakat HM, Shengwu Xiong. Concomitants of generalized order statistics from iterated Farlie–Gumbel–Morgenstern type bivariate distribution. Comm Stat-Theor M. 2022; 5(16): 5488-5504.

Bairamov I, Kotz S. Dependence structure and symmetry of Huang- Kotz FGM distributions and their extensions. Metrika. 2002; 56(1): 55-72.

Barakat HM, Husseiny IA. Some information measures in concomitants of generalized order statistics under iterated FGM bivariate type. Quaest Math. 2021; 44(5): 581-598.

Barakat HM, Nigm EM, Husseiny IA. Measures of information in order statistics and their concomitants for the single iterated Farlie-Gumbel-Morgenstern bivariate distribution. Math Popul Stud. 2020; 28(3): 154-175.

Barakat HM, Nigm EM, Syam AH. Concomitants of order statistics and record values from Bairamov-Kotz-Becki-FGM bivariate-generalized exponential distribution. Filomat. 2018; 32(9): 3313-3324.

Barakat HM, Nigm EM, Syam AH. Concomitants of ordered variables from Huang-Kotz-FGM type bivariate-generalized exponential distribution. Bull. Malays Math Sci Soc. 2019; 42: 337-353.

Barakat HM, Alawady MA, Husseiny IA, Mansour GM. Sarmanov family of bivariate distributions: statistical properties-concomitants of order statisticsinformation measures. Bull Malays Math Sci Soc. 2022; 45(Suppl 1): 49-83.

Barakat HM, Nigm EM, Alawady MA, Husseiny IA. Concomitants of order statistics and record values from iterated FGM type bivariate-generalized exponential distribution. REVSTAT. 2021; 19(2): 291-307.

David HA, Nagaraja HN. Concomitants of Order Statistics, in Order Statistics: Theory & Methods, eds. Balakrishnan, N, Rao, C R, Elsevier, Amsterdam 1998; pp. 487-513.

David HA, Nagaraja HN. Order Statistics. 3rd Ed., John Wiley & Sons, Inc. ISBN: 978-0-471-38926-2; 2003.

Eryilmaz S. On an application of concomitants of order statistics. Comm Stat-Theor M. 2016; 45(19): 5628-5636 .

Farlie DJG. The performance of some correlation coefficients for a general bivariate distribution. Biometrica. 1960; 47: 307-323.

Fischer M, Klein I. Constructing generalized FGM copulas by means of certain univariate distributions. Metrika. 2007; 65: 243-260.

Gradshteyn IS, Ryzhik IM. Table of Integrals, Series, and Products (7th). Amsterdam, New York: Elsevier; 2007.

Gumbel EJ. Bivariate exponential distributions. J Am Stat Assoc. 1960; 55: 698-707.

Huang JS, Kotz S. Correlation structure in iterated Farlie-Gumbel-Morgenstern distributions. Biometrika. 1984; 71: 633-636 .

Huang JS, Kotz S. Modifications of the Farlie-Gumbel-Morgenstern distributions. A tough hill to climb. Metrika. 1999; 49: 135-145.

Husseiny IA, Barakat HM, Mansour GM, Alawady MA. Information measures in records and their concomitants arising from Sarmanov family of bivariate distributions. J Comput Appl Math. 2022; 408, 114-120.

Johnson NL, Kotz S. On some generalized Farlie-Gumbel-Morgenstern distributions. Comm Stat-Theor M. 1975; 4: 415-427.

Johnson NL, Kotz S. On some generalized Farlie-Gumbel-Morgenstern distributions II: Regression, correlation an further generalizations. Comm Stat-Theor Meth. 1977; 6: 485-496.

Kharazmi O, Asadi M. On the time-dependent Fisher information of a density function. Braz J Probab Stat. 2018; 32(4): 795–814.

Morgenstern D. Einfache beispiele zweidimensionaler verteilungen. Mitt Math Stat. 1956; 8: 234-235.

Philip A, Thomas PY. On concomitants of order statistics and its application in defining ranked set sampling from Farlie-Gumbel-Morgenstern bivariate Lomax distribution. J Iran Stat Soc. 2017; 16(02): 67-95.