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Comparing Some Iterative Methods of Parameter Estimation for Progressively Censored Lomax Data

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Abstract

Based on Progressively Type-II censored samples, the maximum likelihood estimator, the uniformly minimum variance unbiased estimator (*UMVU*), and the Bayes estimators for the shape parameter and the hazard function of the Lomax model are derived. The Bayesian estimators are obtained based on symmetric (squared error, absolute difference, and logarithmic loss functions) and asymmetric (*LINEX*, *General Entropy*, and *Logarithmic*) loss functions. A real data example consists of data from Iowa 65+ Rural Health Study (RHS) is used to illustrate the proposed methods

Keywords: Lomax distribution, progressive censoring, maximum likelihood, symmetric and asymmetric loss functions.

1. Introduction

In life testing experiments, it is common to cease testing before all items have failed for a variety of reasons, such as insufficient funds and/or time constraints. Samples that result from such situations are called censored samples. There are different types of censoring schemes, for example, type-I censoring, wherein the test ends at a predetermined time, or type-II censoring, which allows experiments to be terminated based on a predetermined number of failures. Unfortunately, none of these methods allow the removal of active units during the experiment. Thus, in the past few years, progressive censoring has become the norm due to its flexibility, which allows the experimenter to remove active units during the experiment.

In this article, the progressively type-II censoring when n independent items are placed on a life-testing experiment simultaneously and only m ($< n$) failures are completely observed, is considered. The censoring occurs progressively in m stages as follows: When the first failure is observed, a random sample of size R_1 is immediately drawn and removed from the $(n - 1)$ survivals, hence leaving $n - 1 - R_1$ survival items. Then after the failure of the second item, the sample becomes $n - 2 - R_1$ of which R_2 units are randomly selected and removed from the remaining survival units. Continue with this process until m failures are observed and all the remaining $n - m - R_1 - \dots - R_{m-1}$ ($= R_m$) surviving units are removed from the experiment.

Many authors have discussed inference under progressive censoring using different lifetime distributions. Among others, Wingo (1993), Balakrishnan and Sandhu (1995). For recent comprehensive review of progressive censoring, readers may refer to Aggarwala and Balakrishnan (2019).

In this study, we assume that the lifetimes have Lomax distributions. This distribution was introduced by Lomax (1954) as a model for business failure data. It belongs to the class of decreasing failure rate distributions (Chahkandi and Ganjali, 2009). Moreover, it provides a very good alternative to common lifetimes distribution such as exponential, Weibull, or gamma distributions when the experimenter assumes that the population may be a heavy-tailed distribution (see Bryson, 1974). We can find the origin and other aspects of this distribution and application areas in Arnold and Press (1983).

Inferential issues for the Lomax distribution based on censored data have been addressed by Childs et al. (2001). They consider the order statistics from nonidentical right truncated Lomax distribution. Howlader and Hossain (2002) consider Bayesian estimation of the survival function of the Lomax distribution. Abd-Elfattah et al. (2007) derive the Bayesian and the non-Bayesian estimators for the same sample size from the Lomax distribution based on the progressively type-I censoring. Asgharzadeh and Valiollahi (2010) derive the Bayesian estimator of the scale parameter of the Lomax distribution based on progressive type-II censoring. Cramer and Schmiedt (2011) discuss the optimal censoring scheme for estimating the parameters of the Lomax distribution using progressive type-II censoring. As far as we know, no one described the *EM* algorithm for determining the *MLEs* of the parameters of the Lomax distribution based on progressive type-II censoring.

The purpose of this study is to compare different estimators of the shape parameter θ and the corresponding estimators of the failure rate function when Progressively Type-II censoring is considered. We will discuss the maximum likelihood (*MLE*) and the uniformly minimum variance unbiased (*UMVU*) estimators of the shape parameter and their corresponding failure rate functions in Section 2. The Bayes estimators are developed using the gamma conjugate prior, and based on two different types of loss functions, namely symmetric and asymmetric loss functions along with their corresponding failure rate functions, all of which will be discussed in section 3. In Section 4, a simulation study is conducted to compare the suggested estimators, a numerical example to illustrate the estimation results, and some final remarks are also provided. Illustration using real data from RHS is provided in Section 5.

2. Maximum Likelihood Estimators

2.1. Model description

Assume that the lifetime of a particular component has an exponential distribution with a failure rate λ . Let λ follows a gamma distribution with shape parameter θ and scale parameter β . The failure time X of a component selected at random from such a mixed population has a Lomax distribution with probability density function (pdf) given by:

$$f(x; \theta, \beta) = \int_0^{\infty} \lambda e^{-\lambda x} \frac{\lambda^{\theta-1} e^{-\lambda/\beta}}{\beta^{\theta} \Gamma(\theta)} d\lambda = \frac{\beta \theta}{(1 + \beta x)^{\theta+1}}, \quad x > 0, \beta > 0, \theta > 0, \quad (1)$$

[see Johnson et al. (1994)]. The cumulative distribution function (cdf) of the Lomax (θ, β) is given by:

$$F(x; \theta, \beta) = 1 - \frac{1}{(1 + \beta x)^{\theta}}, \quad x > 0, \beta > 0, \theta > 0. \quad (2)$$

Consider the random variable $Y = \beta X$. Then Y has the standard Lomax distribution with *pdf* and *cdf* as follows:

$$f(y) = \frac{1}{\theta(1+y)^{\theta+1}}, \quad y > 0, \theta > 0, \quad (3)$$

$$\text{and} \quad F(y) = 1 - \frac{1}{(1+y)^{\theta}}, \quad y > 0, \theta > 0, \quad (4)$$

respectively. Note that $g(y; c; \theta) = cf(cy)$ and $G(y; c; \theta) = F(cy)$.

Suppose that n independent units are placed on a life-testing experiment whose lifetimes have standard Lomax distribution with pdf as shown in (3), and $(R_1, \dots, R_m)$ denote the corresponding number of units removed from the test. Let $Y_{1:m:n} \leq Y_{2:m:n} \leq \dots \leq Y_{m:m:n}$ denote the above mentioned m progressively type-II censoring. For notation simplicity, we will write Y_i for $Y_{i:m:n}$. The likelihood function based on progressively type-II censored sample (see Balakrishnan and Cramer 2014) is given by:

$$\ell_p = k \prod_{i=1}^m f(y_i; \theta) [1 - F(y_i; \theta)]^{R_i} \quad (5)$$

where

$$k = n(n-1-R_1)(n-2-R_1-R_2)\dots(n-\sum_{i=1}^{m-1}(R_i+1)),$$

substituting (3) and (4), into (5), and take the natural logarithm we get

$$\ell_p = \ln K + m \ln(\theta) - \sum_{i=1}^m [\theta(R_i+1) + 1] \ln(1+y_i). \quad (6)$$

The *MLE* of the parameter θ can be obtained by deriving the likelihood function with respect to θ and equating the normal equations to 0 as follows:

$$\frac{\partial \ell_p}{\partial \theta} = \frac{m}{\theta} - \sum_{i=1}^m (R_i+1) \ln(1+y_i) = 0, \quad (7)$$

Eqn. (7) yields the *MLE* of θ

$$\hat{\theta}_{ML} = \frac{m}{S} \quad (8)$$

where $S = \sum_{i=1}^m (R_i+1) \ln(1+x_i)$. $\hat{\theta}_{MLE}$ exists and unique, (see Wingo (1993)). Since $h(t)$ is strictly monotone, then the corresponding *MLE* of the failure function $h(t)$ is given by

$$\hat{h}_{ML}(t) = \hat{\theta}_{ML} (1+t)^{-1}.$$

2.2. Uniformly minimum unbiased estimator

Let $X_1, X_2, \dots, X_m$ be a Progressive random data from (1). If $Y = 1 + cX$ then it is easy to show that Y is distributed as *Pareto*(θ) with pdf

$$g(y; \theta) = \theta y^{-\theta-1} \quad y > 1, \theta > 0.$$

Let $Y_1, Y_2, \dots, Y_m$ be a Progressively Type-II censored data from *Pareto*(θ) obtained from a sample of size n with censoring scheme $(R_1, R_2, \dots, R_m)$ then

$$\begin{aligned} Z_1 &= Y_1 \sim \text{Pareto}(n\theta) \\ \text{and } Z_i &= \frac{Y_i}{Y_{i-1}} \sim \text{Pareto}(n - \theta \sum_{j=1}^{i-1} (R_j + 1)), \quad i = 2, \dots, m. \end{aligned}$$

Balakrishnan and Aggarwala (2000) proved that $Z_1, Z_2, \dots, Z_m$ are all statistically independent. Since

$$U = 2n\theta \ln(Z_1),$$

and

$$V = 2\theta \sum_{i=2}^m (n - \sum_{j=1}^{i-1} (R_j + 1)) \ln(Z_i),$$

are independent random variables from chi-square distribution with 2 and $2m - 2$ degrees of freedom respectively, then, $U + V = 2S\theta$ is distributed as chi-square distributed with $2m$ degrees of freedom, (see Wu; 2008).

It can easily be seen that $S = \sum_{i=1}^m (R_i + 1) \ln(1 + y_i)$ is a complete sufficient statistic of θ . Hence, using, the above results, $\frac{m-1}{S}$ is an unbiased estimator of θ . It followed by Lehmann-Scheffe's theorem that

$$\hat{\theta}_{UM} = \frac{m-1}{S}$$

is the *UMVU* of θ . In the same manner $g(y) = \frac{m-1}{S-1}(1+y)^{-1}$, is a function of a complete sufficient statistic that is an unbiased estimator of $h(x)$. Hence the *UMVUE* estimator of $h(t)$ is

$$\hat{h}_{UM}(t) = \frac{m-1}{S}(1+t)^{-1}.$$

3. Bayesian Estimation

The performance of the Bayesian estimates depends on the loss functions assumed and on the form of the prior distribution. The prior information can be expressed by the experimenter, who has some beliefs about the parameters of the parametric model.

Several authors consider two types of symmetric loss functions, namely squared and absolute difference error loss functions, where the first loss is used to determine the posterior mean, the second is used to determine the posterior median, and both values are regarded as Bayesian estimates.

However, in practice, the real loss function is often not symmetric. For example, Feynman (1987) claims that in the space shuttle disaster, the management may have overestimated the average life or reliability of a solid-fuel rocket booster. In estimating reliability and failure rate functions, overestimating causes more damage than underestimating. To resolve such a situation asymmetrical loss function could be more appropriate to be utilized. Many authors consider asymmetric loss functions in reliability and failure rate functions such as Schäb (1991), Norstrom (1996), Spring and Yeung (1998), Ren et al. (2006) and Li et al. (2007), Helu and Samawi (2015), Musleh et al. (2020).

This section uses the Bayes procedure to estimate the shape parameter θ based on Progressively Type-II censored data using two types of loss functions. The symmetric loss function includes squared (*SE*) and absolute difference (*AD*) error loss and asymmetric loss functions, including *LINEX* (*LN*), General Entropy (*GE*), and Logarithmic (*LG*) error loss functions.

Let $\underline{x} = x_1, x_2, \dots, x_m$ be a Progressively censored sample from (1). When the parameter c is assumed to be known, and θ has a gamma conjugate prior density of the form

$$\pi(\theta|\beta) = \beta e^{-\beta\theta} \quad \theta > 0, \beta > 0, \quad (9)$$

then, the posterior density of θ is given by

$$\pi(\theta|\underline{x}, c) \propto \theta^m e^{-\theta(\beta + \sum_{i=1}^m T_i)},$$

where $T_i = \ln(1 + x_i)^{(R_i+1)}$, hence the posterior distribution is *gamma* ($m + 1, \beta + S$) and $S = \sum_{i=1}^m T_i$.

3.1. Bayes estimates using symmetric loss function

Let $u(\theta)$ denotes a function of θ , then the squared error loss function is defined as

$$L_{SE}(\tilde{u}(\theta), u(\theta)) = (\tilde{u}(\theta) - u(\theta))^2. \quad (10)$$

Using the fact that, under the squared error loss function the Bayes estimate of $u(\theta)$ is the posterior mean, thus the Bayes estimate of the shape parameter θ is given by

$$\tilde{\theta}_{SE} = \frac{m+1}{\beta+S} \quad (11)$$

with failure rate function:

$$\tilde{h}_{SE}(t) = \tilde{\theta}_{SE}(1+t)^{-1}.$$

The absolute difference loss function is given by

$$L_{AD}(\tilde{u}(\theta), u(\theta)) = |\tilde{u}(\theta) - u(\theta)|. \quad (12)$$

Box and Tia (2011) show that based on the absolute difference loss functions, the Bayes estimate of $u(\theta)$ is the posterior median, and since the posterior distribution is gamma, then

$$\tilde{\theta}_{AD} = \frac{\gamma_{2(m+1)}}{2(\beta+S)} \quad (13)$$

where $\gamma_{2(m+1)}$ is the median of a chi-square distribution with $2(m+1)$ degrees of freedom, [see Asgarzadeh and Valiollahi (2011)].

Furthermore, the Bayes estimate of the failure rate function, under the above error loss is given as

$$\tilde{h}_{AD}(t) = \tilde{\theta}_{AD}(1+t)^{-1}.$$

3.2. Bayes estimates using asymmetric loss function

One of the useful asymmetric loss functions is the *LINEX* loss function (*LN*) which was introduced by Varian (1975) and adopted by Zellner (1986), Pendy et al. (1996) and Pendy (1997). A review of *LINEX* loss function and its properties is given by Parsian and Kirmani (2002). Under the assumption that the minimal loss occurs at $\tilde{u}(\theta) - u(\theta)$, the *LINEX* loss function $u(\theta)$ can be expressed as

$$L(\Delta) \propto e^{a\Delta} - a\Delta - 1; \quad a \neq 0 \quad (14)$$

where $\Delta = \hat{u}(\theta) - u(\theta)$. The sign and magnitude of (a) represent the direction and the degree of symmetry, respectively. Therefore, when $a < 0$ means that underestimation is more serious than overestimation and conversely when $a > 0$. However, when (a) is equal to zero, then the *LINEX* loss is simply the squared error loss (i.e. almost symmetric).

The Bayes estimate under *LINEX* loss function is defined as

$$\begin{aligned} \tilde{\theta}_{LN} &= \frac{-1}{a} \log Ee^{-a\theta} \\ &= \frac{m+1}{a} \log\left(1 + \frac{a}{\beta+S}\right), \end{aligned} \quad (15)$$

with failure rate function

$$\tilde{h}_{LN}(t) = \tilde{\theta}_{LN}(1+t)^{-1}.$$

Another example of asymmetric loss function is General entropy loss (L_{GE})

$$L_{GE}(\tilde{u}(\theta), u(\theta)) \propto \left(\frac{\tilde{u}(\theta)}{u(\theta)}\right)^q - q \log\left(\frac{\tilde{u}(\theta)}{u(\theta)}\right) - 1. \quad (16)$$

When $q = 1$, Eqn. (16) is reduced to basically the entropy loss, which was adopted by many researchers such as James and Stein (1961), who used the entropy loss to estimate the multinormal variance-covariance matrix. Moreover, Brown (1968), Haff (1977, 1979, 1980), Ighodaro and Santner (1982) considered the entropy loss for simultaneous estimation of independent binomial and

multinomial proportions. Dey et al. (1986) consider the entropy loss for simultaneous estimation of the independent gamma scale parameters and their reciprocals, whose minimum occurs at $\tilde{u} - u$, and $q > 0$. Note that, a positive error causes more severe consequences than a negative error.

Thus, the Bayes estimate under General entropy loss is defined as

$$\tilde{u}_{GE} = (E_u [u(\theta)^{-q}])^{-\frac{1}{q}},$$

provided that $E_u [u^{-q}]$ exists. Hence

$$\tilde{\theta}_{GE} = \left[\frac{\Gamma(m - q + 1)}{\Gamma(m + 1)} \right]^{-\frac{1}{q}} (\beta + S)^{-1}, \quad (17)$$

with failure rate function

$$\tilde{h}_{GE}(t) = \tilde{\theta}_{GE} (1 + t)^{-1}.$$

In addition to the above mentioned asymmetric loss function, we will consider the Logarithmic loss function which is defined as

$$L_{LG}(\tilde{u}(\theta), u(\theta)) = \left[\ln \left(\frac{\tilde{u}(\theta)}{u(\theta)} \right) \right]^2. \quad (18)$$

This loss function, introduced by Brown (1968), places a small weight on estimates whose ratios to the true value are close to one and proportionally moves weight on estimates whose ratio to the true value are significantly different from one. The Bayes estimate of the function $u(\theta)$ is

$$\tilde{u}_{LG}(\theta) = \text{Exp}(E \ln(u(\theta)|x)),$$

provided that $E \ln(u(\theta)|x)$ exists. Hence the Bayes estimate of θ is

$$\tilde{\theta}_{LG} = \text{Exp} \left(\int_0^\infty \theta^m \ln(\theta) e^{-\theta(\beta+S)} \frac{(\beta+S)^{(m+1)}}{\Gamma(m+1)} d\theta \right)$$

and the Bayes estimate of the failure rate is

$$\tilde{h}_{LG}(t) = \tilde{\theta}_{LG} (1 + t)^{-1}.$$

Asgharzadeh and Valiollahi (2011) proved that $E \ln(\theta|x) = \Psi(m+1) - \ln(\beta+S)$, where $\Psi(x)$ is the digamma function defined by $\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)}$. Hence we have

$$\begin{aligned} \tilde{\theta}_{LG} &= e^{\Psi(m+1) - \ln(\beta+S)} \\ &= \frac{e^{\Psi(m+1)}}{\beta + S}. \end{aligned} \quad (19)$$

4. Simulation Study

Using the algorithm given in Balakrishnan and Sandhu (1995), and Aggarwala and Balakrishnan (2019) we generate Progressively Type-II censored samples of different sizes from Lomax distribution as follows

1. Generate m independent $U(0, 1)$ random variables $W_1, W_2, \dots, W_m$.
2. For given values of Progressive censoring scheme $R_1, R_2, \dots, R_m$, set $E_i = (i + \sum_{j=m-i+2}^m R_j)$, $i = 1, \dots, m$.

3. $V_i = W_i^{1/E_i}$, for $i = 1, \dots, m$.
4. Let $U_i = 1 - V_m \times V_{m-1} \times \dots \times V_{m-i+1}$, $i = 1, \dots, m$, then $u_1, \dots, u_m$ is a Progressive Type-II censored sample of size m from $U(0, 1)$.
5. A value of θ is generated from the conjugate prior given in (9) with specified parameter β ($\beta = 2$ and $\beta = 4$). This value is the value of the exact θ to which each estimate is compared. The value of θ is also used to calculate the exact value of the failure rate $h(t)$, $[h(t = 1, c = 1; \theta = 0.4986) = 0.374$ and $h(t = 1, c = 1; \theta = 0.2513) = 0.1885]$ to which each estimate of $h(t)$ is compared. The same θ will also be used to generate a Progressive Type-II censored sample of size m .
6. For given values of c, θ, m, n , and $(r_1, \dots, r_m)$ we set $X_{i:m:n} = X_i = F^{-1}(U_i) = \left[(1 - U_i)^{-1/\theta} - 1 \right]^{1/c}$, $i = 1, \dots, m$. Finally, $x_1, x_2, \dots, x_m$ is the required Progressive Type-II censored sample of size m from Lomax(c, θ) distribution.
7. The above steps are repeated 10000 times.
8. A comparison between the estimators of θ is made by comparing their

(a) **Percentage Error %:** which is computed as $PE = \frac{|\bar{\hat{\theta}} - \theta|}{\theta} \times 100$, where $\bar{\hat{\theta}}$ is the average of the 10000 estimates of θ , and θ is the value that is used in the simulation.

(b) **Estimated Risk:** $ER(\hat{\theta}) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{\theta}_i - \theta)^2$, $\hat{\theta}_i$ being the estimate of θ for the i th simulated data set.

9. A comparison between the failure rate functions is done through computing their estimated risks (ER) where $ER(\hat{h}) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{h}_i - h)^2$, where $\hat{h}_i$ being the estimate of h for the i th simulated data set.

4.1. Data analysis and comparison study

This paper describes the mathematical methodology for estimating the shape parameter of the Lomax distribution from Progressively censored sample data based on Bayesian and non-Bayesian methods. Six different cases of the sample size and censoring scheme are shown in Table (3).

Tables (4) - (12) display the estimated values of the shape parameter θ and the corresponding failure rate functions as well as their estimated risks using different censoring schemes and different initials. From those tables, we observe the following:

1. From Tables 4-7, it can be seen that as the sample proportion m/n increases, the PE values and estimated risks of the estimators decrease drastically. Furthermore, for fixed n and m , different schemes do not change our main conclusions, as shown in Tables 12-15.
2. The uniformly minimum variance unbiased estimator ($\hat{\theta}_{UM}$) has the smallest PE value and this is expected. For large n & m , the estimators are quit identical to the exact value.
3. Compared to symmetric loss functions, Bayesian estimators have smaller values of PE and ER than those for asymmetric loss functions.
4. Moreover, the General entropy ($\tilde{\theta}_{GE}$) has PE values that are quit similar to that for the $UMVU$ especially for $n/m \geq 0.75$. Moreover, $\tilde{\theta}_{GE}$ has the smallest estimated risk, which means that $\tilde{\theta}_{GE}$ is not only outperforming $\hat{\theta}_{UM}$, but it is also out performed $\hat{\theta}_{ML}$, symmetric Bayes estimators as well as the remaining two kinds of asymmetric loss functions, namely $LINEX$ and $Logarithmic$ based estimates.

5. Tables (8)-(11) indicate that changing the initial values of θ and β has no effect on the performance of the estimators.

4.2. Illustrative example

To illustrate the use of the estimation method proposed in this article, a Progressively Type-II censored sample of size $m = 8$ are generated from Lomax distribution with parameters $\theta = 0.499$, $c = 1$, $\beta = 2$; and $t = 1$ and 2 for different values of a and q using the algorithm in Section 4. The data are presented in the Tables below, with exact value of $h(t = 1) = 0.374$ and $h(t = 2) = 0.214$.

Table 1 Illustrative example

	i	1	2	2	3	4	5	6	7	8				
	R_i	0	0	0	3	0	3	0	0	5				
	x_i	0.0129	0.0151	0.0151	0.0646	0.1522	0.1812	0.2115	0.4076	1.476				

	Exact	ML	UM	SE	AD	LG	LN				GE			
							a				q			
							0.5	5	-0.5	-2	1	3	-2	-5
$\hat{\theta}$	0.499	0.558	0.448	0.551	0.530	0.520	0.542	0.481	0.559	0.587	0.490	0.425	0.581	0.668
$\hat{h}_{t=1}$	0.374	0.418	0.366	0.413	0.398	0.390	0.407	0.360	0.420	0.441	0.367	0.319	0.435	0.501
$\hat{h}_{t=2}$	0.214	0.239	0.209	0.236	0.227	0.223	0.233	0.206	0.240	0.252	0.210	0.182	0.249	0.286

The illustrative example shows that the Bayes estimates based on asymmetric loss functions, namely $LINEX$ and GE , are sensitive to changes in a and q .

5. Example using Real Life Data

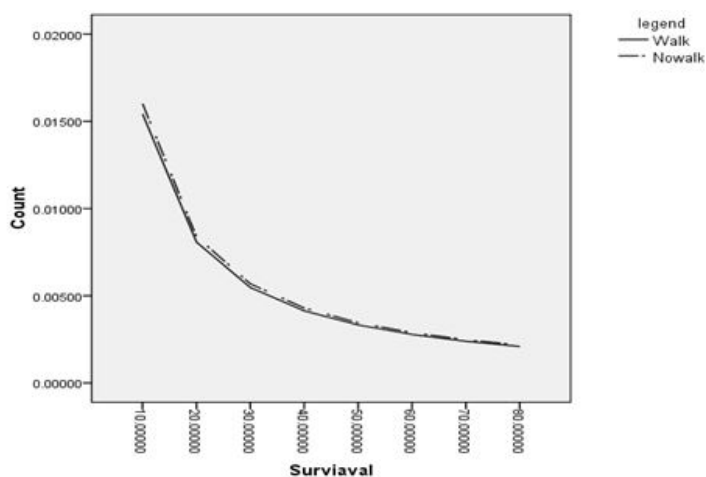
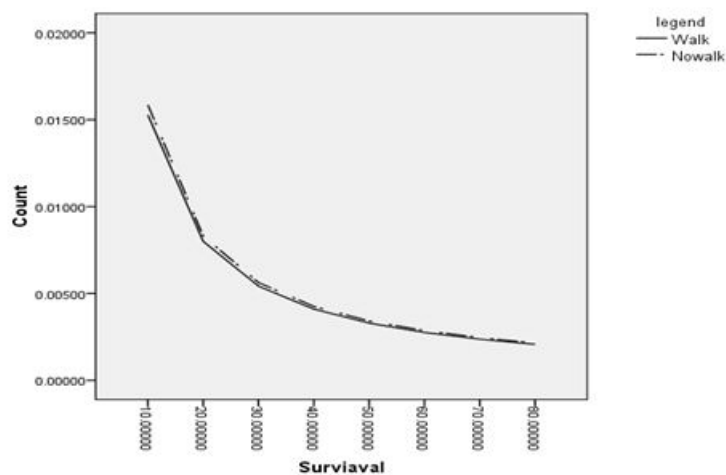
The RHS is a prospective longitudinal cohort study, of 8 years follow-up from 1981 to 1989, of 3,062 individuals (males and female) aged 65 or older living in Washington and Iowa counties of the state of Iowa. This study is one of four supported by the National Institute on Aging and collectively referred to as EPESE, Established Populations for Epidemiological Studies of the Elderly; see Cornoni-Huntley(1986).

The life histories of 3062 non-institutional individuals who could walk across a small room without any help were obtained from RHS and divided into two cohorts; one containing 1243 who exercised daily by walking and the other containing 1819 who did not exercise daily by walking.

The purpose of this illustration is to compare the failure rate curves of the daily walk and non-daily walk elderly. From the given data we created artificial Type II progressively censored data of size $m = 100$ from each group (daily walking and non-daily walking), where $R_1 = 20$, $R_2 = R_3 = \dots = R_{99} = 0$, $R_{100} = 1123$ for the walking group and $R_1 = 20$, $R_2 = R_3 = \dots = R_{99} = 0$, $R_{100} = 1699$ for the non-walking group. Table 14 has all proposed estimators of the shape parameter θ of the Lomax distribution with scale parameter $c = 1$. Figures 1-3 are the failure rate curves of daily walking and non-daily walking groups for MLE , $UMVUE$ and the *Bayes* using squared error loss function estimates of θ . The other loss functions curves used in Bayes estimators are ignored due to the fact that we found all curves are similar. Note that the $Q - Q$ plots showed that the two selected samples are approximately followed Lomax distribution with scale parameter $c = 1$. From Figures 1, 2, and 3, there is no clear significant difference between the two curves.

Table 2 Estimates of the shape parameter when $a = 1$, $c = 1$, $q = 1$, and $\beta = 2$

<i>Estimates</i>	<i>Daily – walking</i>	<i>Non – daily – walking</i>
$\hat{\theta}_{ML}$	0.1697	0.1762
$\hat{\theta}_{UM}$	0.1680	0.1744
$\hat{\theta}_{SE}$	0.1709	0.1773
$\hat{\theta}_{AD}$	0.1703	0.1767
$\hat{\theta}_{LN}$	0.1707	0.1772
$\hat{\theta}_{GE}$	0.1692	0.1756
$\hat{\theta}_{LG}$	0.1700	0.1764

**Figure 1** Failure rate function of daily and non-daily walking groups using MLE.**Figure 2** Failure rate function of daily and non-daily walking groups using MLE.

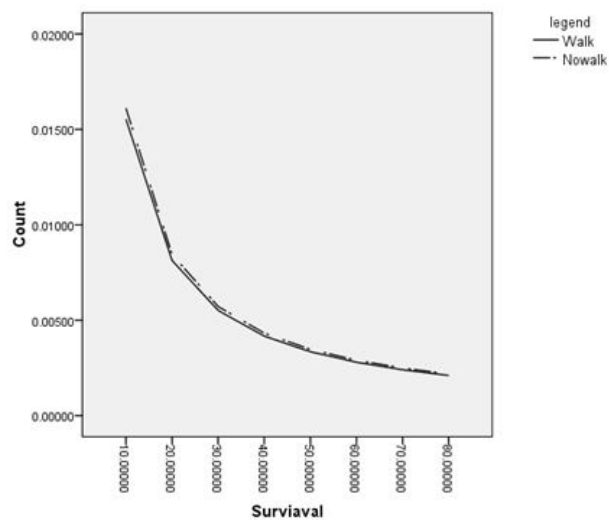


Figure 3 Failure rate function of daily and non-daily walking groups using MLE.

6. Final Remarks

In this study, we examined the estimation of the shape parameter of the *Lomax* distribution using progressively Type-II censored samples based on both Bayesian and non-Bayesian methods. Our analysis revealed that the uniformly minimum variance unbiased estimator ($\hat{\theta}_{UM}$) consistently demonstrated the smallest *PE* values and performed best, especially for larger sample sizes. Bayesian estimators generally showed smaller *PE* and smaller risks than non-Bayesian methods, with the General entropy ($\tilde{\theta}_{GE}$) performing the best and demonstrating the smallest estimated risk.

Based on these findings, we recommend using ($\hat{\theta}_{UM}$) for precise parameter estimates, particularly in large samples. Bayesian estimators are preferred for their robustness, and the (θ_{GE}) is particularly recommended for its superior performance in minimizing estimated risks.

Table 3 Censoring scheme $R = (R_1, \dots, R_m)$

Case	n	m	Scheme
1	20	5	(3,3,3,3,3)
2	20	10	(0,2,1,0,1,1,2,0,0,3)
3	30	20	(0,1,0* ⁴ ,2,0* ³ ,2,0* ² ,3,0* ² ,1,0* ² ,1)
4	40	30	(1,0,2,0* ² ,1,0,2,0* ³ ,1,0* ³ ,1,0* ² ,1,0* ¹⁰ ,1)
5	50	35	(0,1,0* ² ,2,0* ⁴ ,2,0* ² ,1,0* ² ,2,0* ⁴ ,1,0* ² ,1,0* ⁴ ,1,0* ² ,2,0,1,0)
6	100	80	((20,0* ⁸⁰))

Table 4 $PE(\%)$ values of the estimates of θ when $\beta = 2, \theta = 0.4986$

n	m	$PE(\hat{\theta}_{ML})$	$PE(\hat{\theta}_{UM})$	$PE(\tilde{\theta}_{SE})$	$PE(\tilde{\theta}_{AD})$	$PE(\tilde{\theta}_{LN})$	$PE(\tilde{\theta}_{GE})$	$PE(\tilde{\theta}_{LG})$
						a=0.5	q=1	
20	5	25.90	0.72	15.41	9.06	12.32	3.83	8.93
20	10	11.86	0.67	9.49	6.19	8.03	0.47	4.55
30	20	5.11	0.15	4.59	2.94	3.92	0.39	2.11
40	30	3.51	0.06	3.28	2.17	2.84	0.05	1.62
50	35	2.97	0.03	2.81	1.86	2.43	0.05	1.38
100	80	1.27	0.00	1.24	0.82	1.08	0.01	0.61

Table 5 Estimated risk (ER) of estimates of θ when $c = 3, \beta = 2, \theta = 0.4986$

n	m	$ER(\hat{\theta}_{ML})$	$ER(\hat{\theta}_{UM})$	$ER(\tilde{\theta}_{SE})$	$ER(\tilde{\theta}_{AD})$	$ER(\tilde{\theta}_{LN})$	$ER(\tilde{\theta}_{GE})$	$ER(\tilde{\theta}_{LG})$
						a = 0.5	q = 1	
20	5	0.157	0.090	0.059	0.050	0.051	0.037	0.046
20	10	0.043	0.032	0.031	0.028	0.029	0.024	0.027
30	20	0.016	0.014	0.014	0.013	0.014	0.012	0.013
40	30	0.010	0.009	0.009	0.009	0.009	0.008	0.009
50	35	0.008	0.008	0.008	0.007	0.007	0.007	0.007
100	80	0.003	0.003	0.003	0.003	0.003	0.002	0.003

Table 6 Estimates of the failure rate $h(t)$ ($\beta = 2; \theta = 0.4986; t = 1$)

n	m	$\hat{h}_{ML}$	$\hat{h}_{UM}$	$\tilde{h}_{SE}$	$\tilde{h}_{AD}$	$\tilde{h}_{LN}$	$\tilde{h}_{GE}$	$\tilde{h}_{LG}$
						a=0.5	q=1	
20	5	0.471	0.377	0.432	0.408	0.420	0.359	0.396
20	10	0.418	0.376	0.409	0.397	0.404	0.372	0.391
30	20	0.393	0.373	0.391	0.385	0.389	0.372	0.382
40	30	0.387	0.374	0.386	0.382	0.385	0.374	0.380
50	35	0.385	0.374	0.384	0.381	0.383	0.374	0.379
100	80	0.379	0.374	0.379	0.377	0.378	0.374	0.376

Table 7 Estimated risks (ER) of the estimates of $h(t)$; $\beta = 2; \theta = 0.4986; t = 1$

n	m	$ER(\hat{h}_{ML})$	$ER(\hat{h}_{UM})$	$ER(\tilde{h}_{SE})$	$ER(\tilde{h}_{AD})$	$ER(\tilde{h}_{LN})$	$ER(\tilde{h}_{GE})$	$ER(\tilde{h}_{LG})$
						a=0.5	q=1	
20	5	0.0882	0.0504	0.0333	0.0279	0.0288	0.0210	0.0258
20	10	0.0243	0.0181	0.0173	0.0157	0.0161	0.0133	0.0149
30	20	0.0090	0.0078	0.0079	0.0075	0.0076	0.0069	0.0073
40	30	0.0056	0.0051	0.0052	0.0050	0.0050	0.0047	0.0049
50	35	0.0046	0.0042	0.0043	0.0041	0.0042	0.0039	0.0041
100	80	0.0019	0.0018	0.0018	0.0018	0.0018	0.0017	0.0018

Table 8 $PE(\%)$ values of the estimates of θ ($\beta = 2; \theta = 0.2513$)

n	m	$PE(\hat{\theta}_{ML})$	$PE(\hat{\theta}_{UM})$	$PE(\tilde{\theta}_{SE})$	$PE(\tilde{\theta}_{AD})$	$PE(\tilde{\theta}_{LN})$	$PE(\tilde{\theta}_{GE})$	$PE(\tilde{\theta}_{LG})$
						a=0.5	q=1	
20	5	26.25	1.01	15.03	8.71	13.43	4.14	5.59
20	10	10.47	0.58	8.86	5.58	8.17	1.03	3.95
30	20	4.91	0.34	4.37	2.72	4.03	0.60	1.89
40	30	3.68	0.22	3.26	2.15	3.03	0.07	1.60
50	35	2.80	0.14	2.64	1.69	2.45	0.21	1.22
100	80	1.11	0.15	1.08	0.66	0.99	0.17	0.45

Table 9 Estimated risks (ER) of the estimates of θ ($c = 3; \beta = 2; \theta = 0.2513$)

n	m	$ER(\hat{\theta}_{ML})$	$ER(\hat{\theta}_{UM})$	$ER(\tilde{\theta}_{SE})$	$ER(\tilde{\theta}_{AD})$	$ER(\tilde{\theta}_{LN})$	$ER(\tilde{\theta}_{GE})$	$ER(\tilde{\theta}_{LG})$
						a=0.5	q=1	
20	5	0.1568	0.0897	0.0592	0.0496	0.0512	0.0374	0.0458
20	10	0.0431	0.0321	0.0308	0.0278	0.0286	0.0236	0.0266
30	20	0.0160	0.0138	0.0140	0.0133	0.0135	0.0122	0.0130
40	30	0.0100	0.0090	0.0092	0.0089	0.0090	0.0084	0.0087
50	35	0.0081	0.0075	0.0076	0.0074	0.0074	0.0070	0.0073
100	89	0.0008	0.0008	0.0008	0.0008	0.0008	0.0007	0.0008

Table 10 Estimates of the failure rate $h(t)$ ($\beta = 2; \theta = 0.2513; t = 1$)

n	m	$\hat{h}_{ML}$	$\hat{h}_{UM}$	$\hat{h}_{SE}$	$\hat{h}_{AD}$	$\hat{h}_{LN}$	$\hat{h}_{GE}$	$\hat{h}_{LG}$
						a=0.5	q=1	
20	5	0.2434	0.1947	0.2218	0.2096	0.2187	0.1848	0.2034
20	10	0.1966	0.1770	0.1938	0.1879	0.1925	0.1762	0.1850
30	20	0.1973	0.1875	0.1963	0.1932	0.1957	0.1870	0.1917
40	30	0.2040	0.1973	0.2032	0.2011	0.2028	0.1967	0.2000
50	35	0.1924	0.1869	0.1921	0.1903	0.1918	0.1868	0.1895
100	80	0.1906	0.1882	0.1905	0.1897	0.1904	0.1881	0.1893

Table 11 Estimated risks (ER) of the estimates of $h(t)$ ($\beta = 2; \theta = 0.2513; t = 1$)

n	m	$ER(\hat{h}_{ML})$	$ER(\hat{h}_{UM})$	$ER(\hat{h}_{SE})$	$ER(\hat{h}_{AD})$	$ER(\hat{h}_{LN})$	$ER(\hat{h}_{GE})$	$ER(\hat{h}_{LG})$
						a=0.5	q=1	
20	5	0.0220	0.0125	0.0085	0.0071	0.0079	0.0054	0.0066
20	10	0.0051	0.0039	0.0038	0.0034	0.0037	0.0029	0.0033
30	20	0.0023	0.0020	0.0020	0.0019	0.0019	0.0017	0.0018
40	30	0.0015	0.0013	0.0014	0.0013	0.0013	0.0012	0.0013
50	35	0.0011	0.0010	0.0010	0.0010	0.0010	0.0010	0.0010
100	80	0.0005	0.0005	0.0005	0.0004	0.0004	0.0004	0.0004

Table 12 $PE(\%)$ values of estimates of θ when $\beta = 2, \theta = 0.4986, n = 20$ and $m = 5$

R	$PE(\hat{\theta}_{ML})$	$PE(\hat{\theta}_{UM})$	$PE(\tilde{\theta}_{SE})$	$PE(\tilde{\theta}_{AD})$	$PE(\tilde{\theta}_{LN})$	$PE(\tilde{\theta}_{GE})$	$PE(\tilde{\theta}_{LG})$
					a=0.5	q=1	
{3,3,3,3,3}	25.90	0.72	15.41	9.06	12.32	3.83	8.93
{5,10,0,0,0}	26.56	1.25	15.91	9.54	12.81	3.40	6.40
{0,0,0,0,15}	25.26	0.21	15.22	8.89	12.17	3.98	5.77
{15,0,0,0,0}	24.86	0.11	14.74	8.43	11.70	4.38	5.32
{5,5,5,0,0}	25.83	0.66	15.41	9.06	12.33	3.83	5.94
{0,15,0,0,0}	25.06	0.05	15.03	8.70	11.98	4.14	5.59

Table 13 Estimated risk(ER) of estimates of θ when $c = 3, \beta = 2, \theta = 0.4986, n = 20$ and $m = 5$

R	$ER(\hat{\theta}_{ML})$	$ER(\hat{\theta}_{UM})$	$ER(\tilde{\theta}_{SE})$	$ER(\tilde{\theta}_{AD})$	$ER(\tilde{\theta}_{LN})$	$ER(\tilde{\theta}_{GE})$	$ER(\tilde{\theta}_{LG})$
					a=0.5	q=1	
{3,3,3,3,3}	0.157	0.090	0.059	0.050	0.051	0.037	0.046
{5,10,0,0,0}	0.159	0.090	0.059	0.050	0.051	0.037	0.046
{0,0,0,0,15}	0.142	0.081	0.057	0.048	0.050	0.036	0.044
{15,0,0,0,0}	0.152	0.087	0.058	0.048	0.050	0.037	0.045
{5,5,5,0,0}	0.153	0.087	0.059	0.050	0.051	0.037	0.046
{0,15,0,0,0}	0.149	0.085	0.057	0.048	0.049	0.036	0.044

Table 14 Estimates of failure rate $h(t)$ when $c = 3, \beta = 2, \theta = 0.4986, n = 20$ and $m = 5$

R	$\hat{h}_{ML}$	$\hat{h}_{UM}$	$\hat{h}_{SE}$	$\hat{h}_{AD}$	$\hat{h}_{LN}$ a=0.5	$\hat{h}_{GE}$ q=1	$\hat{h}_{LG}$
{3,3,3,3,3}	0.471	0.377	0.432	0.408	0.420	0.359	0.396
{5,10,0,0,0}	0.473	0.473	0.433	0.409	0.422	0.361	0.398
{0,0,0,0,15}	0.468	0.375	0.431	0.407	0.420	0.359	0.396
{15,0,0,0,0}	0.467	0.374	0.429	0.406	0.418	0.358	0.396
{5,5,5,0,0}	0.471	0.376	0.432	0.408	0.420	0.360	0.396
{0,15,0,0,0}	0.468	0.374	0.430	0.407	0.419	0.359	0.395

Table 15 Estimated risk (ER) of estimates of $h(t)$ for fixed n and m ($\beta = 2, \theta = 0.4986, n = 20, m = 5$)

R	ER($\hat{h}_{ML}$)	ER($\hat{h}_{UM}$)	ER($\hat{h}_{SE}$)	ER($\hat{h}_{AD}$)	ER($\hat{h}_{LN}$) a=0.5	ER($\hat{h}_{GE}$) q=1	ER($\hat{h}_{LG}$)
{3,3,3,3,3}	0.088	0.050	0.033	0.028	0.029	0.021	0.026
{5,10,0,0,0}	0.089	0.051	0.034	0.028	0.029	0.021	0.026
{0,0,0,0,15}	0.080	0.045	0.032	0.027	0.028	0.020	0.025
{15,0,0,0,0}	0.085	0.049	0.032	0.027	0.028	0.021	0.025
{5,5,5,0,0}	0.086	0.049	0.033	0.028	0.029	0.021	0.026
{0,15,0,0,0}	0.084	0.048	0.032	0.027	0.028	0.020	0.025

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