Thailand Statistician
July 2024; 22(3): 547-564
http://statassoc.or.th
Contributed paper

Robust Inference for the Skew Normal Regression Model Under Type II Censoring

Iklim Gedik Balay [a]* and Birdal Senoglu [b]
[a]Ankara Yıldırım Beyazıt University, Department of Banking and Finance, 06760, Ankara, Turkey
[b]Ankara University, Department of Statistics, 06100, Ankara, Turkey
*Corresponding author; e-mail: iklimgdk@gmail.com

Received: 25 September 2021
Revised: 14 April 2022
Accepted: 15 April 2022

Abstract

In this paper, we concentrate on statistical inference for the regression model with skew normal (*SN*) distributed error terms under type-II censoring. Iteratively reweighting algorithm (IRA) is used for computing maximum likelihood (ML) estimates of the model parameters, see Arslan (2009). We also use the non-iterative modified maximum likelihood (MML) methodology to obtain the explicit estimators of the model parameters, see Tiku (1967). Additionally, confidence intervals for the model parameters are constructed based on the proposed estimators. Monte Carlo simulation study is used to compare the efficiencies of the ML and MML estimators, and also the performances of the corresponding confidence intervals.

Keywords: Reliability, modified maximum likelihood estimators, Monte Carlo simulation, skewness, censored samples.

1. Introduction

In most of the studies in engineering, the aim is to get reliability information quickly see Escobar and Meeker (2006). Test units of a products are subjected to higher levels of one or more accelerating variables (such as temperature, voltage, pressure, vibration, loads or some combinations of them) than the usual levels, see Nelson (1972, 1975), Kececioglu and Jacks (1984). The results of this procedures are used to make prediction on life of units at use condition see Escobar and Meeker (2006).

A simple, commonly used model to define relationship between failure time and accelerating variables is the regression see Newby (1988), Escobar and Meeker (2006) and Kalbfleisch and Prentice (2011). This approach is very appealing because of its direct physical interpretation see Cox and Oakes (1984) and Reid (1994).

Consider the following regression model

$$Y = \theta_0 + \theta_1 x + \varepsilon, \tag{1}$$

where Y is the response variable related to accelerating variable x, θ_0 and θ_1 are unknown regression parameters and ε is the independent and identically distributed (iid) random error term with mean 0 and variance σ^2, see Rousseeuw and Leroy (2005) and Islam and Tiku (2010).

According to the type of accelerating variable, there are some widely used life-stress relationship models. For example, Arrhenius and Eyring life-stress relationships are used when a single accelerating variable is temperature. Inverse power law (IPL) life-stress relationship is commonly used when the accelerating variable is a single, non-thermal stress (such as voltage, vibration, loading etc.) The combination model is considered for testing combined effect of thermal and non-thermal stresses. For instance, if both temperature and voltage are considered as accelerating variables, Arrhenius model and the IPL model can be combined to obtain a new model, see Kececioglu and Jacks (1984) for more details.

For an illustration, IPL relationship between the failure time τ and accelerating variable V is defined by

$$\tau = A/V^{\gamma_1} \tag{2}$$

where A and γ_1 are positive parameters characteristic of the product, see Kececioglu and Jacks (1984). IPL relationship is linearized using the logarithm of (2) as

$$\ln(\tau) = \ln(A) - \gamma_1 \ln(V), \tag{3}$$

where $Y = ln(\tau)$, $x = ln(V)$ is an accelerating variable, $\theta_0 = ln(A)$ and $\theta_1 = \gamma_1$ are the regression model parameters see Chan et al.(2008) and Nelson (2009).

There exist several studies on modeling life time data using the regression model in the literature. Wei (1992) reviewed some linear regression methods for analyzing failure time observations and stated that this approach had a physical interpretation and could be useful alternative to the Cox model in survival analysis. Lawless (1998) used parametric models to represent the distributions of lifetimes, and their relationship to accelerating variables. Chan et al. (2008) proposed estimators for regression parameters by using IPL relationship under type-II censoring when the failure time has a Weibull distribution. They also obtained confidence intervals based on the proposed estimators.

In this paper, we obtain the estimators of regression parameters by using maximum likelihood (ML) and modified maximum likelihood (MML) methodologies under type-II censoring when the distribution of the error terms is Azzalini (1985)'s skew normal (*SN*). To the best of our knowledge, this is the first study in estimating parameters of the regression model with *SN* distributed error terms under type-II censoring.

To estimate parameters of the regression model, ML estimation method is widely used because of its optimal properties under regularity conditions. However, ML equations cannot be solved analytically when the distribution of the error terms is nonnormal. For this reason, iterative methods are used to derive ML estimates of the unknown regression parameters. In our study, iteratively reweighting algorithm (IRA) is used to obtain ML estimates of regression parameters. *SN* distribution is a skew scale mixture of normal distribution therefore IRA is equivalent to EM type algorithm and its convergence is guarenteed, see Arrellano-Valle et al. (2005), Lachos et al. (2007), Xie et al. (2009), Arslan (2009), Lachos et al. (2010) and Garay et al. (2014). Also, we obtain explicit estimators of model parameters by using a non-iterative methodology known as modified maximum likelihood (MML) as an alternative approach see Tiku (1967,1968) for more details. In the previous works on the MML estimators, it is shown that they are asymptotically equivalent to ML estimators and have high efficiencies even for small sample sizes, see Senoglu and Tiku (2001) and Tiku and Suresh (1992).

In practice, failure time has often a skew distribution see Cox and Oakes (1984). For this reason, using *SN* is convenient for modeling failure time data since it provides flexibility for modeling the data according to different values of skewness parameter. Lachos et al.(2007) presented EM algorithms for estimating parameters of *SN* regression model. Cancho et al. (2010) made inference on *SN* nonlinear regression models following both a classical and Bayesian approach.

The rest of the paper is structured as follows. *SN* distribution is introduced in Section 2. In Section 3, the ML and MML estimators of the regression model parameters are derived, respectively. Confidence intervals based on the proposed estimators for the model parameters are presented in Section 4. Performances of the proposed estimators are compared via Monte Carlo simulation study in Section 5. Coverage probabilities and average lengths of the confidence intervals are also compared in Section 5. Concluding remarks are given in Section 6.

2. Skew-Normal Distribution

A random variable Z is said to have a *SN* distribution with skewness parameter λ, if its probability density function (pdf) is given by

$$\phi(z;\lambda) = 2\phi(z)\Phi(\lambda z), -\infty < z < \infty \tag{4}$$

where $\phi(z)$ and $\Phi(z)$ denote the pdf of the standard normal and the corresponding cumulative distribution function (cdf), respectively. A random variable Z having *SN* distribution with parameter λ is denoted by $Z \sim SN(\lambda)$ see Azzalini (1985, 1986) for more details. The following Owen's function (T)

$$T(z;\lambda) = \int_z^\infty \int_0^{\lambda s} \phi(s)\phi(t)dtds \tag{5}$$

is used to get the cdf of $SN(\lambda)$ as

$$F(z;\lambda) = \Phi(\lambda) - 2T(z;\lambda). \tag{6}$$

It may be noted that $SN(\lambda)$ reduces to the well known standard normal distribution $N(0,1)$ for $\lambda = 0$. Also, it converges to the half-normal density function when $\lambda \to \infty$. For fixed λ, $SN(\lambda)$ is strongly unimodal. It is right skewed and left skewed for positive and negative values of λ, respectively. $SN(\lambda)$ distribution has also the following properties:

i. If $Z \sim SN(\lambda)$ then $-Z \sim SN(-\lambda)$

ii. If $Z \sim SN(\lambda)$ then $Z^2 \sim \chi_1^2$ see Azzalini (2005).

The moment generating function (mgf) of the random variable $Z \sim SN(\lambda)$ is given by

$$M_Z(t) = E[e^{tz}] = 2exp\left(\frac{t^2}{2}\right)\Phi(\delta t). \tag{7}$$

where $\delta = \frac{\lambda}{\sqrt{1+\lambda^2}}$.

By using the mgf of the random variable $Z \sim SN(\lambda)$, we get the expected value $E(Z)$ and variance $V(Z)$ as

$$E(Z) = \sqrt{\frac{2}{\pi}}\delta \quad \text{and} \quad V(Z) = 1 - \frac{2}{\pi}\delta^2. \tag{8}$$

Also, the measures of skewness (γ_1) and kurtosis (γ_2) are given by

$$\gamma_1 = \frac{1}{2}(4-\pi)sign(\lambda)\left(\frac{\lambda^2}{\frac{\pi}{2} + (\frac{\pi}{2}-1)\lambda^2}\right)^{3/2} \quad \text{and} \quad \gamma_2 = 2(\pi - 3)\left(\frac{\lambda^2}{\frac{\pi}{2} + (\frac{\pi}{2}-1)\lambda^2}\right). \tag{9}$$

Azzalini (1985) states that the maximum values of γ_1 and γ_2 are about 0.995 and 0.869, respectively. Table 1 shows the coefficients of skewness and kurtosis for some representative values of λ.

Table 1 The skewness and the kurtosis values of the $SN(\lambda)$ distribution.

λ	0.0	1.0	2.0	3.0	4.0	5.0	10	20	∞
γ_1	0.00	0.14	0.45	0.67	0.78	0.85	0.96	0.99	0.995
γ_2	3.00	3.06	3.31	3.51	3.63	3.71	3.82	3.86	3.869

According to Table 1, it is obvious that the skewness and kurtosis values of the $SN(\lambda)$ distribution increase as λ increases. Skewness values given in Table 1 are calculated for positive values of λ. It should be noted that they take exactly the same values but with negative signs for the negative values of λ, see Celik et al. (2015). Some extensions of this distribution can be found in Martinez-Florez et al. (2009) and Pereira et al. (2012).

3. Estimation

In this section, we obtain the estimators of the regression parameters by using the ML and the MML methodologies, see Gedik-Balay (2014).

3.1. Maximum likelihood estimation

We consider the regression model with type-II censoring extensively used to specify the accelerating variable effects on failure time see Newby (1988) and Kalbfleisch and Prentice (2011). Suppose that n_k units are tested at a single accelerating variable x_k for $k = 1, ..., m$ and k denote the levels of accelerating variable. Under type-II censoring, s_k largest units are removed from the ordered values, in other words $y_{1:n_k} \leq y_{2:n_k} \leq ... \leq y_{n_k - s_k:n_k}$ out of n_k values are observed. The likelihood function for the regression model with $SN(\lambda)$ distributed error terms under type-II censoring is given by

$$L = \prod_{k=1}^{m} \left\{ \left[1 - F\left(z_{n_k - s_k:n_k}\right)\right]^{s_k} \times \left[\prod_{i=1}^{n_k - s_k} \frac{2}{\sigma} \phi\left(z_{i:n_k}\right) \Phi\left(\lambda z_{i:n_k}\right) \right] \right\} \tag{10}$$

where $z_{i:n_k} = \frac{y_{i:n_k} - \theta_0 - \theta_1 x_k}{\sigma}$, $z_{n_k - s_k:n_k} = \frac{y_{n_k - s_k:n_k} - \theta_0 - \theta_1 x_k}{\sigma}$ and $\phi(.)$ is the pdf of the standard normal distribution, as mentioned before.

Then the corresponding log-likelihood (lnL) function is obtained as shown below

$$\ln L = \sum_{k=1}^{m} s_k \ln\left[1 - F\left(z_{n_k - s_k:n_k}\right)\right] + \sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} \left[\ln(2) - \ln(\sigma) - \frac{1}{2}\ln(2\pi) - \frac{1}{2} z_{i:n_k}^2 + \ln\left(\Phi\left(\lambda z_{i:n_k}\right)\right) \right]. \tag{11}$$

The ML estimators of the unknown parameters θ_0, θ_1 and σ are obtained by differentiating the lnL function with respect to the parameters of interest and setting them equal to zero as follows:

$$\begin{aligned} \frac{\partial \ln L}{\partial \theta_0} &= \frac{1}{\sigma} \left[\sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} z_{i:n_k} - \lambda \sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} g_1(z_{i:n_k}) + \sum_{k=1}^{m} s_k g_2(z_{n_k - s_k:n_k}) \right] = 0 \\ \frac{\partial \ln L}{\partial \theta_1} &= \frac{1}{\sigma} \left[\sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} x_k z_{i:n_k} - \lambda \sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} x_k g_1(z_{i:n_k}) + \sum_{k=1}^{m} s_k x_k g_2(z_{n_k - s_k:n_k}) \right] = 0 \end{aligned} \tag{12}$$

and

$$\frac{\partial \ln L}{\partial \sigma} = \frac{1}{\sigma} \left[-n + \sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} z_{i:n_k}^2 - \lambda \sum_{k=1}^{m} \sum_{i=1}^{n_k - s_k} z_{i:n_k} g_1(z_{i:n_k}) + \sum_{k=1}^{m} s_k z_{n_k - s_k:n_k} g_2(z_{n_k - s_k:n_k}) \right] = 0$$

where $g_1(z) = \frac{\phi(\lambda z)}{\Phi(\lambda z)}$, $g_2(z) = \frac{f(z)}{1 - F(z)}$ and $n = \sum_{k=1}^{m} (n_k - s_k)$. Since the terms as $g_1(z_{i:n_k})$ and $g_2(z_{n_k - s_k:n_k})$ are intractable, then the equations given in (12) cannot be solved analytically. For this reason, some iterative procedure must be employed. In this study, we provide IRA to compute the ML estimates of the model parameters for the reason given in Section 1.

The steps of IRA

Step 1: Start with initial values $\theta_0^{(0)}$, $\theta_1^{(0)}$, $\sigma^{(0)}$ for θ_0, θ_1, σ. In this study, we use LS estimates of regression parameters as initial values.

Step 2: Use $\theta_0^{(m)}$, $\theta_1^{(m)}$ and $\sigma^{(m)}$ for $m = 0, 1, 2, 3, ...$, to calculate the values of $g_1^{(m)}(z_{i:n_k})$ and $g_2^{(m)}(z_{n_k - s_k:n_k})$.

Step 3: Solve the following equations

$$\theta_0^{(m+1)} = \overline{y} - \theta_1^{(m)} \overline{x} - \sigma^{(m)} \overline{\gamma} \quad \text{and} \quad \theta_1^{(m+1)} = K - D^{(m)} \sigma^{(m)}$$

to calculate the new estimates $\hat{\theta}_0^{(m)}$, $\hat{\theta}_1^{(m)}$ and $\hat{\sigma}^{(m)}$. Here,

$$K = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n_k-s_k}(x_k-\overline{x})(y_{i:n_k}-\overline{y})}{\sum_{k=1}^{m}\sum_{i=1}^{n_k-s_k}(x_k-\overline{x})^2}, \quad D^{(m)} = \frac{\sum_{k=1}^{m}(x_k-\overline{x})(\gamma_k^{(m)}-\overline{\gamma}^{(m)})}{\sum_{k=1}^{m}(x_k-\overline{x})^2},$$

$$\sigma^{(m+1)} = \frac{-B^{(m)}+\sqrt{B^2-4A^{(m)}C^{(m)}}}{2A^{(m)}}.$$

where

$$A^{(m)} = n-\lambda\sum_{k=1}^{m}B_{n_k-s_k:n_k}^{(m)}\sum_{i=1}^{n_k-s_k}g_1^{(m)}(z_{i:n_k})+\sum_{k=1}^{m}s_kg_2^{(m)}(z_{n_k-s_k:n_k})B_{n_k-s_k:n_k}^{(m)}-\sum_{k=1}^{m}(B_{n_k-s_k:n_k}^{(m)})^2,$$

$$B^{(m)} = \lambda\sum_{k=1}^{m}\sum_{i=1}^{n_k-s_k}g_1^{(m)}(z_{i:n_k})A_{i:n_k}^{(m)}-\sum_{k=1}^{m}s_kg_2^{(m)}(z_{n_k-s_k:n_k})\sum_{i=1}^{n_k-s_k}A_{i:n_k}^{(m)}+2\sum_{k=1}^{m}B_{n_k-s_k:n_k}^{(m)}\sum_{i=1}^{n_k-s_k}A_{i:n_k}^{(m)},$$

$$C^{(m)} = \sum_{i=1}^{n_k-s_k}(A_{i:n_k}^{(m)})^2$$

and $\gamma_k^{(m)} = \sum_{i=1}^{n_k-s_k}g_1^{(m)}(z_{i:n_k})-s_kg_2^{(m)}(z_{n_k-s_k:n_k})$,

$$A_{i:n_k} = (y_{i:n_k}-\overline{y})-K(x_k-\overline{x}), \quad B_{n_k-s_k:n_k} = D(x_k-\overline{x})+\overline{\gamma^{(m)}},$$

$$\overline{y} = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n_k-s_k}y_{i:n_k}}{n}, \quad \overline{x} = \frac{\sum_{k=1}^{m}\sum_{i=1}^{n_k-s_k}x_k}{n} \quad \text{and} \quad \overline{\gamma} = \frac{\sum_{k=1}^{m}\gamma_k}{n}.$$

Step 4: Stop the iterations when the conditions $|\hat{\theta}_0^{(m+1)}-\hat{\theta}_0^{(m)}|<\epsilon$, $|\hat{\theta}_1^{(m+1)}-\hat{\theta}_1^{(m)}|<\epsilon$ and $|\hat{\sigma}^{(m+1)}-\hat{\sigma}^{(m)}|<\epsilon$ are hold simultaneously. Here, ϵ is a predetermined small constant.

3.2. Modified Maximum Likelihood Estimator

We use Tiku (1967)'s MML methodology to obtain the explicit solutions of the likelihood equations in (12), see Tiku (1967, 1968). Based on MML methodology, intractable terms $g_1(z_{i:n_k})$ and $g_2(z_{n_k-s_k:n_k})$ are linearized by using the first two terms of Taylor series expansion around the expected values of the order statistics i.e., $t_{i:n_k} = E(z_{i:n_k})$ and $t_{n_k-s_k:n_k} = E(z_{n_k-s_k:n_k})$, respectively. Linearized forms of the functions $g_1(z_{i:n_k})$ and $g_2(z_{n_k-s_k:n_k})$ are given below:

$$g_1(z_{i:n_k}) = \alpha_{1i}-\beta_{1i}z_{i:n_k} \quad \text{and} \quad g_2(z_{n_k-s_k:n_k}) = \alpha_{2i}-\beta_{2i}z_{n_k-s_k:n_k}. \tag{13}$$

Here,

$$\beta_{1i} = \frac{\lambda^2t_{i:n_k}\phi(\lambda t_{i:n_k})\Phi(\lambda t_{i:n_k})-\lambda\phi(\lambda t_{i:n_k})^2}{\Phi(\lambda t_{i:n_k})^2}, \quad \alpha_{1i} = \frac{\phi(\lambda t_{i:n_k})}{\Phi(\lambda t_{i:n_k})}+t_{i:n_k}\beta_{1i},$$

$$\beta_{2i} = \frac{f'(t_{n_k-s_k:n_k})}{[1-F(t_{n_k-s_k:n_k})]}+\frac{f(t_{n_k-s_k:n_k})^2}{[1-F(t_{n_k-s_k:n_k})]^2}, \quad \alpha_{2i} = \frac{f(t_{n_k-s_k:n_k})}{[1-F(t_{n_k-s_k:n_k})]}-t_{n_k-s_k:n_k}\beta_{2i}.$$

We obtain approximate values of $t_{i:n_k}$ and $t_{n_k-s_k:n_k}$ by solving the following equations

$$\int_{-\infty}^{t_{i:n_k}}f(z_{i:n_k})\,dz = \frac{i}{n_k+1} \quad \text{and} \quad \int_{-\infty}^{t_{n_k-s_k:n_k}}f(z_{n_k-s_k:n_k})\,dz = \frac{n_k-s_k}{n_k+1}, \tag{14}$$

respectively. The modified likelihood equations $\frac{\partial \ln L^*}{\partial\theta_0}$, $\frac{\partial \ln L^*}{\partial\theta_1}$ and $\frac{\partial \ln L^*}{\partial\sigma}$ are obtained by replacing the intractable terms $g_1(z_{i:n_k})$ and $g_2(z_{n_k-s_k:n_k})$ with the corresponding linear approximations in

(13). Solutions of the modified likelihood equations with respect to the unknown parameters are the following MML estimators of the regression parameters

$$\hat{\theta}_0 = a\hat{\sigma} + b, \quad \hat{\theta}_1 = c\hat{\sigma} + d \quad \text{and} \quad \hat{\sigma} = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \tag{15}$$

where
$$A = \sum_{k=1}^{m}(n_k - s_k)$$
$$B = \sum_{k=1}^{m}\{-s_k[y_{n_k-s_k:n_k} - (b + dx_k)][\alpha_{n_k-s_k:n_k} - \beta_{n_k-s_k:n_k}(a + cx_k)] + \sum_{i=1}^{n_k-s_k}[y_{i:n_k} - (b + dx_k)][\lambda\alpha_{i:n_k} + (1 + \lambda\beta_{i:n_k})(a + cx_k)]\},$$
$$C = -\sum_{k=1}^{m}[s_k\beta_{n_k-s_k:n_k}[y_{n_k-s_k:n_k} - (b + dx_k)]^2 + \sum_{i=1}^{n_k-s_k}(1 + \lambda\beta_{i:n_k})[y_{i:n_k} - (b + dx_k)]^2].$$

Here,

$$a = \frac{\sum_{k=1}^{m} Y_k \sum_{k=1}^{m} x_k^2 a_k - \sum_{k=1}^{m} x_k Y_k \sum_{k=1}^{m} x_k a_k}{\sum_{k=1}^{m} x_k^2 a_k \sum_{k=1}^{m} a_k - (\sum_{k=1}^{m} x_k a_k)^2}, \quad b = \frac{\sum_{k=1}^{m} \Delta_k \sum_{k=1}^{m} x_k^2 a_k - \sum_{k=1}^{m} x_k \Delta_k \sum_{k=1}^{m} x_k a_k}{\sum_{k=1}^{m} x_k^2 a_k \sum_{k=1}^{m} a_k - (\sum_{k=1}^{m} x_k a_k)^2},$$

$$c = \frac{\sum_{k=1}^{m} a_k \sum_{k=1}^{m} x_k \Delta_k - \sum_{k=1}^{m} \Delta_k \sum_{k=1}^{m} x_k a_k}{\sum_{k=1}^{m} x_k^2 a_k \sum_{k=1}^{m} a_k - (\sum_{k=1}^{m} x_k a_k)^2}, \quad d = \frac{\sum_{k=1}^{m} a_k \sum_{k=1}^{m} x_k Y_k - \sum_{k=1}^{m} Y_k \sum_{k=1}^{m} x_k a_k}{\sum_{k=1}^{m} x_k^2 a_k \sum_{k=1}^{m} a_k - (\sum_{k=1}^{m} x_k a_k)^2}.$$

and

$$a_k = s_k\beta_{2i} + \sum_{i=1}^{n_k-s_k}(1 + \lambda\beta_{1i}), \quad Y_k = s_k\beta_{2i}y_{n_k-s_k:n_k} + \sum_{i=1}^{n_k-s_k}(1 + \lambda\beta_{1i})y_{i:n_k},$$
$$\Delta_k = s_k\alpha_{2i} - \lambda\sum_{i=1}^{n_k-s_k}\alpha_{1i}.$$

MML estimators are asymptotically equivalent to the ML estimators, therefore, under regularity conditions, they are fully efficient. Also, a remarkable property of MML methodology is that it gives highly efficient estimator even for small sample cases see Bhattacharyya (1985) and Vaughan and Tiku (2000).

4. Confidence Intervals for the Regression Parameters

In this section, we construct confidence intervals (CIs) for the regression parameters θ_0 and θ_1 based on the ML and MML estimators. First, we need to evaluate the accuracy of the standard normal $N(0,1)$ distribution to the percentage points of the following test statistics,

$$z_1 = \frac{\hat{\theta}_{0,ML}}{\sqrt{Var(\hat{\theta}_{0,ML})}}, z_2 = \frac{\hat{\theta}_{1,ML}}{\sqrt{Var(\hat{\theta}_{1,ML})}},$$
$$z_3 = \frac{\hat{\theta}_{0,MML}}{\sqrt{Var(\hat{\theta}_{0,MML})}}, z_4 = \frac{\hat{\theta}_{1,MML}}{\sqrt{Var(\hat{\theta}_{1,MML})}} \tag{16}$$

where $\hat{\theta}_{i,ML}$ and $\hat{\theta}_{i,MML}$ are the ML and MML estimators of the parameter θ_i, $(i = 0, 1)$, respectively. Variances given in the denominator of the test statistics are computed via Monte Carlo simulation study. In the computation of the simulated variances, following equality is used.

$$Var(\hat{\theta}) = \frac{\sum_{i=1}^{r}(\hat{\theta}_i - \bar{\hat{\theta}})^2}{r-1} \tag{17}$$

Here, $\hat{\theta}_i$ is the estimate of the parameter θ at the ith iteration and r is the number of Monte Carlo runs. $\bar{\hat{\theta}}$ denotes the mean of $\hat{\theta}$ and it is shown mathematically as follows:

$$\bar{\hat{\theta}} = \frac{\sum_{i=1}^{r}\hat{\theta}_i}{r}. \tag{18}$$

We therefore simulate the values of the probabilities $P\{|z_1| > z_{\alpha/2}\}$, $P\{|z_2| > z_{\alpha/2}\}$, $P\{|z_3| > z_{\alpha/2}\}$ and $P\{|z_4| > z_{\alpha/2}\}$. Here $z_{\alpha/2}$ is the $100(1-\alpha/2)$ percent point of the standard normal $N(0,1)$ distribution. Simulation results are given in Tables 2-3. Type I errors of the test statistics given in (16) are computed for $\lambda = 0.7$ and $k = 2$ just for an illustration.

Table 2 Simulated type I errors of z_1 and z_3, $\alpha = 0.050$

s_1	0	1	1	2	2	2	3	3	3	3
s_2	0	0	1	0	1	2	0	1	2	3
$n_1 = n_2 = 10$										
(i)	0.053	0.055	0.053	0.054	0.051	0.050	0.050	0.050	0.050	0.053
(ii)	0.053	0.054	0.055	0.054	0.054	0.050	0.055	0.056	0.052	0.053
$n_1 = n_2 = 15$										
(i)	0.049	0.050	0.051	0.050	0.049	0.056	0.049	0.049	0.049	0.048
(ii)	0.050	0.050	0.050	0.051	0.050	0.057	0.051	0.048	0.046	0.044
$n_1 = n_2 = 20$										
(i)	0.051	0.051	0.048	0.047	0.048	0.050	0.048	0.049	0.050	0.050
(ii)	0.051	0.051	0.050	0.051	0.049	0.051	0.049	0.050	0.051	0.051

(i) Simulated type I errors of z_1
(ii) Simulated type I errors of z_3

Table 3 Simulated type I errors of z_2 and z_4, $\alpha = 0.050$

s_1	0	1	1	2	2	2	3	3	3	3
s_2	0	0	1	0	1	2	0	1	2	3
$n_1 = n_2 = 10$										
(i)	0.054	0.053	0.054	0.054	0.055	0.055	0.053	0.056	0.055	0.054
(ii)	0.054	0.052	0.055	0.055	0.057	0.058	0.051	0.055	0.055	0.056
$n_1 = n_2 = 15$										
(i)	0.051	0.049	0.048	0.050	0.050	0.048	0.049	0.049	0.048	0.048
(ii)	0.050	0.052	0.049	0.049	0.049	0.050	0.049	0.050	0.052	0.053
$n_1 = n_2 = 20$										
(i)	0.049	0.048	0.048	0.048	0.045	0.047	0.048	0.046	0.047	0.046
(ii)	0.050	0.048	0.049	0.049	0.047	0.044	0.046	0.047	0.046	0.046

(i) Simulated type I errors of z_2
(ii) Simulated type I errors of z_4

Type I errors have been computed for some other λ values (such as 0, 0.4 and 1) and have been found very close to the nominal level $\alpha = 0.050$. Therefore, we did not reproduce them for the sake of brevity. So it is concluded from Tables 2-3 that test statistics based on the ML and the MML estimators provide accurate approximations to the standard normal $N(0,1)$ distribution. Based on these findings, CIs for regression parameters θ_0 and θ_1 based on ML estimators are constructed as follows:

$$\begin{aligned} CI(\hat{\theta}_{0,ML}) &= (\hat{\theta}_{0,ML} - z_{\alpha/2}\sqrt{Var(\hat{\theta}_{0,ML})}, \hat{\theta}_{0,ML} + z_{\alpha/2}\sqrt{Var(\hat{\theta}_{0,ML})}), \\ CI(\hat{\theta}_{1,ML}) &= (\hat{\theta}_{1,ML} - z_{\alpha/2}\sqrt{Var(\hat{\theta}_{1,ML})}, \hat{\theta}_{1,ML} + z_{\alpha/2}\sqrt{Var(\hat{\theta}_{1,ML})}), \end{aligned} \tag{19}$$

respectively. Corresponding CIs for θ_0 and θ_1 based on MML estimators are constructed in a similar manner as shown below:

$$\begin{aligned} CI(\hat{\theta}_{0,MML}) &= (\hat{\theta}_{0,MML} - z_{\alpha/2}\sqrt{Var(\hat{\theta}_{0,MML})}, \hat{\theta}_{0,MML} + z_{\alpha/2}\sqrt{Var(\hat{\theta}_{0,MML})}), \\ CI(\hat{\theta}_{1,MML}) &= (\hat{\theta}_{1,MML} - z_{\alpha/2}\sqrt{Var(\hat{\theta}_{1,MML})}, \hat{\theta}_{1,MML} + z_{\alpha/2}\sqrt{Var(\hat{\theta}_{1,MML})}), \end{aligned} \tag{20}$$

respectively.

5. Monte Carlo Simulation Study

In this section, we carried out simulation study based on 10,000 Monte Carlo runs to evaluate the performances of the ML and MML estimators. We consider different sample sizes as $n = 10, 15$ and 20. The skewness parameter was determined as $\lambda = 0, 0.4, 0.7$ and 1. We set the simulations with various different degrees of censoring as Chan et al. Chan et al. (2008) suggest. We use two levels accelerating variable with $x_1 = -1$ and $x_2 = 1$ for the sake of brevity. We take $\theta_0 = 0$, $\theta_1 = 1$ and $\sigma = 1$ in simulation setup without loss of generality.

Bias, variance, mean squares error (MSE) and deficiency (Def) criteria are used in the comparisons. Means and variances for $\hat{\theta}_0$, $\hat{\theta}_1$ and $\hat{\sigma}$ are obtained using the equalities in (18) and (17), respectively. Bias is defined for $\hat{\theta}$ as shown below:

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta. \tag{21}$$

MSE is a well-known and widely-used criterion in the literature for comparing estimators with respect to their efficiency. MSE for the estimator $\hat{\theta}$ is obtained by using following equality

$$MSE(\hat{\theta}) = E(\hat{\theta} - \theta)^2. \tag{22}$$

Def criteria is also used to compare the joint efficiencies of the estimators $\hat{\theta}_0$, $\hat{\theta}_1$ and $\hat{\sigma}$. It is formulated as follows

$$Def = MSE(\hat{\theta}_0) + MSE(\hat{\theta}_1) + MSE(\hat{\sigma}). \tag{23}$$

See Tables 4-7 for the simulated means, variances, MSEs and Def values for the ML and MML estimators of the parameters θ_0, θ_1 and σ.

As can be seen from Tables 4-7, the ML estimators show superior performance with smaller bias than the corresponding MML estimators when the sample size is small. For the large sample sizes and small proportion of censoring, the ML and MML estimators have more or less the same bias in most of the cases. Besides, as we expected when the skewness parameter increases, biases of all estimators increase as well. The ML estimators of the parameters θ_0, θ_1 and σ are more efficient than the MML estimators with small MSE values for $\lambda = 0$ and $n = 10, 15$. However, efficiencies of the ML and MML estimators are very close to each other when $n = 20$.

The MML estimator of θ_0 shows better performance than the corresponding ML estimator in terms of the MSE criterion for $\lambda = 0.4$ and $\lambda = 0.7$. This statement is also true for $\lambda = 1$ when the proportion of censoring is $(s_1, s_2) = (2, 2), (3, 2)$,$(3, 3)$ and $n =$ 10, 15. For $\lambda = 1$ and $n =$ 20, the MML estimator of θ_0 shows the best performance for all censoring schemes.

Table 4 Means, variances and MSEs for the ML and MML estimators of θ_0, θ_1 and σ; $(\lambda = 0)$; $(i)Mean(\hat{\theta}_0)$, $(ii)Var(\hat{\theta}_0)$, $(iii)MSE(\hat{\theta}_0)$,$(iv)Mean(\hat{\theta}_1)$, $(v)Var(\hat{\theta}_1)$, $(vi)MSE(\hat{\theta}_1)$, $(vii)Mean(\hat{\sigma})$, $(viii)Var(\hat{\sigma})$, $(ix)MSE(\hat{\sigma})$, $(x)Def$

			(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
s_1	s_2	$n_1 = n_2 = 10$										
0	0	ML	0.0027	0.0506	0.0506	0.9960	0.0490	0.0491	0.9938	0.0271	0.0271	0.1268
		MML	0.0027	0.0506	0.0506	0.9960	0.0490	0.0491	0.9938	0.0271	0.0271	0.1268
1	0	ML	-0.0715	0.0516	0.0568	1.0702	0.0505	0.0555	0.9472	0.0273	0.0301	0.1423
		MML	-0.0738	0.0515	0.0569	1.1534	0.0516	0.0751	0.9402	0.0268	0.0304	0.1624
1	1	ML	-0.1531	0.0530	0.0765	0.9924	0.0507	0.0508	0.8822	0.0253	0.0391	0.1664
		MML	-0.1537	0.0528	0.0765	0.9935	0.0533	0.0534	0.8689	0.0243	0.0415	0.1713
2	0	ML	-0.1380	0.0536	0.0726	1.1212	0.0530	0.0677	0.9278	0.0290	0.0343	0.1745
		MML	-0.1495	0.0539	0.0763	1.3178	0.0554	0.1564	0.9103	0.0279	0.0360	0.2687
2	1	ML	-0.2208	0.0541	0.1028	1.0489	0.0537	0.0561	0.8580	0.0269	0.0471	0.2060
		MML	-0.2262	0.0543	0.1055	1.1544	0.0560	0.0799	0.8335	0.0253	0.0531	0.2385
2	2	ML	-0.2765	0.0551	0.1315	0.9967	0.0556	0.0556	0.8348	0.0285	0.0558	0.2430
		MML	-0.2821	0.0558	0.1354	0.9966	0.0584	0.0584	0.7968	0.0260	0.0673	0.2611
3	0	ML	-0.1567	0.0558	0.0804	1.1400	0.0563	0.0759	0.9445	0.0325	0.0356	0.1919
		MML	-0.1936	0.0571	0.0946	1.5043	0.0591	0.3135	0.9095	0.0305	0.0387	0.4468
3	1	ML	-0.2453	0.0560	0.1162	1.0739	0.0574	0.0628	0.8692	0.0303	0.0474	0.2265
		MML	-0.2698	0.0573	0.1301	1.3290	0.0587	0.1669	0.8247	0.0274	0.0582	0.3552
3	2	ML	-0.3024	0.0568	0.1483	1.0241	0.0593	0.0599	0.8447	0.0324	0.0565	0.2646
		MML	-0.3227	0.0588	0.1629	1.1661	0.0603	0.0879	0.7824	0.0280	0.0754	0.3263
3	3	ML	-0.3261	0.0583	0.1647	0.9961	0.0623	0.0623	0.8582	0.0374	0.0576	0.2845
		MML	-0.3593	0.0620	0.1911	0.9960	0.0622	0.0622	0.7626	0.0301	0.0865	0.3399
s_1	s_2	$n_1 = n_2 = 15$										
0	0	ML	0.0031	0.0301	0.0301	0.9981	0.0326	0.0326	0.9890	0.0181	0.0182	0.0809
		MML	0.0031	0.0301	0.0301	0.9981	0.0326	0.0326	0.9890	0.0181	0.0182	0.0809
1	0	ML	-0.0536	0.0304	0.0333	1.0549	0.0332	0.0362	0.9481	0.0179	0.0206	0.0900
		MML	-0.0543	0.0304	0.0333	1.1054	0.0339	0.0450	0.9444	0.0176	0.0207	0.0991
1	1	ML	-0.1117	0.0310	0.0435	0.9983	0.0334	0.0334	0.9009	0.0172	0.0270	0.1038
		MML	-0.1115	0.0309	0.0433	0.9985	0.0346	0.0346	0.8941	0.0168	0.0280	0.1060
2	0	ML	-0.0954	0.0309	0.0400	1.0966	0.0337	0.0430	0.9334	0.0185	0.0230	0.1059
		MML	-0.0986	0.0308	0.0405	1.2120	0.0351	0.0801	0.9243	0.0180	0.0238	0.1444
2	1	ML	-0.1552	0.0314	0.0555	1.0411	0.0339	0.0355	0.8809	0.0176	0.0318	0.1228
		MML	-0.1559	0.0313	0.0556	1.1029	0.0356	0.0462	0.8689	0.0170	0.0342	0.1360
2	2	ML	-0.2012	0.0321	0.0726	0.9972	0.0344	0.0344	0.8556	0.0178	0.0387	0.1456
		MML	-0.2009	0.0320	0.0723	0.9975	0.0365	0.0365	0.8382	0.0169	0.0431	0.1519
3	0	ML	-0.1265	0.0316	0.0476	1.1278	0.0344	0.0508	0.9318	0.0197	0.0244	0.1228
		MML	-0.1363	0.0316	0.0502	1.3252	0.0364	0.1422	0.9151	0.0189	0.0261	0.2185
3	1	ML	-0.1887	0.0321	0.0677	1.0743	0.0347	0.0402	0.8748	0.0187	0.0344	0.1423
		MML	-0.1938	0.0320	0.0696	1.2124	0.0367	0.0818	0.8552	0.0177	0.0386	0.1900
3	2	ML	-0.2360	0.0327	0.0884	1.0313	0.0352	0.0362	0.8467	0.0188	0.0423	0.1669
		MML	-0.2381	0.0326	0.0893	1.1053	0.0374	0.0485	0.8212	0.0175	0.0494	0.1873
3	3	ML	-0.2728	0.0334	0.1078	0.9963	0.0359	0.0360	0.8346	0.0198	0.0471	0.1909
		MML	-0.2751	0.0335	0.1092	0.9964	0.0380	0.0380	0.8001	0.0179	0.0578	0.2050
s_1	s_2	$n_1 = n_2 = 20$										
0	0	ML	0.0067	0.0231	0.0231	0.9988	0.0210	0.0210	0.9914	0.0126	0.0127	0.0568
		MML	-0.0052	0.0232	0.0232	0.9989	0.0216	0.0216	1.0320	0.0137	0.0147	0.0595
1	0	ML	-0.0387	0.0264	0.0279	1.0447	0.0244	0.0264	0.9561	0.0125	0.0144	0.0687
		MML	-0.0390	0.0263	0.0278	1.0807	0.0248	0.0313	0.9537	0.0124	0.0145	0.0736
1	1	ML	-0.0857	0.0265	0.0339	0.9986	0.0247	0.0247	0.9151	0.0118	0.0190	0.0775
		MML	-0.0855	0.0264	0.0337	0.9986	0.0254	0.0254	0.9108	0.0116	0.0195	0.0787
2	0	ML	-0.0744	0.0265	0.0321	1.0804	0.0247	0.0312	0.9395	0.0130	0.0166	0.0799
		MML	-0.0758	0.0264	0.0322	1.1609	0.0254	0.0513	0.9337	0.0127	0.0171	0.1006
2	1	ML	-0.1224	0.0267	0.0417	1.0349	0.0250	0.0262	0.8949	0.0121	0.0231	0.0910
		MML	-0.1224	0.0265	0.0415	1.0777	0.0260	0.0320	0.8873	0.0118	0.0245	0.0980
2	2	ML	-0.1601	0.0269	0.0525	0.9985	0.0253	0.0253	0.8717	0.0122	0.0286	0.1065
		MML	-0.1594	0.0268	0.0522	0.9986	0.0266	0.0266	0.8611	0.0117	0.0310	0.1098
3	0	ML	-0.1036	0.0268	0.0375	1.1096	0.0252	0.0372	0.9326	0.0136	0.0182	0.0929
		MML	-0.1076	0.0267	0.0383	1.2433	0.0261	0.0853	0.9222	0.0132	0.0193	0.1429
3	1	ML	-0.1528	0.0269	0.0503	1.0650	0.0254	0.0297	0.8848	0.0126	0.0259	0.1058
		MML	-0.1544	0.0268	0.0506	1.1582	0.0266	0.0517	0.8729	0.0121	0.0283	0.1306
3	2	ML	-0.1913	0.0272	0.0638	1.0290	0.0258	0.0266	0.8593	0.0126	0.0324	0.1228
		MML	-0.1911	0.0270	0.0636	1.0783	0.0272	0.0333	0.8444	0.0120	0.0362	0.1331
3	3	ML	-0.2234	0.0275	0.0774	0.9983	0.0262	0.0262	0.8450	0.0129	0.0369	0.1405
		MML	-0.2227	0.0274	0.0772	0.9982	0.0279	0.0279	0.8255	0.0121	0.0426	0.1476

Table 5 Means, variances and MSEs for the ML and MML estimators of θ_0, θ_1 and σ; $(\lambda = 0.4)$; $(i) Mean(\hat{\theta}_0)$, $(ii) Var(\hat{\theta}_0)$, $(iii) MSE(\hat{\theta}_0)$, $(iv) Mean(\hat{\theta}_1)$, $(v) Var(\hat{\theta}_1)$, $(vi) MSE(\hat{\theta}_1)$, $(vii) Mean(\hat{\sigma})$, $(viii) Var(\hat{\sigma})$, $(ix) MSE(\hat{\sigma})$, $(x) Def$

s_1	s_2		(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
s_1	s_2	$n_1 = n_2 = 10$										
0	0	ML	-0.0162	0.0449	0.0452	1.0049	0.0431	0.0431	1.0023	0.0298	0.0298	0.1181
		MML	-0.0280	0.0451	0.0459	1.0052	0.0444	0.0444	1.0411	0.0322	0.0339	0.1242
1	0	ML	-0.0652	0.0496	0.0539	1.0682	0.0436	0.0482	0.9773	0.0290	0.0296	0.1316
		MML	-0.0514	0.0483	0.0509	1.1033	0.0436	0.0543	0.9367	0.0257	0.0297	0.1349
1	1	ML	-0.1210	0.0508	0.0655	0.9980	0.0440	0.0440	0.9154	0.0284	0.0355	0.1450
		MML	-0.1050	0.0489	0.0599	0.9712	0.0442	0.0450	0.8743	0.0244	0.0402	0.1452
2	0	ML	-0.1077	0.0510	0.0626	1.1123	0.0451	0.0577	0.9716	0.0319	0.0327	0.1530
		MML	-0.0928	0.0486	0.0572	1.2072	0.0464	0.0893	0.9232	0.0273	0.0332	0.1797
2	1	ML	-0.1644	0.0522	0.0792	1.0452	0.0457	0.0477	0.8992	0.0305	0.0406	0.1676
		MML	-0.1406	0.0491	0.0689	1.0723	0.0466	0.0518	0.8536	0.0254	0.0469	0.1676
2	2	ML	-0.2086	0.0534	0.0970	0.9976	0.0471	0.0472	0.8757	0.0325	0.0480	0.1921
		MML	-0.2023	0.0476	0.0885	0.9516	0.0488	0.0512	0.8389	0.0295	0.0554	0.1952
3	0	ML	-0.1287	0.0525	0.0691	1.1269	0.0475	0.0636	0.9266	0.0298	0.0352	0.1690
		MML	-0.1237	0.0493	0.0646	1.3137	0.0502	0.1486	0.9942	0.0363	0.0363	0.2485
3	1	ML	-0.1890	0.0536	0.0893	1.0663	0.0484	0.0528	0.9138	0.0346	0.0420	0.1842
		MML	-0.1660	0.0495	0.0771	1.1736	0.0499	0.0800	0.8508	0.0276	0.0498	0.2069
3	2	ML	-0.2339	0.0547	0.1094	1.0213	0.0499	0.0504	0.8867	0.0370	0.0499	0.2096
		MML	-0.2545	0.0515	0.1163	1.0201	0.0508	0.0512	0.9062	0.0427	0.0515	0.2191
3	3	ML	-0.2570	0.0558	0.1219	0.9979	0.0522	0.0522	0.8985	0.0422	0.0525	0.2266
		MML	-0.2451	0.0480	0.1081	0.9237	0.0522	0.0580	0.8263	0.0348	0.0650	0.2312
s_1	s_2	$n_1 = n_2 = 15$										
0	0	ML	0.0028	0.0300	0.0300	1.0028	0.0295	0.0295	0.9882	0.0182	0.0184	0.0778
		MML	-0.0090	0.0301	0.0302	1.0028	0.0304	0.0304	1.0279	0.0197	0.0205	0.0811
1	0	ML	-0.0502	0.0306	0.0331	1.0556	0.0307	0.0338	0.9870	0.0194	0.0196	0.0864
		MML	-0.0368	0.0300	0.0313	1.0729	0.0304	0.0357	0.9467	0.0173	0.0201	0.0871
1	1	ML	-0.0907	0.0311	0.0394	1.0028	0.0312	0.0312	0.9387	0.0186	0.0224	0.0929
		MML	-0.0765	0.0303	0.0362	0.9817	0.0311	0.0314	0.8988	0.0163	0.0265	0.0941
2	0	ML	-0.0850	0.0311	0.0384	1.0947	0.0313	0.0402	0.9719	0.0200	0.0208	0.0994
		MML	-0.0687	0.0301	0.0348	1.1431	0.0316	0.0521	0.9291	0.0174	0.0225	0.1094
2	1	ML	-0.1255	0.0317	0.0475	1.0429	0.0318	0.0336	0.9175	0.0189	0.0257	0.1068
		MML	-0.1059	0.0304	0.0416	1.0509	0.0323	0.0349	0.8770	0.0161	0.0312	0.1077
2	2	ML	-0.1607	0.0322	0.0581	1.0024	0.0324	0.0324	0.8915	0.0193	0.0310	0.1215
		MML	-0.1389	0.0306	0.0499	0.9657	0.0330	0.0342	0.8500	0.0162	0.0387	0.1228
3	0	ML	-0.1131	0.0318	0.0446	1.1227	0.0320	0.0470	0.9724	0.0212	0.0220	0.1136
		MML	-0.0952	0.0303	0.0393	1.2126	0.0330	0.0782	0.9237	0.0180	0.0238	0.1413
3	1	ML	-0.1545	0.0324	0.0563	1.0731	0.0325	0.0379	0.9129	0.0197	0.0273	0.1214
		MML	-0.1301	0.0306	0.0475	1.1186	0.0335	0.0475	0.8683	0.0165	0.0338	0.1288
3	2	ML	-0.1959	0.0307	0.0691	1.0456	0.0290	0.0311	0.8922	0.0208	0.0324	0.1327
		MML	-0.1605	0.0307	0.0565	1.0327	0.0341	0.0351	0.8389	0.0164	0.0424	0.1340
3	3	ML	-0.2266	0.0313	0.0827	1.0132	0.0301	0.0302	0.8800	0.0215	0.0359	0.1489
		MML	-0.1879	0.0308	0.0661	0.9491	0.0345	0.0371	0.8229	0.0169	0.0482	0.1515
s_1	s_2	$n_1 = n_2 = 20$										
0	0	ML	0.0067	0.0231	0.0231	0.9988	0.0210	0.0210	0.9914	0.0126	0.0127	0.0568
		MML	-0.0052	0.0232	0.0232	0.9989	0.0216	0.0216	1.0320	0.0137	0.0147	0.0595
1	0	ML	-0.0376	0.0234	0.0249	1.0414	0.0219	0.0236	0.9964	0.0136	0.0136	0.0621
		MML	-0.0247	0.0231	0.0238	1.0517	0.0216	0.0242	0.9558	0.0121	0.0141	0.0621
1	1	ML	-0.0697	0.0237	0.0286	0.9983	0.0222	0.0222	0.9545	0.0129	0.0149	0.0657
		MML	-0.0563	0.0232	0.0264	0.9810	0.0220	0.0223	0.9145	0.0113	0.0186	0.0673
2	0	ML	-0.0661	0.0238	0.0281	1.0746	0.0223	0.0278	0.9800	0.0141	0.0145	0.0705
		MML	-0.0508	0.0231	0.0256	1.1041	0.0223	0.0331	0.9385	0.0123	0.0161	0.0749
2	1	ML	-0.0980	0.0240	0.0336	1.0320	0.0226	0.0236	0.9341	0.0132	0.0176	0.0748
		MML	-0.0809	0.0232	0.0298	1.0329	0.0227	0.0238	0.8945	0.0113	0.0225	0.0760
2	2	ML	-0.1264	0.0243	0.0403	0.9980	0.0229	0.0229	0.9099	0.0132	0.0214	0.0845
		MML	-0.1081	0.0233	0.0350	0.9674	0.0231	0.0241	0.8701	0.0112	0.0281	0.0872
3	0	ML	-0.0912	0.0241	0.0324	1.1014	0.0226	0.0329	0.9739	0.0149	0.0155	0.0809
		MML	-0.0735	0.0231	0.0285	1.1565	0.0231	0.0476	0.9297	0.0127	0.0176	0.0937
3	1	ML	-0.1233	0.0244	0.0396	1.0599	0.0229	0.0265	0.9244	0.0138	0.0195	0.0856
		MML	-0.1023	0.0233	0.0337	1.0844	0.0234	0.0306	0.8834	0.0116	0.0252	0.0895
3	2	ML	-0.1517	0.0246	0.0476	1.0262	0.0232	0.0239	0.8975	0.0138	0.0243	0.0958
		MML	-0.1281	0.0233	0.0399	1.0186	0.0238	0.0242	0.8573	0.0114	0.0318	0.0958
3	3	ML	-0.1774	0.0249	0.0564	0.9976	0.0236	0.0236	0.8824	0.0141	0.0279	0.1078
		MML	-0.1520	0.0234	0.0465	0.9548	0.0241	0.0261	0.8410	0.0115	0.0367	0.1094

Table 6 Means, variances and MSEs for the ML and MML estimators of θ_0, θ_1 and σ; $(\lambda = 0.7)$; $(i)Mean(\hat{\theta}_0)$, $(ii)Var(\hat{\theta}_0)$, $(iii)MSE(\hat{\theta}_0)$,$(iv)Mean(\hat{\theta}_1)$, $(v)Var(\hat{\theta}_1)$, $(vi)MSE(\hat{\theta}_1)$, $(vii)Mean(\hat{\sigma})$, $(viii)Var(\hat{\sigma})$, $(ix)MSE(\hat{\sigma})$, $(x)Def$

			(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
s_1	s_2	$n_1 = n_2 = 10$										
0	0	ML	0.0289	0.0407	0.0416	0.9997	0.0319	0.0319	0.9871	0.0270	0.0272	0.1006
		MML	0.0598	0.0401	0.0437	1.0000	0.0373	0.0373	1.0770	0.0322	0.0381	0.1191
1	0	ML	-0.0095	0.0438	0.0439	1.0581	0.0337	0.0371	0.9423	0.0259	0.0292	0.1102
		MML	-0.0277	0.0444	0.0452	1.0614	0.0379	0.0417	1.0299	0.0334	0.0343	0.1212
1	1	ML	-0.0568	0.0441	0.0474	0.9436	0.0341	0.0373	0.9641	0.0330	0.0342	0.1189
		MML	-0.0358	0.0435	0.0448	0.9987	0.0383	0.0383	0.8813	0.0248	0.0389	0.1220
2	0	ML	-0.0481	0.0447	0.0470	1.0993	0.0389	0.0487	0.9324	0.0278	0.0324	0.1282
		MML	-0.0446	0.0447	0.0466	1.1185	0.0361	0.0501	1.0271	0.0370	0.0378	0.1345
2	1	ML	-0.0861	0.0509	0.0583	1.0031	0.0362	0.0363	0.9482	0.0358	0.0385	0.1362
		MML	-0.0708	0.0503	0.0554	1.0397	0.0394	0.0410	0.8651	0.0263	0.0445	0.1361
2	2	ML	-0.1236	0.0484	0.0637	0.9982	0.0407	0.0407	0.9215	0.0380	0.0442	0.1485
		MML	-0.0990	0.0473	0.0571	0.8991	0.0365	0.0467	0.8387	0.0272	0.0532	0.1569
3	0	ML	-0.0745	0.0438	0.0493	1.1081	0.0405	0.0522	0.9384	0.0303	0.0341	0.1357
		MML	-0.0929	0.0440	0.0527	1.1766	0.0396	0.0707	1.0583	0.0422	0.0455	0.1689
3	1	ML	-0.1180	0.0520	0.0659	1.0589	0.0394	0.0429	0.9681	0.0404	0.0414	0.1502
		MML	-0.1108	0.0516	0.0639	1.0551	0.0413	0.0443	0.8656	0.0286	0.0467	0.1548
3	2	ML	-0.1527	0.0476	0.0709	0.9539	0.0394	0.0416	0.9346	0.0426	0.0469	0.1595
		MML	-0.1268	0.0464	0.0625	1.0165	0.0426	0.0428	0.8366	0.0296	0.0563	0.1615
3	3	ML	-0.1812	0.0500	0.0829	0.9983	0.0444	0.0444	0.9440	0.0483	0.0515	0.1787
		MML	-0.1374	0.0482	0.0671	0.8564	0.0396	0.0603	0.8315	0.0321	0.0605	0.1878
s_1	s_2	$n_1 = n_2 = 15$										
0	0	ML	-0.0056	0.0251	0.0251	1.0044	0.0175	0.0175	1.009	0.0182	0.0182	0.0609
		MML	0.0407	0.0279	0.0295	1.0007	0.0217	0.0217	0.9887	0.0183	0.0184	0.0697
1	0	ML	0.0154	0.0287	0.0289	1.0377	0.0224	0.0238	0.9497	0.0168	0.0194	0.0721
		MML	-0.0010	0.0290	0.0290	1.0479	0.0255	0.0278	1.0402	0.0217	0.0234	0.0801
1	1	ML	-0.0502	0.0311	0.0336	1.0007	0.0258	0.0258	0.9900	0.0209	0.0210	0.0805
		MML	-0.0333	0.0307	0.0318	0.9581	0.0228	0.0246	0.9029	0.0158	0.0253	0.0817
2	0	ML	-0.0326	0.0309	0.0319	1.0775	0.0231	0.0291	0.9354	0.0170	0.0212	0.0823
		MML	-0.0405	0.0312	0.0328	1.0822	0.0259	0.0326	1.0259	0.0225	0.0232	0.0886
2	1	ML	-0.0747	0.0333	0.0389	1.0360	0.0262	0.0275	0.9688	0.0213	0.0222	0.0886
		MML	-0.0627	0.0330	0.0370	0.9977	0.0235	0.0235	0.8853	0.0158	0.0289	0.0894
2	2	ML	-0.0960	0.0322	0.0414	1.0003	0.0266	0.0266	0.9416	0.0217	0.0251	0.0932
		MML	-0.0805	0.0319	0.0384	0.9245	0.0237	0.0294	0.8601	0.0158	0.0354	0.1031
3	0	ML	-0.0644	0.0333	0.0375	1.1162	0.0243	0.0378	0.9323	0.0177	0.0223	0.0975
		MML	-0.0625	0.0333	0.0372	1.1058	0.0264	0.0376	1.0287	0.0239	0.0248	0.0996
3	1	ML	-0.0853	0.0324	0.0397	1.0617	0.0268	0.0306	0.9653	0.0224	0.0236	0.0938
		MML	-0.0799	0.0321	0.0385	1.0357	0.0246	0.0259	0.8793	0.0163	0.0308	0.0953
3	2	ML	-0.1262	0.0327	0.0486	1.0269	0.0272	0.0279	0.9335	0.0227	0.0271	0.1036
		MML	-0.1119	0.0323	0.0448	0.9624	0.0248	0.0262	0.8522	0.0162	0.0380	0.1090
3	3	ML	-0.1446	0.0356	0.0565	0.9994	0.0277	0.0277	0.9202	0.0238	0.0302	0.1145
		MML	-0.1241	0.0351	0.0505	0.8925	0.0247	0.0363	0.8386	0.0166	0.0426	0.1294
s_1	s_2	$n_1 = n_2 = 20$										
0	0	ML	0.0128	0.0224	0.0225	1.0083	0.0197	0.0198	0.9666	0.0127	0.0138	0.0561
		MML	0.0347	0.0222	0.0234	1.0087	0.0199	0.0199	0.9654	0.0126	0.0138	0.0571
1	0	ML	-0.0103	0.0230	0.0231	0.9699	0.0203	0.0213	0.9213	0.0131	0.0193	0.0637
		MML	0.0051	0.0228	0.0228	0.9361	0.0206	0.0247	0.9238	0.0132	0.0190	0.0665
1	1	ML	-0.0317	0.0224	0.0234	1.0020	0.0177	0.0177	1.0045	0.0145	0.0146	0.0556
		MML	-0.0157	0.0221	0.0223	1.0018	0.0178	0.0178	0.9152	0.0111	0.0183	0.0584
2	0	ML	-0.0106	0.0217	0.0218	1.0564	0.0178	0.0209	0.9405	0.0118	0.0153	0.0581
		MML	-0.0203	0.0219	0.0223	1.0683	0.0201	0.0248	1.0314	0.0156	0.0166	0.0637
2	1	ML	-0.0477	0.0237	0.0260	0.9769	0.0209	0.0215	0.8690	0.0125	0.0297	0.0807
		MML	-0.0363	0.0234	0.0247	0.9410	0.0212	0.0247	0.8575	0.0122	0.0325	0.0783
2	2	ML	-0.0669	0.0230	0.0275	1.0002	0.0205	0.0205	0.9586	0.0150	0.0168	0.0647
		MML	-0.0549	0.0228	0.0258	0.9380	0.0181	0.0220	0.8746	0.0110	0.0267	0.0745
3	0	ML	-0.0395	0.0241	0.0256	1.0855	0.0184	0.0257	0.9341	0.0121	0.0165	0.0678
		MML	-0.0420	0.0242	0.0260	1.0914	0.0204	0.0288	1.0265	0.0164	0.0171	0.0718
3	1	ML	-0.0682	0.0243	0.0289	0.9518	0.0194	0.0217	0.9746	0.0156	0.0162	0.0669
		MML	-0.0619	0.0241	0.0280	0.8807	0.0197	0.0340	0.8896	0.0114	0.0236	0.0855
3	2	ML	-0.0899	0.0242	0.0323	0.9779	0.0224	0.0229	0.9463	0.0155	0.0184	0.0736
		MML	-0.0795	0.0241	0.0304	0.9412	0.0225	0.0259	0.8649	0.0112	0.0294	0.0858
3	3	ML	-0.1155	0.0222	0.0356	0.9998	0.0210	0.0210	0.9305	0.0159	0.0207	0.0773
		MML	-0.1021	0.0220	0.0324	0.9124	0.0188	0.0264	0.8500	0.0112	0.0337	0.0926

Table 7 Means, variances and MSEs for the ML and MML estimators of θ_0, θ_1 and σ; $(\lambda = 1)$; $(i)Mean(\hat{\theta}_0)$, $(ii)Var(\hat{\theta}_0)$, $(iii)MSE(\hat{\theta}_0)$,$(iv)Mean(\hat{\theta}_1)$, $(v)Var(\hat{\theta}_1)$, $(vi)MSE(\hat{\theta}_1)$, $(vii)Mean(\hat{\sigma})$, $(viii)Var(\hat{\sigma})$, $(ix)MSE(\hat{\sigma})$, $(x)Def$

			(i)	(ii)	(iii)	(iv)	(v)	(vi)	(vii)	(viii)	(ix)	(x)
s_1	s_2	$n_1 = n_2 = 10$										
0	0	ML	0.0514	0.0201	0.0227	1.0017	0.0157	0.0157	0.9751	0.0122	0.0128	0.0512
		MML	0.0544	0.0200	0.0230	1.0015	0.0161	0.0161	0.8803	0.0100	0.0243	0.0634
1	0	ML	-0.0019	0.0436	0.0436	0.9696	0.0170	0.0179	0.9696	0.0170	0.0179	0.0794
		MML	-0.0191	0.0440	0.0444	0.9381	0.0171	0.0210	0.9455	0.0244	0.0274	0.0927
1	1	ML	0.0015	0.0378	0.0378	0.9947	0.0338	0.0339	1.0052	0.0348	0.0349	0.1065
		MML	0.0049	0.0381	0.0381	0.9941	0.0343	0.0343	0.8849	0.0230	0.0363	0.1087
2	0	ML	0.0054	0.0412	0.0412	0.9205	0.0368	0.0431	1.0756	0.0401	0.0458	0.1301
		MML	-0.0122	0.0417	0.0418	0.7872	0.0386	0.0838	1.0837	0.0408	0.0478	0.1734
2	1	ML	-0.0261	0.0442	0.0449	0.9633	0.0348	0.0361	0.9898	0.0381	0.0382	0.1192
		MML	-0.0279	0.0443	0.0451	0.8950	0.0357	0.0467	0.8728	0.0246	0.0408	0.1326
2	2	ML	-0.1391	0.0425	0.0619	0.9946	0.0352	0.0353	0.9605	0.0395	0.0411	0.1382
		MML	-0.1384	0.0426	0.0617	0.8597	0.0251	0.0448	0.8480	0.0249	0.0480	0.1545
3	0	ML	0.0149	0.0188	0.0190	1.0697	0.0275	0.0324	0.8766	0.0285	0.0437	0.0951
		MML	0.0220	0.0186	0.0191	1.0864	0.0351	0.0425	0.7882	0.0238	0.0686	0.1303
3	1	ML	0.0311	0.0200	0.0210	0.9734	0.0279	0.0286	0.8738	0.0270	0.0430	0.0926
		MML	0.0385	0.0201	0.0216	1.0407	0.0357	0.0374	1.0137	0.0434	0.0436	0.1026
3	2	ML	-0.0779	0.0419	0.0480	0.9898	0.0387	0.0388	0.9748	0.0447	0.0453	0.1321
		MML	-0.0669	0.0414	0.0459	0.9268	0.0379	0.0433	0.8463	0.0275	0.0511	0.1402
3	3	ML	-0.0774	0.0407	0.0467	0.9898	0.0362	0.0363	0.8435	0.0295	0.0540	0.1371
		MML	-0.1056	0.0421	0.0532	0.9876	0.0380	0.0382	0.9816	0.0499	0.0502	0.1416
s_1	s_2	$n_1 = n_2 = 15$										
0	0	ML	0.0393	0.0249	0.0264	1.0011	0.0211	0.0211	0.8743	0.0138	0.0296	0.0788
		MML	-0.0302	0.0265	0.0275	1.0009	0.0219	0.0219	1.0760	0.0254	0.0312	0.0819
1	0	ML	0.0626	0.0242	0.0281	0.9629	0.0230	0.0244	0.9994	0.0212	0.0212	0.0720
		MML	0.0683	0.0240	0.0287	0.9199	0.0239	0.0304	0.9586	0.0188	0.0205	0.0774
1	1	ML	-0.0245	0.0258	0.0264	1.0017	0.0458	0.0458	1.0320	0.0225	0.0235	0.1015
		MML	-0.0361	0.0252	0.0265	0.9750	0.0467	0.0473	0.9031	0.0150	0.0244	0.1101
2	0	ML	-0.0795	0.0259	0.0322	1.1078	0.0264	0.0380	0.9837	0.0217	0.0219	0.1022
		MML	-0.0982	0.0288	0.0384	0.9750	0.0467	0.0473	0.9402	0.0188	0.0224	0.1135
2	1	ML	-0.1139	0.0293	0.0422	1.0548	0.0274	0.0304	1.0108	0.0229	0.0230	0.1009
		MML	-0.1202	0.0293	0.0438	1.0621	0.0277	0.0315	0.8887	0.0149	0.0273	0.1067
2	2	ML	-0.1329	0.0298	0.0475	0.9989	0.0230	0.0230	0.9816	0.0233	0.0237	0.1003
		MML	-0.1344	0.0298	0.0479	0.8968	0.0162	0.0269	0.8641	0.0149	0.0334	0.1201
3	0	ML	-0.1523	0.0304	0.0536	1.0900	0.0229	0.0310	0.9841	0.0225	0.0228	0.0824
		MML	-0.1703	0.0309	0.0599	1.1365	0.0271	0.0457	0.9351	0.0190	0.0232	0.1061
3	1	ML	-0.0554	0.0256	0.0286	1.0858	0.0280	0.0354	1.0085	0.0241	0.0241	0.1528
		MML	-0.1054	0.0261	0.0372	1.1312	0.0284	0.0456	0.8854	0.0154	0.0286	0.1555
3	2	ML	-0.2076	0.0502	0.0933	1.0202	0.0508	0.0512	0.9737	0.0243	0.0250	0.1695
		MML	-0.1849	0.0470	0.0812	1.0503	0.0506	0.0532	0.8592	0.0154	0.0352	0.1696
3	3	ML	-0.2801	0.0533	0.1317	0.9944	0.0529	0.0529	0.8800	0.0215	0.0359	0.0888
		MML	-0.2451	0.0481	0.1082	0.9237	0.0522	0.0580	0.8462	0.0157	0.0394	0.1223
s_1	s_2	$n_1 = n_2 = 20$										
0	0	ML	-0.0029	0.0249	0.0249	0.9888	0.0222	0.0223	0.9896	0.0128	0.0130	0.0601
		MML	0.0088	0.0248	0.0248	0.9891	0.0229	0.0230	1.0177	0.0130	0.0133	0.0612
1	0	ML	-0.0343	0.0253	0.0265	1.0303	0.0230	0.0239	0.9831	0.0129	0.0132	0.0637
		MML	-0.0215	0.0249	0.0254	1.0414	0.0227	0.0244	1.0833	0.0154	0.0224	0.0722
1	1	ML	-0.0668	0.0255	0.0300	0.9853	0.0234	0.0236	0.9357	0.0118	0.0160	0.0696
		MML	-0.0534	0.0250	0.0278	0.9696	0.0231	0.0241	0.9168	0.0108	0.0177	0.0696
2	0	ML	-0.0613	0.0258	0.0295	1.0612	0.0233	0.0271	0.9696	0.0136	0.0145	0.0711
		MML	-0.0465	0.0250	0.0271	1.0926	0.0236	0.0322	0.9435	0.0116	0.0148	0.0741
2	1	ML	-0.1034	0.0206	0.0313	1.0236	0.0201	0.0207	1.0144	0.0151	0.0153	0.0675
		MML	-0.0768	0.0250	0.0309	1.0202	0.0241	0.0245	0.9018	0.0107	0.0204	0.0758
2	2	ML	-0.1177	0.0210	0.0348	0.9961	0.0203	0.0203	0.9834	0.0147	0.0150	0.0703
		MML	-0.0529	0.0189	0.0217	0.9156	0.0145	0.0217	0.8626	0.0097	0.0286	0.0721
3	0	ML	-0.1070	0.0217	0.0332	1.0855	0.0235	0.0308	0.9661	0.0142	0.0153	0.0793
		MML	-0.0679	0.0251	0.0297	1.1435	0.0242	0.0449	0.9389	0.0119	0.0157	0.0902
3	1	ML	-0.1172	0.0265	0.0402	1.0423	0.0239	0.0257	1.0200	0.0172	0.0176	0.0835
		MML	-0.1177	0.0220	0.0358	1.0700	0.0246	0.0295	0.8955	0.0110	0.0219	0.0873
3	2	ML	-0.1462	0.0265	0.0479	1.0072	0.0240	0.0241	0.8789	0.0128	0.0274	0.0994
		MML	-0.1229	0.0251	0.0402	1.0031	0.0249	0.0249	0.8395	0.0107	0.0364	0.1016
3	3	ML	-0.1720	0.0267	0.0563	0.9780	0.0242	0.0246	0.8616	0.0129	0.0320	0.1129
		MML	-0.1468	0.0252	0.0467	0.8831	0.0125	0.0262	0.8214	0.0107	0.0426	0.1155

Table 8 Average Lengths and Coverage Probabilities of the $\%95$ confidence intervals for the ML and MML estimators of θ_0 and θ_1; $(\lambda = 0)$.

	s_1	s_2		Average Lengths $\hat{\theta}_0$	Average Lengths $\hat{\theta}_1$	Coverage Probabilities % $\hat{\theta}_0$	Coverage Probabilities % $\hat{\theta}_1$
$n_1 = n_2 = 10$							
	0	0	ML	0.8133	0.8724	92.96	96.98
			MML	0.8133	0.8724	92.96	96.98
	1	0	ML	0.8158	0.8785	92.96	95.98
			MML	0.8143	0.8836	92.96	95.48
	1	1	ML	0.8255	0.8893	92.96	95.48
			MML	0.8242	0.9112	92.96	95.98
	2	0	ML	0.8896	0.8900	93.79	95.30
			MML	0.8910	0.9047	93.69	95.10
	2	1	ML	0.8371	0.9016	92.96	95.48
			MML	0.8372	0.9119	92.46	94.97
	2	2	ML	0.8393	0.9195	92.96	95.98
			MML	0.8438	0.9529	93.46	94.97
	3	0	ML	0.9127	0.9145	94.39	94.89
			MML	0.9207	0.9241	94.39	94.79
	3	1	ML	0.9169	0.9248	94.59	95.10
			MML	0.9246	0.9239	94.79	94.69
	3	2	ML	0.9196	0.9416	94.59	94.99
			MML	0.9338	0.9428	94.99	94.69
	3	3	ML	0.9281	0.9682	94.29	95.30
			MML	0.9544	0.9683	94.09	95.40
$n_1 = n_2 = 15$							
	0	0	ML	0.6810	0.7084	94.80	95.05
			MML	0.6810	0.7084	94.80	95.05
	1	0	ML	0.6859	0.7170	94.75	95.20
			MML	0.6852	0.7245	94.70	95.30
	1	1	ML	0.6954	0.7196	94.65	95.20
			MML	0.6939	0.7328	94.65	95.35
	2	0	ML	0.6902	0.7232	94.90	95.40
			MML	0.6897	0.7388	94.90	95.40
	2	1	ML	0.6994	0.7260	94.65	95.50
			MML	0.6982	0.7446	94.75	95.30
	2	2	ML	0.7097	0.7313	94.55	95.25
			MML	0.7086	0.7543	94.55	95.55
	3	0	ML	0.6966	0.7317	94.75	95.40
			MML	0.6972	0.7520	94.70	95.30
	3	1	ML	0.7052	0.7349	94.95	95.40
			MML	0.7051	0.7552	94.95	95.30
	3	2	ML	0.7148	0.7405	94.65	95.60
			MML	0.7148	0.7627	94.45	95.50
	3	3	ML	0.7235	0.7485	94.40	95.45
			MML	0.7250	0.7683	94.30	95.35
$n_1 = n_2 = 20$							
	0	0	ML	0.6235	0.5965	94.65	94.90
			MML	0.6235	0.5965	94.65	94.90
	1	0	ML	0.6229	0.5994	94.60	94.75
			MML	0.6222	0.6032	94.60	94.75
	1	1	ML	0.6236	0.6032	94.95	94.75
			MML	0.6223	0.6116	94.95	94.75
	2	0	ML	0.6249	0.6034	94.75	94.65
			MML	0.6238	0.6099	94.65	94.50
	2	1	ML	0.6249	0.6075	94.70	94.70
			MML	0.6234	0.6177	94.80	94.60
	2	2	ML	0.6254	0.6128	94.70	94.75
			MML	0.6240	0.6271	94.65	94.75
	3	0	ML	0.6292	0.6099	94.55	94.80
			MML	0.6280	0.6183	94.60	94.75
	3	1	ML	0.6285	0.6141	94.45	94.70
			MML	0.6270	0.6254	94.55	94.40
	3	2	ML	0.6284	0.6196	94.45	94.85
			MML	0.6270	0.6344	94.60	94.50
	3	3	ML	0.6311	0.6261	94.50	94.90
			MML	0.6301	0.6439	94.55	94.90

Table 9 Average Lengths and Coverage Probabilities of the $\%95$ confidence intervals for the ML and MML estimators of θ_0 and θ_1; $(\lambda = 0.4)$.

	s_1	s_2		Average Lengths		Coverage Probabilities %	
				$\hat{\theta}_0$	$\hat{\theta}_1$	$\hat{\theta}_0$	$\hat{\theta}_1$
$n_1 = n_2 = 10$							
	0	0	ML	0.8372	0.8322	94.05	96.20
			MML	0.8357	0.8194	94.15	96.25
	1	0	ML	0.8458	0.8382	94.10	96.45
			MML	0.8350	0.8391	94.20	96.15
	1	1	ML	0.8570	0.8427	94.10	96.00
			MML	0.8414	0.8474	94.25	96.10
	2	0	ML	0.8575	0.8499	93.95	96.10
			MML	0.8390	0.8612	94.20	96.20
	2	1	ML	0.8685	0.8559	94.10	95.70
			MML	0.8437	0.8655	94.15	95.80
	2	2	ML	0.8784	0.8669	94.30	95.55
			MML	0.8472	0.8751	94.15	95.70
	3	0	ML	0.8705	0.8713	94.00	95.70
			MML	0.8464	0.8908	93.90	95.80
	3	1	ML	0.8808	0.8796	93.90	95.55
			MML	0.8485	0.8902	94.00	95.75
	3	2	ML	0.8888	0.8912	94.00	95.30
			MML	0.8491	0.8964	94.10	95.60
	3	3	ML	0.8988	0.9096	93.90	95.75
			MML	0.8505	0.9082	94.25	95.80
$n_1 = n_2 = 15$							
	0	0	ML	0.6761	0.6674	95.50	94.85
			MML	0.6749	0.6573	95.50	94.75
	1	0	ML	0.6816	0.6706	95.50	94.50
			MML	0.6752	0.6683	95.55	94.55
	1	1	ML	0.6880	0.6777	95.40	94.85
			MML	0.6791	0.6780	95.40	94.75
	2	0	ML	0.6873	0.6757	95.45	94.40
			MML	0.6763	0.6796	95.45	94.40
	2	1	ML	0.6937	0.6826	95.50	94.70
			MML	0.6798	0.6883	95.50	94.4
	2	2	ML	0.6992	0.6898	95.40	94.60
			MML	0.6820	0.6969	95.65	94.55
	3	0	ML	0.6944	0.6829	95.40	94.40
			MML	0.6784	0.6926	95.55	94.25
	3	1	ML	0.7008	0.6896	95.45	94.60
			MML	0.6816	0.6995	95.65	94.65
	3	2	ML	0.7061	0.6967	95.45	94.60
			MML	0.6832	0.7067	95.65	94.65
	3	3	ML	0.7109	0.7052	95.55	94.45
			MML	0.6837	0.7116	95.55	94.70
$n_1 = n_2 = 20$							
	0	0	ML	0.5551	0.5947	92.96	95.47
			MML	0.5543	0.5847	92.96	95.97
	1	0	ML	0.5580	0.5988	91.45	94.97
			MML	0.5532	0.5945	91.45	95.47
	1	1	ML	0.5598	0.5993	91.45	95.97
			MML	0.5537	0.5966	91.45	96.48
	2	0	ML	0.5627	0.6020	91.95	95.47
			MML	0.5534	0.6054	91.95	94.97
	2	1	ML	0.5644	0.6027	91.95	94.97
			MML	0.5541	0.6074	91.95	94.47
	2	2	ML	0.5640	0.6020	91.95	94.97
			MML	0.5529	0.6084	90.95	95.47
	3	0	ML	0.5706	0.6050	91.95	95.48
			MML	0.5565	0.6134	91.46	95.48
	3	1	ML	0.5720	0.6058	91.96	94.97
			MML	0.5570	0.6143	91.96	94.47
	3	2	ML	0.5713	0.6052	91.45	95.47
			MML	0.5556	0.6145	91.45	94.97
	3	3	ML	0.5709	0.6075	91.95	95.98
			MML	0.5546	0.6154	91.45	95.47

Table 10 Average Lengths and Coverage Probabilities of the $\%95$ confidence intervals for the ML and MML estimators of θ_0 and θ_1; $(\lambda = 0.7)$.

	s_1	s_2		Average Lengths $\hat{\theta}_0$	Average Lengths $\hat{\theta}_1$	Coverage Probabilities % $\hat{\theta}_0$	Coverage Probabilities % $\hat{\theta}_1$
$n_1 = n_2 = 10$							
	0	0	ML	0.8215	0.7761	94.45	95.20
			MML	0.8139	0.7177	94.55	95.00
	1	0	ML	0.8322	0.7809	94.15	95.30
			MML	0.8081	0.7355	94.60	95.30
	1	1	ML	0.8424	0.7836	94.45	95.05
			MML	0.8137	0.7395	94.60	95.00
	2	0	ML	0.8437	0.7905	94.40	95.25
			MML	0.8039	0.7588	94.75	95.05
	2	1	ML	0.8540	0.7948	94.80	95.15
			MML	0.8084	0.7604	94.60	94.75
	2	2	ML	0.8656	0.8054	94.80	95.00
			MML	0.8127	0.7626	94.85	94.65
	3	0	ML	0.8544	0.8080	94.95	95.40
			MML	0.8003	0.7915	94.55	95.35
	3	1	ML	0.8644	0.8144	95.00	95.25
			MML	0.8022	0.7898	94.70	94.80
	3	2	ML	0.8747	0.8254	94.85	95.00
			MML	0.8026	0.7902	94.75	94.90
	3	3	ML	0.8826	0.8406	94.75	95.05
			MML	0.7984	0.7927	94.75	95.05
$n_1 = n_2 = 15$							
	0	0	ML	0.6658	0.6119	94.40	94.35
			MML	0.6594	0.5653	94.45	94.60
	1	0	ML	0.6713	0.6152	94.60	94.85
			MML	0.6567	0.5754	94.60	94.70
	1	1	ML	0.6770	0.6189	94.80	94.90
			MML	0.6598	0.5810	94.70	94.80
	2	0	ML	0.6775	0.6192	94.70	94.95
			MML	0.6544	0.5870	94.60	94.95
	2	1	ML	0.6833	0.6231	94.75	94.85
			MML	0.6575	0.5917	94.70	94.85
	2	2	ML	0.6894	0.6276	94.65	95.10
			MML	0.6601	0.5943	94.75	94.95
	3	0	ML	0.6846	0.6250	94.70	94.90
			MML	0.6526	0.6000	94.75	94.75
	3	1	ML	0.6903	0.6290	94.65	95.05
			MML	0.6555	0.6039	94.80	94.75
	3	2	ML	0.6963	0.6335	94.70	94.80
			MML	0.6577	0.6058	94.90	94.60
	3	3	ML	0.7020	0.6389	94.65	94.75
			MML	0.6582	0.6047	95.15	94.60
$n_1 = n_2 = 20$							
	0	0	ML	0.5848	0.5548	94.95	95.30
			MML	0.5793	0.5127	95.00	95.25
	1	0	ML	0.5881	0.5573	94.95	95.40
			MML	0.5768	0.5196	94.95	95.30
	1	1	ML	0.5927	0.5595	95.10	95.50
			MML	0.5798	0.5233	95.05	95.15
	2	0	ML	0.5928	0.5601	95.20	95.45
			MML	0.5751	0.5276	95.10	95.40
	2	1	ML	0.5973	0.5625	95.20	95.70
			MML	0.5785	0.5312	95.25	95.50
	2	2	ML	0.6011	0.5647	95.05	95.65
			MML	0.5806	0.5330	95.15	95.70
	3	0	ML	0.5980	0.5625	95.20	95.45
			MML	0.5730	0.5371	95.15	95.55
	3	1	ML	0.6025	0.5650	95.25	95.45
			MML	0.5767	0.5405	95.20	95.65
	3	2	ML	0.6062	0.5673	95.20	95.40
			MML	0.5787	0.5420	95.15	95.55
	3	3	ML	0.6094	0.5700	95.20	95.40
			MML	0.5796	0.5420	95.20	95.55

Table 11 Average Lengths and Coverage Probabilities of the $\%95$ confidence intervals for the ML and MML estimators of θ_0 and θ_1; $(\lambda = 1)$.

	s_1	s_2		Average Lengths $\hat{\theta}_0$	Average Lengths $\hat{\theta}_1$	Coverage Probabilities % $\hat{\theta}_0$	Coverage Probabilities % $\hat{\theta}_1$
$n_1 = n_2 = 10$							
	0	0	ML	0.7556	0.7202	94.65	95.15
			MML	0.7423	0.5892	94.60	95.05
	1	0	ML	0.7658	0.7248	94.75	94.80
			MML	0.7327	0.6034	94.55	95.05
	1	1	ML	0.7750	0.7263	94.75	94.90
			MML	0.7378	0.6105	94.90	95.00
	2	0	ML	0.7763	0.7319	94.55	95.10
			MML	0.7214	0.6200	94.85	95.10
	2	1	ML	0.7851	0.7350	94.55	95.30
			MML	0.7278	0.6256	94.90	95.40
	2	2	ML	0.7959	0.7427	94.85	95.25
			MML	0.7343	0.6269	94.85	95.40
	3	0	ML	0.7853	0.7447	94.55	95.10
			MML	0.7086	0.6544	94.35	94.90
	3	1	ML	0.7941	0.7501	94.85	95.05
			MML	0.7144	0.6576	94.75	95.05
	3	2	ML	0.8044	0.7590	94.55	95.10
			MML	0.7201	0.6584	94.85	94.95
	3	3	ML	0.8111	0.7718	95.05	95.35
			MML	0.7221	0.6569	94.70	95.20
$n_1 = n_2 = 15$							
	0	0	ML	0.6545	0.5780	94.90	95.15
			MML	0.6402	0.4734	94.90	95.00
	1	0	ML	0.6608	0.5792	95.05	95.10
			MML	0.6367	0.4807	94.75	94.70
	1	1	ML	0.6671	0.5820	95.15	95.25
			MML	0.6401	0.4870	95.10	94.90
	2	0	ML	0.6672	0.5813	95.15	94.95
			MML	0.6321	0.4869	94.75	94.85
	2	1	ML	0.6738	0.5843	95.35	95.25
			MML	0.6365	0.4929	95.05	94.95
	2	2	ML	0.6803	0.5880	95.25	94.85
			MML	0.6404	0.4958	95.10	95.30
	3	0	ML	0.6739	0.5865	95.15	94.75
			MML	0.6265	0.4895	94.65	94.70
	3	1	ML	0.6802	0.5898	95.25	95.00
			MML	0.6314	0.4951	94.95	95.25
	3	2	ML	0.6867	0.5935	95.15	94.95
			MML	0.6356	0.4972	94.90	95.35
	3	3	ML	0.6926	0.5982	95.10	95.00
			MML	0.6381	0.4948	95.10	95.30
$n_1 = n_2 = 20$							
	0	0	ML	0.5578	0.5192	95.10	95.45
			MML	0.5462	0.4228	95.05	95.25
	1	0	ML	0.5607	0.5206	95.05	95.35
			MML	0.5431	0.4282	95.00	95.35
	1	1	ML	0.5651	0.5228	95.25	95.35
			MML	0.5455	0.4331	95.05	95.05
	2	0	ML	0.5647	0.5217	94.90	95.35
			MML	0.5403	0.4341	95.15	95.10
	2	1	ML	0.5688	0.5241	95.25	95.30
			MML	0.5430	0.4390	95.30	95.20
	2	2	ML	0.5728	0.5268	95.10	95.30
			MML	0.5450	0.4422	95.15	95.25
	3	0	ML	0.5688	0.5226	95.05	95.20
			MML	0.5369	0.4394	95.30	95.15
	3	1	ML	0.5731	0.5252	95.20	95.25
			MML	0.5399	0.4441	95.20	95.25
	3	2	ML	0.5769	0.5280	95.25	95.10
			MML	0.5420	0.4470	95.00	95.55
	3	3	ML	0.5803	0.5301	95.15	94.90
			MML	0.5439	0.4469	95.00	95.40

The ML estimators of θ_1 and σ outperform the corresponding MML estimators with respect to the MSE criterion in all cases. Overall, it is observed that efficiencies of the ML and MML estimators of θ_1 and σ are very similar to each other when the sample size is large and the proportion of censoring is small.

Finally, we can conclude that the ML estimators show stronger performance than the corresponding MML estimators with low deficiencies in all cases.

We also conducted a simulation study to compare performance of CIs based on the ML and MML estimators. We presented average lengths and coverage probabilities of CIs based on the ML and MML estimators given in (19) and (20), see Tables 8-11.

It is obvious from simulation results that the ML estimates produce a little bit wider CIs than the MML estimators on the average. However, coverage probabilities of the CIs based on the ML and MML estimators are not significantly different from each other.

6. Conclusion

In this study, we obtain the ML and MML estimates of the regression model parameters under Type-II censoring when the distribution of the error terms is $SN(\lambda)$. We then compare the performances of the ML estimators obtained using IRA with the MML estimators via Monte Carlo simulation study. Simulation study shows that ML estimators are more efficient than the MML estimators for most of the cases. Additionally, we construct CIs based on the proposed ML and MML estimators and compare their converge probabilities and average lengths via Monte Carlo simulation study. Average lengths of CIs based on the ML estimators are wider than the average lengths of the corresponding CIs based on the MML estimators. Coverage probabilities of the ML and MML estimators are almost same for all cases. It can also be concluded that the MML estimators are the explicit functions of the sample observations and therefore they are easy to compute. If our concern is to have efficient estimators together with the smaller CPU time needed for the computation process, the MML estimators can also be preferred.

References

Azais JM, Bardet JM. Le modle linaire par l'exemple: rgression, analyse de la variance et plans d'expriences illustres avec R, SAS et Splus. Dunod; 2005.

Bangdiwala SI. Regression: simple linear. Int J Inj Control Sa. 2018; 25(1):113-115.

Djaballah-Djeddour K, Tazerouti M. Test for Linearity in Non-Parametric Regression Models. Aust J Stat. 2015; 51(1):16-34.

Diaz-Garcia JA, Galea Rojas M, Leiva-Sanchez V. Influence diagnostics for elliptical multivariate linear regression models. Commun Stat Theory. 2003; 32(3):625-641.

Ferreira JT, Steel FJ. A new class of skewed multivariate distributions with applications to regression analysis. Stat Sinica. 2007; 505-529.

Galea M, Paula GA, Bolfarine H. Local influence in elliptical linear regression models. The Statistician. 1997; 46(1):71-79.

Gaso DV, Berger AG, Ciganda VS. Predicting wheat grain yield and spatial variability at field scale using a simple regression or a crop model in conjunction with Landsat images. Comput Electron Agr. 2019; 159(1): 75-83

Huber PJ. Robust Statistics. Wiley, New York. 1981.

Qamarul Islam M, Tiku ML. Multiple linear regression model under nonnormality. Commun Stat Theory. 2004; 33(10):2443-2467.

Sazak HS. Regression analysis with a stochastic design variable. Int Stat Rev. 2006; 74(1): 77-88.

Sutradhar BC, Mir MA. Estimation of the parameters of a regression model with a multivariate t error variable. Commun Stat Theory. 1986; 15(2):429-450.

Tiku ML, Tan WY, Balakrishnan N. Robust Inference. New York Marcel Dekker. 1986.

Tiku ML, Qamarul Islam M, Sazak HS. Estimation in bivariate nonnormal distributions with stochastic variance functions. Comput Stat Data An. 2008; 52(3): 1728-1745.

Tiku ML, Islam MQ, Selcuk AS. Non-normal regression.II.Symmetric distributions. Commun Stat Theory. 2001; 30(1):1021-1045.

Zellner A. Bayesian and non-Bayesian analysis of the regression model with multivariate Student-t error terms. J Am Stat Assoc. 1976; 71(354):400-405.

Lim KS. Canadian lynx data. Encyclopedia of Mathematics. Available from: http://encyclopediaofmath.org/index.php?title=Canadian_lynx_data&oldid=46191.