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## On the New Mixture Sushila and Rayleigh Distributions: Mathematical Properties and Application

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### Abstract

In this study, a newly proposed mixture of Sushila and Rayleigh distributions is presented, introducing a right-skewed pattern in the distribution of lifetime data. The fundamental properties of the distribution have been derived. Parameter estimation for the new mixture of Rayleigh and Sushila distributions is conducted using the maximum likelihood method. The application of this new mixture in modeling remission times for bladder cancer is explored. Results indicate that the Sushila and Rayleigh distributions mixture provides a better fit for bladder cancer remission times compared to other existing distributions.

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**Keywords:** Lifetime distribution, right skewed distribution, maximum likelihood estimation, mixture distribution.

### 1. Introduction

Modeling and analysing lifetime data is extremely important in many applied sciences (Shanker et al. 2016), including medicine, engineering, and others. Building a new probability model from an existing one can be accomplished using a variety of techniques. One of these techniques is described in the literature as a mixture of distributions (Marin et al. 2005). Sometimes it is possible to assume that the observed data came from a combination of two or more populations. In these circumstances, mixture distributions are extremely adaptable and effective for analysing random phenomena. If the variable is random  $X$ , it has a mixture distribution with  $n$  components, and the probability density function (pdf) and cumulative distribution function (cdf) can be represented as, respectively:

$$f(x) = \alpha_1 f_1(x) + \alpha_2 f_2(x) + \dots + \alpha_n f_n(x); \quad \alpha_i > 0$$
$$F(x) = \alpha_1 F_1(x) + \alpha_2 F_2(x) + \dots + \alpha_n F_n(x),$$

where  $0 \leq \alpha_i \leq 1$  and  $\sum_{i=1}^n \alpha_i = 1, i = 1, 2, \dots, n$ .

The Rayleigh distribution is one of the most widely known probability distributions. The Rayleigh distribution, a special case of the Weibull distribution, was first introduced by Rayleigh (1880). In areas like survival and reliability analysis, physical sciences, and clinical research, it is crucial for modeling and analyzing life-time data. Numerous researchers have developed extensions of

the Rayleigh distribution, such as the Rayleigh-Rayleigh distribution (Al-Noor and Assi 2020), the Maxwell-Rayleigh distribution (Mohammed and Al-Aidi 2021), and the Topp-Leone G-Rayleigh Rayleigh distribution (Al Sabah et al. 2022). Shanker et al. (2013) introduced a new lifetime distribution called the Sushila distribution. Being a modified Lindley distribution, this distribution combines the exponential and gamma distributions. The Sushila distribution was discussed because it reveals greater flexibility than the Lindley and exponential distributions in terms of failure rate function and mean residual life function. Additionally, the moment generating function of the Sushila distribution can be expressed in closed form.

The purpose of this research is to present a new mixture of Sushila and Rayleigh distributions (MSRD), which is a combination of two distributions that are frequently utilised. The new Mixture Sushila Rayleigh Distribution combines elements from both the Sushila distribution and the Rayleigh distribution. The Sushila distribution is characterized by its ability to model data with positive skewness and the Rayleigh distribution is known for its ability to model data with positive kurtosis, the Mixture Sushila Rayleigh Distribution offers a more flexible approach by incorporating both skewness and kurtosis in its modeling capabilities. This unique combination allows researchers to better capture the complexities of real-world data that may exhibit characteristics of both distributions simultaneously. Additionally, along with an exploration of its structural statistical properties and the maximum likelihood estimation of its parameters in Section 3. Section 4 focuses on a numerical illustration through a simulation study. The application of the new mixture, the Sushila and Rayleigh distributions, to a lifetime dataset is presented in Section 5.

## 2. Sushila and Rayleigh Distribution

Sushila distribution was proposed by Shanker et al. (2013). This is a continuous distribution, a combination of exponential and gamma distributions. Sushila distribution has two parameters. The corresponding cumulative distribution function (CDF) is given by

$$F_S(x) = 1 - \frac{\theta(\lambda + 1) + \lambda x}{\theta(\lambda + 1)} \exp\left(-\frac{\lambda}{\theta}x\right). \quad (1)$$

This formula represents the probability density function (PDF):

$$f_S(x) = \frac{\lambda^2}{\theta(\lambda + 1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right); \quad x, \theta, \lambda > 0. \quad (2)$$

The Rayleigh distribution used in communication theory, is also a special case of the Weibull distribution with one scale parameter. The corresponding cumulative distribution function (CDF) is given by

$$F_R(x) = 1 - \exp\left(-\frac{x^2}{2\beta^2}\right). \quad (3)$$

This formula represents the probability density function (PDF):

$$f_R(x) = \frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right); \quad x > 0, \beta > 0. \quad (4)$$

## 3. Theoretical Result

### 3.1. Mixture of Sushila and Rayleigh distributions (MSRD)

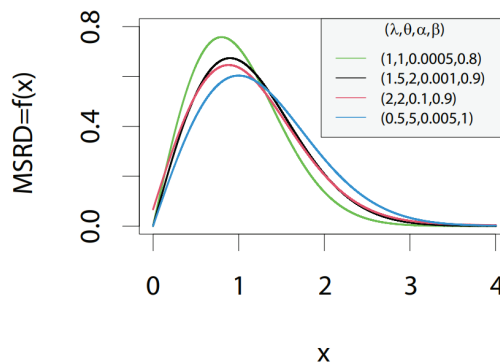
With mixing proportions of  $\alpha$  and  $1 - \alpha$ , the MSRD is a two-component mixture of the Sushila and Rayleigh distributions. The probability density function (PDF) of the new mixture is defined as follows:

$$f(x) = \alpha f_S(x) + (1 - \alpha) f_R(x). \quad (5)$$

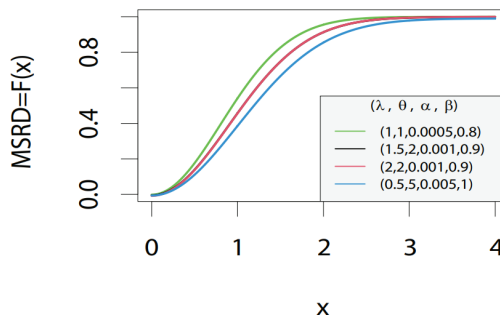
Insert (2) and (4) in (5) which provide the PDF expression, which is

$$f(x) = \frac{\alpha\lambda^2}{\theta(\lambda+1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right) + (1-\alpha) \frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right). \quad (6)$$

Figure 1 shows the MSRD's PDF for a few different parameter values. It reveals the MSRD's apparent positive skewedness.



**Figure 1** PDF of the new mixture of the Sushila and Rayleigh distributions



**Figure 2** CDF of the new mixture of the Sushila and Rayleigh distributions

The corresponding cumulative distribution function (CDF) of the new mixture is defined as follows:

$$F(x) = \int_0^x f(t) dt$$

$$F(x) = \alpha F_S(x) + (1-\alpha) F_R(x). \quad (7)$$

Inserting Eqns. (1) and (3) in (7) provides the CDF expression

$$F(x) = \alpha \left(1 - \frac{\theta(\lambda+1) + \lambda x}{\theta(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right)\right) + (1-\alpha) \left(1 - \exp\left(-\frac{x^2}{2\beta^2}\right)\right). \quad (8)$$

Figure 2 shows the CDF of the MSRD for some values of parameters.

The survival function  $S(x)$  of the new mixture can be written in this form:

$$S(x) = 1 - F(x). \quad (9)$$

The survival function is obtained by inserting Eqn. (8) into Eqn. (9)

$$S(x) = 1 - \left[ \alpha \left( 1 - \frac{\theta(\lambda+1) + \lambda x}{\theta(\lambda+1)} \exp \left( -\frac{\lambda}{\theta} x \right) \right) + (1 - \alpha) \left( 1 - \exp \left( -\frac{x^2}{2\beta^2} \right) \right) \right]. \quad (10)$$

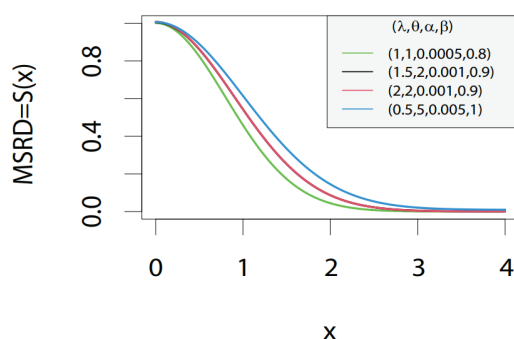
The hazard rate function  $hrf(x)$  of the new mixture can be defined in this form:

$$hrf(x) = \frac{f(x)}{S(x)}. \quad (11)$$

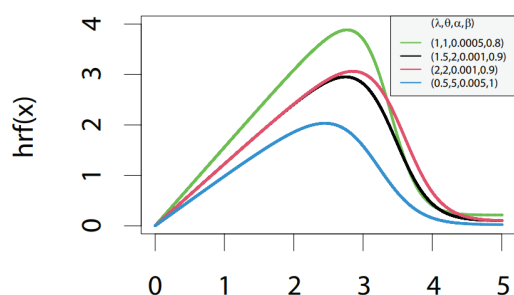
Inserting Eqns. (6) and (10) into Eqn. (11) provides the hazard rate function

$$hrf(x) = \frac{\frac{\alpha\lambda^2}{\theta(\lambda+1)} \left( 1 + \frac{x}{\theta} \right) \exp \left( -\frac{\lambda}{\theta} x \right) + (1 - \alpha) \frac{x}{\beta^2} \exp \left( -\frac{x^2}{2\beta^2} \right)}{1 - \left[ \alpha \left( 1 - \frac{\theta(\lambda+1) + \lambda x}{\theta(\lambda+1)} \exp \left( -\frac{\lambda}{\theta} x \right) \right) + (1 - \alpha) \left( 1 - \exp \left( -\frac{x^2}{2\beta^2} \right) \right) \right]}.$$

Figures 3 and 4 show the survival and the hazard rate functions for some values of parameters.



**Figure 3** The survival function of the new mixture of the Sushila and Rayleigh distributions



**Figure 4** The hazard rate function of the new mixture of the Sushila and Rayleigh distributions

### 3.2. The raw moments of the new mixture of Sushila and Rayleigh distribution

The  $r^{th}$  moment of random variable  $X$  having MSRD is defined by

$$\mu'_r = E(X^r) = \int_0^{\infty} x^r f(x) dx. \quad (12)$$

Insert Eqn. (6) in (12) yields

$$\mu'_r = \int_0^{\infty} \left( \frac{x^r \alpha \lambda^2}{\theta(\lambda+1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right) + (1-\alpha) \frac{x^{r+1}}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right) \right) dx. \quad (13)$$

Integrate Eqn. (13) with respect to  $x$ , which provides the  $r^{th}$  moment of the new mixture of Sushila and Rayleigh distribution

$$\mu'_r = \frac{\alpha \theta^r (\lambda + r + 1) \Gamma(r+1)}{\lambda^r (\lambda + 1)} + (1-\alpha) \beta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right), \quad r = 1, 2, 3, \dots \quad (14)$$

The mean of the new mixture of Sushila and Rayleigh distribution takes the form:

$$\mu'_1 = \frac{\alpha \theta (\lambda + 2)}{\lambda (\lambda + 1)} + (1-\alpha) \beta \sqrt{\frac{\pi}{2}}. \quad (15)$$

$\mu_2$  is the second moment of the new mixture Sushila and Rayleigh distribution is obtained from Eqn. (14) when  $r = 2$ ;

$$\mu'_2 = \frac{2\alpha \theta^2 (\lambda + 3)}{\lambda^2 (\lambda + 1)} + (1-\alpha) 2\beta^2. \quad (16)$$

The variance of the new mixture Sushila and Rayleigh distribution is obtained by inserting Eqns. (15) and (16) into Eqn. (17)

$$Var(X) = \mu'_2 - [\mu'_1]^2. \quad (17)$$

$$Var(X) = \frac{2\alpha \theta^2 (\lambda + 3)}{\lambda^2 (\lambda + 1)} + (1-\alpha) 2\beta^2 - \left[ \frac{\alpha \theta (\lambda + 2)}{\lambda (\lambda + 1)} + (1-\alpha) \beta \sqrt{\frac{\pi}{2}} \right]^2. \quad (18)$$

The standard deviation of the new mixture Sushila and Rayleigh distribution takes the form:

$$SD(X) = \sqrt{Var(X)} = \sqrt{\frac{2\alpha \theta^2 (\lambda + 3)}{\lambda^2 (\lambda + 1)} + (1-\alpha) 2\beta^2 - \left[ \frac{\alpha \theta (\lambda + 2)}{\lambda (\lambda + 1)} + (1-\alpha) \beta \sqrt{\frac{\pi}{2}} \right]^2}.$$

The coefficient of variation of the new mixture Sushila and Rayleigh distribution take the form:

$$C.V = \frac{SD(X)}{E(X)} = \frac{\sqrt{\frac{2\alpha \theta^2 (\lambda + 3)}{\lambda^2 (\lambda + 1)} + (1-\alpha) 2\beta^2 - \left[ \frac{\alpha \theta (\lambda + 2)}{\lambda (\lambda + 1)} + (1-\alpha) \beta \sqrt{\frac{\pi}{2}} \right]^2}}{\frac{\alpha \theta (\lambda + 2)}{\lambda (\lambda + 1)} + (1-\alpha) \beta \sqrt{\frac{\pi}{2}}}.$$

### 3.3. The moments generating function of the new mixture of Sushila and Rayleigh distribution

The moment generating function of the new mixture of Sushila and Rayleigh distribution is given by

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r \mu'_r}{r!}. \quad (19)$$

Insert Eqn. (14) into (19), which provide the moment generating function of the new mixture of Sushila and Rayleigh distribution in Eqn. (20).

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[ \frac{\alpha \theta^r (\lambda + r + 1) \Gamma(r + 1)}{\lambda^r (\lambda + 1)} + (1 - \alpha) \beta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \right]. \quad (20)$$

### 3.4. The characteristic function of the new mixture of Sushila and Rayleigh distribution

The characteristic function of the new mixture of Sushila and Rayleigh distribution is given by

$$\phi_X(t) = E(e^{itX}) = \sum_{r=0}^{\infty} \frac{(it)^r \mu'_r}{r!}. \quad (21)$$

In Eqn. (21) insert Eqn. (14) to provides the characteristic function of the new Sushila and Rayleigh distribution mixture in Eqn. (22)

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(it)^r}{r!} \left[ \frac{\alpha \theta^r (\lambda + r + 1) \Gamma(r + 1)}{\lambda^r (\lambda + 1)} + (1 - \alpha) \beta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \right]. \quad (22)$$

### 3.5. The cumulant generating function of the new mixture of Sushila and Rayleigh distribution

The cumulant function of the new mixture of Sushila and Rayleigh distribution is given by

$$K_X(t) = \log [E(e^{tX})] = \log \sum_{r=0}^{\infty} \frac{t^r \mu'_r}{r!}. \quad (23)$$

Insert Eqn. (14) into (23), which provide the cumulant function of the new mixture of Sushila and Rayleigh distribution in Eqn. (24)

$$K_X(t) = \log \sum_{r=0}^{\infty} \frac{t^r}{r!} \left[ \frac{\alpha \theta^r (\lambda + r + 1) \Gamma(r + 1)}{\lambda^r (\lambda + 1)} + (1 - \alpha) \beta^r 2^{\frac{r}{2}} \Gamma\left(\frac{r}{2} + 1\right) \right]. \quad (24)$$

### 3.6. The Incomplete moments of the new mixture of Sushila and Rayleigh distribution

The incomplete moments of the new mixture of Sushila and Rayleigh distribution is given by

$$A_r(t) = \int_0^t x^r f(x) \partial x. \quad (25)$$

Substituting Eqn. (6) into (25) yields

$$A_r(t) = \int_0^t \left( \frac{x^r \alpha \lambda^2}{\theta (\lambda + 1)} \left( 1 + \frac{t}{\theta} \right) \exp \left( -\frac{\lambda}{\theta} x \right) + (1 - \alpha) \frac{x^{r+1}}{\beta^2} \exp \left( -\frac{x^2}{2\beta^2} \right) \right) \partial x. \quad (26)$$

Integrate Eqn. (26) with respect to  $t$ , provides the  $r^{th}$  incomplete moment of the new mixture of Sushila and Rayleigh distribution in Eqn. (27)

$$A_r(t) = \frac{\alpha \lambda}{\theta (\lambda + 1)} \left( \frac{\theta}{\lambda} \right)^r \left[ \theta \gamma \left( r + 1, \frac{\lambda}{\theta} t \right) + \frac{\theta}{\lambda} \gamma \left( r + 2, \frac{\lambda}{\theta} t \right) \right] + (1 - \alpha) \beta^r 2^{\frac{r}{2}} \gamma \left( \frac{r + 2}{2}, \frac{t^2}{2\beta^2} \right). \quad (27)$$

If  $r = 1, 2$ , then Eqn. (27) gives the first and the second incomplete moments of the new mixture of Sushila and Rayleigh distribution.

### 3.7. Mean residual life (MRL) of the new mixture of Sushila and Rayleigh distribution

The life expectation at age  $t$  defined by  $m_1(t) = E[(X - t)|X > t]$  and for MSRD is given by

$$m_1(t) = \frac{\alpha\theta(\lambda+2)}{\lambda(\lambda+1)} + (1-\alpha)\beta\sqrt{\frac{\pi}{2}} - \frac{\alpha\lambda}{\theta(\lambda+1)} \left(\frac{\theta}{\lambda}\right)^r \left[ \theta\gamma\left(r+1, \frac{\lambda}{\theta}x\right) + \frac{\theta}{\lambda}\gamma\left(r+2, \frac{\lambda}{\theta}x\right) \right] \\ + (1-\alpha)\beta^r 2^{\frac{r}{2}} \gamma\left(\frac{r+2}{2}, \frac{x^2}{2\beta^2}\right) / \\ \left[ 1 - \alpha \left( 1 - \frac{\theta(\lambda+1) + \lambda x}{\theta(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right) \right) - (1-\alpha) \left( 1 - \exp\left(-\frac{x^2}{2\beta^2}\right) \right) \right].$$

### 3.8. Mean deviations (MD) of the new mixture of Sushila and Rayleigh distribution

The MD about mean of the new mixture of Sushila and Rayleigh distribution is defined as follows:

$$MD(\mu) = 2\mu F(\mu) - 2 \int_0^\mu xf(x)\partial x,$$

where  $\mu$  is the population mean of the Sushila and Rayleigh distribution is given in Eqn. (15).

$$MD(\mu) = 2 \left[ \alpha \left( 1 - \frac{\theta(\lambda+1) + \lambda\mu}{\theta(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}\mu\right) \right) + (1-\alpha) \left( 1 - \exp\left(-\frac{\mu^2}{2\beta^2}\right) \right) \right] \times \\ \left[ \frac{\alpha\theta(\lambda+2)}{\lambda(\lambda+1)} + (1-\alpha)\beta\sqrt{\frac{\pi}{2}} \right] - (1-\alpha)\beta^r 2^{\frac{r+1}{2}} \gamma\left(\frac{r+2}{2}, \frac{\mu^2}{2\beta^2}\right) \\ - 2 \left[ \frac{\alpha\lambda}{\theta(\lambda+1)} \left(\frac{\theta}{\lambda}\right)^r \left[ \theta\gamma\left(r+1, \frac{\lambda}{\theta}\mu\right) + \frac{\theta}{\lambda}\gamma\left(r+2, \frac{\lambda}{\theta}\mu\right) \right] \right].$$

Similarly, the MD about Median is

$$MD(M) = \mu - 2 \int_0^M xf(x)\partial x \\ MD(M) = \left[ \frac{\alpha\theta(\lambda+2)}{\lambda(\lambda+1)} + (1-\alpha)\beta\sqrt{\frac{\pi}{2}} \right] - (1-\alpha)\beta^r 2^{\frac{r+1}{2}} \gamma\left(\frac{r+2}{2}, \frac{M^2}{2\beta^2}\right) \\ - 2 \left[ \frac{\alpha\lambda}{\theta(\lambda+1)} \left(\frac{\theta}{\lambda}\right)^r \left[ \theta\gamma\left(r+1, \frac{\lambda}{\theta}M\right) + \frac{\theta}{\lambda}\gamma\left(r+2, \frac{\lambda}{\theta}M\right) \right] \right].$$

### 3.9. Parameter estimation of the new mixture of Sushila and Rayleigh distribution

In this section, the parameters of the new mixture of Sushila and Rayleigh distribution will be estimated using maximum likelihood estimators. The likelihood function of the new mixture of Sushila and Rayleigh distribution is given by:

$$\ell(\lambda, \theta, \alpha, \beta) = \prod_{i=1}^n \left\{ \frac{\alpha\lambda^2}{\theta(\lambda+1)} \left( 1 + \frac{x_i}{\theta} \right) \exp\left(-\frac{\lambda}{\theta}x_i\right) + (1-\alpha) \frac{x_i}{\beta^2} \exp\left(-\frac{x_i^2}{2\beta^2}\right) \right\}.$$

The log-likelihood function of the new mixture of the Sushila and Rayleigh distribution is given by:

$$L(\lambda, \theta, \alpha, \beta) = \ln \prod_{i=1}^n \left\{ \frac{\alpha\lambda^2}{\theta(\lambda+1)} \left( 1 + \frac{x_i}{\theta} \right) \exp\left(-\frac{\lambda}{\theta}x_i\right) + (1-\alpha) \frac{x_i}{\beta^2} \exp\left(-\frac{x_i^2}{2\beta^2}\right) \right\}. \quad (28)$$

Take the derivative of Eqn. (28) with respect to  $\lambda, \theta, \alpha, \beta$  and solve for  $\lambda, \theta, \alpha, \beta$  to get the maximum likelihood estimators.

$$\frac{\partial L(\lambda, \theta, \alpha, \beta)}{\partial \lambda} = \sum_{i=1}^n \frac{\frac{2\alpha\lambda(1+\frac{x}{\theta})}{\theta(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right) - \frac{\alpha\lambda^2(1+\frac{x}{\theta})}{\theta(\lambda+1)^2} \exp\left(-\frac{\lambda}{\theta}x\right) - \frac{\alpha\lambda^2x(1+\frac{x}{\theta})}{\theta^2(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right)}{\frac{\alpha\lambda^2}{\theta(\lambda+1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right) + (1-\alpha)\frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right)} = 0$$
$$\frac{\partial L(\lambda, \theta, \alpha, \beta)}{\partial \theta} = \sum_{i=1}^n \frac{\frac{\alpha\lambda^3x(1+\frac{x}{\theta})}{\theta^3(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right) - \frac{\alpha\lambda^2(1+\frac{x}{\theta})}{\theta^2(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right) - \frac{\alpha\lambda^2x(1+\frac{x}{\theta})}{\theta^3(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right)}{\frac{\alpha\lambda^2}{\theta(\lambda+1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right) + (1-\alpha)\frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right)} = 0$$
$$\frac{\partial L(\lambda, \theta, \alpha, \beta)}{\partial \alpha} = \sum_{i=1}^n \frac{\frac{\lambda^2(1+\frac{x}{\theta})}{\theta(\lambda+1)} \exp\left(-\frac{\lambda}{\theta}x\right) - \frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right)}{\frac{\alpha\lambda^2}{\theta(\lambda+1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right) + (1-\alpha)\frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right)} = 0$$
$$\frac{\partial L(\lambda, \theta, \alpha, \beta)}{\partial \beta} = \sum_{i=1}^n \frac{\frac{x^3(1-\alpha)}{\beta^5} \exp\left(-\frac{x^2}{2\beta^2}\right) - \frac{2(1-\alpha)x}{\beta^3} \exp\left(-\frac{x^2}{2\beta^2}\right)}{\frac{\alpha\lambda^2}{\theta(\lambda+1)} \left(1 + \frac{x}{\theta}\right) \exp\left(-\frac{\lambda}{\theta}x\right) + (1-\alpha)\frac{x}{\beta^2} \exp\left(-\frac{x^2}{2\beta^2}\right)} = 0$$

These equations cannot be solved analytically because they are nonlinear, but they can be solved numerically using iterative methods. Applying the optim function in the R programming language to conduct Maximum Likelihood Estimation (MLE) using the simulated annealing approach.

4. Numerical Illustrations

Using the optim function in the R programming language for the maximum likelihood estimation (MLE) via the simulated annealing method. This paper generated random numbers of the new mixture of Sushila and Rayleigh distributions by the inverse transform method, and the simulation was repeated 5,000 times for each scenario. The sample size  $n = 20, 50, 100$  with parameters  $\alpha = 0.5, \theta = 1, 2, \lambda = 1, 2, \beta = 1, 2$ . The initial values of the parameters are  $c(0.5, \bar{x}, \bar{x}, \bar{x})$ . Tables 1-3 present the computation results for each scenario. Regarding the simulation results, it was found that the MLE via simulated annealing method works well for all parameter, especially  $\alpha$  and  $\beta$ . It should be noted that the bias will be lower than the results in Tables 1-3 if appropriate initial parameters are chosen for each dataset, but in this paper, we set the same initial parameter for all datasets.

Table 1 The average estimates, the bias, the mean squared errors for  $n = 20$ .

| $\beta$                   | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{\beta}$ | Bias ( $\hat{\alpha}$ ) | Bias ( $\hat{\theta}$ ) | Bias ( $\hat{\lambda}$ ) | Bias ( $\hat{\beta}$ ) | MSE ( $\hat{\alpha}$ ) | MSE ( $\hat{\theta}$ ) | MSE ( $\hat{\lambda}$ ) | MSE ( $\hat{\beta}$ ) |
|---------------------------|----------------|----------------|-----------------|---------------|-------------------------|-------------------------|--------------------------|------------------------|------------------------|------------------------|-------------------------|-----------------------|
| $\theta = 1, \lambda = 1$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.4383         | 0.8946         | 1.2998          | 0.9464        | -0.0617                 | -0.1054                 | 0.2998                   | -0.0536                | 0.0038                 | 0.0111                 | 0.0899                  | 0.0029                |
| 2                         | 0.6158         | 0.5745         | 0.6298          | 2.2978        | 0.1158                  | -0.4255                 | -0.3702                  | 0.2978                 | 0.0134                 | 0.1811                 | 0.1370                  | 0.0887                |
| $\theta = 1, \lambda = 2$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.3491         | 0.5100         | 2.0582          | 0.8810        | -0.1509                 | -0.4900                 | 0.0582                   | -0.1190                | 0.0248                 | 0.3667                 | 1.0021                  | 0.0212                |
| 2                         | 0.4902         | 0.6815         | 1.1414          | 2.0572        | -0.0098                 | -0.3185                 | -0.8586                  | 0.0572                 | 0.0112                 | 0.5659                 | 2.0410                  | 0.0108                |
| $\theta = 2, \lambda = 1$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.5228         | 2.2544         | 1.3030          | 0.8383        | 0.0228                  | 0.2544                  | 0.3030                   | -0.1617                | 0.0005                 | 0.0647                 | 0.0918                  | 0.0262                |
| 2                         | 0.5494         | 3.1566         | 1.2363          | 2.5457        | 0.0494                  | 1.1566                  | 0.2363                   | 0.5457                 | 0.0500                 | 5.6948                 | 0.1058                  | 0.5376                |
| $\theta = 2, \lambda = 2$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.5621         | 2.4387         | 3.1086          | 0.5813        | 0.0621                  | 0.4387                  | 1.1086                   | -0.4187                | 0.0039                 | 0.1925                 | 1.2291                  | 0.1753                |
| 2                         | 0.4829         | 1.4067         | 1.2993          | 2.0364        | -0.0171                 | -0.5933                 | -0.7007                  | 0.0364                 | 0.0178                 | 1.7657                 | 1.3615                  | 0.0851                |

5. Application

In this section, we have shown that the new mixture Sushila and Rayleigh distributions can be a better model using a real data set. The data set is an uncensored data set that corresponds to the number of months that a randomly selected sample of 128 bladder cancer patients experienced remission, as reported in Lawless (2011) and Merovci (2013). This data has been applied in various survival analysis. The data set consist of the following observations.

**Table 2** The average estimates, the bias, the mean squared errors for  $n = 50$ .

| $\beta$                   | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{\beta}$ | Bias ( $\hat{\alpha}$ ) | Bias ( $\hat{\theta}$ ) | Bias ( $\hat{\lambda}$ ) | Bias ( $\hat{\beta}$ ) | MSE ( $\hat{\alpha}$ ) | MSE ( $\hat{\theta}$ ) | MSE ( $\hat{\lambda}$ ) | MSE ( $\hat{\beta}$ ) |
|---------------------------|----------------|----------------|-----------------|---------------|-------------------------|-------------------------|--------------------------|------------------------|------------------------|------------------------|-------------------------|-----------------------|
| $\theta = 1, \lambda = 1$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.5999         | 1.6542         | 1.2667          | 1.0199        | 0.0999                  | 0.6542                  | 0.2667                   | 0.0199                 | 0.0846                 | 4.5260                 | 1.9910                  | 0.0101                |
| 2                         | 0.4382         | 1.0549         | 1.3633          | 1.8941        | -0.0618                 | 0.0549                  | 0.3633                   | -0.1059                | 0.0966                 | 1.0678                 | 1.6288                  | 0.2048                |
| $\theta = 1, \lambda = 2$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.3260         | 0.7839         | 1.4746          | 0.9724        | -0.1740                 | -0.2161                 | -0.5254                  | -0.0276                | 0.0553                 | 1.8805                 | 2.9668                  | 0.0078                |
| 2                         | 0.6036         | 1.5108         | 2.0941          | 1.9870        | 0.1036                  | 0.5108                  | 0.0941                   | -0.0130                | 0.0107                 | 0.2609                 | 0.0089                  | 0.0002                |
| $\theta = 2, \lambda = 1$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.4647         | 3.1225         | 1.5847          | 1.2766        | -0.0353                 | 1.1225                  | 0.5847                   | 0.2766                 | 0.0012                 | 1.2600                 | 0.3419                  | 0.0765                |
| 2                         | 0.5246         | 1.8340         | 1.1707          | 2.3695        | 0.0246                  | -0.1660                 | 0.1707                   | 0.3695                 | 0.2455                 | 3.1766                 | 1.3087                  | 0.2518                |
| $\theta = 2, \lambda = 2$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.3699         | 4.3723         | 3.6242          | 1.0538        | -0.1301                 | 2.3723                  | 1.6242                   | 0.0538                 | 0.0295                 | 8.5901                 | 3.4511                  | 0.0251                |
| 2                         | 0.5191         | 2.5532         | 2.7544          | 2.2442        | 0.0191                  | 0.5532                  | 0.7544                   | 0.2442                 | 0.0198                 | 3.6478                 | 4.0322                  | 0.0598                |

**Table 3** The average estimates, the bias, the mean squared errors for  $n = 100$ .

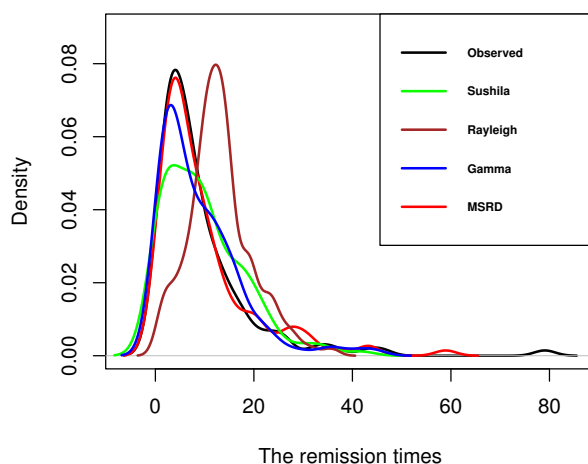
| $\beta$                   | $\hat{\alpha}$ | $\hat{\theta}$ | $\hat{\lambda}$ | $\hat{\beta}$ | Bias ( $\hat{\alpha}$ ) | Bias ( $\hat{\theta}$ ) | Bias ( $\hat{\lambda}$ ) | Bias ( $\hat{\beta}$ ) | MSE ( $\hat{\alpha}$ ) | MSE ( $\hat{\theta}$ ) | MSE ( $\hat{\lambda}$ ) | MSE ( $\hat{\beta}$ ) |
|---------------------------|----------------|----------------|-----------------|---------------|-------------------------|-------------------------|--------------------------|------------------------|------------------------|------------------------|-------------------------|-----------------------|
| $\theta = 1, \lambda = 1$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.6865         | 0.9936         | 0.9137          | 0.7821        | 0.1865                  | -0.0064                 | -0.0863                  | -0.2179                | 0.0607                 | 0.4465                 | 0.2869                  | 0.1390                |
| 2                         | 0.3944         | 1.1138         | 1.2328          | 1.9015        | -0.1056                 | 0.1138                  | 0.2328                   | -0.0985                | 0.0335                 | 1.5881                 | 1.4838                  | 0.1057                |
| $\theta = 1, \lambda = 2$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.4189         | 0.5609         | 2.6248          | 1.0075        | -0.0811                 | -0.4391                 | 0.6248                   | 0.0075                 | 0.0520                 | 0.9936                 | 24.9510                 | 0.0104                |
| 2                         | 0.4807         | 1.0437         | 2.6153          | 1.9385        | -0.0193                 | 0.0437                  | 0.6153                   | -0.0615                | 0.0013                 | 0.5754                 | 3.8051                  | 0.0148                |
| $\theta = 2, \lambda = 1$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.4472         | 1.9168         | 0.9372          | 0.9742        | -0.0528                 | -0.0832                 | -0.0628                  | -0.0258                | 0.0028                 | 0.0069                 | 0.0040                  | 0.0007                |
| 2                         | 0.5606         | 2.0823         | 1.1168          | 2.0271        | 0.0606                  | 0.0823                  | 0.1168                   | 0.0271                 | 0.0493                 | 4.5490                 | 1.4056                  | 0.1479                |
| $\theta = 2, \lambda = 2$ |                |                |                 |               |                         |                         |                          |                        |                        |                        |                         |                       |
| 1                         | 0.6454         | 2.2521         | 2.5082          | 0.8380        | 0.1454                  | 0.2521                  | 0.5082                   | -0.1620                | 0.0211                 | 0.0636                 | 0.2583                  | 0.0262                |
| 2                         | 0.3848         | 1.9681         | 2.3507          | 2.0476        | -0.1152                 | -0.0319                 | 0.3507                   | 0.0476                 | 0.0189                 | 16.3933                | 18.1821                 | 0.0135                |

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 5.49

It is evident from Figure 5 that the data are right-skewed. Consequently, the Sushila, the Rayleigh, the gamma, and the new mixture Sushila and Rayleigh distributions have all been fitted to data by MLE. The parameter estimates, the log likelihood (LL), and Akaike's information criterion (AIC) are all presented in Table 4. Based on the results, it is obvious that the new mixture Sushila and Rayleigh distributions better fits the data than the others.

**Table 4** Results of fitting the bladder cancer data set.

| Distribution                 | Estimate parameters |          |           |         | LL         | AIC       |
|------------------------------|---------------------|----------|-----------|---------|------------|-----------|
|                              | $\alpha$            | $\theta$ | $\lambda$ | $\beta$ |            |           |
| Sushila                      | -                   | 52.650   | 484.078   | -       | -414.347   | 832.694   |
| Rayleigh                     | -                   | -        | -         | 9.932   | -1,634.523 | 3,271.046 |
| Gamma                        | 1.173               | -        | -         | 0.125   | -413.368   | 830.736   |
| Mixture Sushila and Rayleigh | 0.618               | 14.968   | 1.698     | 3.975   | -410.428   | 828.857   |



**Figure 5** Plot of the observed and fitted distribution of the remission times of bladder cancer patients

## 6. Conclusions

The new Sushila and Rayleigh mixture distributions are proposed in this research (MSRD). Various MSRD properties have been derived. Positive skew is present in the MSRD. Utilizing the maximum likelihood method, parameter is estimated. For performing the MLE through the simulated annealing, a simulation study was used. The MLE via simulated annealing method was found to work well for all parameters. The model has been used to describe the remission times of bladder cancer patients and compared to existing distributions to demonstrate its usefulness. The conclusion is that among the competing distributions, MSRD shows better fit to the data.

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