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Parameters Estimation of Bayesian and E-Bayesian Methods on the Generalized Order Statistics under Exponential Family

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Abstract
In this paper, we have estimated the parameter of Generalized Order Statistics (GOS) of Exponentiated Distribution Family using Bayesian and E-Bayesian method for computing estimates. To find the estimates, we have employed various loss function viz. Square Error Loss Function (SELF), LINEX and General Entropy Loss Function (GELF). The estimates are derived considering the conjugate prior. Furthermore, the relation among E-Bayesian under different prior distribution of hyper-parameters have been established. In the last section, the comparison have been made of derived estimates using Monte Carlo Simulation. To support and validate the obtained result a real Data set is analysed.

Keywords: Generalized Order Statistics (GOS), exponentiated distribution family, E-Bayesian, square error loss function (SELF), LINEX, general entropy loss function (GELF), Monte Carlo simulation.

1. Introduction

Generalized Order Statistics models plays an important roles in Statistics and are commonly used in Reliability theory and life testing experiments. The concept of Generalized Order Statistics was developed by Kamps (1995), as generalized framework for models of ordered random variables. Moreover, many other models like, upper order statistics, record values, sequential order statistics and progressively Type-II censoring order statistics are seen to be a particular cases of GOS Sarhan (2007).

Let $X(1,n,m,k), X(2,n,m,k), \dots X(r,n,m,k)$ be the Generalized Order Statistics. The joint density function of GOS based on absolutely continuous distribution function F with density function f is given by

$$f_{X(1,n,m,k),X(2,n,m,k)\dots X(r,n,m,k)}(x_{(1)},x_{(2)},\dots,x_{(r)}) = C_{r-1}\left[\prod_{i=1}^{r-1}\Big(1-F\big(x_{(i)}\big)\Big)^{m} f\big(x_{(i)}\big)\right] \Big(1-F\big(x_{(n)}\big)\Big)^{k-1} f\big(x_{(n)}\big) \quad (1)$$

for $F^{-1}(0+) < x_{(1)} \leq x_{(2)} \cdots \leq x_{(n)} < F^{-1}(1)$ where $\gamma_i = k+(n-r)(m+1) > 0$ and $C_{r-1} = \prod_{i=1}^{r}\gamma_i \quad r = 1,2\dots n,\ \gamma = k$ with $n\,\epsilon N, k = 1$ and $m\,\epsilon R, X(r,n,m,k)$ reduces to the ordinary r^{th} order statistics and (1) reduces to joint pdf of r^{th} order statistics. For study of various distributions and the properties of ordinary order statistics one may refer David (1998) and Arnold et al. (1998). The record values were firstly introduced by Chandler (1952). Record values or Record Statistics are defined as the largest or smallest values obtained from the sequence of the random variables. The theory of the record values ware closely related to order statistics. For detailed properties of record values one can refer Ahsanulla (2000) and Arnold et al. (1998). If m= -1 and k=1 then (1) reduced to joint pdf of the r^{th} upper record values. Also, if $m_i = R_i, i = 1, 2 \ldots, n-1$ and $k = R_n + 1$, then (1) reduced to the progressive Type-II censored data. In life testing experiment, most of Type-I and Type-II censoring schemes are used. In Type-I censoring scheme, experiment continues up to a predefined time T and any failure that occur after T will not be observed. In Type-II censoring, the experiment terminated after occurrence of pre-specified number of items $(r \leqslant n)$ failure. Unfortunately, in any of these censoring schemes, it is not possible to withdraw live item during the experiment. To overcome this, Progressive Type-II censoring scheme was introduced which is the generalization of the both censoring scheme, in which it is possible to withdraw live items during the experiment.

The statistical properties based on Generalized Order Statistics (GOS) for some life time distributions have been studied by many researchers. Kampus and Gather (1997) developed distributional properties of GOS of the Exponential distribution. Properties of GOS of two parametric exponential distribution studied by Ahsanallah (2000). Habibullah (2000) derived the estimators of the parameters of Pareto Type-II distribution for the GOS model. Cramer and Kampus (2000) derived relations for the expectations of function of GOS class of distributions which include Exponential, Uniform, Pareto, Lomax and Pearson are special cases. Moghadam et al. (2012) obtained Bayesian estimates of the unknown parameters of GOS of Lomax distribution. Gupta and Jamal (2019) studied the Weibull generalized exponential distribution based on generalized order statistics. Classical estimation of Exponential family based on Generalized Order Statistics derived by Khan and Khatoon (2020). Azhad et al. (2021) estimated the parameters of several heterogeneous exponential populations based on generalized order statistics. The work in recent years on Generalized Order Statistics is more or less has been extended using the Bayesian inference tools as well as employed the censoring Scheme.

Bayesian estimation based on k record data for exponential family of distribution was studied by Ahmadi et al. (2009). Kim et al. (2011) derived the Bayesian estimates of shape parameters and reliability function of Exponentiated family based on Type-II right censored data. The reliability function estimate for a family of Exponentiated Distributions studied by Chaturvedi and Pathak (2014). Moreover, Kim and Han (2014) obtained the estimators and credible interval for scale parameters of Rayleigh distribution based on GOS. A new approach of Bayesian estimation was introduced by Han (1997), named E-Bayesian estimation. In recent years, there has been a growing interest of researchers in the study of E-Bayesian estimation. Han (2019) studied E-Bayesian estimate of the Exponentiated distribution family parameter under LINEX loss function. E-Bayesian estimates of the unknown parameter and the reliability function for the generalized half Logistic distribution and their relations has been derived by Reyad (2016). E-Bayesian estimate of the parameter of truncated Geometric distribution has been studied by Devi et al. (2019). E-Bayesian estimated for the proportional hazard and reversed hazard rate models based on record values have been developed by Kizilaslan (2017). Sharma and Kumar (2020) studied E-Bayesian and Bayesian estimate of parameters of inverse Lomax distribution obtained under different loss function.

In Section 2, we have introduced the model of Exponentiated Family of distribution under Generalized Order Statistics (GOS). In Section 3, we have obtained Bayes estimators of parameters of proposed distribution family using gamma conjugate prior under symmetric and asymmetric loss function. In Section 4, Expected-Bayes (E-Bayes) estimators has obtained for the unknown parameter θ using three different prior distributions under three different loss functions viz; SELF, LINEX and Generalized entropy Loss function (GELF). In Section 5, we have derived the properties of E-Bayesian estimators. In section 6, comparison and computations between Bayes and E-Bayes estimators have been made using Monte Carlo simulation. Furthermore, analysis of a real data set is presented.

2. The Model of Exponentiated Family of Distribution under GOS

The cumulative distribution function of exponentiated family of distribution (cdf) is defined as

$$F(x;\theta) = 1 - \big[g(x)\big]^{\theta}, \qquad where\, A \le x \le B, \theta > 0 \tag{2}$$

and the corresponding the probability density function (pdf) is

$$f(x;\theta) = -\theta\big[g(x)\big]^{'}\big[g(x)\big]^{\theta-1}, \qquad A \le x \le B, \theta > 0 \tag{3}$$

where $g(x)$ is monotone decreasing function such that g(A)=1, g(B)=0 and θ is an unknown shape parameter.

If $X = (x_1, x_2 \dots x_n)$ are the sample observation from distribution family (3), then the likelihood function is obtained by substituting Eqns. (2) and (3) in (1),

$$L(\theta/\underline{x}) \propto (-1)^r\theta^r \prod_{i=1}^{r}\left[\frac{g(x_i)^{'}}{g(x_i)}\right] exp\Big[-\theta\Big\{(m+1)\sum_{i=1}^{r-1} log(g(x)) + k\, log(g(x))\Big\}\Big]. \tag{4}$$

Let $Z_r = \Big[(m+1)\sum_{i=1}^{r-1} log\big(g(x)\big) + k\, log\big(g(x)\big)\Big]$ then

$$L(\theta/\underline{x}) = (-1)^r\theta^r \prod_{i=1}^{r}\left[\frac{g(x_i)^{'}}{g(x_i)}\right] exp\big[-\theta Z_r\big]. \tag{5}$$

It may be noted that (2) represents a family of exponentiated distributions as it cover the following distributions as particular cases:

1. If g(x)= e^{-x}, A = 0, B= $+\infty$ then it reduces to exponential distribution with pdf f(x) = $\theta e^{-\theta x}$.
2. If g(x)=$e^{-(x-\mu)}$, A = $\mu > 0$, B= $+\infty$ then it reduces to with pdf two parameter exponential distribution f(x)= $\theta e^{-\theta(x-\mu)}$.
3. If g(x)= $e^{-x^{\lambda}}$, A = 0, B= $+\infty$ then it reduces to two parameter Weibull distribution with pdf f(x) = $\theta\lambda x^{\lambda-1}e^{-\theta x^{\lambda}}$, where λ is another unknown parameter.
4. If $g(x) = (1+\frac{x}{\lambda})^{-1}, A = 0, B = +\infty$ then it reduces to two parameter Lomax distribution with pdf f(x) = $\frac{\theta}{\lambda}(1+\frac{x}{\lambda})^{-(\theta+1)}$.
5. If $g(x) = (1+x^{\lambda})^{-1} A = 0, B = +\infty$ then it reduces to two parameter Burr type XII distribution having pdf f(x)=$\lambda\theta x^{\lambda-1}(1+x^{\lambda})^{-(\theta+1)}$.
6. If g(x)= $\frac{\lambda}{x}$, A =λ, B= $+\infty$ then it reduces to two parameter Pareto distribution having pdf $f(x) = \theta\lambda^{\theta}x^{-(\lambda+1)}$.

3. Bayesian Estimation of Parameter under GOS

In this section, we have discussed and derived the expression for the Bayes estimators of the unknown parameter (θ) of the model (2) using symmetric and asymmetric loss functions. In most practical situations the statistician possesses some subjective *a prior* information concerning the probable values of the parameter. This prior information depends on the hyper-parameters. On the other, Loss function is an important function in Bayesian analysis. In the Bayesian analysis, the Squared Error Loss function (SELF) is the most commonly used loss function as it is symmetrical and gives equal weight to over estimation as well as under estimation. Being symmetric loss functions it is inappropriate to use it in many circumstances, particularly when positive and negative errors have different consequences. In such cases, the most commonly used asymmetric loss function is the LINEX (linear exponential) loss function. It was introduced by Varian (1975) and became popular due to Zellner (1986). As the LINEX loss function is suitable only for estimation of location parameter. In many practical situations, it appears to be more realistic to express the loss in terms of the ratio. For such cases, Calabria and Pulcini (1996) point out that another a useful asymmetric loss function is the Generalized Entropy loss function (GELF). In this section, we have obtained the Bayesian estimates of the parameters of GOS of Exponential Family by considering SELF, LINEX and GELF loss functions.

We have taken conjugate prior of θ, namely gamma (a,b) the pdf given as

$$\pi(a,b) = \frac{b^a \theta^{(a-1)} exp(-\theta b)}{\Gamma(a)}; \quad \theta > 0 \tag{6}$$

where $\Gamma(a) = \int_0^\infty x^{a-1} exp(-x)dx$ is the gamma function, $a > 0$ and $b > 0$ are the hyper-parameters.

From (5) and (6), the posterior density of θ is given as

$$\begin{aligned} h(\theta/a,b) &= \frac{\pi(a,b)L(\theta/\underline{x})}{\int_0^\infty \pi(a,b)L(\theta/\underline{x})d\theta} \\ &= \frac{(Z_r+b)^{(r+a)}}{\Gamma(r+a)}\theta^{(r+a-1)} exp\Big(-\theta(Z_r+b)\Big) \end{aligned} \tag{7}$$

where $Z_r = \Big[(m+1)\sum_{i=1}^{r-1} log\big(g(x)\big) + k\, log\big(g(x)\big)\Big]$ as defined in Section 2.

3.1. Bayesian estimation under SELF

Mood (1974) introduced the SELF and defined it as

$$L(\hat{\theta},\theta) = (\hat{\theta}-\theta)^2; \quad \theta > 0$$

where $\hat{\theta}$ is an estimator of θ .

The Bayesian estimate of θ is obtained as

$$\begin{aligned} E(\theta/\underline{x}) &= \int_0^\infty exp(-\theta)\, h(\theta/a,b)d\theta \\ &= \frac{(Z_r+b)^{(r+a)}}{\Gamma(r+a)} \int_0^\infty \theta^{(r+a)} exp\Big(-\theta(Z_r+b)\Big)d\theta \\ \hat{\theta}_{BSE} &= \frac{r+a}{Z_r+b}\ . \end{aligned} \tag{8}$$

3.2. Bayesian estimation under LINEX

The LINEX loss function is defined as

$$L(\hat{\theta},\theta) = exp\big[p(\hat{\theta}-\theta)\big] - p(\hat{\theta}-\theta) - 1$$

where $p \neq 0$ is known parameter, sign and magnitude of p reflects the direction and degree of asymmetry.

The Bayesian estimator of θ is given as

$$E(-p\theta/\underline{x}) = \frac{-1}{k} lnE\Big[exp(-p\theta/x)\Big]$$

we have

$$\begin{aligned} E(-p\theta/\underline{x}) &= \int_0^\infty exp(-p\theta)\, h(\theta/a,b) d\theta \\ &= \frac{(Z_r+b)^{(r+a)}}{\Gamma(r+a)} \int_0^\infty \theta^{(r+a-1)} exp\Big(-\theta(Z_r+b+p)\Big) d\theta \\ &= \left(\frac{Z_r+b}{Z_r+b+p}\right)^{r+a}. \end{aligned}$$

On simplification, we get Bayesian estimate of θ under LINEX is

$$\hat{\theta}_{BLLF} = \left(\frac{r+a}{p}\right) ln\left(1+\frac{p}{Z_r+b}\right) \quad . \tag{9}$$

3.3. Bayesian estimation under GELF

The Generalized Entropy loss function is defined as

$$L(\theta,\hat{\theta}) = \left[\left(\frac{\hat{\theta}}{\theta}\right)^c - c\, ln\left(\frac{\hat{\theta}}{\theta}\right) - 1\right] \; c \neq 0 \tag{10}$$

where, $\hat{\theta}$ is the estimator of the θ. For $c > 0$ is the over estimation has more serious effect then a negative error and for $c < 0$, a under estimation has a more serious effect than a positive error.

The Bayesian estimator of θ is given as

$$\begin{aligned} \hat{\theta}_{BGELF} &= E(\theta^{-c}/\underline{x})^{\frac{-1}{c}} \\ E(\theta^{-c}/\underline{x}) &= \int_0^\infty exp(\theta^{-c})\, h(\theta/a,b) d\theta \\ &= \frac{(Z_r+b)^{(r+a)}}{\Gamma(r+a)} \int_0^\infty \theta^{(r+a-c-1)} exp\Big(-\theta(Z_r+b+p)\Big) d\theta \\ \hat{\theta}_{BGELF} &= \Big(Z_r+b\Big)^{-1} \left[\frac{\Gamma(r+a)}{\Gamma(r+a-c)}\right]^{\frac{1}{c}} \end{aligned} \tag{11}$$

when $c = 1$ then generalized entropy loss function reduces to entropy loss function given as

$$\hat{\theta}_{BELF} = \frac{(r+a-1)}{(Z_r+b)} \; . \tag{12}$$

4. Expected-Bayesian (E-Bayesian) Estimation of Parameter under GOS

In this section, we have derived the expression for the E-Bayes estimators of θ under different loss functions and using three different prior distributions. E-Bayesian is the Expected-Bayesian (E-Bayes), which is a new approach of Bayesian estimation introduced by Han (1997). According to Han (1997), in Eqn. (6) a and b should be selected to guarantee that $\pi(\theta/a,b)$ is a decreasing function of θ.

The derivative of $\pi(\theta/a,b)$ with respect to θ is,

$$\frac{d\pi(\theta/a,b)}{d\theta} = \frac{b^a \theta^{a-2} exp(-b\theta)}{\Gamma a} [(a-1)-b\theta] \; . \tag{13}$$

Note that $a > 0, b > 0$ and $\theta > 0$. It follows $0 < a < 1, b > 0$ due to $\frac{d\pi(\theta/a,b)}{d\theta} < 0$, and therefore $\pi(\theta/a,b)$ is a decreasing function of θ. Given $0 < a < 1$, the larger b is, the thinner will be the tail of the gamma density function. Considering the robustness of Bayesian estimate Berger (1985), the narrower tailed prior distribution often leads to worse robustness. Accordingly, it is better to choose b below some given upper bound c (c is a positive constant). Thus, we consider the scope of hyper parameter a and b as $0 < b < c$ and $a < 1$.

The E-Bayes estimate of θ (expectation of the Bayes estimate) can be written as

$$\hat{\theta} = \int\int_D \hat{\theta}(a,b)\pi(a,b)dadb$$

where D is the domain of a, b and $\hat{\theta}(a,b)$ is the Bayesian estimator of θ with hyper parameters a , b and $\pi(a,b)$ is the density function of a and b.

E-Bayes estimate of θ is obtained by using three different prior distributions of the hyper parameters a and b. These distributions are used to observe the influence of different prior distributions on the E-Bayes estimate of θ. We have used following distributions of a and b

$$\pi_1(a,b) = \frac{1}{cB(u,v)}a^{u-1}(1-a)^{v-1}, \qquad 0 < a < 1, 0 < b < c \tag{14}$$

$$\pi_2(a,b) = \frac{2}{c^2B(u,v)}(c-b)a^{u-1}(1-a)^{v-1}, \qquad 0 < a < 1, 0 < b < c \tag{15}$$

$$\pi_3(a,b) = \frac{2b}{c^2B(u,v)}a^{u-1}(1-a)^{v-1}, \qquad 0 < a < 1, 0 < b < c \tag{16}$$

where $B(u,v)$ is the beta function.

4.1. E-Bayesian estimate of θ under SELF

Under SELF, E-Bayesian estimate of θ based on $\pi_1(a,b)$ is obtained by using (8) and (14) as

$$\begin{aligned}\hat{\theta}_{ESE_1} &= \int_0^1\int_0^c \left(\frac{r+a}{Z_r+b}\right)\cdot\frac{1}{cB(u,v)}a^{u-1}(1-a)^{v-1}dadb \\ &= \frac{1}{c}\left(r+\frac{u}{u+v}\right)\int_0^c \frac{1}{Z_r+b}db \\ &= \frac{1}{c}\left(r+\frac{u}{u+v}\right)ln\left(\frac{Z_r+c}{Z_r}\right).\end{aligned} \tag{17}$$

Similarly, E-Bayesian estimate of θ based on $\pi_2(a,b)$ is obtained by using (8) and (15) as

$$\begin{aligned}\hat{\theta}_{ESE_2} &= \int_0^1\int_0^c \left(\frac{r+a}{Z_r+b}\right)\cdot\frac{2}{c^2B(u,v)}(c-b)a^{u-1}(1-a)^{v-1}dadb \\ &= \frac{2}{c^2}\left(r+\frac{u}{u+v}\right)\int_0^c \frac{c-b}{Z_r+b}db \\ &= \frac{2}{c}\left(r+\frac{u}{u+v}\right)\left[\left(\frac{Z_r+c}{c}\right)ln\left(\frac{Z_r+c}{c}\right)-1\right].\end{aligned} \tag{18}$$

Similarly, E-Bayesian estimate of θ based on $\pi_3(a,b)$ is obtained by using (8) and (16) as

$$\begin{aligned}\hat{\theta}_{ESE_3} &= \int_0^1\int_0^c \left(\frac{r+a}{Z_r+b}\right)\frac{2b}{c^2B(u,v)}a^{u-1}(1-a)^{v-1}dadb \\ &= \frac{2}{c^2}\left(r+\frac{u}{u+v}\right)\int_0^c \frac{b}{Z_r+b}db \\ &= \frac{2}{c}\left(r+\frac{u}{u+v}\right)\left[1-\left(\frac{Z_r}{c}\right)ln\left(\frac{Z_r+c}{Z_r}\right)\right].\end{aligned} \tag{19}$$

4.2. E-Bayesian estimate of θ under LINEX

Under LINEX Loss function, E-Bayesian estimate of θ based on $\pi_1(a,b)$ is obtained by using (9) and (14) as

$$\begin{aligned}\hat{\theta}_{ELL_1} &= \int_0^1\int_0^c \left(\frac{r+a}{p}\right) ln\left(1+\frac{p}{Z_r+b}\right)\frac{1}{cB(u,v)}a^{u-1}(1-a)^{v-1}dadb\\ &= \frac{1}{c}\left(r+\frac{u}{u+v}\right)\int_0^c \frac{1}{p}\,ln\left(1+\frac{p}{Z_r+b}\right)db\\ &= \frac{1}{c}\left(r+\frac{u}{u+v}\right)\Bigg[c\,ln\left(1+\frac{p}{Z_r+c}\right)+\left(Z_r+p\right)ln\left(1+\frac{c}{Z_r+p}\right)\\ &\qquad - Z_r\,ln\left(\frac{Z_r+c}{Z_r}\right)\Bigg]. \qquad (20)\end{aligned}$$

Similarly, E-Bayesian estimate of θ based on $\pi_2(a,b)$ is obtained by using (9) and (15) as

$$\begin{aligned}\hat{\theta}_{ELL_2} &= \int_0^1\int_0^c \left(\frac{r+a}{p}\right) ln\left(1+\frac{p}{Z_r+b}\right)\frac{2}{c^2B(u,v)}(c-b)a^{u-1}(1-a)^{v-1}dadb\\ &= \left(r+\frac{u}{u+v}\right)\Bigg[\frac{1}{p}\,ln\left(1+\frac{p}{Z_r}\right)-\frac{(Z_r+c)^2}{c^2p}\,ln\left(1+\frac{c}{Z_r}\right)\\ &+\frac{\left(Z_r+p+c\right)^2}{c^2p}\,ln\left(1+\frac{c}{Z_r+p}\right)-\frac{1}{c}\Bigg]. \qquad (21)\end{aligned}$$

Similarly, E-Bayesian estimate of θ based on $\pi_3(a,b)$ is obtained by using (9) and (16) as

$$\begin{aligned}\hat{\theta}_{ELL_3} &= \int_0^1\int_0^c \left(\frac{r+a}{p}\right) ln\left(1+\frac{p}{Z_r+b}\right)\frac{2b}{c^2B(u,v)}a^{u-1}(1-a)^{v-1}dadb\\ &= \left(r+\frac{u}{u+v}\right)\Bigg[\frac{1}{p}\,ln\left(1+\frac{p}{Z_r+c}\right)+\frac{Z_r^2}{c^2p}\,ln\left(1+\frac{p}{Z_r}\right)\\ &-\frac{\left(Z_r+p\right)^2}{c^2p}\,ln\left(1+\frac{c}{Z_r+p}\right)+\frac{1}{c}\Bigg]. \qquad (22)\end{aligned}$$

4.3. E-Bayesian estimate of θ under ELF

Under Entropy Loss function, E-Bayesian estimate of θ based on $\pi_1(a,b)$ is obtained by using (12) and (14) as

$$\begin{aligned}\hat{\theta}_{EEL_1} &= \int_0^1\int_0^c \left(\frac{r+a-1}{Z_r+b}\right)\frac{1}{cB(u,v)}a^{u-1}(1-a)^{v-1}dadb\\ &= \frac{1}{c}\left(r+\frac{u}{u+v}-1\right)\int_0^c \frac{1}{Z_r+b}db\\ &= \frac{1}{c}\left(r+\frac{u}{u+v}-1\right)ln\left(\frac{Z_r+c}{Z_r}\right). \qquad (23)\end{aligned}$$

Similarly, E-Bayesian estimate of θ based on $\pi_2(a,b)$ is obtained by using (12) and (15) as

$$\begin{aligned}\hat{\theta}_{EEL_2} &= \int_0^1\int_0^c \left(\frac{r+a-1}{Z_r+b}\right)\frac{2}{c^2B(u,v)}(c-b)a^{u-1}(1-a)^{v-1}dadb\\ &= \frac{2}{c^2}\left(r+\frac{u}{u+v}-1\right)\int_0^c \frac{c-b}{Z_r+b}db\\ &= \frac{2}{c}\left(r+\frac{u}{u+v}-1\right)\left[\left(\frac{Z_r+c}{c}\right)ln\left(\frac{Z_r+c}{c}\right)-1\right]. \qquad (24)\end{aligned}$$

Similarly, E-Bayesian estimate of θ based on $\pi_3(a,b)$ is obtained by using (12) and (16) as

$$\begin{aligned}\hat{\theta}_{EEL_3} &= \int_0^1\int_0^c\left(\frac{r+a-1}{Z_r+b}\right)\frac{2b}{c^2B(u,v)}a^{u-1}(1-a)^{v-1}dadb\\ &= \frac{2}{c^2}\left(r+\frac{u}{u+v}-1\right)\int_0^c\frac{b}{Z_r+b}db\\ &= \frac{2}{c}\left(r+\frac{u}{u+v}-1\right)\left[1-\left(\frac{Z_r}{c}\right)ln\left(\frac{Z_r+c}{Z_r}\right)\right]. \end{aligned} \tag{25}$$

5. Relation among Different E-Bayesian Estimators

In this section, we have derived the relationship among three E-Bayesian estimators $\hat{\theta}_{ESE_i}$, $\hat{\theta}_{ELL_i}$, and $\hat{\theta}_{EEL_i}$,where i=1,2,3, which represents different loss function as defined in previous section 4. We have established the relationship among estimators in the form of theorems as follows

Theorem 1 *E-Bayes estimator of parameter $\theta(\hat{\theta}_{ESE_i}, i=1,2,3)$ under Squared Error loss function follows the following relation*

$$i)\quad \hat{\theta}_{ESE_3}<\hat{\theta}_{ESE_2}<\hat{\theta}_{ESE_1}$$
$$ii)\ \lim_{Z_r\to\infty}\hat{\theta}_{ESE_1}=\lim_{Z_r\to\infty}\hat{\theta}_{ESE_2}=\lim_{Z_r\to\infty}\hat{\theta}_{ESE_3}$$

Proof (i): From (17) and (18) , we have

$$\hat{\theta}_{ESE_2}-\hat{\theta}_{ESE_3}=\frac{1}{c}\left(r+\frac{u}{u+v}\right)\left[\left(\frac{c+2Z_r}{c}\right)log\left(\frac{Z_r+c}{Z_r}\right)-2\right]. \tag{26}$$

From (18) and (19)

$$\hat{\theta}_{ESE_1}-\hat{\theta}_{ESE_3}=\frac{1}{c}\left(r+\frac{u}{u+v}\right)\left[\left(\frac{c+2Z_r}{c}\right)log\left(\frac{Z_r+c}{Z_r}\right)-2\right]. \tag{27}$$

From (26) and (27), we get

$$\begin{aligned}\hat{\theta}_{ESE_2}-\hat{\theta}_{ESE_1} &= \hat{\theta}_{ESE_1}-\hat{\theta}_{ESE_3}\\ &= \frac{1}{c}\left(r+\frac{u}{u+v}\right)\left[\left(\frac{c+2Z_r}{c}\right)log\left(\frac{Z_r+c}{Z_r}\right)-2\right]. \end{aligned}\tag{28}$$

The difference in (28) is positive for any $c>0$ and $Z_r>0$.
Let x= c/Z_r. Eqn. (28) can be rewritten and represented as

$$f(x)=\frac{1}{c}\left(r+\frac{u}{u+v}\right)f_1(x) \tag{29}$$

where

$$f_1(x)=\left(1+2/x\right)ln(1+x)-2, x>0.$$

Then we have $\lim_{x\to 0}f_1(x)=0, \lim_{x\to\infty}f_1(x)=\infty$, and $f_1'(x)>0$ for $x>0$, that is $f_1(x)$ is an increasing function, and $f_1(x)>0$. Hence from (29), we have $f(x)>0$ for any $c>0$ and $Z_r>0$ which implies $\hat{\theta}_{ESE_3}<\hat{\theta}_{ESE_2}<\hat{\theta}_{ESE_1}$. □

Proof (ii): As $Z_r\to\infty$ then $x\to 0$ for fixed value of c. From (28) and (29), we have

$$\lim_{Z_r\to\infty}\left(\hat{\theta}_{ESE_1}-\hat{\theta}_{ESE_2}\right)=\lim_{Z_r\to\infty}\left(\hat{\theta}_{ESE_2}-\hat{\theta}_{ESE_3}\right)$$

$$= \frac{1}{c}\left(r + \frac{u}{u+v}\right) f_1(x) = 0 \,. \tag{30}$$

Thus,

$$\lim_{Z_t \to \infty} \hat{\theta}_{ESE_1} = \lim_{Z_r \to \infty} \hat{\theta}_{ESE_2} = \lim_{Z_r \to \infty} \hat{\theta}_{ESE_3} \,. \qquad \square$$

Theorem 2 *E-Bayes estimator of parameter* $\theta(\hat{\theta}_{ELL_i}, i = 1, 2, 3)$ *under LINEX loss function follows the following relation*

$$i) \quad \hat{\theta}_{ELL_1} < \hat{\theta}_{ELL_2} < \hat{\theta}_{ELL_3}$$
$$ii) \lim_{Z_r \to \infty} \hat{\theta}_{ELL_1} = \lim_{Z_r \to \infty} \hat{\theta}_{ELL_2} = \lim_{Z_r \to \infty} \hat{\theta}_{ELL_3}$$

Proof (i): From (20) and (21) , we have

$$\hat{\theta}_{ELL_1} - \hat{\theta}_{ELL_2} = \frac{1}{cp}\left(r + \frac{u}{u+v}\right)\left[\left(\frac{Z_r^2}{c} + Z_r\right) ln\left(1 + \frac{c}{Z_r}\right) - \left(\frac{(Z_r+p)^2}{c} + Z_r + p\right) ln\left(1 + \frac{c}{Z_r + p}\right) + p\right] . \tag{31}$$

From (21) and (22), we have

$$\hat{\theta}_{ELL_3} - \hat{\theta}_{ELL_1} = \frac{1}{cp}\left(r + \frac{u}{u+v}\right)\left[\left(\frac{Z_r^2}{c} + Z_r\right) ln\left(1 + \frac{c}{Z_r}\right) - \left(\frac{(Z_r+p)^2}{c} + Z_r + p\right) ln\left(1 + \frac{c}{Z_r + p}\right) + p\right] . \tag{32}$$

From (31) and (32) , we have

$$\hat{\theta}_{ELL_1} - \hat{\theta}_{ELL_2} = \hat{\theta}_{ELL_3} - \hat{\theta}_{ELL_1} = \frac{1}{cp}\left(r + \frac{u}{u+v}\right)\left[\left(\frac{Z_r^2}{c} + Z_r\right) ln\left(1 + \frac{c}{Z_r}\right) - \left(\frac{(Z_r+p)^2}{c} + Z_r + p\right) ln\left(1 + \frac{c}{Z_r + p}\right) + p\right] . \tag{33}$$

Let $x = c/Z_r$ and $f(c, Z_r, w) = \hat{\theta}_{ELL_1} - \hat{\theta}_{ELL_2} = \hat{\theta}_{ELL_3} - \hat{\theta}_{ELL_1}$. Then, Eqn. (33) can be rewritten and represented as

$$f(x) = \frac{1}{cp}\left(r + \frac{u}{u+v}\right) f_1(x) \tag{34}$$

where

$$f_1(x) = \frac{Z_r(1+x)}{x} ln(1+x) - \left(\frac{Z_r + 2w}{x} + \frac{w^2}{c} + Z_r + w\right) ln\left(1 + \frac{xc}{c + xw}\right) + w \,.$$

As the limits of $f(x)$ are investigated as $x \to 0$ and $x \to \infty$. For any $c, f(x) = \lim_{x\to 0} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$. It is clear that $f_1'(x) > 0$ for $p > 0$ that is $f_1(x)$ is an increasing function for $f_1(x) > 0$ for any $c > 0, c > 0$ and $Z_r > 0$. Hence from (34), we have $\hat{\theta}_{ELL_1} < \hat{\theta}_{ELL_2} < \hat{\theta}_{ELL_3}$. $\square$

Proof (ii): As $Z_r \to \infty$ then $x \to 0$ for fixed value of c. From (33) and (34), we have

$$\lim_{Z_r \to \infty}\left(\hat{\theta}_{ELL_1} - \hat{\theta}_{ELL_2}\right) = \lim_{Z_r \to \infty}\left(\hat{\theta}_{ELL_2} - \hat{\theta}_{ELL_3}\right) = \frac{1}{cp}\left(n + \frac{u}{u+v}\right) f_1(x) = 0 \tag{35}$$

Thus,

$$\lim_{Z_t \to \infty} \hat{\theta}_{ELL_1} = \lim_{Z_r \to \infty} \hat{\theta}_{ELL_2} = \lim_{Z_r \to \infty} \hat{\theta}_{ELL_3} . \qquad \square$$

Theorem 3 *E-Bayes estimator of parameter* $\theta(\hat{\theta}_{EEL_i}, i = 1, 2, 3)$ *under Entropy loss function follows the following relation*

$$i) \quad \hat{\theta}_{EEL_3} < \hat{\theta}_{EEL_2} < \hat{\theta}_{EEL_1}$$
$$ii) \lim_{Z_r \to \infty} \hat{\theta}_{EEL_1} = \lim_{Z_r \to \infty} \hat{\theta}_{EEL_2} = \lim_{Z_r \to \infty} \hat{\theta}_{EEL_3} .$$

Proof (i): From (23) and (24) , we have

$$\hat{\theta}_{EEL_2} - \hat{\theta}_{EEL_3} = \frac{1}{c}\left(r + \frac{u}{u+v} - 1\right)\left[\left(\frac{c+2Z_r}{c}\right)log\left(\frac{Z_r+c}{Z_r}\right) - 1\right]. \qquad (36)$$

From (24) and (25)

$$\hat{\theta}_{EEL_1} - \hat{\theta}_{EEL_3} = \frac{1}{c}\left(r + \frac{u}{u+v} - 1\right)\left[\left(\frac{c+2Z_r}{c}\right)log\left(\frac{Z_r+c}{Z_r}\right) - 1\right]. \qquad (37)$$

From (36) and (37), we get

$$\begin{aligned}\hat{\theta}_{EEL_2} - \hat{\theta}_{EEL_1} &= \hat{\theta}_{EEL_1} - \hat{\theta}_{EEL_3} \\ &= \frac{1}{c}\left(r + \frac{u}{u+v} - 1\right)\left[\left(\frac{c+2Z_r}{c}\right)log\left(\frac{Z_r+c}{Z_r}\right) - 1\right]. \qquad (38)\end{aligned}$$

It will be shown that the difference in (38) is positive for any $c > 0$ and $Z_r > 0$. Let x= c/Z_r. Eqn. (38) can be rewritten and represented as

$$f(x) = \frac{1}{c}\left(r + \frac{u}{u+v} - 1\right)f_1(x) \qquad (39)$$

where

$$f_1(x) = \left(1 + 2/x\right)ln(1+x) - 1, x > 0.$$

Then we have $\lim_{x\to 0} f_1(x) = 0, \lim_{x\to\infty} f_1(x) = \infty$, and $f_1^{'}(x) > 0$ for $x > 0$, that is $f_1(x)$ is an increasing function ($f_1(x) > 0$). Hence from (29), we have $f(x) > 0$ for any $c > 0$ and $Z_r > 0$ we have $\hat{\theta}_{EEL_3} < \hat{\theta}_{EEL_2} < \hat{\theta}_{EEL_1}$. $\square$

Proof (ii): As $Z_r \to \infty$ then $x \to 0$ for fixed value of c. From (38) and (39), we have

$$\lim_{Z_r \to \infty}\left(\hat{\theta}_{EEL_1} - \hat{\theta}_{EEL_2}\right) = \lim_{Z_r \to \infty}\left(\hat{\theta}_{EEL_2} - \hat{\theta}_{EEL_3}\right)$$

$$= \frac{1}{c}\left(r + \frac{u}{u+v}\right)f_1(x) = 0 . \qquad (40)$$

Thus,

$$\lim_{Z_t \to \infty} \hat{\theta}_{EEL_1} = \lim_{Z_r \to \infty} \hat{\theta}_{EEL_2} = \lim_{Z_r \to \infty} \hat{\theta}_{EEL_3} . \qquad \square$$

6. Monte Carlo Simulation Study

In this section, We have carried out Monte Carlo simulations to compare the Bayesian and E-Bayesian estimates. To illustrate the methods proposed in the paper. We have analysed a real data set of golfer income used by Arnold (2015). We choose Pareto distribution $P(\lambda, \theta)$, defined as

$$F(x,\theta) = 1 - \left(\frac{\lambda}{x}\right)^{\theta} \quad x, \theta, \lambda > 0 \tag{41}$$

as definition family as per (2). The algorithm used for above purpose is defined as

1. Set the default values $u = 1.5, v = 0.5, k = 2, \lambda = 6, c = 6.5, m = 1, p = 0.5, n = 150, 100$ and r (100,120,130,140) and (60,70,80,90), with respect to n.
2. Generate a and b from Beta hyper-prior and Gamma hyper-prior distribution, respectively.
3. Use generated values of a and b, to generate θ from Gamma prior using Eqn. (6)
4. Draw random sample of size n from Pareto distribution for known values of λ.
5. Compute Bayes estimates and E-Bayes estimates of the unknown shape parameter under different loss functions using $(8), (9), (12), (17), (18), (19)(20), (21), (22), (23), (24)$ and (25) respectively.
6. Repeat the steps (1 to 5) 10,000 times and compute the MSE for all the estimates for different sample sizes and termination sample numbers (r). where ,

$$MSE = \frac{1}{N}\sum_i (\hat{\theta}_i - \theta)^2$$

and $\hat{\theta}$ stands for the estimator of θ.

To implement this algorithm, we have written a program in R(ver 3.6.3) well as Matlab 2016a. The obtained results are compared on the basis of their MSE and bias. We have illustrated the conclusion through tables for different combinations of influencing parameters.

Table 1 Bias and MSE of Bayes and E-Bayes estimates of θ for different values of r and fixed $n = 150, \theta = 0.21220$

$Loss function \rightarrow$		SEF				LLF				ELF		
$r \downarrow$	$\hat{\theta}_{BSE}$	$\hat{\theta}_{ESE_1}$	$\hat{\theta}_{ESE_2}$	$\hat{\theta}_{ESE_3}$	$\hat{\theta}_{BLL}$	$\hat{\theta}_{ELL_1}$	$\hat{\theta}_{ELL_2}$	$\hat{\theta}_{ELL_3}$	$\hat{\theta}_{BEL}$	$\hat{\theta}_{EEL_1}$	$\hat{\theta}_{EEL_2}$	$\hat{\theta}_{EEL_3}$
100	0.16327	0.16426	0.16454	0.16397	0.16320	0.16419	0.16448	0.16390	0.16204	0.16264	0.16292	0.16235
	-0.048937	-0.04794	-0.04765	-0.04823	-0.04900	-0.04801	-0.04772	-0.04830	-0.05015	-0.04956	-0.04927	-0.04984
	(0.00239)	(0.00229)	**(0.00227)**	(0.00232)	(0.00240)	(0.00230)	**(0.00227)**	(0.00233)	(0.00251)	(0.00245)	**(0.00242)**	(0.00248)
120	0.16506	0.16589	0.16613	0.16564	0.16500	0.16583	0.16608	0.16559	0.16403	0.16452	0.16477	0.16428
	-0.04714	-0.04631	-0.04606	-0.04650	-0.04720	-0.04637	-0.04612	-0.04661	-0.04817	-0.04768	-0.04743	-0.04792
	(0.00222)	(0.00214)	**(0.00212)**	(0.00216)	(0.00222)	(0.00215)	**(0.00212)**	(0.00217)	(0.00232)	(0.00227)	**(0.00225)**	(0.00229)
130	0.16529	0.16606	0.16629	0.16583	0.16524	0.16601	0.16623	0.16578	0.16434	0.16480	0.16502	0.16457
	-0.04690	-0.04614	-0.04591	-0.04637	-0.04696	-0.04619	-0.04596	-0.04642	-0.04785	-0.04740	-0.04718	-0.04763
	(0.00220)	(0.00212)	**(0.00210)**	(0.00215)	(0.00220)	(0.00213)	**(0.00211)**	(0.00215)	(0.00229)	(0.00224)	**(0.00222)**	(0.00226)
140	0.16674	0.16745	0.16767	0.16724	0.16669	0.16740	0.16762	0.16719	0.16585	0.16627	0.16648	0.16606
	-0.04546	-0.04474	-0.04453	-0.04496	-0.04551	-0.04479	-0.04458	-0.04501	-0.04634	-0.04593	-0.04571	-0.04614
	(0.00206)	(0.00200)	**(0.00198)**	(0.00202)	(0.00207)	(0.00200)	**(0.00198)**	(0.00202)	(0.00214)	(0.00210)	**(0.00209)**	(0.00212)

Table 2 Bias and MSE of Bayes and E-Bayes estimates of θ for different values of r and fixed $n = 100, \theta = 0.21776$

$Loss function \rightarrow$		SEF				LLF				ELF		
$r \downarrow$	$\hat{\theta}_{BSE}$	$\hat{\theta}_{ESE_1}$	$\hat{\theta}_{ESE_2}$	$\hat{\theta}_{ESE_3}$	$\hat{\theta}_{BLL}$	$\hat{\theta}_{ELL_1}$	$\hat{\theta}_{ELL_2}$	$\hat{\theta}_{ELL_3}$	$\hat{\theta}_{BEL}$	$\hat{\theta}_{EEL_1}$	$\hat{\theta}_{EEL_2}$	$\hat{\theta}_{EEL_3}$
60	0.16475	0.16626	0.16675	0.16577	0.16463	0.16615	0.16665	0.16566	0.16267	0.16356	0.16404	0.16308
	-0.05301	-0.05149	-0.05100	-0.05198	-0.05312	-0.05160	-0.05112	-0.05209	-0.05508	-0.05419	-0.05372	-0.05467
	(0.00281)	(0.00265)	**(0.00260)**	(0.00270)	(0.002822)	(0.00266)	**(0.00261)**	(0.00271)	(0.00303)	(0.00293)	**(0.00288)**	(0.00298)
70	0.16584	0.16714	0.16757	0.16672	0.16574	0.16704	0.16662	0.16566	0.16405	0.16480	0.16522	0.16439
	-0.05191	-0.05061	-0.05019	-0.05103	-0.05201	-0.05071	-0.05028	-0.05113	-0.05370	-0.05295	-0.05253	-0.05336
	(0.00269)	(0.00256)	**(0.00251)**	(0.00260)	(0.00270)	(0.00257)	**(0.00252)**	(0.00261)	(0.00278)	(0.00271)	**(0.00267)**	(0.00275)
80	0.16547	0.16769	0.16806	0.16731	0.16646	0.16760	0.16797	0.16723	0.16497	0.16563	0.16600	0.16526
	-0.05121	-0.05007	-0.04969	-0.05044	-0.05130	-0.05015	-0.04978	-0.05053	-0.05278	-0.05212	-0.05176	-0.05249
	(0.00262)	(0.00250)	**(0.00246)**	(0.00254)	(0.00263)	(0.00251)	**(0.00247)**	(0.00253)	(0.00263)	(0.00251)	**(0.00247)**	(0.00253)
90	0.16741	0.16842	0.16876	0.16809	0.16733	0.16835	0.16868	0.16801	0.16600	0.16658	0.16692	0.16625
	-0.05035	-0.04933	-0.04899	-0.04966	-0.05042	-0.04941	-0.04907	-0.04974	-0.05175	-0.05117	-0.05084	-0.05150
	(0.00253)	(0.00243)	**(0.00240)**	(0.00246)	(0.00254)	(0.00244)	**(0.00240)**	(0.00267)	(0.00261)	(0.00258)	**(0.00265)**	(0.00247)

In Table 1, SEF , LLF and ELF represents the Squared Error Loss Function, LINEX Loss Function and Entropy Loss Function respectively. The values in column represents the estimated values of θ under corresponding loss function. The first row values represents the estimated values, second row values represents the bias and third row values in parenthesis represents the MSE.

In Table 2, SEF, LLF and ELF represents the Squared Error Loss Function, LINEX Loss Function and Entropy Loss Function respectively. The values in column represents the estimated values of θ under corresponding loss function. The first row values represents the estimated values, second row values represents the bias and third row values in parenthesis represents the MSE.

In the Tables 1 and 2, we observed their MSE of the estimators decrease as the sample size n increase. The MSE under SEF are ordered as $\hat{\theta}_{ESE_2} < \hat{\theta}_{ESE_1} < \hat{\theta}_{ESE_3} < \hat{\theta}_{BSE}$. Similarly, MSE under LINEX and Entropy losss function are ordered as $\hat{\theta}_{ELL_2} < \hat{\theta}_{ELL_1} < \hat{\theta}_{ELL_3} < \hat{\theta}_{BLL}$ and $\hat{\theta}_{EEL_2} < \hat{\theta}_{EEL_1} < \hat{\theta}_{EEL_3} < \hat{\theta}_{BEL}$. Moreover all the MSE are close to each other as n increases. Bayesian and E-Bayesian estimates under Squared loss function given better estimates than LINEX and Entropy loss function.

Real Data Set :- A real life data set, of golfers income is considered to illustrate the use of the proposed estimators. This data set is a secondary data and has been used by Arnold (2015) for the observation of the 50 golfers earning more than 70000 dollar, their income by the end of the 1980 are year data as given below.

```
3581,1690, 1433, 1184, 1066, 1005, 883, 841, 778, 753, 2474, 1684, 1410,
1171, 1056, 1001, 878, 825, 778, 746, 2202, 1627, 1374, 1109, 1051, 965,
871, 820, 771, 729, 1858, 1537, 1338, 1095, 1031, 944, 849, 816, 769, 712,
1829, 1519, 1208, 1092, 1016, 912, 844, 814, 759, 708
```

Similar algorithm followed to obtained the proposed estimators. Also performance of Bayesian and E-Bayesian estimators of the unknown parameter θ are compared for different values of c and λ. Then generated a and b from Uniform and Beta priors. Let $u = 1.5$, $v = 0.5$, $k = 1$, $n = 50$, $m = 1$, $p = 0.5$, $\lambda = 15, 20, 30$ and $c = 20, 25, 35$ respectively.

In Table 3, SEF , LLF and ELF represents the Squared Error Loss Function, LINEX Loss Function and Entropy Loss Function respectively. The values in column represents the estimated values of θ under corresponding loss function. The first row values represents the estimated values, second row values represents the bias and third row values in parenthesis represents the MSE.

In Table 4, SEF, LLF and ELF represents the Squared Error Loss Function, LINEX Loss Function and Entropy Loss Function respectively. The values in column represents the estimated values of θ under corresponding loss function. The first row values represents the estimated values, second row values represents the bias and third row values in parenthesis represents the MSE.

In Table 5, SELF, LLF and ELF represents the Squared Error Loss Function, LINEX Loss Function and Entropy Loss Function respectively. The values in column represents the estimated values of θ under corresponding loss function. The first row values represents the estimated values, second row values represents the bias and third row values in parenthesis represents the MSE. From the Tables 3, 4 and 5, the MSE of the estimators decrease as the sample size n increase. The MSE under SEF are ordered as $\hat{\theta}_{ESE_3} < \hat{\theta}_{ESE_2} < \hat{\theta}_{BSE} < \hat{\theta}_{ESE_1}$. Similarly MSE under LINEX and Entropy loss function are ordered as $\hat{\theta}_{ELL_3} < \hat{\theta}_{ELL_2} < \hat{\theta}_{BLL} < \hat{\theta}_{ELL_3}$ and $\hat{\theta}_{EEL_3} < \hat{\theta}_{ELL_2} < \hat{\theta}_{BEL} < \hat{\theta}_{EEL_1}$. Moreover all the MSE are close to each other as n increases. Bayesian and E- Bayesian estimates under Entropy loss function given better estimates than LINEX and Square error loss function.

Table 3 Bias and MSE of Bayes and E-Bayes Estimates of θ for different values r and fixed $n = 50, \theta = 0.09826$

$Loss function \rightarrow$		SEF				LLF				ELF		
$r \downarrow$	$\hat{\theta}_{BSE}$	$\hat{\theta}_{ESE_1}$	$\hat{\theta}_{ESE_2}$	$\hat{\theta}_{ESE_3}$	$\hat{\theta}_{BLL}$	$\hat{\theta}_{ELL_1}$	$\hat{\theta}_{ELL_2}$	$\hat{\theta}_{ELL_3}$	$\hat{\theta}_{BEL}$	$\hat{\theta}_{EEL_1}$	$\hat{\theta}_{EEL_2}$	$\hat{\theta}_{EEL_3}$
10	0.15834	0.15470	0.16148	0.14791	0.15775	0.154191	0.16092	0.14745	0.14620	0.14153	0.14773	0.13533
	0.06007	0.05643	0.06321	0.04965	0.05949	0.05592	0.06299	0.04919	0.04793	0.04371	0.04947	0.03706
	(0.00360)	(0.00318)	(0.00399)	**(0.00246)**	(0.00353)	(0.00312)	(0.00392)	**(0.00241)**	(0.00229)	(0.00187)	(0.00244)	**(0.00137)**
20	0.13999	0.13907	0.14203	0.13610	0.13975	0.13884	0.14180	0.13589	0.13436	0.13267	0.13550	0.12985
	0.04173	0.04080	0.04377	0.03784	0.04149	0.04058	0.04353	0.03763	0.03610	0.03441	0.03724	0.03158
	(0.00174)	(0.00166)	(0.00191)	**(0.00143)**	(0.00172)	(0.00164)	(0.00189)	**(0.00141)**	(0.00130)	(0.00118)	(0.00138)	**(0.0099)**
30	0.13267	0.13230	0.13414	0.13047	0.13252	0.13217	0.13400	0.13033	0.12905	0.12814	0.12992	0.12636
	0.03440	0.03404	0.03588	0.03220	0.03426	0.03390	0.03574	0.03207	0.03078	0.02987	0.03165	0.02809
	(0.00118)	(0.00115)	(0.00128)	**(0.00103)**	(0.00117)	(0.00114)	(0.00127)	**(0.00102)**	(0.00094)	(0.00089)	(0.00100)	**(0.00078)**
40	0.12768	0.12754	0.12884	0.12624	0.12758	0.12744	0.12874	0.12615	0.124504	0.12449	0.12575	0.12322
	0.02942	0.02928	0.03058	0.02798	0.03051	0.02588	0.02896	0.02766	0.02677	0.02622	0.02749	0.02496
	(0.00086)	(0.00085)	(0.00093)	**(0.00078)**	(0.00093)	(0.00067)	(0.00083)	**(0.00052)**	(0.00071)	(0.00068)	(0.00075)	**(0.00062)**

Table 4 Bias and MSE of Bayes and E-Bayes estimates of θ for different values r and and fixed $n = 50, \theta = 0.10304$

$Loss function \rightarrow$		SEF				LLF				ELF		
$r \downarrow$	$\hat{\theta}_{BSE}$	$\hat{\theta}_{ESE_1}$	$\hat{\theta}_{ESE_2}$	$\hat{\theta}_{ESE_3}$	$\hat{\theta}_{BLL}$	$\hat{\theta}_{ELL_1}$	$\hat{\theta}_{ELL_2}$	$\hat{\theta}_{ELL_3}$	$\hat{\theta}_{BEL}$	$\hat{\theta}_{EEL_1}$	$\hat{\theta}_{EEL_2}$	$\hat{\theta}_{EEL_3}$
10	0.16824 0.06520 (0.00425)	0.16031 0.05727 (0.00328)	0.16941 0.06637 (0.00440)	0.15122 0.04818 **(0.00232)**	0.16757 0.06453 (0.00416)	0.15976 0.05672 (0.00321)	0.16880 0.06576 (0.00432)	0.15073 0.04769 **(0.002274)**	0.15563 0.05259 (0.00276)	0.14667 0.04363 (0.00190)	0.15499 0.045195 (0.00269)	0.13835 0.03531 **(0.00124)**
20	0.14974 0.04670 (0.00218)	0.14684 0.04380 (0.00191)	0.15097 0.04793 (0.00229)	0.14271 0.03967 **(0.00157)**	0.14947 0.04643 (0.00215)	0.14660 0.043560 (0.00189)	0.15071 0.04767 (0.00227)	0.14248 0.03944 **(0.00155)**	0.14388 0.04084 (0.00166)	0.14009 0.03705 (0.00137)	0.14403 0.04099 (0.00168)	0.13615 0.03311 **(0.00109)**
30	0.14208 0.03904 (0.00152)	0.14050 0.03745 (0.00140)	0.14309 0.04000 (0.00160)	0.13790 0.03486 **(0.00121)**	0.14192 0.03888 (0.00151)	0.14034 0.03730 (0.00139)	0.14292 0.03988 (0.00159)	0.13776 0.03472 **(0.00120)**	0.13831 0.03527 (0.00124)	0.13607 0.03303 (0.00109)	0.13858 0.03554 (0.00126)	0.13356 0.03052 **(0.00093)**
40	0.13671 0.03367 (0.00113)	0.13572 0.03268 (0.00106)	0.13756 0.03452 (0.00119)	0.13388 0.03084 **(0.00095)**	0.13659 0.03051 (0.00093)	0.13561 0.02588 (0.00067)	0.13744 0.02896 (0.00083)	0.13377 0.02766 **(0.00052)**	0.13396 0.03092 (0.00095)	0.13247 0.02943 (0.00086)	0.13426 0.03122 (0.00095)	0.13067 0.02763 **(0.00076)**

Table 5 Bias and MSE of Bayes and E-Bayes estimates of θ for different values r and and fixed $n = 50, \theta = 0.12244$

$Loss function \rightarrow$	SEF				LLF				ELF			
$r \downarrow$	$\hat{\theta}_{BSE}$	$\hat{\theta}_{ESE_1}$	$\hat{\theta}_{ESE_2}$	$\hat{\theta}_{ESE_3}$	$\hat{\theta}_{BLL}$	$\hat{\theta}_{ELL_1}$	$\hat{\theta}_{ELL_2}$	$\hat{\theta}_{ELL_3}$	$\hat{\theta}_{BEL}$	$\hat{\theta}_{EEL_1}$	$\hat{\theta}_{EEL_2}$	$\hat{\theta}_{EEL_3}$
10	0.19054	0.16626	0.17993	0.15259	0.18969	0.16567	0.17924	0.15209	0.17686	0.15211	0.16462	0.13961
	0.06810	0.04381	0.05748	0.03014	0.06724	0.04322	0.05679	0.02964	0.05441	0.02966	0.04217	0.01716
	(0.00463)	(0.00191)	(0.00330)	**(0.00090)**	(0.00452)	(0.00186)	(0.00322)	**(0.00087)**	(0.00296)	(0.00088)	(0.00177)	**(0.00294)**
20	0.16881	0.15792	0.16461	0.15124	0.16847	0.15764	0.16429	0.15098	0.16243	0.15066	0.15704	0.14429
	0.04636	0.03547	0.04216	0.02879	0.04602	0.03519	0.04184	0.02853	0.03998	0.02821	0.03459	0.02184
	(0.00215)	(0.00125)	(0.00177)	**(0.00082)**	(0.00211)	(0.00123)	(0.00175)	**(0.00018)**	(0.00159)	(0.00079)	(0.00119)	**(0.00477)**
30	0.15963	0.15229	0.15722	0.14863	0.15943	0.15274	0.15703	0.14846	0.15552	0.148117	0.15227	0.14395
	0.03718	0.03048	0.03477	0.02618	0.03698	0.03029	0.03458	0.02601	0.03308	0.02566	0.02982	0.02150
	(0.00138)	(0.00092)	(0.00120)	**(0.00068)**	(0.00136)	(0.00091)	(0.00119)	**(0.00067)**	(0.00109)	(0.00065)	(0.00088)	**(0.00046)**
40	0.15311	0.14847	0.15155	0.14539	0.15296	0.14833	0.15141	0.14526	0.15011	0.14491	0.14792	0.14190
	0.04636	0.03547	0.04216	0.02879	0.03051	0.02588	0.02896	0.02281	0.02766	0.02246	0.02547	0.01946
	(0.00094)	(0.00067)	(0.00084)	**(0.00052)**	(0.00093)	(0.00067)	(0.00083)	**(0.00052)**	(0.00076)	(0.00050)	(0.00064)	**(0.000378)**

7. Conclusions

In this paper, we have used Bayesian and E-Bayesian technique to obtain the estimates of the unknown parameters of generalized order statistics of Exponential Family. We have used Pareto distribution for the explanation of Exponential Family. We have used three different loss functions such as SELF, LINEX and Generalized Entropy Loss Function to obtain Bayes and E-Bayes estimators of the unknown parameters. We have made comparison between the performance of the obtained estimators by using Monte Carlo simulation study and using real data set.

Furthermore, we have been compared of the Bayes and E-Bayes estimator in terms their bias and MSE. from the Simulation study, we have observed that their bias and MSE decreases as sample size say r (order statistics) increases and estimator improved. In the numerical study, we seen that the proposed E-Bayes estimators have better performance in terms of their bias and MSE than Bayes estimators. So, E-Bayesian technique is a good choice instead of Bayesian method.

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