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Wrapped Length Biased Exponential Distribution

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Abstract

In this article, a new circular distribution called wrapped length biased exponential distribution is introduced. The properties of the new distribution are discussed and explicit expressions are derived for the characteristic function, trigonometric moments, and other statistical measures like resultant length, mean, circular variance, standard deviation, coefficient of skewness, and kurtosis. The maximum likelihood estimation is used to evaluate the model parameter and simulation study is conducted to investigate the performance of the estimator. Finally, an application of the model to a real data set is presented and compared with the fit attained by some other well-known models in the literature.

Keywords: circular model, estimation, length biased exponential distribution, parameters, wrapped Lindley distribution, skewness and kurtosis.

1. Introduction

Lévy (1939) proposed a novel method for introducing a wrapped variable to the symmetric and asymmetric distributions, which is known in the literature as wrapped distribution. Many useful and interesting circular distributions can be generated from known probability distributions on the real line or plane, by a variety of techniques, a few such general methods like (i) By wrapping a linear distribution around the unit circle. (ii) Through characterizing properties such as maximum entropy. (iii) By transforming a bivariate linear random variable to just its directional component, such distributions are called offset distributions. (iv) By applying inverse stereographic projection on distributions on real line are mentioned in Jammalamadaka and Sengupta (2001). Pewsey (2000) developed wrapped skew-normal distribution on the circle for modeling data by adopting the Azzalini (1985) method of generating skew distributions. Jammalamadaka and Kozubowski (2001, 2003, 2004) discuss circular models obtained by wrapping the classical exponential and Laplace distributions. Rao et al. (2007) discuss some wrapped versions of life testing distributions such as wrapped Weibull, logistic, extreme-value, lognormal distributions and their population characteristics. Abe et al. (2010) discussed a four-parameter family of symmetric unimodel distributions which extends both the Minh and Farnum (2003) and Jones and Pewsey (2005) families. Abe and Pewsey (2011) developed Sine-skewed circular distributions generated by perturbation of symmetric circular distributions. Roy and Adnan (2012a) developed a new circular model called wrapped weighted exponential distribution. Roy and Adnan (2012b) introduced a class of wrapped generalized Gompertz distribution and is used to analyse data on the headings of orientation of the nests of noisy scrub birds. Adnan and Roy (2014) proposed a wrapped variance gamma distribution. Yedlapalli et al. (2013) proposed and studied various properties of stereographic semicircular exponential and Weibull distributions by applying inverse stereographic projection on linear exponential and Weibull variables. Rao et al. (2016) developed stereographic logistic distributions by applying inverse stereographic projection on linear distribution. Joshi and Jose (2018) explored wrapped Lindley distribution. Rao et al. (2018) derived wrapped Lomax distribution and studied its characteristics. Bhattacharjee and Borah (2019) proposed two parameter wrapped length biased weighted exponential distribution and applied it to long-axis orientation of 164 feldspar laths data set. Yedlapalli et al. (2020) proposed a new family of Semi-circular and circular arc tan-exponential type distributions. Ahmad and Ayat (2021a, 2021b), Ahmad and Shawkat (2019), and Ayat and Ahmad (2021) developed and studied some new wrapped distributions like, wrapped quasi Lindley, wrapped Akash, wrapped Shanker and wrapped Ishita probability distributions. Bhattacharjee et al. (2021) proposed wrapped version of two-parameter Lindley distribution for modelling circular data.

In the recent past, many researchers showed attention and motivated in developing new flexible statistical probability models from that of the existing classical models. These flexible new models are having many applications in several applied areas such as lifetime analysis, demography, actuarial science, economics and finance, environmental, engineering, biological and medical sciences (Azzalini 1985, Marshall and Olkin 1997 to name a few). In this connection, we mention a popular distribution, exponential, which has been extensively applied to analyse lifetime data in the field of reliability and survival analysis. Due to the wide range of applications of exponential model many extensions have been proposed and studied by several researchers. One among them is length biased exponential (LBE) distribution (also called moment exponential distribution) proposed by Dara and Ahmad (2012) through assigning weights to the exponential distribution by adapting the idea of Fisher (1934) and showed that the length biased exponential distribution is more flexible than the exponential distribution.

In this article, we motivated to proposed and derived a circular version (wrapped) of length biased exponential distribution, studied some of its properties, fitted the new model to turtle data set and further observed its relative performance with two of its competing distributions namely wrapped exponential (WE) and wrapped Lindley (WL) distributions.

The rest of the article is organized as follows. Section 2 describes a method of synthesis for circular version of length biased exponential distribution following the methodology of wrapping of univariate density and we obtain the explicit form of probability density function for proposed model. The distribution function, characteristic function and some statistical properties of the proposed model are derived in Section 3. Maximum likelihood estimation method is discussed to estimate the model parameter in Section 4. Simulation study is carry out in Section 5 to show the consistency of the estimate. In Section 6, we apply the proposed model to the turtle dataset and compare its performance with wrapped exponential (Jammalamadaka and Kozubowski 2004) and wrapped Lindley (Joshi and Jose 2018) distributions. Finally, the findings of the article are summarized in Section 7.

2. Definition and Synthesis of Wrapped Length Biased Exponential Distribution

In this section, we describe a method of synthesis for circular version of length biased exponential distribution following the methodology of wrapping of univariate density (Jammalamadaka and Kozubowski 2001). In this approach, a known distribution is taken on real line and wrapped around a unit circle. Taking a real random variable X and wrapped it around the circle by transformation $X \pmod{2\pi}$. Here we call the following definition of length biased exponential distribution (Dara and Ahmad 2012).

Definition 1 A random variable X (on real line) is said to have the length biased exponential distribution with scale parameter λ if its probability density function (pdf) is defined as

$$f_X = \frac{x}{\lambda^2} e^{-\frac{x}{\lambda}}, \text{ where } x > 0, \lambda > 0.$$

The corresponding cumulative distribution function is

$$F_X = 1 - \left(1 + \frac{x}{\lambda}\right) e^{-\frac{x}{\lambda}}, \text{ where } x > 0, \lambda > 0.$$

Now, the wrapped length biased exponential random variable θ is defined as $\theta = X\left(\operatorname{mod} 2\pi\right)$ such that for $\theta \in [0, 2\pi)$, the probability density function of θ is given by

$$\begin{aligned} g(\theta) &= \sum_{m=0}^{\infty} f_X(\theta + 2\pi m) = \sum_{m=0}^{\infty}\left(\frac{\theta + 2\pi m}{\lambda^2}\right) e^{-\left(\frac{\theta + 2\pi m}{\lambda}\right)} = \frac{e^{-\frac{\theta}{\lambda}}}{\lambda^2}\sum_{m=0}^{\infty}(\theta + 2\pi m) e^{-\left(\frac{2\pi m}{\lambda}\right)} \\ &= \frac{e^{-\frac{\theta}{\lambda}}}{\lambda^2}\left(\theta \sum_{m=0}^{\infty} e^{-\left(\frac{2\pi m}{\lambda}\right)} + 2\pi \sum_{m=0}^{\infty} m e^{-\left(\frac{2\pi m}{\lambda}\right)}\right) = \frac{e^{-\frac{\theta}{\lambda}}}{\lambda^2}\left(\frac{\theta}{\left(1 - e^{-\frac{2\pi}{\lambda}}\right)} + \frac{2\pi e^{-\frac{2\pi}{\lambda}}}{\left(1 - e^{-\frac{2\pi}{\lambda}}\right)^2}\right) \\ &= \frac{e^{-\frac{1}{\lambda}(\theta - 2\pi)}}{\lambda^2\left(e^{\frac{2\pi}{\lambda}} - 1\right)}\left(\theta + \frac{2\pi}{\left(e^{\frac{2\pi}{\lambda}} - 1\right)}\right). \end{aligned}$$

Definition 2 A circular random variable θ is said to follow the wrapped length biased exponential distribution with scale parameter λ, if its probability density function is of the form

$$g(\theta) = \frac{e^{-\frac{1}{\lambda}(\theta - 2\pi)}}{\lambda^2\left(e^{\frac{2\pi}{\lambda}} - 1\right)}\left(\theta + \frac{2\pi}{\left(e^{\frac{2\pi}{\lambda}} - 1\right)}\right), \text{where } \theta \in [0, 2\pi) \text{ and } \lambda > 0.$$

It is denoted by $\theta \sim WLBE(\lambda)$.

Figure 1 depict the shapes of probability density function and Figure 2 is the circular representation density function of the wrapped length biased exponential distribution for various values of parameter.

3. Distributional Properties

In this section, we study distributional properties, such as cumulative distribution function, characteristic function, trigonometric moments and other statistical measure like resultant length, mean, circular variance, standard deviation, skewness and kurtosis of wrapped length biased exponential distribution.

3.1. Cumulative distribution function

The cumulative distribution function of $\theta \sim WLBE(\lambda)$ is obtained as follows (Jammalamadaka and Sengupta 2001).

$$G(\theta) = \sum_{m=0}^{\infty}\left(F_X(\theta + 2\pi m) - F_X(2\pi m)\right)$$

$$= \sum_{m=0}^{\infty}\left(1-\left(1+\left(\frac{\theta+2\pi m}{\lambda}\right)\right)e^{-\left(\frac{\theta+2\pi m}{\lambda}\right)}-1+\left(1+\frac{2\pi m}{\lambda}\right)e^{-\left(\frac{2\pi m}{\lambda}\right)}\right)$$

$$= \sum_{m=0}^{\infty}\left(-e^{-\frac{\theta}{\lambda}}\left(1+\left(\frac{\theta+2\pi m}{\lambda}\right)\right)e^{-\left(\frac{2\pi m}{\lambda}\right)}+\left(1+\frac{2\pi m}{\lambda}\right)e^{-\left(\frac{2\pi m}{\lambda}\right)}\right)$$

$$= -\frac{e^{-\frac{\theta}{\lambda}}}{\left(1-e^{-\frac{2\pi}{\lambda}}\right)}-\left(\frac{\theta e^{-\frac{\theta}{\lambda}}}{\lambda\left(1-e^{-\frac{2\pi}{\lambda}}\right)}\right)-\left(\frac{2\pi e^{-\frac{2\pi}{\lambda}}e^{-\frac{\theta}{\lambda}}}{\lambda\left(1-e^{-\frac{2\pi}{\lambda}}\right)^2}\right)+\frac{1}{\left(1-e^{-\frac{2\pi}{\lambda}}\right)}+\left(\frac{2\pi e^{-\frac{2\pi}{\lambda}}}{\lambda\left(1-e^{-\frac{2\pi}{\lambda}}\right)^2}\right)$$

$$= \frac{e^{\frac{2\pi}{\lambda}}}{\lambda\left(e^{\frac{2\pi}{\lambda}}-1\right)}\left(\lambda-(\lambda+\theta)e^{-\frac{\theta}{\lambda}}+2\pi\left(\frac{1-e^{-\frac{\theta}{\lambda}}}{e^{\frac{2\pi}{\lambda}}-1}\right)\right), \quad (1)$$

where $\theta\in[0,2\pi)$ and $\lambda>0$.

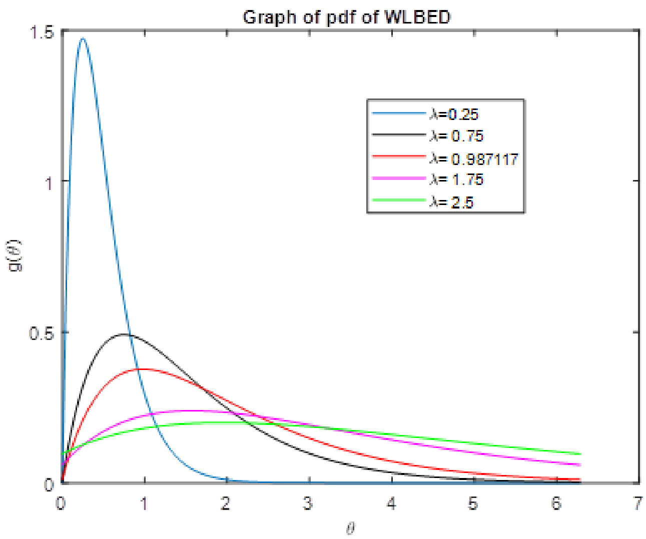

Figure 1 The plots of probability density function of wrapped length biased exponential distribution

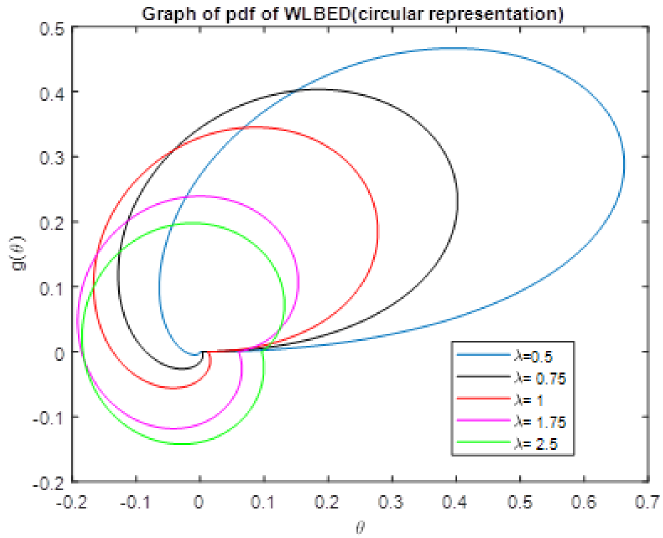

Figure 2 The plots of probability density function of wrapped length biased exponential distribution (circular representation)

Figure 3 depict the shapes of cumulative distribution and Figure 4 is the circular representation of cumulative distribution function of the wrapped length biased exponential distribution for various values of parameter.

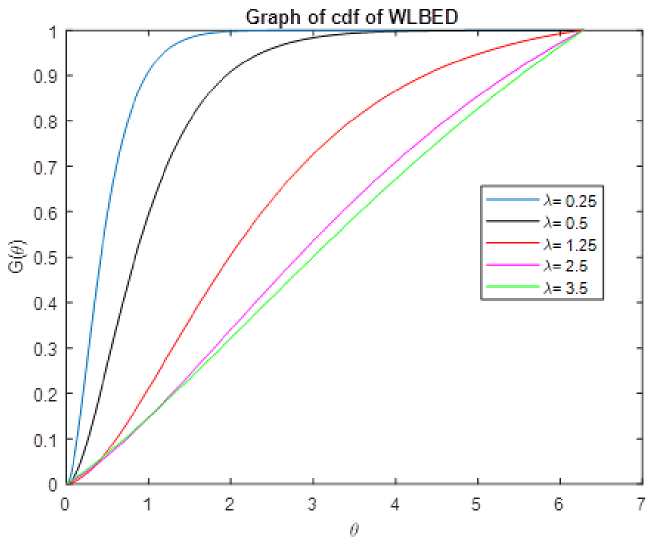

Figure 3 The plots of cumulative distribution function of wrapped length biased exponential distribution

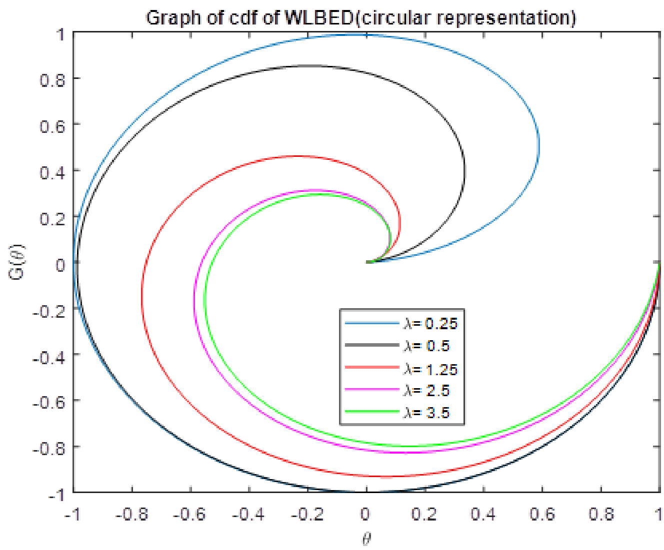

Figure 4 The plots of cumulative distribution function of wrapped length biased exponential distribution

Theorem 1 *If $\theta_1 \sim WLBE(\lambda)$ and $\theta_2 \sim WLBE(-\lambda)$ then $g_{\theta_1}(\theta) = g_{\theta_2}(2\pi-\theta)$ or $g_{\theta_2}(\theta) = g_{\theta_1}(2\pi-\theta)$, i.e., wrapped length biased exponential distribution have probability density function is mirrored with respect to origin.*

Proof: From the probability density function of linear length biased exponential distribution, we have

$$g(x;\lambda) = \frac{1}{\lambda^2} x \times e^{-\frac{x}{\lambda}} = \frac{1}{(-\lambda)^2}(-x)\times e^{-\left(\frac{-x}{-\lambda}\right)} = -g(-x;-\lambda),\ \lambda > 0.$$

This shows that the pdf of LBE (λ) is mirrored with respect to origin. Now,

$$\begin{aligned} g_\theta(2\pi-\theta,-\lambda) &= \sum_{t=-1}^{-\infty} -g(2\pi-\theta+2\pi t;-\lambda) = \sum_{t=-1}^{-\infty} -g(-\theta+2\pi(t+1);-\lambda) \\ &= \sum_{t=0}^{-\infty} -g(-\theta+2\pi t;-\lambda) = \sum_{s=0}^{\infty} -g(-\theta-2\pi s;-\lambda),\ \text{where } t=-s \\ &= \sum_{s=0}^{\infty} g(\theta+2\pi s;-\lambda) = g_\theta(\theta;-\lambda) \\ &= g_\theta(\theta;-\lambda),\ \lambda>0. \end{aligned}$$

Hence, probability density functions of wrapped length biased exponential distribution are mirrored with respect to origin.

3.2. Characteristic function

According to Jammalamadaka and Sengupta (2001), the trigonometric moment of order p for a wrapped circular distribution corresponds to the value of the characteristic function of the unwrapped random variable X, say $\phi_X(t)$ at the integer value p, i.e., $\varphi_p = \phi_X(p)$.

$$\begin{aligned}\varphi_p &= \int_0^\infty e^{ipx} d\left(F_X(x)\right) = \frac{1}{\lambda^2}\int_0^\infty e^{ipx} x e^{-\frac{x}{\lambda}} dx,\ p\in \mathrm{Z} = \frac{1}{\lambda^2}\int_0^\infty x e^{-\left(\frac{1}{\lambda}-ip\right)x} dx \\ &= \frac{1}{\lambda^2}\left(x\frac{e^{-\left(\frac{1}{\lambda}-ip\right)x}}{-\left(\frac{1}{\lambda}-ip\right)}\Bigg|_0^\infty - \frac{e^{-\left(\frac{1}{\lambda}-ip\right)x}}{\left(\frac{1}{\lambda}-ip\right)^2}\Bigg|_0^\infty \right) = \frac{1}{\lambda^2}\left(\frac{1}{\left(\frac{1}{\lambda}-ip\right)^2}\right) \\ &= \frac{1}{\left(1-ip\lambda\right)^2} = \frac{1}{1+\lambda^2 p^2}\times e^{i\left(2\tan^{-1}(\lambda p)\right)},\ p\in Z,\end{aligned}$$

which is the characteristic function of wrapped length biased exponential distribution.

3.3. Trigonometric moments and related measures

An alternative expression for φ_p is $\varphi_p = \rho_p e^{i\mu_p}$ hence, we have

$$\rho_p = \frac{1}{1+\lambda^2 p^2},\ \text{and}\ \mu_p = 2\tan^{-1}(\lambda p). \tag{2}$$

The p^{th} order non-central trigonometric moments of $\theta \sim WLBE(\lambda)$ are given by $\varphi_p = \alpha_p + i\beta_p$, where $\alpha_p = \rho_p \cos(\mu_p),\ \beta_p = \rho_p \sin(\mu_p)$

$$\alpha_p = \frac{1}{\left(1+\lambda^2 p^2\right)}\cos\left(2\tan^{-1}(\lambda p)\right), \tag{3}$$

$$\beta_p = \frac{1}{\left(1+\lambda^2 p^2\right)}\sin\left(2\tan^{-1}(\lambda p)\right). \tag{4}$$

Now according to Carslaw (1930) an alternative expression for the pdf of wrapped length biased exponential distribution can be obtained using the trigonometric moments as

$$g(\theta) = \frac{1}{2\pi}\left[1+2\sum_{p=1}^{\infty}\left(\alpha_p\cos(p\theta)+\beta_p\sin(p\theta)\right)\right]. \tag{5}$$

From Equations (3), (4) and (5) we have

$$\begin{aligned}g(\theta) &= \frac{1}{2\pi}\left(1+2\sum_{p=1}^{\infty}\left(\frac{1}{\left(1+\lambda^2p^2\right)}\cos\left(2\tan^{-1}(\lambda p)\right)\cos(p\theta) - \frac{1}{\left(1+\lambda^2p^2\right)}\sin\left(2\tan^{-1}(\lambda p)\right)\sin(p\theta)\right)\right) \\ &= \frac{1}{2\pi}\left(1+2\sum_{p=1}^{\infty}\left(\frac{1}{\left(1+\lambda^2p^2\right)}\left(\cos\left(2\tan^{-1}(\lambda p)\right)\cos(p\theta) - \sin\left(2\tan^{-1}(\lambda p)\right)\sin(p\theta)\right)\right)\right)\end{aligned}$$

$$g(\theta) = \frac{1}{2\pi}\left(1+2\sum_{p=1}^{\infty}\left(\frac{1}{\left(1+\lambda^2p^2\right)}\left(\cos\left(2\tan^{-1}(\lambda p)-p\theta\right)\right)\right)\right).$$

The resultant length, the mean direction, the circular variance and the circular standard deviation of $\theta \sim WLBE(\lambda)$, respectively, given by

Resultant length: $\rho = \rho_1 = \dfrac{1}{1+\lambda^2}$,

Mean direction: $\mu = \mu_1 = 2\tan^{-1}(\lambda)$,

Circular variance: $V_0 = 1-\rho = 1-\dfrac{1}{1+\lambda^2} = \dfrac{\lambda^2}{1+\lambda^2}$, (6)

Circular standard deviation: $\sigma_0 = \sqrt{-2\log(1-V_0)} = \sqrt{2\log(1+\lambda^2)}$.

The central trigonometric moments of the $WLBE(\lambda)$ are given by

$$\overline{\alpha_p} = \rho_p \cos(\mu_p - p\mu) = \frac{1}{1+\lambda^2 p^2}\cos\left(2\tan^{-1}(\lambda p) - 2p\tan^{-1}(\lambda)\right), \tag{7}$$

$$\overline{\beta_p} = \rho_p \sin(\mu_p - p\mu) = \frac{1}{1+\lambda^2 p^2}\sin\left(2\tan^{-1}(\lambda p) - 2p\tan^{-1}(\lambda)\right). \tag{8}$$

Skewness: One common measure of skewness of a circular distribution is

$$\xi_1^0 = \overline{\beta_2} / V_0^{3/2}. \tag{9}$$

To compute skewness for the wrapped length biased exponential distribution we need its second central trigonometric moment $\overline{\beta_2}$. From Equation (8), we have

$$\overline{\beta_2} = \rho_2 \sin(\mu_2 - 2\mu) = \frac{1}{1+4\lambda^2}\sin\left(2\tan^{-1}(2\lambda) - 4\tan^{-1}(\lambda)\right).$$

From Equations (6), (8) and (9), the skewness of the $WLBE(\lambda)$ distribution is

$$\xi_1^0 = \frac{1}{1+4\lambda^2}\sin\left(2\tan^{-1}(2\lambda) - 4\tan^{-1}(\lambda)\right) \times \left(\frac{1+\lambda^2}{\lambda^2}\right)^{3/2}.$$

The kurtosis of circular distribution is

$$\xi_2^0 = \frac{\left(\overline{\alpha_2} - (1-V_0)^4\right)}{V_0^2}. \tag{10}$$

To calculate kurtosis for the wrapped length biased distribution we need its second central trigonometric moment $\overline{\alpha_2}$. From Equation (7), we have

$$\overline{\alpha_2} = \rho_2 \cos(\mu_2 - 2\mu) = \frac{1}{1+4\lambda^2}\cos\left(2\tan^{-1}(2\lambda) - 4\tan^{-1}(\lambda)\right). \tag{11}$$

From Equations (6), (8), and (11), the kurtosis of the $WLBE(\lambda)$ distribution is

$$\xi_2^0 = \left(\frac{1}{1+4\lambda^2}\cos\left(2\tan^{-1}(2\lambda) - 4\tan^{-1}(\lambda)\right) - \left(\frac{1}{1+\lambda^2}\right)^4\right) \times \left(\frac{1+\lambda^2}{\lambda^2}\right)^2.$$

Table 1 Values of different characteristics of wrapped length biased exponential distribution for varying λ

Characteristics of WLBE distribution	λ				
	0.25	0.75	1.25	1.75	2.25
Mean direction (μ)	0.4900	1.2870	1.7921	2.1033	2.3051
Resultant length (ρ)	0.9412	0.6400	0.3902	0.2462	0.6149
Circular variance (V_0)	0.0588	0.3600	0.6098	0.7538	0.8351
Circular standard deviation (σ_0)	–0.3482	–0.9448	1.3718	1.6744	1.8985
Non-central (α_1)	0.8304	0.1792	–0.0857	–0.1250	–0.1105
Trigonometric (α_2)	0.4800	–0.1183	–0.0999	–0.0641	–0.0426
Moments (β_1)	0.4429	0.6144	0.3807	0.2121	0.1224
Moments (β_2)	0.6400	0.2840	0.0951	0.0399	0.0199
Central $(\overline{\alpha_1})$	0.9412	0.6400	0.3902	0.2462	0.6149
Trigonometric $(\overline{\alpha_2})$	0.7989	0.2525	0.0495	–0.0038	–0.0155
Moments $(\overline{\beta_1})$	0	0	0	0	0
Moments $(\overline{\beta_2})$	–0.0421	–0.1759	–0.1287	–0.0754	–0.0444
Coefficient of Skewness (ξ_1^0)	–2.9494	–0.8142	–0.2704	–0.1152	–0.0582
Coefficient of Kurtosis (ξ_2^0)	–25.1200	–1.2124	–0.2764	–0.1134	–0.0612

4. Maximum Likelihood Estimation

In this section, we discuss the method of maximum likelihood estimation to estimate the model parameter λ. Let $\theta_1,\theta_2,\theta_3,\ldots,\theta_n$ be random sample of size n from $WLBE(\lambda)$. Then, the likelihood function is given by

$$L(\theta;\lambda)=\prod_{i=1}^{n} g(\theta_i;\lambda)== \prod_{i=1}^{n}\frac{e^{-\frac{1}{\lambda}(\theta_i-2\pi)}}{\lambda^2\left(e^{\frac{2\pi}{\lambda}}-1\right)}\left(\theta_i+\frac{2\pi}{\left(e^{\frac{2\pi}{\lambda}}-1\right)}\right).$$

The log-likelihood function is given by

$$l \;=\; \log_e(L) \;=\; -\frac{1}{\lambda}\sum_{i=1}^{n}(\theta_i-2\pi)-2n\log_e(\lambda)-n\log_e\left(e^{\frac{2\pi}{\lambda}}-1\right)+\sum_{i=1}^{n}\log_e\left(\theta_i+\frac{2\pi}{\left(e^{\frac{2\pi}{\lambda}}-1\right)}\right). \quad (12)$$

Taking first order partial differentiation of the log-likelihood function with respect to λ then equating them to zero, we get the following equation

$$\frac{\partial l}{\partial \lambda} \;=\; \frac{1}{\lambda^2}\sum_{i=1}^{n}(\theta_i-2\pi)-\frac{2n}{\lambda}+\frac{2n\pi e^{\frac{2\pi}{\lambda}}}{\lambda^2\left(e^{\frac{2\pi}{\lambda}}-1\right)}+\frac{4\pi^2 e^{-\frac{2\pi}{\lambda}}}{\lambda^2\left(e^{\frac{2\pi}{\lambda}}-1\right)}\sum_{i=1}^{n}\frac{1}{\left(2\pi+\left(e^{\frac{2\pi}{\lambda}}-1\right)\theta_i\right)} \;=\; 0.$$

The above normal equation is nonlinear; we are not able to get the closed form maximum likelihood estimator, therefore we employ numerical technique to get the value for the parameter λ.

5. Simulation Study

In this part, we carry out a simulation study to generate data from the proposed distribution and to verify for the consistency of its maximum likelihood estimate. Samples of size 25, 50, 100, and 500 are generated from $WLBE(\lambda)$ for different values of λ. Next, MLE of the parameter is obtained by solving the Equation (12) for each of the respective samples. Finally, the average values of bias and mean square error (MSE) of the estimate are computed by the Monte Carlo approximation method, by considering N =10,000 replications. The detailed procedure is given below.

Step I. Generating a random sample from $WLBE(\lambda)$

Step (a) A random variable is generated from the $U(0,1)$ distribution, say u.

Step (b) Find the expression for the quantile function for the given distribution (i.e., find the inverse of cumulative distribution function). For this, expression of distribution function given in (1) is equated to u and is solve for θ which is a random variable from $WLBE(\lambda)$.

Steps (a) and (b) are repeated as many times as the desirable sample size (i.e., replications, say $N=$ 10,000). This method is called inverse transform technique for generating θ.

Step II. Obtain maximum likelihood estimates of the parameters

The MLE of the parameter λ is obtained by substituting the value of θ obtained from Step I in (12) log-likelihood function and maximizing this with respect to λ.

Step III. Calculate the average bias and MSE of the maximum likelihood estimate

Let the true value of the parameter be λ and the MLE be λ^*. Then the absolute average bias and MSE of λ^* are given by

$$\left|Bias\left(\lambda^*\right)\right| = \frac{1}{N}\sum_{i=1}^{N}\left(\left|\lambda_i-\lambda^*\right|\right), \quad MSE\left(\lambda^*\right) = \frac{1}{N}\sum_{i=1}^{N}\left(\lambda_i-\lambda^*\right)^2.$$

The MLE is said to be consistent if the bias and MSE approaches to zero with an increase in the sample size. Table 2 given below shows average values of the bias and MSE of the MLE of λ for various sample sizes and for various set of values for λ.

Table 2 Absolute averages bias and MSE for λ^*

	$\lambda=1$		$\lambda=1.5$	
n	$\lvert Bias(\lambda)\rvert$	$MSE(\lambda)$	$\lvert Bias(\lambda)\rvert$	$MSE(\lambda)$
25	0.11961	0.02457	0.17941	0.05528
50	0.08648	0.00938	0.11972	0.02131
100	0.06544	0.00679	0.09812	0.01529
500	0.02227	0.00089	0.03741	0.00209
	$\lambda=2$		$\lambda=2.5$	
n	$\lvert Bias(\lambda)\rvert$	$MSE(\lambda)$	$\lvert Bias(\lambda)\rvert$	$MSE(\lambda)$
25	0.26250	0.12483	0.33884	0.20709
50	0.18111	0.04523	0.22353	0.06667
100	0.14108	0.03145	0.18297	0.05241
500	0.04401	0.00323	0.05567	0.00555

From above Table 2, it is clear that the absolute bias and MSE of the MLE of λ approaches towards zero with an increase in the sample size. Which shows that the estimate of the parameter λ is accurate, precise and hence, consistent.

6. Application to a Real Data

In order to demonstrate the modeling behavior of the wrapped length biased exponential distribution, we analyses popular turtle data set from the literature. This data consists of orientations of 76 female turtles after laying eggs (Jammalamadaka and Sengupta 2001) and is given in Table 3.

Table 3 Orientations of 76 turtles after laying eggs

Direction (in degrees) clockwise from north												
8	9	13	13	14	18	22	27	30	34	38	38	40
44	45	47	48	48	48	48	50	53	56	57	58	58
61	63	64	64	64	65	65	68	70	73	78	78	78
83	83	88	88	88	90	92	92	93	95	96	98	100
103	106	113	118	138	153	153	155	204	215	223	226	237
238	243	244	250	251	256	268	285	319	343	350		

This dataset was used by Joshi and Jose (2018) as an application of the wrapped Lindley $WL(\beta)$ distribution. The authors reported the maximum log-likelihood, AIC, and BIC values for $WL(\beta)$ distribution are −119.7089, 241.4178, and 243.7485, respectively. In the same study the maximum likelihood, AIC, and BIC values for wrapped exponential $WE(\alpha)$ distribution (Jammalamadaka and Kozubowski 2004) are −120.6474, 243.2948, and 245.6255, respectively. We apply the wrapped length biased exponential $WLBE(\lambda)$ distribution to this dataset and estimated the parameter using Mathematica's NMaximize command. We also computed various statistics like log-likelihood, AIC, and BIC statistic for the proposed model along with wrapped exponential $WE(\alpha)$ and wrapped Lindley $WL(\beta)$ distributions and summarized the outcomes for these models in Table 4.

Table 4 Summary of statistics

Model	MLE	Log-likelihood	AIC	BIC
$WE(\alpha)$	(0.4229)	−120.64	243.280	249.941
$WL(\beta)$	(0.7484)	−119.70	241.398	243.729
$WLBE(\lambda)$	(0.9872)	−118.88	239.753	242.083

The larger value of log-likelihood statistic and the smaller values of AIC and BIC statistic indicate that the wrapped length biased exponential distribution gives better fit to the turtle dataset than the wrapped Lindley distribution.

7. Conclusions

In this article, wrapped length biased exponential $WLBE(\lambda)$ distribution is introduced. The proposed distribution inherits the flexibility properties of length biased exponential distribution. We also obtained the explicit forms of density, distribution, characteristic functions, trigonometric moments, and other statistical measure like resultant length, mean, circular variance, standard deviation, skewness and kurtosis. To estimate the model parameter, the method of maximum likelihood estimation is employed. The performance is compared with wrapped exponential $WE(\alpha)$ and wrapped Lindley $WL(\beta)$ distributions using log-likelihood, AIC, and BIC statistics. Up on the critical observation on summary of statistics, it is concluded that the proposed model $WLBE(\lambda)$ gives better fit to the turtle dataset than the wrapped exponential $WE(\alpha)$ and wrapped Lindley $WL(\beta)$ distributions.

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