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# Estimation of Stress-Strength Reliability of Power Distribution under Type-II Censored Data

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**Abstract**

In the statistical literature, there are many lifetime distributions used in reliability analysis, including exponential, normal, gamma, and Weibull distributions. Power distribution is also useful in many scientific contexts, with significant consequences for our understanding of natural and man-made phenomena. This expository paper presents the evaluation of reliability when stress and strength follow power distribution with a common scale and different shape parameters. We obtain maximum likelihood (ML) estimates of stress-strength reliability with their confidence intervals. Furthermore, to compare the performance of various procedures, we apply statistical simulation. Finally, an analysis of a real dataset is given for illustrative purposes.

**Keywords**: Maximum likelihood estimator; P[Y<X], right censoring, asymptotic confidence interval, real data.

## 1. Introduction

The stress-strength reliability of a system can be defined as an assessment of reliability in terms of stress, represented by a random variable $Y$, experienced by a component and the strength which is represented by the random variable $X$ of a component. In other words, we can say that if the stress on a system exceeds the strength of the component, then the system will fail. The literature of system stress strength reliability and applications has been discussed by Birnbaum (1956). Firstly, Birnbaum and McCarty (1958) proposed the stress strength reliability of a system and obtained the confidence bound based distribution free sample for the stress strength reliability. Cheng and Chao (1984) obtained the distribution free confidence interval for probability of $Y < X$ of a system. Surles and Padgett (1998) obtained an estimate of stress-strength reliability when both components follow the Burr type $X$ distribution. Al-Mutairi (2013) discussed the inferential procedure for stress strength reliability when both variables follow the Lindley distribution. Kumar et al. (2015) considered estimation process of stress and strength reliability when both follow the Lindely distribution under progressively first failure censoring. Sharma et al. (2015) discussed stress-strength reliability when both are the inverse Lindley lifetime and obtained a maximum likelihood (ML) estimate of the reliability function. They also obtained the Bayes estimate of parametric function by using Markov chain Monte Carlo (MCMC) method and Lindley approximation under informative and non-informative prior. Chaudhary et al. (2017) obtained the stress strength reliability when both strength and stress follows the Maxwell distribution. They also found the Bayes estimate of stress-strength reliability under the square error loss function by using the MCMC technique. Kumar and Kumar (2021) obtained the estimation of the stress-strength reliability for the inverse Pareto distribution under progressively censored data. Saini et al. (2021) discussed the classical and Bayesian estimate of the stress-strength reliability for generalized Maxwell failure distribution under progressive first failure censoring. In the past few decades, the literature has reviewed several papers on the application of stress strength reliability for various models. A few names for reference are cited (Dhillon 1980, Govindarajulu 1967, Juvairiyya and Kumar 2019, Pham and Almhana 1995, Weerahandi and Johnson 1992).

Power distribution has wide applications in many fields of survival and reliability analysis. The following methods are commonly used for analyzing the power-law data: least-squares fitting and parameter estimation in different situations for the power distributions. Even in many cases, such methods provide an accurate answer, but they are not satisfactory because they do not show the indication to obey the power distribution. Gaudoin (2003) discussed some related transformations and tests for goodness of fit for power distribution. Goldstein et al. (2004a) showed the same basic characteristics of discrete and continuous power-law distribution and discussed the application for scientific importance, which has significant consequences for our understanding of survival and reliability. They also found the ML estimate of the scale parameter of the power distribution and the estimating procedure of the lower bound on power-law behavior. They also used the Kolmogorov-Smirnov (K-S) test for goodness of fit and pointed out the powerful application of this distribution for twenty-four real data points. Cordeiro and Brito (2012) showed that the power distribution is the inverse of the Pareto distribution and obtained the basic characteristics of this distribution. They also derived the ML function for some real data. Okorie et al. (2017) discussed some statistical properties of modified power function distribution and obtained the ML estimate of the parameter of modified power distribution using some real data sets (see Koen and Kondlo 2009, Meniconi and Barry 1996, Rigdon 1989).

There are many situations where power distribution is widely used in many areas of reliability. In this paper, we have considered the estimation procedures and application of the stress-strength reliability when strength and stress and both components are followed to the power distribution with the same scale and different shape parameters. We organize the remainder of this paper as follows: In Section 2, we discussed some statistical properties of the power distribution. In Section 3, we evaluate the ML estimate of stress-strength reliability for complete and censored data. In Section 4, we also calculated the asymptotic confidence and boot-p intervals for both cases. Section 5 discusses the analysis of the simulation study and a real data set, while Section 6 concludes with some concluding remarks.

## 2. The Model

Let $X$ be a random variable follow to power distribution and having probability density function (pdf) is given as

$$f(x)=\frac{\alpha}{\theta^{\alpha}}x^{\alpha-1}, \quad 0<x<\theta, \alpha>0, \theta>0. \tag{1}$$

where $\alpha$ is the shape and $\theta$ is the scale parameter and the cumulative distribution function (cdf) of power distribution is

$$F\left(x\right)=\left(\frac{x}{\theta}\right)^{\alpha},\theta>0,\alpha>0,0<x<\theta. \quad (2)$$

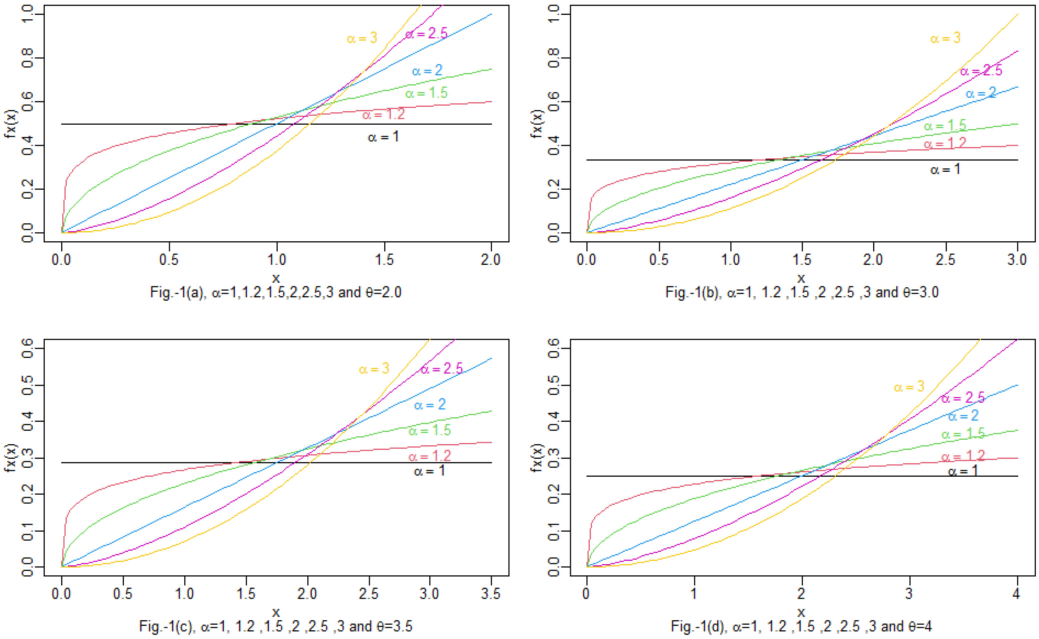


**Figure 1** The pdf plots of the power distribution

Figure 1 shows the behavior of the pdf based on different value of $\theta$ and $\alpha$. The $m^{\text{th}}$ moment of the power distribution is

$$E\left(X^{m}\right)=\frac{\beta\,\theta^{m}}{m+\alpha}.$$

So, the mean $(\mu)$ and variance $(\sigma^2)$ of the power distribution are

$$\mu=\frac{\theta\alpha}{\alpha+1} \text{ and } \sigma^{2}=\frac{\alpha\theta^{2}}{(\alpha+2)(\alpha+1)^{2}}.$$

The corresponding hazard function $(h(t))$ and survival function $(S(t))$ are given by

$$h(t)=\frac{\alpha}{\theta^{\alpha}-t^{\alpha}}\,t^{\alpha-1} \text{ and } S(t)=\left(\theta^{\alpha}-t^{\theta}\right)\theta^{-\alpha},$$

where $t$ is the pre-define time.

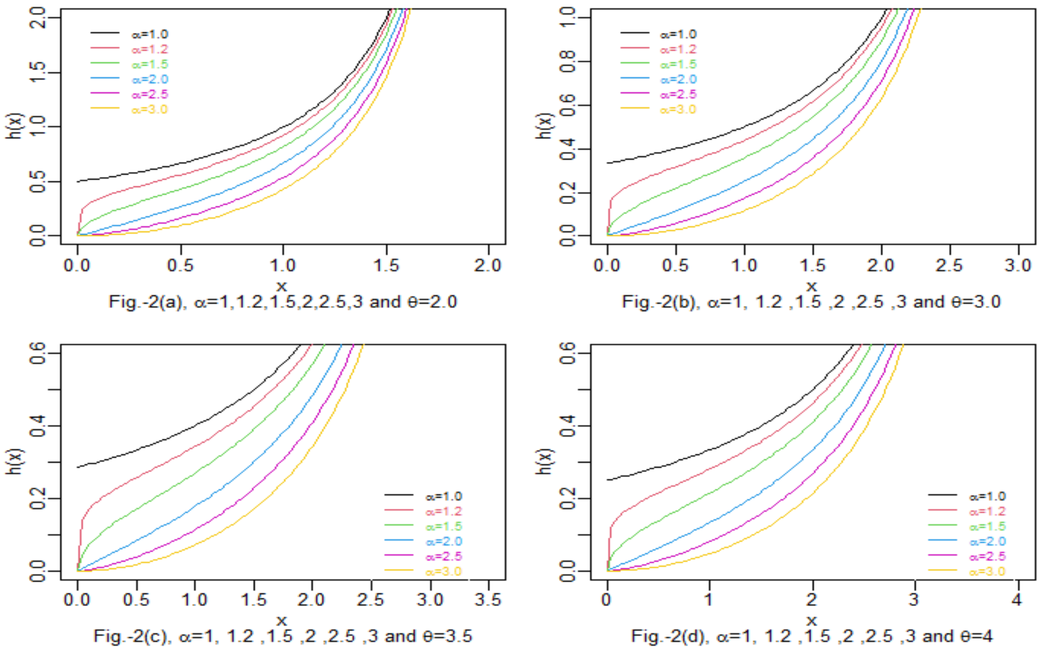


**Figure 2** The hazard function plot of the power distribution

The time to failure of given distribution is given by

$$E(t)=1-\exp\left[-\int_0^t h(y)\,dy\right]=1-\exp\left[-\int_0^t \frac{\alpha x^{\alpha-1}}{\theta^\alpha - x^\alpha}dx\right].$$

On substituting $\theta^\alpha - x^\alpha = u$ and integrating, we get

$$E(t)=1-\exp\left[-\int_{\theta^\alpha}^{\theta^\alpha - t^\alpha} \frac{1}{u}du\right]=\frac{t^\alpha}{\theta^\alpha}.$$

Figure 2 shows the behavior of the $h(t)$ based on different value of $\theta$ and $\alpha$. The quantile function of the power distribution is $Q(p)=\theta\, p^{1/\alpha}$.

### 2.1. The moment generating and characteristics function

The moment generating function $(M_X(t))$ and characteristics function ( $(\phi_X(t))$, respectively, of the power distribution are as follows

$$M_x(t)=\frac{\alpha\left[\Gamma(\alpha)-\Gamma(\alpha,-t\theta)\right]}{(-t\theta)^\alpha} \text{ and } \phi_X(t)=\frac{\Gamma(\alpha)-\Gamma(\alpha,-it\theta)}{(-i\,t)^\alpha}\frac{\alpha}{\theta^\alpha}.$$

### 2.2. Order statistic

Let $X_1, X_2,\ldots,X_n$ be the independent and identically distributed random variables of size $n$ and each variate having cdf given in Equation (2). If these variables are put in ascending order with theirs magnitude and these written in the form $X_{(1)} \le X_{(2)} \le \ldots \le X_{(n)}$. We say $X_{(r)}$ as the $r^{\text{th}}$ order statistic, now the pdf of $r^{\text{th}}$ order statistic is

$$f(x_{(r)})=\frac{\alpha}{x}\frac{(\theta^{\alpha}-x^{\alpha})^{n-r}}{B(r,n-r+1)}\left(\frac{x^{r}}{\theta^{n}}\right)^{\alpha}.$$

Since $X_{(1)}=\min(X_1, X_2,...,X_n)$ and $X_{(n)}=\max(X_1,X_2,...,X_n)$ are define as the first and last order statistic, respectively. The pdf of $X_{(1)}$ and $X_{(n)}$ are obtained by putting $r=1$ and $r=n$ in $f(x_{(r)})$ and define as

$$f(x_{(1)})=\frac{n\alpha}{\theta^{n\alpha}}x^{r-1}\frac{(\theta^{\alpha}-x^{\alpha})^{n}}{(\theta^{\alpha}-x^{\alpha})},$$

and

$$f(x_{(n)})=\frac{n}{x}\left(\frac{x}{\theta}\right)^{n\alpha}.$$

The pdf of joint order statistic say $X_{(r)}=x$ and $X_{(s)}=y,\ X<Y,$ is given by

$$f(X_r=x,X_s=y)=\frac{n!\ y^{\alpha-1}x^{\alpha r-1}}{(r-1)!(n-s)!(n-s-1)!}\frac{\left(y^{\alpha}-x^{\alpha}\right)^{s-r-1}}{\theta^{n\alpha}\left(\theta^{\alpha}-y^{\alpha}\right)^{s-n}}.$$

The pdf of range, say $w$, is define as

$$f(w)=\frac{(n-1)n\alpha^{2}}{\theta^{n\alpha}}\int_{0}^{\theta}x^{\alpha-1}\left(w+x\right)^{\alpha-1}\left\{\left(x+w\right)^{\alpha}-x^{\alpha}\right\}^{n-2}dx,\quad 0\le w\le\theta.$$

Since the above integral is not closed form so we use some numerical method i.e., MCMC method to solve it.

### 2.3. Parameter estimates by method of moment

One perceptive method of estimation of parameters is the method of moment, which compares the sample moments with corresponding population moments, which are the values expressed in terms of the parameters of the given distribution. Here, the mean and variance of given sample are $\overline{x}=n^{-1}\sum_{i=1}^{n}x_i$ and $s^2=(n-1)^{-1}\sum_{i=1}^{n}(x_i-\overline{x})^2$, respectively. On equating the sample moments to corresponding population moment, we have

$$\overline{x}=\frac{\alpha\theta}{\alpha+1}\ \text{and}\ s^2=\frac{\alpha\theta^2}{(\alpha+2)(\alpha+1)^2}.$$

After solving these equations and we get the estimate of parameters of $\theta$ and $\alpha$, say $\hat{\theta}_m$ and $\hat{\alpha}_m$ are define as

$$\hat{\alpha}_m=\frac{\sqrt{s^2-\overline{x}^2}-s}{s}\ \text{and}\ \hat{\theta}_m=\frac{\overline{x}\sqrt{s^2-\overline{x}^2}}{\sqrt{s^2-\overline{x}^2}-s}.$$

### 2.4. Lorenz curve

The Lorenz curve is defined the graphical representation of the cumulative income distribution. The Lorenz curve (Lorenz 1905) for a positive real random variable $X$ is defined as the graph of the ratio

$$L\left(F(x)\right)=\frac{E\left(X\mid X\le x\right)}{E(X)}=\frac{\int_0^x xf(x)\,dx}{\int_0^\infty xf(x)\,dx}.$$

For the power distribution, the denominator and nominator of above equation are

$$\int_0^x xf(x)\,dx=\int_0^x x\frac{\alpha}{\theta^\alpha}x^{\alpha-1}\,dx=\frac{x^{\alpha+1}\alpha}{\theta^\alpha(\alpha+1)} \text{ and } \int_0^\theta x\,f(x)\,dx=\int_0^\theta x\frac{\alpha}{\theta^\alpha}x^{\alpha-1}\,dx=\frac{\alpha\,\theta}{\alpha+1}.$$

So the Lorenz curve is defined as $L\left(F(x)\right)=\dfrac{x^{\alpha+1}}{\theta^{\alpha+1}}.$

## 3. Stress-Strength Reliability Computation

Let $X$ and $Y$ be represents the strength and stress for a system and having the densities $f(x)$ and $f(y)$, respectively. Since $X$ follows to the power distribution having shape parameter $\alpha_1$ and scale parameter $\theta$, and the pdf of $X$ is given by

$$f(x)=\frac{x^{\alpha_1-1}\alpha_1}{\theta^{\alpha_1}},\ \alpha_1>0, 0<x<\theta,\ \theta>0. \tag{3}$$

Since $Y$ also follows to the power distribution having shape parameter $\alpha_2$ and common scale parameter $\theta$, and the pdf of $Y$ is given by

$$f(y)=\frac{\alpha_2\,y^{\alpha_2-1}}{\theta^{\alpha_2}},\ \alpha_2>0,\ 0<y<\theta,\ \theta>0. \tag{4}$$

So that the stress-strength reliability is define as

$$\Re=P(Y<X)=\int_0^\theta f\left(x\right)F_Y(x)\,dx.$$

Using (1) and (2), the Equation (5) is

$$\Re=\int_0^\theta\frac{\alpha_1}{\theta^{\alpha_1}}x^{\alpha_1-1}\left\{\int_0^x\frac{\alpha_2}{\theta^{\alpha_2}}y^{\alpha_2-1}dy\right\}dx.$$

After simplifying, we obtain stress strength reliability in this form

$$\Re=\frac{\alpha_1}{\alpha_1+\alpha_2}. \tag{5}$$

### 3.1. Maximum likelihood estimator (MLE) of $\Re$ based on complete sample

Let $x_1,x_2,...x_{n_1}$ be a random sample of size $n_1$ from $f(x,\theta,\alpha_1)$ and $y_1,y_2,...y_{n_2}$ be a random sample of size $n_2$ from $f(y,\theta,\alpha_2)$, respectively. Since $X$ and $Y$ follows to the power distribution, then the likelihood function is given by

$$L\left(\alpha_1,\alpha_2,\theta\mid x,y\right)=\prod_{i=1}^{n_1}\prod_{j==1}^{n_2}f(x_i)f(y_j).$$

Using (3) and (4), the above equation is as the form

$$L\left(\alpha_1,\alpha_2,\theta\mid x,y\right)=\prod_{i=1}^{n_1}\left[\frac{\alpha_1}{\theta^{\alpha_1}}x_i^{\alpha_1-1}\right]\prod_{j=1}^{n_2}\left[\frac{\alpha_2}{\theta^{\alpha_2}}y_j^{\alpha_2-1}\right]. \tag{6}$$

Taking the logarithm both side of Equation (7) and the log-likelihood function is as follows

$$\log L = n_1 \ln \alpha_1 - n_1\alpha_1 \ln\theta + (\alpha_1 - 1)\sum_{i=1}^{n_1} \ln x_i - n_2\alpha_2 \ln\theta + n_2 \ln\alpha_2 + (\alpha_2 - 1)\sum_{j=1}^{n_2} \ln y_j. \tag{7}$$

Therefore, the ML estimates of $\theta$, $\alpha_1$ and $\alpha_2$ and which maximizes Equation (8). The normal equations are given by

$$\left.\begin{aligned} \frac{\partial \ln L(\alpha_1, \alpha_2, \theta)}{\partial \alpha_1} &= \frac{n_1}{\alpha_1} - n_1 \ln\theta + \sum_{i=1}^{n_1} \ln x_i = 0, \\ \frac{\partial \ln L(\alpha_2, \alpha_1, \theta)}{\partial \alpha_2} &= \frac{n_2}{\alpha_2} - n_2 \ln\theta + \sum_{j=1}^{n_2} \ln y_j = 0, \\ \frac{\partial \ln L(\theta, \alpha_1, \alpha_2)}{\partial \theta} &= -\frac{n_1\alpha_1 + n_2\alpha_2}{\theta} = 0. \end{aligned}\right\} \tag{8}$$

Then we obtain the ML estimate of $\alpha_1$ and $\alpha_2$ are the function of $\theta$, respectively, given by

$$\hat{\alpha}_1 = \frac{n_1}{n_1 \ln\theta - \sum_{i=1}^{n_1} \ln x_i} \quad \text{and} \quad \hat{\alpha}_2 = \frac{n_2}{n_2 \ln\theta - \sum_{j=1}^{n_2} \ln y_j}. \tag{9}$$

From (7), we estimate the ML estimate of $\theta$ as follows

$$-\frac{n_1\alpha_1 + n_2\alpha_2}{\theta} = 0 \;\Rightarrow\; \hat{\theta} = \infty.$$

We cannot obtain the ML estimate of $\theta$ directly. So, in this case, the ML estimates of $\theta$ as follows: we have to choose $\theta$ for $L(\alpha_1, \alpha_2, \theta)$ in (8) is maximum. Now $L(\alpha_1, \alpha_2, \theta)$ is maximum if $\theta$ is minimum. Let $x_1, x_2, \ldots, x_{n_2}$ and $y_1, y_2, \ldots, y_{n_2}$ be a random samples of size $n_1$ and $n_2$ independent observation from the given population so that $0 \le x_{(1)} \le x_{(2)} \le \ldots \le x_{(n_1)} \le \theta$ and $0 \le y_{(1)} \le y_{(2)} \le \ldots \le y_{(n_2)} \le \theta.$ Since the minimum value of $\theta$ consistent with the sample is $x_{(n_1)}$ and $y_{(n_2)}$, the largest sample observation, $\hat{\theta} = \max(x_{(n_1)}, y_{(n_2)}).$ We obtain the ML estimate of stress-strength reliability, say $\Re$, from (6), by using invariance property of the ML estimate, is $\hat{\Re} = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 + \hat{\alpha}_2}.$

### 3.2. Maximum likelihood estimation for censored sample

Censoring refers to lifetime data analysis for mechanistic or natural systems. Life testing experiments, usually consume more time as well as these are very expensive due to their destructive nature. In some situations, it is neither possible nor desirable to observe each and every unit under test. In such circumstances, only a portion of the sample is studied, and we call the experiment censored. The important factor that affects the life-time experiment is the amount of time required to obtain the complete sample. To limit this factor, we may put some items into a test and have the have the test terminate at a pre-defined time. The sample obtained from this type of experiment is called a time-censored sample or type-I censoring. In another way, we may put the same items into a test, and the test terminates when a pre-defined number of failed items is reached. The samples obtained from this experiment are called 'failure censored samples' or type-II censoring. Failure-censored samples are most useful in dealing with high-cost items such as color television tubes, submarines, jet plane engines, etc. Now we use failure or type-II censoring for the following analysis (see more about type-II censoring in Panahi and Asadi (2011), Kumar and Tomer (2016), Banerjee and Kundu (2008)).

Let $(X, n_1, m_1)$ and $(Y, n_2, m_2)$ be the combinations of two type-II censored data such that $(X, n_1, m_1) = X_{1:m_1:n_1}, X_{2:m_1:n_1}, \ldots, X_{m_1:m_1:n_1}$ and $(Y, n_2, m_2) = Y_{1:m_2:n_2}, Y_{2:m_2:n_2}, \ldots, Y_{m_2:m_2:n_2}$. Thus the combined likelihood function of the parameters $(\alpha_1, \alpha_2, \theta)$ given observed data is written as

$$L(\alpha_1, \theta, \alpha_2 | data) = \frac{n_1!\, n_2!}{(n_1 - m_1)!(n_2 - m_2)!} \prod_{i=1}^{m_1} f(x_i, \alpha_1, \theta)\left[1 - F(x_{m_1}, \alpha_1, \theta)\right]^{n_1 - m_1}$$

$$\times \prod_{j=1}^{m_2} f(y_j, \alpha_2, \theta)\left[1 - F(y_{m_2}, \alpha_2, \theta)\right]^{n_2 - m_2}. \tag{10}$$

The likelihood function is modified by using (1), (2) and (10), we have

$$\propto \frac{\alpha_1^{m_1} \alpha_2^{m_2}}{\theta^{n_1\alpha_1 + n_1\alpha_2}} \left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)^{n_1 - m_1} \left(\theta^{\alpha_2} - y_{m_2}^{\alpha_2}\right)^{n_2 - m_2} \prod_{i=1}^{m_1} x_i^{\alpha_1 - 1} \prod_{j=1}^{m_2} y_j^{\alpha_2 - 1}.$$

Taking logarithm both side of above equation, we have

$$\log L \mu\, m_1 \log\alpha_1 + m_2 \log\alpha_2 - (n_1\alpha_1 + n_2\alpha_2)\log\theta + (n_1 - m_1)\log\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right) + (n_2 - m_2)\log\left(\theta^{\alpha_2} - y_{m_2}^{\alpha_2}\right)$$

$$+ (\alpha_1 - 1)\sum_{i=1}^{m_1} \log x_i + (\alpha_2 - 1)\sum_{j=1}^{m_2} \log y_j. \tag{11}$$

On partial differentiation of (11) with respect to $\alpha_1, \alpha_2$ and $\theta$, we have

$$\left.\begin{aligned}
\frac{\partial \log L}{\partial \alpha_1} &= \frac{m_1}{\alpha_1} - n_1 \log\theta + \frac{(n_1 - m_1)\left(\theta^{\alpha_1}\log\theta - x_{m_1}^{\alpha_1}\log x_{m_1}\right)}{\theta^{\alpha_1} - x_{m_1}^{\alpha_1}} + \sum_{i=1}^{m_1}\log x_i, \\
\frac{\partial \log L}{\partial \alpha_2} &= \frac{m_2}{\alpha_2} - n_2 \log\theta + \frac{(n_2 - m_2)\left(\theta^{\alpha_2}\log\theta - y_{m_2}^{\alpha_2}\log y_{m_2}\right)}{\theta^{\alpha_2} - y_{m_2}^{\alpha_2}} + \sum_{j=1}^{m_1}\log y_j, \\
\frac{\partial \log L}{\partial \theta} &= \frac{-(n_1\alpha_1 + n_2\alpha_2)}{\theta} + \frac{\alpha_1(n_1 - m_1)\theta^{\alpha_1 - 1}}{\theta^{\alpha_1} - x_{m_1}^{\alpha_1}} + \frac{(n_2 - m_2)\alpha_2\theta^{\alpha_2 - 1}}{\theta^{\alpha_2} - y_{m_2}^{\alpha_2}}.
\end{aligned}\right\}$$

The above differential equations are put equal to zero. After solving these normal equations, we get the ML estimate of $\alpha_1, \alpha_2$ and $\theta$ say $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\theta}$, respectively, which are defined as

$$\left.\begin{aligned}
\hat{\alpha}_1 &= m_1\left[n_1\log\theta - \frac{(n_1 - m_1)\left(\theta^{\alpha_1}\log\theta - x_{m_1}^{\alpha_1}\log x_{m_1}\right)}{\theta^{\alpha_1} - x_{m_1}^{\alpha_1}} - \sum_{i=1}^{m_1}\log x_i\right]^{-1}, \\
\hat{\alpha}_2 &= m_2\left[n_2\log\theta - \frac{(n_2 - m_2)\left(\theta^{\alpha_2}\log\theta - x_{m_2}^{\alpha_2}\log x_{m_2}\right)}{\theta^{\alpha_2} - x_{m_2}^{\alpha_2}} - \sum_{j=1}^{m_2}\log y_j\right]^{-1}, \\
\hat{\theta} &= (n_1\alpha_1 + n_2\alpha_2)\left[\frac{(n_2 - m_2)\alpha_2\theta^{\alpha_2 - 1}}{\theta^{\alpha_2} - y_{m_2}^{\alpha_2}} + \frac{\alpha_1(n_1 - m_1)\theta^{\alpha_1 - 1}}{\theta^{\alpha_1} - x_{m_1}^{\alpha_1}}\right]^{-1}.
\end{aligned}\right\} \tag{12}$$

Since the above system of normal equations is not closed form so we used some numerical iteration method for solving these equations. Let $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\theta}$ are the MLE of the $\alpha_1, \alpha_2$ and $\theta$. By using the invariance property of MLE, the ML estimator of $\Re$ is obtained as

$$\hat{\Re} = \frac{\hat{\alpha}_1}{\hat{\alpha}_1 + \hat{\alpha}_2}. \tag{13}$$

## 4. Interval Estimation

We have obtained the asymptotic confidence (AC) interval for parametric functions. Since the distribution of the ML estimate is not completely specified in the distribution, we construct the AC interval for the parameter by using the asymptotic property of the ML estimate. The AC intervals of $\alpha_1$, $\alpha_2$ and $\theta$ are find out by fisher information matrix. Since $\Re$ is the function of $\alpha_1$ and $\alpha_2$, so we cannot obtained the AC interval directly. In this case, we use the delta method to obtain the AC interval of $\Re$.

### 4.1. AC interval by delta method

Now, we use the delta method (Qehlert 1992) to obtain the asymptotic confidence interval (ACI) of $\Re$. The delta method is a standard technique in statistics, and it is based on a truncated Taylor series expansion. The delta method allows a normal approximation for a continuous and differentiable function of a sequence of random variables that already has a normal limit in distribution. According to the delta method, the variance of $\Re$ is estimated by $\sqrt{n}\left[\Re - \hat{\Re}\right] \rightarrow N\left(0, V(\Re)\left(\Re'\right)^2\right)$. So the ACI of $\Re$ is obtain as follows

$$\hat{\Re} \pm z_{\alpha/2} S.E.(\hat{\Re}), \tag{14}$$

where $z_{\alpha/2}$ is upper $100(\alpha/2)^{\text{th}}\%$ of standard normal variate and $S.E.(\hat{\Re})$ is the standard error of $\hat{\Re}$ (Qehlert 1992). The ACI of $\Re$ will be found in case of complete and censored data.

### 4.2. Bootstrap confidence interval

Since we deal with censored sample observations obtained from the life-testing experiments. The cognitive sample observations from such an experiment may not be large, so ACI may not be a proper choice. In such situation, we discuss the procedure the bootstrap confidence interval (CI) for $\Re$ as discussed by Efron and Tibshirani (1994). The necessary steps for obtained the parametric bootstrap method for $\hat{\Re}$ are follows.

1. Generate the random samples $x_1, x_2, ... x_{n_1}$ and $y_1, y_2, ... y_{n_2}$ form $f(x, \theta, \alpha_1)$ and $f(y, \theta, \alpha_2)$, respectively.

2. Compute the MLE $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\theta}$.

3. Using $\hat{\alpha}_1, \hat{\alpha}_2$ and $\hat{\theta}$, generate the samples of size $n_1$ and $n_2$ under similar conditions, as in step 1, take $\left\{x_1^*, x_2^*, ..., x_{m_1}^*\right\}$ and $\left\{y_1^*, y_2^*, ..., y_{m_2}^*\right\}$ samples under type-II censored for pre-define values of $m_1$ and $m_2$.

4. Now based on the samples obtained from step 3 and obtained the bootstrap estimate of $\Re$ say $\hat{\Re}^*$ by using delta method.

5. Repeat the Step 3, M times, to obtained the set of bootstrap estimates $(\hat{\Re}_k^*;\ k = 1, 2, ..., M)$.

6. Arrange $(\hat{\Re}_k^*;\ k = 1, 2, ..., M)$ in ascending order, say $(\hat{\Re}_{[1]}^*, \hat{\Re}_{[2]}^*, ..., \hat{\Re}_{[M]}^*)$.

7. A two-sided $100(1-\alpha)$ percentile boot-p CI are given by

$$(\hat{\Re}_L^*, \hat{\Re}_U^*) = (\hat{\Re}_{[M(\alpha/2)]}^*, \hat{\Re}_{[M(1-\alpha/2)]}^*), \tag{15}$$

where $[q]$ represents the $q^{\text{th}}$ integer part of the sequence $(\hat{\Re}_{[1]}^*, \hat{\Re}_{[2]}^*, ..., \hat{\Re}_{[M]}^*)$.

## 5. Numerical Application

Here, we have considered some numerical data to show the application of the power distribution in a real-life situation. We present some illustrations based on the simulation study. Now, we concentrate the sample generation procedure on the power distribution with complete and censored sample data. Sample generation procedure is as follows:

1. We use inverse cdf sampling technique for generating random numbers from the power distribution i.e. generate $U \sim Unif(0,1)$ and $X = F_X^{-1}(U)$.

2. Chose the parametric initial values $(\alpha_1, \alpha_2, \theta)$.

3. Generate the random sample $X_1, X_2, ..., X_{n_1}$ of size $n_1$ form $f_1(\alpha_1, \theta)$.

4. Generate a random sample $Y_1, Y_2, ..., Y_{n_2}$ of size $n_2$ form $f_2(\alpha_2, \theta)$.

5. Assume pre-fixed values $m_1$ and $m_2$ and get two ordered sample $\underline{X} = X_{(1)}, X_{(2)}, ..., X_{(m_1)}$ and $\underline{Y} = Y_{(1)}, Y_{(2)}, ..., Y_{(m_2)}$.

For the simulation study, we assume that

1. Assume $\alpha_1 = \{2,3\}$, $\alpha_2 = \{2,3\}$ and $\theta = \{2,3\}$.

2. Take $\{m_1, m_2\} = \{(10,10), (20,20), (30,30)\}$ for particular value of $(n_1, n_2) = \{30,30\}$.

3. Again assume $\{m_1, m_2\} = \{(20,20), (40,40), (50,50)\}$ for $(n_1, n_2) = \{50, 50\}$.

4. Firstly obtained the ML estimates of $\alpha_1$, $\alpha_2$, and $\theta$ say $\hat{\alpha}_1$, $\hat{\alpha}_2$, and $\hat{\theta}$.

5. Obtained the ML estimate of $\Re$, by using invariance property of the ML estimates, for the different combination of parameters values based with complete and censored data.

6. Asymptotic confidence and boot-p intervals of $\Re$ are also calculated at α level of significance.

7. The ML estimate of $\Re$ with respected mean square error (MSE) is shown in Tables 1-4.

8. The AC and boot-p intervals length with respected coverage probability (CP) are also calculated in Tables 5-8.

We generate repeated samples using these parametric values and enumerate average ML estimate of $\Re$ along with its MSE and also enumerate AC and boot-p intervals along with their CP based on 5000 iteration. From all the tables, we observe that MSEs of stress strength reliability based on the observed sample size. In the simulation study, we observed that

1. ML estimate of stress-strength reliability based on given sample size.

2. ML estimate of parameters are calculated for complete and type-II censored data and we see that the ML estimate based on complete sample are more reliable to the ML estimate obtained by censored sample.

3. In the life time experiment with censored sample, the MSE of ML estimate parallel decreases with observed sample size increase.

4. While comparing boot-p and AC intervals of $\Re$, the CP of boot-p interval is better than the CP of AC interval for given values of $(n, m)$.

5. The length of asymptotic and boot-p confidence intervals for censored data is larger than the complete data. Their length and CP also decrease as $m$ increases for a given value of $n$ for the censored sample.

**Table 1** Average ML estimate values of $\Re$ and their MSEs for $\alpha_2 = 2,\ \theta = 2$ and different values of $\alpha_1, n_1, m_1, n_2,$ and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | $\alpha_1 = 1.50$ | $\alpha_1 = 1.75$ | $\alpha_1 = 2.00$ | $\alpha_1 = 2.25$ | $\alpha_1 = 2.50$ |
|---|---|---|---|---|---|---|---|
| | | $\Re$ | 0.42857 | 0.46667 | 0.50000 | 0.52941 | 0.55556 |
| (30, 30) | (10, 10) | $\hat{\Re}$ | 0.42682 | 0.44758 | 0.50193 | 0.52193 | 0.55193 |
| | | MSE | 0.12878 | 0.14527 | 0.16452 | 0.14276 | 0.17626 |
| | (20, 20) | $\hat{\Re}$ | 0.42902 | 0.45428 | 0.50621 | 0.53621 | 0.54621 |
| | | MSE | 0.12105 | 0.12140 | 0.13251 | 0.14210 | 0.14425 |
| | (30, 30) | $\hat{\Re}$ | 0.43138 | 0.44093 | 0.51058 | 0.51058 | 0.55308 |
| | | MSE | 0.09854 | 0.10142 | 0.11542 | 0.10254 | 0.10982 |
| (50, 50) | (20, 20) | $\hat{\Re}$ | 0.42480 | 0.43436 | 0.52114 | 0.52114 | 0.56114 |
| | | MSE | 0.10141 | 0.14122 | 0.14152 | 0.14685 | 0.15562 |
| | (30, 30) | $\hat{\Re}$ | 0.43979 | 0.44592 | 0.49267 | 0.52267 | 0.55067 |
| | | MSE | 0.09845 | 0.12457 | 0.13485 | 0.12362 | 0.13826 |
| | (50, 50) | $\hat{\Re}$ | 0.42982 | 0.43615 | 0.50461 | 0.52192 | 0.54192 |
| | | MSE | 0.0856 | 0.09475 | 0.10451 | 0.11204 | 0.10241 |

**Table 2** Average ML estimate values of $\Re$ and their MSEs for $\alpha_2 = 3,\ \theta = 3$ and different values of $\alpha_1, n_1, m_1, n_2,$ and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | $\alpha_1 = 1.50$ | $\alpha_1 = 1.75$ | $\alpha_1 = 2.00$ | $\alpha_1 = 2.25$ | $\alpha_1 = 2.50$ |
|---|---|---|---|---|---|---|---|
| | | $\Re$ | 0.33333 | 0.36842 | 0.40000 | 0.42857 | 0.45455 |
| (30, 30) | (10, 10) | $\hat{\Re}$ | 0.33479 | 0.38211 | 0.42541 | 0.42281 | 0.47505 |
| | | MSE | 0.16652 | 0.17142 | 0.18214 | 0.16254 | 0.16251 |
| | (20, 20) | $\hat{\Re}$ | 0.324591 | 0.37619 | 0.41248 | 0.43909 | 0.46718 |
| | | MSE | 0.13150 | 0.14512 | 0.15146 | 0.14215 | 0.16245 |
| | (30, 30) | $\hat{\Re}$ | 0.34523 | 0.36422 | 0.42040 | 0.43122 | 0.45243 |
| | | MSE | 0.10120 | 0.11521 | 0.12541 | 0.10574 | 0.11542 |
| (50, 50) | (20, 20) | $\hat{\Re}$ | 0.39384 | 0.33254 | 0.41542 | 0.42704 | 0.46412 |
| | | MSE | 0.16844 | 0.15212 | 0.17511 | 0.16524 | 0.17265 |
| | (30, 30) | $\hat{\Re}$ | 0.32411 | 0.35245 | 0.41228 | 0.42247 | 0.45248 |
| | | MSE | 0.12104 | 0.13254 | 0.14214 | 0.12548 | 0.14251 |
| | (50, 50) | $\hat{\Re}$ | 0.32438 | 0.35535 | 0.40145 | 0.42545 | 0.45441 |
| | | MSE | 0.09451 | 0.10541 | 0.10547 | 0.09544 | 0.09851 |

**Table 3** Average ML estimate values of $\Re$ and their MSEs for $\alpha_1 = 2$, $\theta = 2$ and different values of $\alpha_2, n_1, m_1, n_2$, and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | $\alpha_2 = 1.50$ | $\alpha_2 = 1.75$ | $\alpha_2 = 2.00$ | $\alpha_2 = 2.25$ | $\alpha_2 = 2.50$ |
|---|---|---|---|---|---|---|---|
| | | $\Re$ | 0.57143 | 0.53333 | 0.50000 | 0.47059 | 0.44444 |
| (30, 30) | (10, 10) | $\hat{\Re}$ | 0.57842 | 0.52548 | 0.50525 | 0.46854 | 0.45854 |
| | | MSE | 0.13256 | 0.14425 | 0.14384 | 0.13845 | 0.14522 |
| | (20, 20) | $\hat{\Re}$ | 0.58542 | 0.53245 | 0.50785 | 0.47592 | 0.43585 |
| | | MSE | 0.11454 | 0.11247 | 0.12549 | 0.13554 | 0.12457 |
| | (30, 30) | $\hat{\Re}$ | 0.58445 | 0.54841 | 0.51548 | 0.46854 | 0.44586 |
| | | MSE | 0.09451 | 0.09842 | 0.10745 | 0.10112 | 0.10845 |
| (50, 50) | (20, 20) | $\hat{\Re}$ | 0.59240 | 0.55245 | 0.49254 | 0.48562 | 0.45471 |
| | | MSE | 0.10461 | 0.12451 | 0.13251 | 0.12548 | 0.13542 |
| | (30, 30) | $\hat{\Re}$ | 0.56754 | 0.53254 | 0.49525 | 0.46422 | 0.46251 |
| | | MSE | 0.09411 | 0.10124 | 0.11555 | 0.10144 | 0.10515 |
| | (50, 50) | $\hat{\Re}$ | 0.58452 | 0.54215 | 0.51541 | 0.47625 | 0.45421 |
| | | MSE | 0.09215 | 0.08214 | 0.10624 | 0.10024 | 0.09451 |

**Table 4** Average ML estimate values of $\Re$ and their MSEs for $\alpha_1 = 3.0$, $\theta = 3$ and different values of $\alpha_2, n_1, m_1, n_2$, and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | $\alpha_2 = 1.50$ | $\alpha_2 = 1.75$ | $\alpha_2 = 2.00$ | $\alpha_2 = 2.25$ | $\alpha_2 = 2.50$ |
|---|---|---|---|---|---|---|---|
| | | $\Re$ | 0.66667 | 0.63158 | 0.60000 | 0.57147 | 0.54545 |
| (30, 30) | (10, 10) | $\hat{\Re}$ | 0.66542 | 0.63558 | 0.61451 | 0.57842 | 0.55145 |
| | | MSE | 0.14515 | 0.14685 | 0.15842 | 0.13944 | 0.13745 |
| | (20, 20) | $\hat{\Re}$ | 0.65411 | 0.64251 | 0.60125 | 0.58542 | 0.54268 |
| | | MSE | 0.12488 | 0.12985 | 0.13845 | 0.12575 | 0.11452 |
| | (30, 30) | $\hat{\Re}$ | 0.67541 | 0.64555 | 0.62834 | 0.58445 | 0.55044 |
| | | MSE | 0.09458 | 0.09842 | 0.11652 | 0.09764 | 0.10441 |
| (50, 50) | (20, 20) | $\hat{\Re}$ | 0.65284 | 0.62549 | 0.62485 | 0.59240 | 0.52417 |
| | | MSE | 0.16844 | 0.15212 | 0.17511 | 0.10461 | 0.17265 |
| | (30, 30) | $\hat{\Re}$ | 0.66454 | 0.63877 | 0.60124 | 0.56754 | 0.54218 |
| | | MSE | 0.11254 | 0.12484 | 0.12410 | 0.11542 | 0.12493 |
| | (50, 50) | $\hat{\Re}$ | 0.66458 | 0.62581 | 0.61245 | 0.58452 | 0.56241 |
| | | MSE | 0.08245 | 0.09541 | 0.10435 | 0.09841 | 0.09552 |

**Table 5** Average length with coverage probability of $\Re$ and their MSEs for $\alpha_2 = 2, \ \theta = 2$ and different values of $\alpha_1, n_1, m_1, n_2,$ and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | | $\alpha_1 = 1.50$ | $\alpha_1 = 1.75$ | $\alpha_1 = 2.00$ | $\alpha_1 = 2.25$ | $\alpha_1 = 2.50$ |
|---|---|---|---|---|---|---|---|---|
| (30, 30) | (10, 10) | ACI | Length | 0.13952 | 0.13535 | 0.13456 | 0.13521 | 0.13821 |
| | | | CP | 0.92514 | 0.92541 | 0.93541 | 0.92541 | 0.93581 |
| | | Boot-p | Length | 0.14582 | 0.14755 | 0.13284 | 0.14785 | 0.13284 |
| | | | CP | 0.93584 | 0.93258 | 0.92541 | 0.93250 | 0.93284 |
| | (20, 20) | ACI | Length | 0.11310 | 0.12417 | 0.12427 | 0.11427 | 0.12127 |
| | | | CP | 0.95447 | 0.93519 | 0.94514 | 0.93851 | 0.94514 |
| | | Boot-p | Length | 0.12545 | 0.12548 | 0.11325 | 0.10244 | 0.11715 |
| | | | CP | 0.94582 | 0.94257 | 0.93540 | 0.94157 | 0.93684 |
| | (30, 30) | ACI | Length | 0.10292 | 0.10235 | 0.10540 | 0.10521 | 0.11101 |
| | | | CP | 0.96742 | 0.96541 | 0.96684 | 0.94566 | 0.97564 |
| | | Boot-p | Length | 0.11457 | 0.11254 | 0.11547 | 0.12154 | 0.12451 |
| | | | CP | 0.95847 | 0.95487 | 0.97554 | 0.95686 | 0.96854 |
| (50, 50) | (20, 20) | ACI | Length | 0.11284 | 0.11557 | 0.12541 | 0.12117 | 0.11457 |
| | | | CP | 0.94154 | 0.94259 | 0.95484 | 0.94521 | 0.95842 |
| | | Boot-p | Length | 0.12144 | 0.10124 | 0.12548 | 0.11845 | 0.12144 |
| | | | CP | 0.95487 | 0.95424 | 0.96523 | 0.95458 | 0.96484 |
| | (30, 30) | ACI | Length | 0.10872 | 0.10745 | 0.10581 | 0.10584 | 0.11854 |
| | | | CP | 0.95215 | 0.96631 | 0.96480 | 0.95614 | 0.98450 |
| | | Boot-p | Length | 0.11124 | 0.10125 | 0.11545 | 0.11024 | 0.10155 |
| | | | CP | 0.954515 | 0.96636 | 0.96845 | 0.96171 | 0.97215 |
| | (50, 50) | ACI | Length | 0.09457 | 0.09102 | 0.09115 | 0.08215 | 0.10542 |
| | | | CP | 0.98541 | 0.97518 | 0.98412 | 0.98635 | 0.99015 |
| | | Boot-p | Length | 0.10214 | 0.09552 | 0.10548 | 0.15484 | 0.09845 |
| | | | CP | 0.97252 | 0.98258 | 0.97892 | 0.97895 | 0.98514 |

**Table 6** Average length with coverage probability of $\Re$ and their MSEs for $\alpha_2 = 3, \ \theta = 3$ and different values of $\alpha_1, n_1, m_1, n_2,$ and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | | $\alpha_1 = 1.50$ | $\alpha_1 = 1.75$ | $\alpha_1 = 2.00$ | $\alpha_1 = 2.25$ | $\alpha_1 = 2.50$ |
|---|---|---|---|---|---|---|---|---|
| (30, 30) | (10, 10) | ACI | Length | 0.12254 | 0.13335 | 0.13254 | 0.13845 | 0.12458 |
| | | | CP | 0.93154 | 0.93854 | 0.94154 | 0.93824 | 0.94251 |
| | | Boot-p | Length | 0.11815 | 0.12515 | 0.11557 | 0.12844 | 0.11854 |
| | | | CP | 0.94845 | 0.94854 | 0.95458 | 0.94549 | 0.95540 |
| | (20, 20) | ACI | Length | 0.12547 | 0.11415 | 0.12484 | 0.12454 | 0.11545 |
| | | | CP | 0.94875 | 0.94549 | 0.9546 | 0.94545 | 0.95148 |
| | | Boot-p | Length | 0.11155 | 0.12145 | 0.12842 | 0.13151 | 0.12845 |
| | | | CP | 0.95485 | 0.95284 | 0.96484 | 0.96584 | 0.96748 |
| | (30, 30) | ACI | Length | 0.10941 | 0.10548 | 0.10654 | 0.11745 | 0.10541 |
| | | | CP | 0.97256 | 0.96844 | 0.97245 | 0.96242 | 0.96451 |
| | | Boot-p | Length | 0.12584 | 0.10258 | 0.15554 | 0.11450 | 0.11021 |
| | | | CP | 0.97455 | 0.98154 | 0.98214 | 0.97251 | 0.98151 |

**Table 6** (Continued)

| $(n_1,n_2)$ | $(m_1,m_2)$ | | | $\alpha_1=1.50$ | $\alpha_1=1.75$ | $\alpha_1=2.00$ | $\alpha_1=2.25$ | $\alpha_1=2.50$ |
|---|---|---|---|---|---|---|---|---|
| (50, 50) | (10, 10) | ACI | Length | 0.13254 | 0.12547 | 0.13898 | 0.13185 | 0.12457 |
| | | | CP | 0.95414 | 0.95414 | 0.94521 | 0.95650 | 0.96245 |
| | | Boot-p | Length | 0.13845 | 0.13545 | 0.12545 | 0.12545 | 0.13254 |
| | | | CP | 0.96854 | 0.97541 | 0.96554 | 0.96848 | 0.97815 |
| | (20, 20) | ACI | Length | 0.101542 | 0.11545 | 0.10581 | 0.11184 | 0.09456 |
| | | | CP | 0.96845 | 0.97545 | 0.96484 | 0.96554 | 0.97255 |
| | | Boot-p | Length | 0.11541 | 0.12454 | 0.10545 | 0.10451 | 0.10241 |
| | | | CP | 0.97842 | 0.98454 | 0.97815 | 0.97851 | 0.98121 |
| | (30, 30) | ACI | Length | 0.08262 | 0.08245 | 0.09245 | 0.09254 | 0.10822 |
| | | | CP | 0.99423 | 0.98264 | 0.97515 | 0.98544 | 0.98245 |
| | | Boot-p | Length | 0.09852 | 0.09451 | 0.10242 | 0.09125 | 0.09451 |
| | | | CP | 0.98955 | 0.98851 | 0.98451 | 0.99151 | 0.99151 |

**Table 7** Average length with coverage probability of $\Re$ and their MSEs for $\alpha_1 = 2,\ \theta = 2$ and different values of $\alpha_2, n_1, m_1, n_2,$ and $m_2$

| $(n_1,n_2)$ | $(m_1,m_2)$ | | | $\alpha_2=1.50$ | $\alpha_2=1.75$ | $\alpha_2=2.00$ | $\alpha_2=2.25$ | $\alpha_2=2.50$ |
|---|---|---|---|---|---|---|---|---|
| (30, 30) | (10, 10) | ACI | Length | 0.12155 | 0.12584 | 0.11455 | 0.12584 | 0.13454 |
| | | | CP | 0.95421 | 0.95416 | 0.95641 | 0.96540 | 0.94895 |
| | | Boot-p | Length | 0.11544 | 0.12145 | 0.11041 | 0.12685 | 0.12181 |
| | | | CP | 0.96854 | 0.96455 | 0.95451 | 0.95474 | 0.95484 |
| | (20, 20) | ACI | Length | 0.10154 | 0.11454 | 0.11845 | 0.12854 | 0.10154 |
| | | | CP | 0.97564 | 0.96425 | 0.97564 | 0.97541 | 0.96842 |
| | | Boot-p | Length | 0.11520 | 0.10151 | 0.10415 | 0.11514 | 0.10451 |
| | | | CP | 0.97584 | 0.97841 | 0.96844 | 0.96945 | 0.97155 |
| | (30, 30) | ACI | Length | 0.09424 | 0.09425 | 0.10451 | 0.102324 | 0.09121 |
| | | | CP | 0.98685 | 0.98421 | 0.98642 | 0.97684 | 0.98368 |
| | | Boot-p | Length | 0.10254 | 0.09542 | 0.10984 | 0.09454 | 0.10125 |
| | | | CP | 0.99841 | 0.98547 | 0.98454 | 0.98152 | 0.98574 |
| (50, 50) | (20, 20) | ACI | Length | 0.10544 | 0.10224 | 0.11248 | 0.11015 | 0.10544 |
| | | | CP | 0.96854 | 0.97244 | 0.96852 | 0.96844 | 0.97651 |
| | | Boot-p | Length | 0.11458 | 0.10215 | 0.11515 | 0.10455 | 0.11515 |
| | | | CP | 0.97815 | 0.98154 | 0.97245 | 0.97155 | 0.96154 |
| | (30, 30) | ACI | Length | 0.09234 | 0.09854 | 0.10281 | 0.09254 | 0.08521 |
| | | | CP | 0.98241 | 0.98236 | 0.98564 | 0.98256 | 0.98754 |
| | | Boot-p | Length | 0.10254 | 0.09185 | 0.10054 | 0.10189 | 0.09154 |
| | | | CP | 0.98945 | 0.98514 | 0.98045 | 0.98012 | 0.98045 |
| | (50, 50) | ACI | Length | 0.07541 | 0.08542 | 0.08634 | 0.08254 | 0.09515 |
| | | | CP | 0.99284 | 0.99254 | 0.99636 | 0.99754 | 0.99854 |
| | | Boot-p | Length | 0.09155 | 0.09415 | 0.09151 | 0.10082 | 0.08424 |
| | | | CP | 0.98915 | 0.98915 | 0.98151 | 0.99540 | 0.99550 |

**Table 8** Average length with coverage probability of $\Re$ and their MSEs for $\alpha_1 = 3$, $\theta = 3$ and different values of $\alpha_2, n_1, m_1, n_2$, and $m_2$

| $(n_1, n_2)$ | $(m_1, m_2)$ | | | $\alpha_2 = 1.50$ | $\alpha_2 = 1.75$ | $\alpha_2 = 2.00$ | $\alpha_2 = 2.25$ | $\alpha_2 = 2.50$ |
|---|---|---|---|---|---|---|---|---|
| (30, 30) | (10, 10) | ACI | Length | 0.11424 | 0.12584 | 0.12154 | 0.11548 | 0.12471 |
| | | | CP | 0.94521 | 0.94852 | 0.94856 | 0.95846 | 0.95474 |
| | | Boot-p | Length | 0.12515 | 0.12511 | 0.11515 | 0.12155 | 0.12454 |
| | | | CP | 0.95451 | 0.95447 | 0.95485 | 0.961544 | 0.95051 |
| | (20, 20) | ACI | Length | 0.10124 | 0.09548 | 0.10451 | 0.10254 | 0.10514 |
| | | | CP | 0.96485 | 0.96474 | 0.96744 | 0.95743 | 0.97154 |
| | | Boot-p | Length | 0.11454 | 0.10545 | 0.11054 | 0.11545 | 0.11745 |
| | | | CP | 0.96654 | 0.96478 | 0.96051 | 0.97215 | 0.96845 |
| | (30, 30) | ACI | Length | 0.08245 | 0.08456 | 0.09842 | 0.09012 | 0.08545 |
| | | | CP | 0.98254 | 0.98415 | 0.97545 | 0.97584 | 0.98544 |
| | | Boot-p | Length | 0.10215 | 0.09455 | 0.09545 | 0.10455 | 0.09845 |
| | | | CP | 0.98451 | 0.978411 | 0.97145 | 0.98545 | 0.97155 |
| (50, 50) | (20, 20) | ACI | Length | 0.11754 | 0.10451 | 0.11452 | 0.10498 | 0.10145 |
| | | | CP | 0.96154 | 0.96745 | 0.95421 | 0.96521 | 0.96481 |
| | | Boot-p | Length | 0.11455 | 0.11415 | 0.12815 | 0.11545 | 0.11451 |
| | | | CP | 0.96584 | 0.96421 | 0.96585 | 0.97245 | 0.96845 |
| | (30, 30) | ACI | Length | 0.09842 | 0.09254 | 0.09454 | 0.09854 | 0.09245 |
| | | | CP | 0.98242 | 0.98514 | 0.98754 | 0.97154 | 0.98655 |
| | | Boot-p | Length | 0.10254 | 0.09844 | 0.10545 | 0.10515 | 0.10245 |
| | | | CP | 0.97515 | 0.97554 | 0.97215 | 0.98454 | 0.97154 |
| | (50, 50) | ACI | Length | 0.08215 | 0.08124 | 0.09214 | 0.082451 | 0.08625 |
| | | | CP | 0.99255 | 0.99284 | 0.98954 | 0.99251 | 0.99285 |
| | | Boot-p | Length | 0.09155 | 0.09012 | 0.09415 | 0.09112 | 0.08954 |
| | | | CP | 0.99845 | 0.99011 | 0.98455 | 0.99545 | 0.99478 |

### 5.1. Simulation data

In this section, we deal the analysis of a simulated data and show how can use the result in real life problem. We generate a simulated sample from the power distribution population having $\alpha_1 = 2$ and $\alpha_2 = 2.5$ and $\theta = 3$ with same sample size $n_1 = n_2 = 25$ presented in Table 9. The observed sample of stress $(Y)$ and strength $(X)$ are as

**Table 9** Simulated data of $X$ and $Y$

| |
|---|
| Data $X$: 0.73517, 0.76286, 0.94939, 1.29551, 1.47085, 1.50888, 1.71101, 1.79574, 1.82376, 1.86437, 1.97652, 2.04804, 2.07875, 2.10856, 2.13789, 2.19393, 2.42172, 2.42760, 2.43914, 2.53862, 2.64788, 2.68165, 2.73583, 2.88319, 2.93943 |
| Data $Y$: 0.57674, 0.85571, 0.96919, 1.09143, 1.59470, 1.76226, 1.84869, 2.14430, 2.17365, 2.18479, 2.29608, 2.30037, 2.35548, 2.40995, 2.55089, 2.55180, 2.65471, 2.65852, 2.66172, 2.66583, 2.79449, 2.79636, 2.80176, 2.95940, 2.96953 |

Using the (9), the ML estimates of $\alpha_1, \alpha_2$, and $\theta$ are 2.1601582, 2.5773875 and 2.9695278, respectively. So, the ML estimate of $\hat{\Re}$ (by using the invariance property of MLE) is 0.45596.

**Table 10** The ML estimate of $\Re$ with ACI and boot-p for simulated data

| $(m_1, m_2)$ | $\hat{\Re}$ | ACI | Boot-p |
|---|---|---|---|
| (10, 10) | 0.41154 | (0.35242, 0.45182) | (0.34038, 0.45230) |
| (20, 20) | 0.42614 | (0.37619, 0.46102) | (0.37885, 0.4723) |
| (25, 25) | 0.45596 | (0.41051, 0.52847) | (0.40867, 0.53135) |

### 5.2. Real data application

For numerical purposes, we show two real data sets to analysis of the strength of system. The data sets are represented and studied by Xie et. al (2009). The data show the Jute fiber breaking strength at two different gauge lengths where $X$ represents the strength of 5mm fiber and $Y$ is the strength of 15mm fiber.

**Table 11** The Jute fiber breaking strength at two different gauge lengths

| Data set 1 of length 5 mm: $X$ ($n_1 = 30$) | Data set 2 of length 15 mm: $Y$ ($n_2 = 30$) |
|---|---|
| 566.31, 270.79, 516.48, 823.03, 226.53, | 594.40, 202.75, 68.37, 574.86, 225.65, |
| 367.70, 185.42,441.87, 618.57, 546.11, | 76.38, 156.67, 127.81, 813.87, 562.39, |
| 268.20, 315.33, 809.23, 218.86, 583.97, | 468.47, 135.09, 72.24, 497.94, 355.56, |
| 304.84, 129.08, 537.45, 496.28, 167.87, | 569.07, 640.48, 200.76, 550.42, 748.75, |
| 306.99, 178.25, 370.02, 168.20, 554.61, | 489.66, 678.06, 457.71, 106.73, 716.30, |
| 360.80, 260.97, 254.29, 495.51, 187.68. | 42.66, 80.40, 339.22, 70.09, 193.42. |

First, we use the Kolmogorov-Smirnov (K-S) test to check whether the power distribution fits the given data. We calculate the ML estimate of given parameters $\alpha_1, \alpha_2$, and $\theta$ of the power distribution for stress and strength model. The K-S distance for data I is 0.2227 with p-value 0.087 whereas for data II the K-S distance is 0.1432 with associated p-value 0.524. It shows that the power distribution is appropriate for both data. The ML estimates of reliability for the real data set are 0.567. The ML estimate of $\Re$ and related confidence intervals based on different sample size are defines in Table 12.

**Table 12** The ML estimate of $\Re$ with ACI and boot-p for the real data set

| $(m_1, m_2)$ | $\Re$ | ACI | Boot-p |
|---|---|---|---|
| (10, 10) | 0.481 | (0.442, 0.512) | (0.438, 0.523) |
| (20, 20) | 0.512 | (0.461, 0.561) | (0.458, 0.572) |
| (30, 30) | 0.567 | (0.532, 0.612) | (0.541, 0.620) |

## 6. Conclusions

In this study, we see that the power distribution is used as a simple model to evaluate system reliability and obtain its statistical properties like mean, variance, order statistic, moment generating function, characteristic function, $m^{\text{th}}$ moment, method of moments estimate and Lorenz curve also examined. Since this study is based on the reliability so the stress-strength reliability and hazard rate function also calculated. The ML estimate of the stress-strength reliability also calculated when both stress and strength of a system also follows to the power distribution. It is suggested that the power distribution be given consideration when analyzing failure experiments. We hope that this work will aid in future research.

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**Appendix**

The second derivatives of the log likelihood function with respect to $\alpha_1, \alpha_2$, and $\theta$ are as follow:

$$\frac{\partial^2 \log L}{\partial \alpha_1^2} = \frac{m_1}{\alpha_1^2} + \frac{(n_1 - m_1)}{\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)^2}\left[\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)\left\{\theta^{\alpha_1}(\log\theta)^2 - x_{m_1}^{\alpha_1}\left(\log x_{m_1}\right)^2\right\} - \left\{\theta^{\alpha_1}(\log\theta) - x_{m_1}^{\alpha_1}\left(\log x_{m_1}\right)\right\}^2\right]$$

$$\frac{\partial^2 \log L}{\partial \alpha_1 \partial \theta} = -\frac{n_1}{\theta} + \frac{(n_1 - m_1)\theta^{\alpha_1 - 1}}{\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)^2}\left[\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)\{\log\theta + 1\} - \alpha_1\left\{\theta^{\alpha_1}\log\theta - x_{m_1}^{\alpha_1}\log x_{m_1}\right\}\right]$$

$$\frac{\partial^2 \log L}{\partial \alpha_1 \partial \alpha_2} = \frac{\partial^2 \log L}{\partial \alpha_2 \partial \alpha_1} = 0$$

$$\frac{\partial^2 \log L}{\partial \alpha_2^2} = \frac{m_2}{\alpha_2^2} + \frac{(n_2 - m_2)}{\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)^2}\left[\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)\left\{\theta^{\alpha_2}(\log\theta)^2 - x_{m_2}^{\alpha_2}\left(\log x_{m_2}\right)^2\right\} - \left\{\theta^{\alpha_2}(\log\theta) - x_{m_2}^{\alpha_2}\left(\log x_{m_2}\right)\right\}^2\right]$$

$$\frac{\partial^2 \log L}{\partial \alpha_2 \partial \theta} = -\frac{n_2}{\theta} + \frac{(n_2 - m_2)\theta^{\alpha_2 - 1}}{\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)^2}\left[\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)\{\log\theta + 1\} - \alpha_2\left\{\theta^{\alpha_2}\log\theta - x_{m2}^{\alpha_2}\log x_{m_2}\right\}\right]$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha_1} = \frac{n_1}{\theta} + \frac{(n_1 - m_1)}{\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)^2}\left[\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)\left\{\theta^{\alpha_1 - 1} + \alpha_1(\alpha_1 - 1)\theta^{\alpha_1 - 2}\right\} - \alpha_1\theta^{\alpha_1 - 1}\left\{\theta^{\alpha_1}(\log\theta) - x_{m_1}^{\alpha_1}\left(\log x_{m_1}\right)\right\}\right]$$

$$\frac{\partial^2 \log L}{\partial \theta \partial \alpha_2} = \frac{n_2}{\theta} + \frac{(n_2 - m_2)}{\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)^2}\left[\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)\left\{\theta^{\alpha_2 - 1} + \alpha_2(\alpha_2 - 1)\theta^{\alpha_2 - 2}\right\} - \alpha_2\theta^{\alpha_2 - 1}\left\{\theta^{\alpha_2}(\log\theta) - x_{m_2}^{\alpha_2}\left(\log x_{m_2}\right)\right\}\right]$$

$$\frac{\partial^2 \log L}{\partial \theta^2} = \frac{n_1\alpha_1 + n_2\alpha_2}{\theta^2} + \frac{(n_1 - m_1)}{\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)^2}\left[\left(\theta^{\alpha_1} - x_{m_1}^{\alpha_1}\right)\left\{\alpha_1(\alpha_1 - 1)\theta^{\alpha_1 - 2}\right\} - \alpha_1\theta^{\alpha_1 - 1}\right]$$
$$+ \frac{(n_2 - m_2)}{\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)^2}\left[\left(\theta^{\alpha_2} - x_{m_2}^{\alpha_2}\right)\left\{\alpha_2(\alpha_2 - 1)\theta^{\alpha_2 - 2}\right\} - \alpha_2\theta^{\alpha_2 - 1}\right].$$

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