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A New Two-Parameter Extension of Half-Logistic distribution: Properties, Applications and Different Method of Estimations

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Abstract

In this paper, we define a new two-parameter lifetime distribution, which is called the new Half-Logistic (NHL) distribution. Theoretical properties of this model, including the hazard function, quantile function, asymptotic, extreme value, moments, conditional moments, mean residual life, mean past lifetime, residual entropy, and order statistics, are derived and studied in detail. The maximum likelihood estimates of parameters are compared with various methods of estimation by conducting a simulation study. Finally, two real data sets illustrate the purposes.

Keywords: Entropy, maximum likelihood, moments, moments generating function.

1. Introduction

Modelling and analysing lifetimes are important in engineering, medicine, economics, etc. In many applied areas such as lifetime analysis, finance and insurance, we need extended forms of distributions. So, several methods for generating new families of distributions have been proposed in literature. Some attempts have been made to define new families of probability distributions that extend well-known families of distributions and with great flexibility in modeling data in practice. Among them, the generalized G-classes of distributions say G are used in which one or more parameter(s) are added to a baseline distribution.

Balakrishnan (1985), proposed the standard half-logistic (SHL) distribution as a lifetime increasing hazard rate function. The cumulative distribution function (CDF) of SHL is given by

$$G(x)=\frac{1-e^{-x}}{1+e^{-x}},\ x>0.$$

The probability density function (PDF) of SHL distribution is

$$g(x)=\frac{2e^{-x}}{(1+e^{-x})^2},\ x>0.$$

In this paper, we introduce a new two-parameter lifetime distribution which contain half-logistic distribution as special case. The CDF of new distribution is given by

$$F(x)=\frac{1-e^{-\beta x}}{(1+e^{-x})^{\alpha}},\quad x>0,\alpha>0,\beta>0 \tag{1}$$

We denote it by new half logistic (NHL(α,β)). The related PDF and the hazard rate function (HRF) are given by

$$f(x)=\frac{\alpha e^{-x}+\beta e^{-\beta x}+(\beta-\alpha)e^{-(\beta+1)x}}{(1+e^{-x})^{\alpha+1}} \text{ and } h(x)=\frac{\alpha e^{-x}+\beta e^{-\beta x}+(\beta-\alpha)e^{-(\beta+1)x}}{(1+e^{-x})\left[(1+e^{-x})^{\alpha}+e^{-\beta x}-1\right]}.$$

For $\alpha=\beta=1$, NHL(α,β) reduces to half-logistic distribution. Figures 1 and 2 show the PDF and the HRF of NHL(α,β) for selected parameter values of parameters. These graphs show that the PDF of NHL(α,β) is unimodal, right skew or almost symmetric. The HRF of NHL(α,β) can be decreasing, increasing, bathtub and upside down shape.

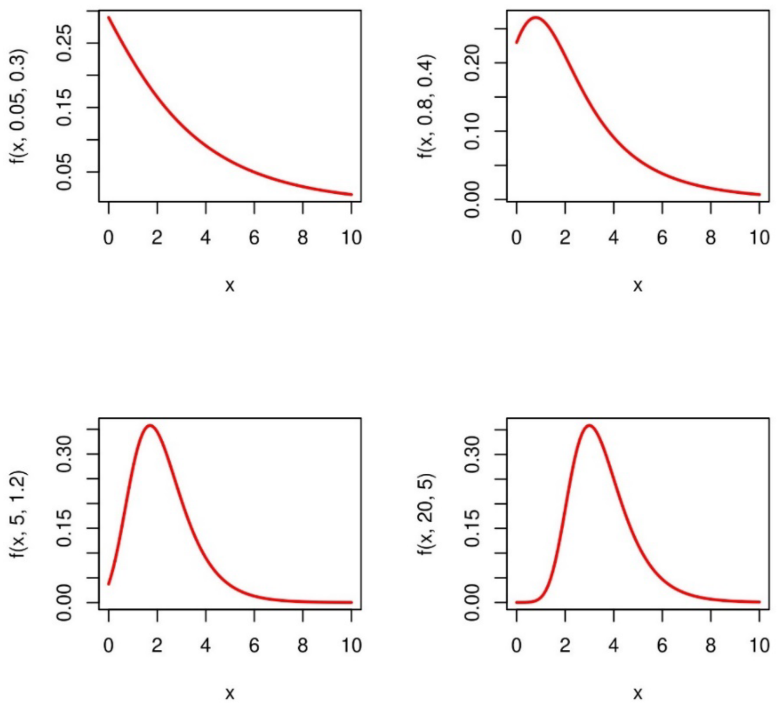

Figure 1 The sample curves of density function of NHL(α,β)

The rest of this paper is organized as follows: In the above, new family of distributions was proposed. Various properties of the proposed distribution are explored in Section 2. These properties include asymptotic, extreme value, quantile function, mixture for PDF, moments, conditional moments, mean residual (past) lifetime, residual entropy and order statistics. The maximum likelihood estimation of parameters are compared with various methods of estimations by conducting simulation study in Section 3. Real data sets are analyzed to show the performance of the new family in Section 4. In Section 5, some concluding remarks are considered.

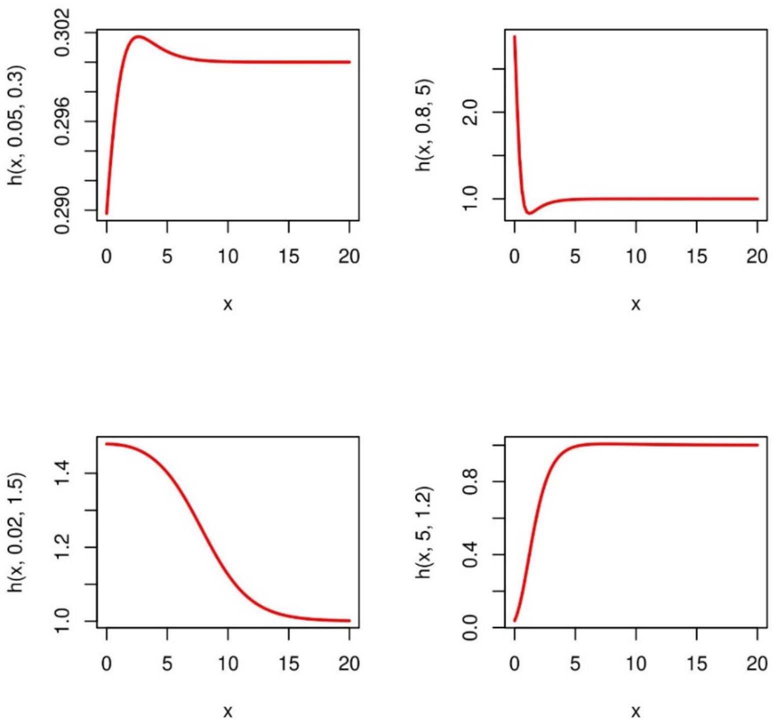

Figure 2 The sample curves of hazard rate function of $\mathrm{NHL}(\alpha,\beta)$

2. Main Properties

In this section, we study some basic properties of $\mathrm{NHL}(\alpha,\beta)$ distribution.

2.1. Quantile function

The quantile function (qf) of $\mathrm{NHL}(\alpha,\beta)$ have not closed form. If $U \sim U(0,1)$, then the qf of $\mathrm{NHL}(\alpha,\beta)$ can obtain by solving non-linear equation $F(x)=u$.

2.2. Asymptotic for CDF, PDF and HRF

Note that $1-e^{-\beta x} \sim \beta x$, $1+e^{-x} \sim 2$ as $x \to 0^{+}$ and $1+e^{-x} \sim 1$ as $x \to \infty$. The asymptotic of CDF, PDF and HRF as $x \to 0^{+}$ are given by $F(x) \sim \beta 2^{\alpha} x$, $f(x) \sim \beta 2^{\alpha}$, and $h(x) \sim \dfrac{\beta 2^{\alpha}}{1-\beta 2^{\alpha} x}$. The asymptotic of CDF, PDF and HRF as $x \to \infty$ are given by $1-F(x) \sim \mathrm{e}^{-\beta x}$, $f(x) \sim \beta \mathrm{e}^{-\beta x}$, and $h(x) \sim \beta$. These equations show the effect of parameters on tails of NHL distribution.

2.3. Mixture for PDF

Using generalized binomial expansion for any $x \in \mathbf{R}$, $(1+e^{-x})^{-\alpha-1} = \sum_{i=0}^{\infty} \binom{-\alpha-1}{i} e^{-i x}$. Then,

$$f(x) = \frac{\alpha \mathrm{e}^{-x} + \beta \mathrm{e}^{-\beta x} + (\beta-\alpha)\mathrm{e}^{-(\beta+1)x}}{(1+\mathrm{e}^{-x})^{\alpha+1}} = \sum_{i=0}^{\infty} \binom{-\alpha-1}{i} \mathrm{e}^{-ix}\left[\alpha \mathrm{e}^{-x} + \beta \mathrm{e}^{-\beta x} + (\beta-\alpha)\mathrm{e}^{-(\beta+1)x}\right]$$

$$=\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\left[\frac{\alpha}{i+1}f_{i+1}(x)+\frac{\beta}{\beta+i}f_{\beta+i}(x)+\frac{\beta-\alpha}{\beta+i+1}f_{\beta+i+1}(x)\right], \tag{2}$$

where $f_c(x)=ce^{-cx}$ denote the exponential distribution with parameter $c>0$. Equation (2), show that we can write the pdf of NHL(α,β) distribution as a linear combination of exponential distributions. Using Equation (2), we can easily obtain

$$E(X^n)=\int_0^{\infty}x^n f(x)dx=\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\int_0^{\infty}x^n \mathrm{e}^{-ix}\left[\alpha\mathrm{e}^{-x}+\beta\mathrm{e}^{-\beta x}+(\beta-\alpha)\mathrm{e}^{-(\beta+1)x}\right]dx$$

$$=\Gamma(n+1)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\left[\frac{\alpha}{(i+1)^{n+1}}+\frac{\beta}{(i+\beta)^{n+1}}+\frac{\beta-\alpha}{(i+\beta+1)^{n+1}}\right],$$

where $\Gamma(n)=\int_0^{\infty}t^{n-1}\mathrm{e}^{-t}dt$ denote the gamma function. The moment generating function (MGF) of NHL(α,β), after using (2) can obtain as follow

$$E(\mathrm{e}^{tX})=\int_0^{\infty}\mathrm{e}^{tx}f(x)dx=\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\int_0^{\infty}\mathrm{e}^{tx}\,\mathrm{e}^{-ix}\left[\alpha\mathrm{e}^{-x}+\beta\mathrm{e}^{-\beta x}+(\beta-\alpha)\mathrm{e}^{-(\beta+1)x}\right]dx$$

$$=\Gamma(n+1)\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\left[\frac{\alpha}{i+1-t}+\frac{\beta}{i+\beta-t}+\frac{\beta-\alpha}{i+\beta+1-t}\right].$$

Here we show that all moments of NHL(α,β) are exist. All moments of NHL are exist.

Proof: First note that $1\le 1+\mathrm{e}^{-}\le 2$ for any $x>0$, then

$$E(X^n)=\int_0^{\infty}x^n f(x)dx=\int_0^{\infty}x^n\frac{\alpha e^{-x}+\beta e^{-\beta x}+(\beta-\alpha)e^{-(\beta+1)x}}{(1+e^{-x})^{\alpha+1}}dx$$

$$<\int_0^{\infty}x^n\left[\alpha e^{-x}+\beta e^{-\beta x}+(\beta-\alpha)e^{-(\beta+1)x}\right]dx=\Gamma(n+1)\left[\alpha+\beta^{-n}+(\beta-\alpha)(\beta+1)^{-n-1}\right]dx\ <\in\infty$$

It conclude that all moments of NHL are exist.

Now, we obtain first four moments for some selected value of parameters, and then we can compute mean, variance, skewness and kurtosis. Figure 3 shows these measures as a function of α,β.

2.4. Conditional moments

Here, we intend to determine the conditional moments of the new family. Therefore,

$$E(X^n\mid X\le x)=\frac{1}{F(x)}\int_0^{x}t^n f(t)dt.$$

After some simple algebraic manipulation one can obtain

$$E(X^n\mid X\le x)=\frac{1}{F(x)}\sum_{i=0}^{\infty}\binom{-\alpha-1}{i}\left[\frac{\alpha\gamma(n+1,\frac{x}{i+1})}{(i+1)^{n+1}}+\frac{\beta\gamma(n+1,\frac{x}{i+\beta})}{(i+1)^{n+1}}+\frac{(\beta-\alpha)\gamma(n+1,\frac{x}{i+\beta+1})}{(i+\beta+1)^{n+1}}\right],$$

where $\gamma(z,a)=\int_0^{z}t^{a-1}\mathrm{e}^{-t}dt$ denote the lower incomplete gamma function. Also,

$$E(X^n\mid X\ge x)=\frac{1}{1-F(x)}\int_x^{\infty}t^n f(t)dt.$$

With same way one can obtain

$$E(X^n \mid X \geq x) = \frac{1}{1-F(x)} \sum_{i=0}^{\infty} \binom{-\alpha-1}{i} \left[\frac{\alpha\Gamma(n+1, \frac{x}{i+1})}{(i+1)^{n+1}} + \frac{\beta\Gamma(n+1, \frac{x}{i+\beta})}{(i+1)^{n+1}} + \frac{(\beta-\alpha)\Gamma(n+1, \frac{x}{i+\beta+1})}{(i+\beta+1)^{n+1}} \right],$$

where $\Gamma(z,a) = \int_z^{\infty} t^{a-1} e^{-t} dt$ denotes the upper incomplete gamma function.

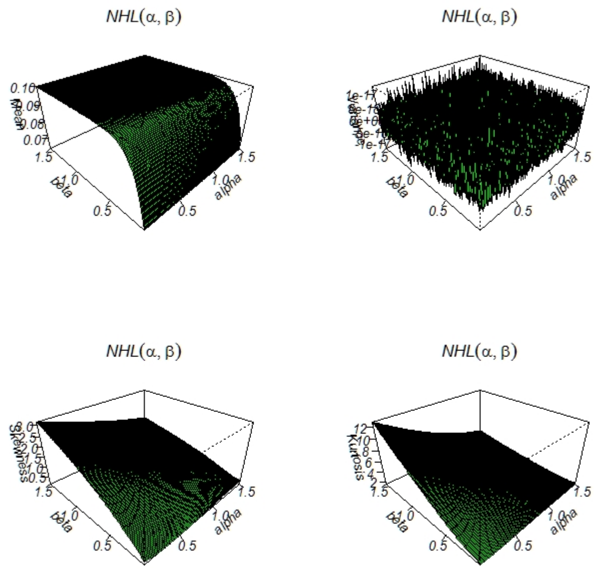

Figure 3 3D plots of mean, variance, skewness and kurtosis as a function of (α, β)

2.5. Mean residual life

In life testing situations, the expected additional lifetime given that a component has survived until time x is a function of x, called the mean residual life. More specifically, if the random variable X represents the life of a component, then the mean residual life is given by $m(x) = E(X - x \mid X > x)$ and can be expressed as

$$m(x) = E(X - x \mid X \geq x) = \frac{\int_x^{\infty} [1-F(u)] du}{1-F(x)}.$$

2.6. Mean past lifetime

In many realistic situations, the random variables are not necessarily related only to the future, but they can also refer to the past. In fact, in many reliability problems, it is of interest to consider variables of the kind $(x - X \mid X \leq x)$ for fixed x, called the past lifetime, which denotes the time elapsed after failure till time x given that the unit has already failed by time x defined for a nonnegative random variable X. The mean past lifetime of non negative random variable X is defined as $m^*(x) = E(x - X \mid X \leq x)$ and is equal to

$$m^*(x) = E(x - X \mid X \le x) = \frac{\int_0^x F(u)du}{F(x)}.$$

2.7. Extreme value

If $\bar{X} = (X_1 + ... + X_n)/n$ denotes the sample mean, then by the central limit theorem, $\sqrt{n}(\bar{X} - E(X))/\sqrt{Var(X)}$ approaches the standard normal distribution as $n \to \infty$. One may be interested in the asymptotic of the extreme values $M_n = \max(X_1, ..., X_n)$ and $m_n = \min(X_1, ..., X_n)$. Let $\tau(x) = 1/\beta$, we obtain following equations for the Equation (1),

$$\lim_{t\to 0} \frac{F(tx)}{F(t)} = x, \text{ and } \lim_{t\to\infty} \frac{1 - F(t + x\tau(t))}{1 - F(t)} = e^{-x}.$$

Thus, from Theorem 1.6.2 in Leadbetter et al. (2012), there must be norming constants $a_n > 0$, b_n, $c_n > 0$ and d_n such that

$$P\left[a_n(M_n - b_n) \le x\right] \to \mathrm{e}^{-\mathrm{e}^{-x}}, \text{ and } P\left[c_n(m_n - d_n) \le x\right] \to 1 - \mathrm{e}^{-x},$$

as $n \to \infty$. The form of the norming constants can also be determined.

2.8. Order statistics

Order statistics are among the most fundamental tools in non-parametric statistics and inference. Suppose $X_1, X_2, \cdots, X_n$ is a random sample from (1). Let $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ denote the corresponding order statistics. The pdf of $X_{i:n}$ is given by

$$F_{i:n}(x) = \sum_{k=i}^{n} \binom{n}{k} F(x)^k [1 - F(x)]^{n-k} = \sum_{k=i}^{n}\sum_{j=0}^{n-k} (-1)^j \binom{n}{k}\binom{n-k}{j} F(x)^{k+j}.$$

By differentiating $F_{i:n}(x)$ with respect to x, the density function of the i^{th} order statistic of any NHL(α, β) distribution can be expressed as

$$\begin{aligned} f_{i:n}(x) &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} (-1)^j \binom{n}{k}\binom{n-k}{j} (k+j) f(x) F(x)^{k+j-1} \\ &= \sum_{k=i}^{n}\sum_{j=0}^{n-k} (-1)^j \binom{n}{k}\binom{n-k}{j} \frac{(k+j)(1-\mathrm{e}^{-\beta x})^{k+j-1}\left[\alpha \mathrm{e}^{-x} + \beta \mathrm{e}^{-\beta x} + (\beta-\alpha)\mathrm{e}^{-(\beta+1)x}\right]}{(1+\mathrm{e}^{-x})^{\alpha(k+j)+1}} \\ &= \sum_{l=0}^{\infty}\sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{s=0}^{k+j-1} w_{j,k,l,s} \mathrm{e}^{-x(l+\beta s)} \left[\alpha \mathrm{e}^{-x} + \beta \mathrm{e}^{-\beta x} + (\beta-\alpha)\mathrm{e}^{-(\beta+1)x}\right], \end{aligned} \tag{4}$$

where $w_{j,k,l,s} = (-1)^{j+s} \binom{n}{k}\binom{n-k}{j}\binom{-\alpha(k+j)-1}{l}\binom{k+j-1}{s}$.

The r^{th} moment of $X_{i:n}$ is given by

$$\begin{aligned} E(X_{i:n}^r) &= \int_0^\infty x^r f_{i:n}(x) dx \\ &= \sum_{l=0}^{\infty}\sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{s=0}^{k+j-1} w_{j,k,l,s} \int_0^\infty x^r \, \mathrm{e}^{-x(l+\beta s)} \left[\alpha \mathrm{e}^{-x} + \beta \mathrm{e}^{-\beta x} + (\beta-\alpha)\mathrm{e}^{-(\beta+1)x}\right] dx \end{aligned}$$

$$=\sum_{l=0}^{\infty}\sum_{k=i}^{n}\sum_{j=0}^{n-k}\sum_{s=0}^{k+j-1} w_{j,k,l,s}\left[\frac{\alpha\Gamma(r+1)}{(l+\beta s+1)^{r+1}}+\frac{\beta\Gamma(r+1)}{(\beta(s+1)+l)^{r+1}}+\frac{(\beta-\alpha)\Gamma(r+1)}{(l+\beta(s+1)+1)^{r+1}}\right].$$

2.9. Residual entropy

Residual entropy is important measure of information. The residual entropy of X is given by

$$c\varepsilon(X)=-\int_0^{\infty}F(x)\log(F(x))dx.$$

After some simple algebra using geometric expansion and generalized binomial expansion, for $\mathrm{NHL}(\alpha,\beta)$ we can obtain

$$-F(x)\log(F(x))=\sum_{i,j=0}^{\infty}\frac{\binom{-\alpha}{j}}{i+1}(1-\mathrm{e}^{-\beta x})\left[\mathrm{e}^{-x(j+\beta(i+1))}+\alpha(-1)^i\mathrm{e}^{-x(i+j+1)}\right],$$

and

$$c\varepsilon(X)=\sum_{i,j=0}^{\infty}\frac{\binom{-\alpha}{j}}{i+1}\left[\frac{1}{j+\beta(i+1)}+\frac{\alpha(-1)^i}{i+j+1}-\frac{1}{j+\beta(i+2)}-\frac{\alpha(-1)^i}{i+j+\beta+1}\right].$$

3. Estimation

3.1. Maximum likelihood estimation

We determine the maximum likelihood estimates (MLEs) of the parameters of the NHL distribution from complete samples only. Let $x_1,\ldots,x_n$ be a random sample of size n from the $\mathrm{NHL}(\alpha,\beta)$ distribution. The log-likelihood function for the vector of parameters $\Theta=(\alpha,\beta)^T$ can be written as

$$l(\Theta)=\sum_{i=1}^{n}\log(\alpha\mathrm{e}^{-x_i}+\beta\mathrm{e}^{-\beta x_i}+(\beta-\alpha)\mathrm{e}^{-(\beta+1)x_i})-(\alpha+1)\sum_{i=1}^{n}\log(1+\mathrm{e}^{-x_i}). \tag{5}$$

The log-likelihood can be maximized either directly by using the SAS (Procedure NLMixed) or by solving the nonlinear likelihood equations obtained by differentiating Equation (5). The components of the score vector $U(\cdot)$ are given by

$$U_\alpha(\Theta)=\sum_{i=1}^{n}\frac{\mathrm{e}^{-x_i}-\mathrm{e}^{-(\beta+1)x_i}}{\alpha\mathrm{e}^{-x_i}+\beta\mathrm{e}^{-\beta x_i}+(\beta-\alpha)\mathrm{e}^{-(\beta+1)x_i}}-\sum_{i=1}^{n}\log(1+\mathrm{e}^{-x_i})$$

and

$$U_\beta(\Theta)=\sum_{i=1}^{n}\frac{(1-\beta x_i)\mathrm{e}^{-\beta x_i}+\left[1+x_i(\alpha-\beta)\right]\mathrm{e}^{-(\beta+1)x_i}}{\alpha\mathrm{e}^{-x_i}+\beta\mathrm{e}^{-\beta x_i}+(\beta-\alpha)\mathrm{e}^{-(\beta+1)x_i}}.$$

3.2. The other estimation methods

There are several approaches to estimate the parameters of distributions that each of them has its characteristic features and benefits. In this subsection five of those methods are briefly introduced and will be numerically investigated in the simulation study. A useful summary of these methods can be seen in Dey et al. (2018). Here $\{t_i; i=1,2,\ldots,n\}$ and $\{t_{i:n}; i=1,2,\ldots,n\}$ is the random sample and associated order statistics and F is the distribution function of NHL distribution.

Least squares and weighted least squares estimators

Least squares and weighted least squares estimators The Least Squares (LSE) and weighted Least Squares Estimators (WLSE) are introduced by Swain et al. (1988). The LSE's and WLSE's are obtained by minimizing the following functions:

$$S_{\mathrm{LSE}}(\alpha,\beta)=\sum_{i=1}^{n}\left(F_{NHL}(t_{i:n};\alpha,\beta)-\frac{i}{n+1}\right)^{2},$$

and

$$S_{\mathrm{WLSE}}(\alpha,\beta)=\sum_{i=1}^{n}\frac{(n+1)^{2}(n+2)}{i(n-i+1)}\left(F_{NHL}(t_{i:n};\alpha,\beta)-\frac{i}{n+1}\right)^{2}.$$

Cramér-von-Mises estimator

Cramér-von-Mises Estimator (CME) is introduced by Choi and Bulgren (1968). The CMEs is obtained by minimizing the following function:

$$S_{\mathrm{CME}}(\alpha,\beta)=\frac{1}{12n}+\sum_{i=1}^{n}\left(F_{NHL}(t_{i:n};\alpha,\beta)-\frac{2i-1}{2n}\right)^{2}.$$

Anderson-Darling and right-tailed Anderson-Darling

The Anderson Darling (ADE) and Right-Tailed Anderson Darling Estimators (RTADE) are introduced by Anderson and Darling (1952). The ADE's and RTADE's are obtained by minimizing the following functions:

$$S_{\mathrm{ADE}}(\alpha,\beta)=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\{\log F_{NHL}(t_{i};\alpha,\beta)+\log\overline{F}_{NHL}(t_{n+1-i};\alpha,\beta)\},$$

and

$$S_{\mathrm{RTADE}}(\alpha,\beta)=\frac{n}{2}-2\sum_{i=1}^{n}F_{NHL}(t_{i};\alpha,\beta)-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\log\overline{F}_{NHL}(t_{n+1-i};\alpha,\beta),$$

where $\overline{F}(\cdot)=1-F(\cdot)$.

3.3. Simulation study

In order to explore the estimators introduced above we consider the one model that have been used in this section , and investigate MSE of those estimators for different samples. For instance according to what has been mentioned above, for $(\alpha,\beta)=(0.9,0.6),(2,1),(3.1,0.4)$.

The performance of each method of parameters estimations for the NHL distribution with respect to sample size n is considered. To do this, a simulation study is done based on following steps:

Step 1. Generate one thousand samples of size n from (1). This work is done simply by quantile function and generated data from uniform distribution .

Step 2. Compute the estimates for the one thousand samples, say $(\hat{\alpha}_i,\hat{\beta}_i)$ for $i=1,2,\ldots,10^4$.

Step 3. Compute the biases and mean squared errors by

$$Bias_{\varepsilon}(n)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\varepsilon}_i-\varepsilon),$$

and

$$MSE_{\varepsilon}(n)=\frac{1}{10000}\sum_{i=1}^{10000}(\hat{\varepsilon}_i-\varepsilon)^{2}.$$

for $\varepsilon = \alpha, \beta$. We repeated these steps for $n = 30, 80, 130, \cdots, 500$ with mentioned special case of parameters. So computing $bias_\varepsilon(n)$ and $MSE_\varepsilon(n)$ for $\varepsilon = \alpha, \beta$ and $n = 30, 80, ..., 500$. To obtain the value of the estimators, we have used the optim function and the Nelder-Mead method in the statistical package R version 3.4.4. The results are shown in Figures 4-6.

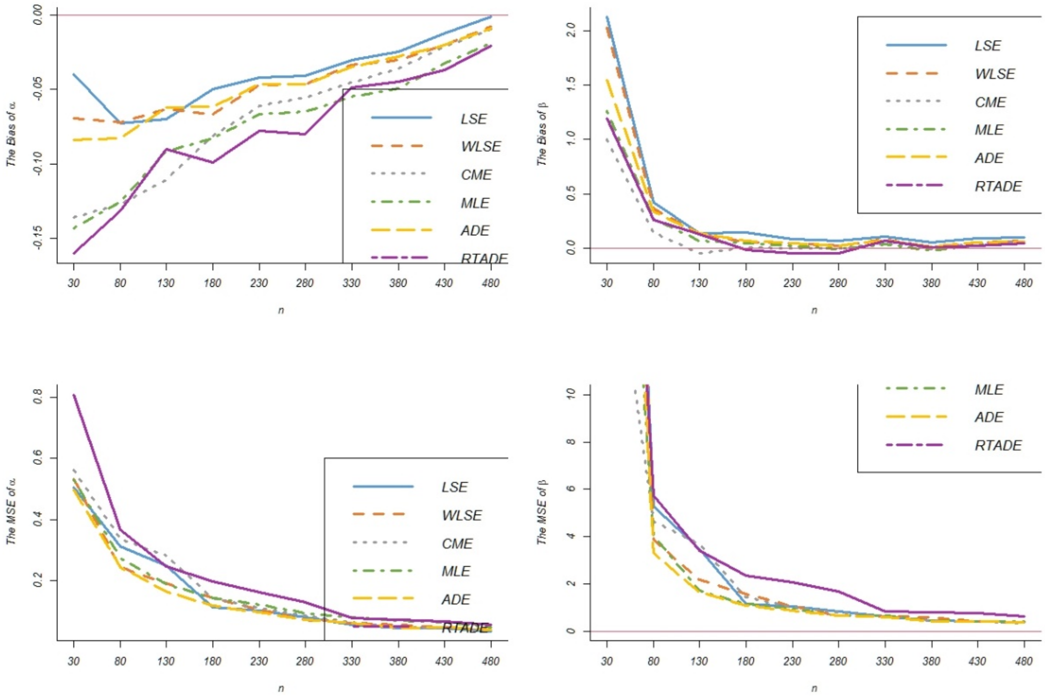

Figure 4 Bias, absolute bias and MSE of estimations for parameter values $(\alpha, \beta) = (0.9, 0.6)$

One can see that MSE plots for two parameters with the increase in the volume of the sample all methods will approach to zero and this verifies the validity of the these estimation methods and numerical calculations for the distribution parameters NHL. Also,

• for estimating α, LSE method has the minimum amount of bias, however for large sample size, all methods have almost same behaviour.

• for estimating β, RTADE method has the minimum amount of bias, however for large sample size, all methods have almost same behaviour.

• for estimating α, RTADE method has the minimum amount of MSE, however for large sample size, all methods have almost same behaviour.

• for estimating β, RTADE method has the minimum amount of MSE, however for large sample size, all methods have almost same behaviour.

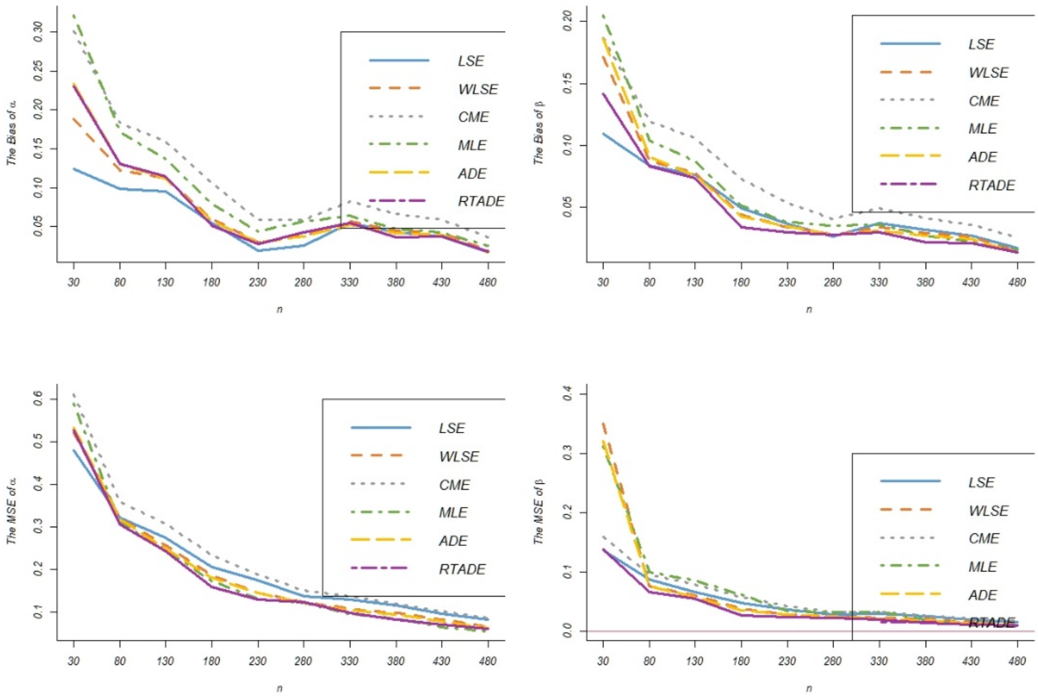

Figure 5 Bias, absolute bias and MSE of estimations for parameter values $(\alpha, \beta) = (2,1)$

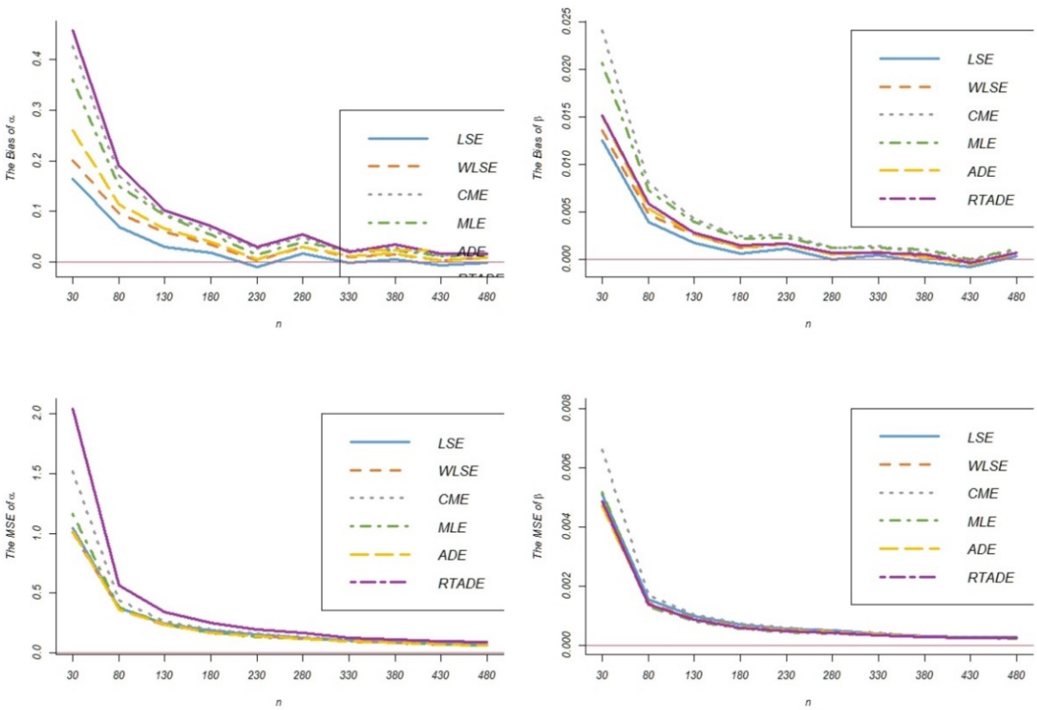

Figure 6 Bias, absolute bias and MSE of estimations for parameter values $(\alpha, \beta) = (3.1, 0.4)$

4. Applications

In this section, we present two applications by fitting the MHL model and some famous models. The Cramér-von Mises (W^*), Anderson-Darling (A^*) have been chosen for comparison the models for the first two examples.

The exponentiated half-logistic (ESHL) distribution due to Kang and Seo (2011), Kumaraswamy standard Half-Logistic distribution (KwSHL) due to Cordeiro and de Castro (2011) , the Beta standard Half-Logistic (BSHL) due to Jones (2004), McDonald standard Half-Logistic (McSHL)distributin due to Oliveira et al. (2016), New Odd log-logistic standard Half-Logistic (NOLL-SHL) distribution due to Alizadeh et al. (2019) weibull distrbution (W), Generalized Exponential (GE) distribution due to Gupta and Kundu (1999), Log Normal (LN) distribution, Gamma (Ga) distribution, Lindley (Li) distribution due to Ghitany et al. (2008) and Power Lindley (PL) distribution due to Ghitany et al. (2013) have been selected for comparison in two examples. The pdf of these models are given in Appendix. The parameters of models have been estimated by the MLE method.

4.1. Data set I

The first data set is related to lifetimes of 20 electronic components given by Murthy et al. (2004). The data are: 0.03, 0.12, 0.22, 0.35, 0.73, 0.79, 1.25, 1.41, 1.52, 1.79, 1.80, 1.94, 2.38, 2.40, 2.87, 2.99, 3.14, 3.17, 4.72, 5.09 .

In the Table 1, a summary of the fitted information criteria and estimated MLE's for this data with different models have come, respectively. One can see , the $\text{NHL}(\alpha,\beta)$ distribution is selected as the best model with more criteria. The histogram of the data set I and the plots of fitted PDF are displayed in Figure 7. Figure 8 show the unimodality of profile likelihood functions of parameters for data set I.

4.2. Data set II

The second real data set is failure time of 50 items given by Murthy et al. (2004). The data are: 0.008, 0.017, 0.058, 0.061, 0.084, 0.090, 0.134, 0.238, 0.245, 0.353, 0.374, 0.480, 0.495, 0.535, 0.564, 0.681, 0.686, 0.688, 0.921, 0.959, 1.022, 1.092, 1.260, 1.284, 1.295, 1.373, 1.395, 1.414, 1.760, 1.858, 1.892, 1.921, 1.926, 1.933, 2.135, 2.169, 2.301, 2.320, 2.405, 2.506, 2.598, 2.808, 2.971, 3.087, 3.492, 3.669, 3.926, 4.446, 5.119, 8.596.

Similar to the previous application example, we have Tables 2 . As it is clear, the $\text{NHL}(\alpha,\beta)$ is selected as the best model with more criteria. The histogram of the data set II and the plots of fitted PDF are displayed in Figure 9. Figure 10 shows the unimodality of profile likelihood functions of parameters for data set II.

Table 1 Result for data set I

Model	Estimated Parameters (SE)			W^*	A^*
NHL(α,β)	3.783 (0.932)	9.952 (8.569)		0.023	0.168
NOLL-SHL(α,β)	0.939 (0.296)	0.697 (0.172)		0.067	0.396
ESHL(α)	1.228 (0.274)			0.057	0.345
KwSHL(α,β)	0.904 (0.302)	0.644 (0.177)		0.058	0.351
BSHL(α,β)	0.920 (0.278)	0.647 (0.180)		0.058	0.354
McSHL(α,β,c)	3.871 (29.956)	0.652 (0.175)	0.215 (1.734)	0.056	0.339
Li(α)	0.803 (0.133)			0.064	1.381
PL(α,β)	0.761 (0.174)	1.063 (0.187)		0.059	0.354
GE(α,β)	0.559 (0.154)	1.139 (0.332)		0.084	0.493
GL(α,β)	0.785 (0.179)	0.955 (0.287)		0.063	0.378
LN(α,β)	0.172 (0.286)	1.279 (0.202)		0.203	1.152
Ga(α,β)	1.162 (0.328)	0.600 (0.210)		0.083	0.486
W(α,β)	0.426 (0.135)	1.196 (0.224)		0.071	0.418

Table 2 Result for data set II

Model	Estimated Parameters (SE)			W^*	A^*
NHL(α,β)	2.991 (0.486)	11.922 (8.918)		0.022	0.159
NOLL-SHL(α,β)	0.796 (0.147)	0.743 (0.126)		0.041	0.256
ESHL(α)	0.946 (0.133)			0.035	0.224
KwSHL(α,β)	0.719 (0.149)	0.668 (0.116)		0.037	0.230
BSHL(α,β)	0.738 (0.136)	0.664 (0.119)		0.038	0.239
McSHL(α,β,c)	91.722 (78.442)	0.677 (0.115)	0.007 (0.005)	0.034	0.218
Li(α)	0.910 (0.096)			0.049	0.300
PL(α,β)	0.994 (0.127)	0.882 (0.096)		0.063	0.375
GE(α,β)	0.560 (0.105)	0.903 (0.162)		0.076	0.448
GL(α,β)	0.790 (0.122)	0.767 (0.142)		0.046	0.285
LN(α,β)	–0.123 (0.205)	1.452 (0.145)		0.329	1.902
Ga(α,β)	0.914 (0.159)	0.546 (0.125)		0.077	0.455
W(α,β)	0.610 (0.105)	0.976 (0.111)		0.079	0.468

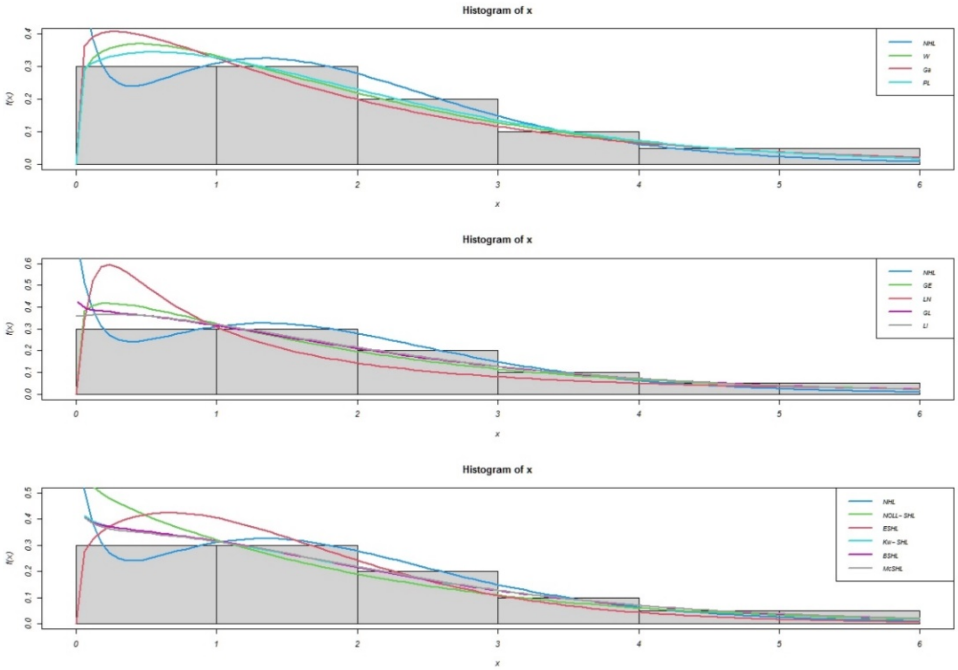

Figure 7 Histogram and fitted pdfs for data set I

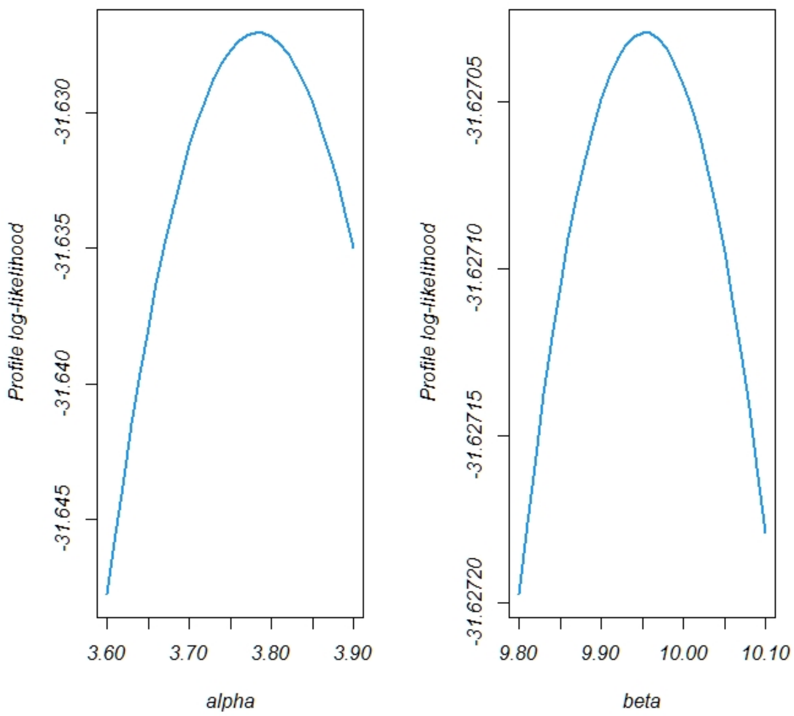

Figure 8 Unimodality of profile likelihood functions of parameters for data set I

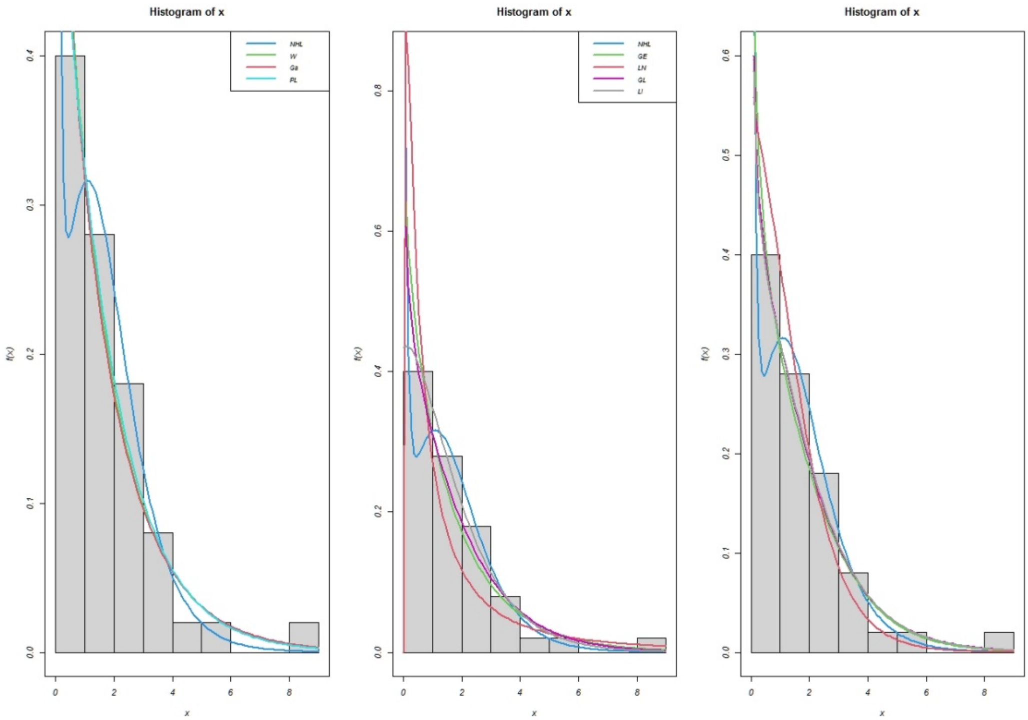

Figure 9 Histogram and fitted pdfs for data set II

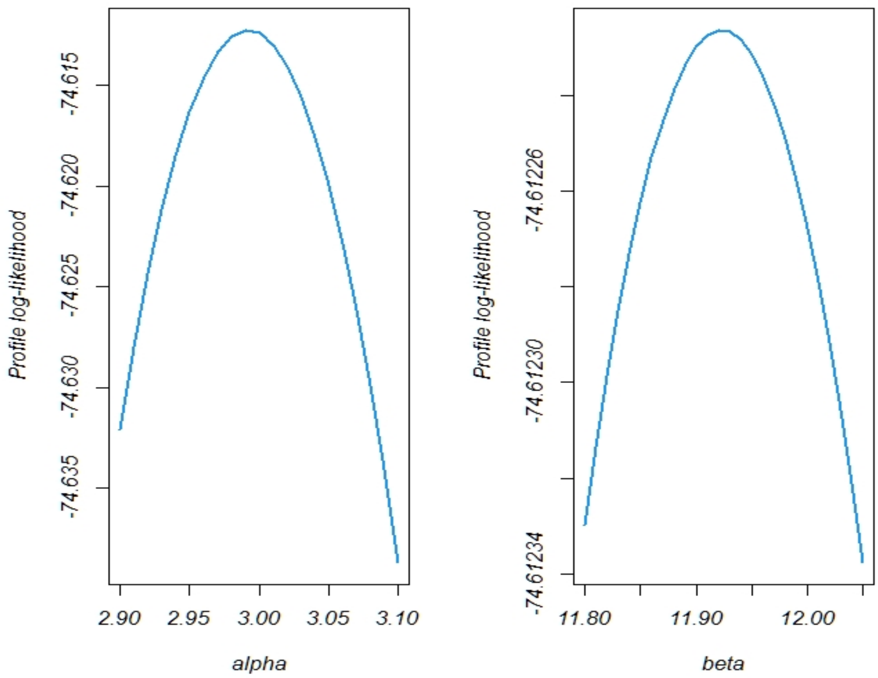

Figure 10 Unimodality of profile likelihood functions of parameters for data set II

5. Conclusions

We introduce a new two-parameter extension of half-logistic distributions called the new half-logistic (NHL) distribution. Some properties of the new family, such as moments, mean past lifetime, residual entropy, order statistics, asymptotic and extreme value properties are obtained. We estimate the parameters using maximum likelihood and other different methods. The bias and MSE plots of parameters for all methods, will approach to zero with the increase in the volume of the sample which verifies the validity of these estimation methods. The flexibility of this distribution is assessed by applying it to real data sets and comparing purpose distribution with others. The results of tables and figures illustrate the new models provide consistently better fits than other competitive models for these data sets. So applications demonstrate the importance of the new family.

Appendix: Pdf of Competitive Models in Application Section

$$f_{ESHL}(x;\alpha)=\frac{2\alpha e^{-x}(1-e^{-x})^{\alpha-1}}{(1+e^{-x})^{\alpha+1}},\ x>0,\alpha>0,$$

$$f_{NOLL-SHL}(x;\alpha,\beta)=\frac{\dfrac{2e^{-x}}{(1+e^{-x})^2}\left[\dfrac{1-e^{-x}}{1+e^{-x}}\right]^{\alpha-1}\left[1-\left(\dfrac{1-e^{-x}}{1+e^{-x}}\right)\right]^{\beta-1}\left[\alpha+(\beta-\alpha)\left(\dfrac{1-e^{-x}}{1+e^{-x}}\right)\right]}{\left\{\left[\dfrac{1-e^{-x}}{1+e^{-x}}\right]^{\alpha}+\left[1-\left(\dfrac{1-e^{-x}}{1+e^{-x}}\right)\right]^{\beta}\right\}^2},x>0,\alpha>0,\beta>0,$$

$$f_{KwSHL}(x;\alpha,\beta)=\frac{2\alpha\beta e^{-x}(1-e^{-x})^{\alpha-1}}{(1+e^{-x})^{\alpha+1}}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{\alpha}\right]^{\beta-1},\ x>0,\alpha>0,\beta>0,$$

$$f_{BSHL}(x;\alpha,\beta)=\frac{2^{\beta}e^{-\beta x}(1-e^{-x})^{\alpha-1}}{B(\alpha,\beta)(1+e^{-x})^{\alpha+\beta}},\ x>0,\alpha>0,\beta>0,$$

$$f_{McSHL}(x;\alpha,\beta,c)=\frac{2ce^{-x}(1-e^{-x})^{\alpha c-1}}{B(\alpha,\beta)(1+e^{-x})^{\alpha c+1}}\left[1-\left(\frac{1-e^{-x}}{1+e^{-x}}\right)^{c}\right]^{\beta-1},\ x>0,\alpha>0,\beta>0,c>0,$$

$$f_{Li}(x;\alpha)=\frac{\alpha^2}{1+\alpha}(1+x)e^{-\alpha x},\ x>0,\alpha>0,$$

$$f_{PL}(x;\alpha,\beta)=\frac{\alpha^2\beta}{1+\alpha}x^{\beta-1}(1+x^{\beta})e^{-\alpha x^{\beta}},\ x>0,\alpha>0,\beta>0,$$

$$f_{GE}(x;\alpha,\beta)=\alpha\beta\mathrm{e}^{-\alpha x}(1-\mathrm{e}^{-\alpha x})^{\beta-1},\ x>0,\alpha>0,\beta>0,$$

$$f_{GL}(x;\alpha,\beta)=\frac{\beta\alpha^2}{1+\alpha}(1+x)e^{-\alpha x}\left[1-\left(1+\frac{\alpha x}{1+\alpha}\right)e^{-\alpha x}\right]^{\beta-1},\ x>0,\alpha>0,$$

$$f_{LN}(x;\alpha,\beta)=\frac{1}{x\beta\sqrt{2\pi}}\mathrm{e}^{\frac{-(\log(x)-\alpha)^2}{2\beta^2}},\ x>0,\alpha\in\mathbf{R},\beta>0,$$

$$f_{Ga}(x;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}x^{\alpha-1}\mathrm{e}^{-\beta x},\ x>0,\alpha>0,\beta>0,$$

$$f_{W}(x;\alpha,\beta)=\alpha\beta x^{\beta-1}\mathrm{e}^{-\alpha x^{\beta}},\ x>0,\alpha>0,\beta>0.$$

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