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Record Values from the Gumbel and q- Gumbel Distributions with Applications

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Abstract

In the present study we investigate the problem of estimating the inherent parameters of the Gumbel and q-Gumbel distributions using record breaking data. We presented the coefficients of the best linear unbiased estimators (BLUE) for location and scale parameters of the Gumbel and q-Gumbel distributions. Finally, the usefulness of our result is illustrated using a simulation study.

Keywords: Maximum likelihood estimates, best linear unbiased estimators, best linear invariant estimators, Akaike information criterion, corrected Akaike information criterion.

1. Introduction

Records are very important when observations are difficult to obtain or when observations are being destroyed when subjected to an experimental test. Chandler (1952) was first to introduce the concept of record values, record times and inter record times for analyzing the breaking strength data of certain material. He proved the result that for any given probability distribution function of a random variable, the expected value of the inter record time is infinite. Feller (1965) gave some examples of record values with respect to gambling problems.

Suppose that $X_1, X_2, ..., X_n$ are a sequence of independent and identically distributed random variables with cumulative probability distribution function $F(x)$. Let $X_n = \min\{X_1, X_2, ..., X_n\}$ for $n \geq 1$. We say, X_j is a lower record value of $\{X_n, n \geq 1\}$, if $X_j < X_{j-1}, j > 1$. An analogous definition exist for upper record values. By definition, X_1 is a lower as well as upper record value. The indices at which the lower record values occur are given by the record times $\{L(r); r > 0\}$, where $L(r) = \min\{j | j > L(r-1), X_j < X_{L(r-1)}; r > 1\}$, and $L(1) = 1$. The probability density function of $X_{L(r)}$ is given by:

$$f_r(x) = \frac{1}{\Gamma(r)}(-\ln(F(x)))^{r-1} f(x, \mu, \beta), \quad x \in (-\infty, \infty). \tag{1}$$

and the cumulative probability distribution function of $X_{L(r)}$ is:

$$F_r(x) = \frac{1}{\Gamma(r)} \int_{-\infty}^{x} (-\ln(F(x)))^{r-1} f(x, \mu, \beta) dx, \quad x \in (-\infty, \infty).$$

The joint probability density function of two lower record values $X_{L(r)}$ and $X_{L(s)}$ is given by

$$f(x_r, x_s) = \frac{(-\ln(F(x_r)))^{r-1}[\ln(F(x_r)) - \ln(F(x_s))]^{s-r-1}}{\Gamma(r)\Gamma(s-r)} \frac{f(x_r)f(x_s)}{F(x_r)}, -\infty < x_s < x_r < \infty. \quad (2)$$

Samaniego and Whitaker (1986) introduced the problem of parametric inference for record-breaking data. The features of maximum likelihood estimates of the mean of an underlying exponential probability distribution were investigated. The work of Samaniego and Whitaker, was extended to the Weibul probability distribution by Gulati and Padgett (2003). The Maximum likelihood and Bayesian estimation of parameters and prediction of future records for Weibull distribution using δ-record data were discussed by Raul, et al.(2019). Prediction the s^{th} record value based on the first m record values $(s > m)$ when the observations from exponential distribution is investigated by Ahsanuallah (1980).

Nigm (2007) introduced the record values from inverse Weibull distribution (IW) and the explicit expressions for its means, variances and covariances. Some simple recurrence relations satisfied by the single and product moments of record values from IW distribution are obtained. The best linear unbiased estimators for the scale and location parameters are derived. Some associated inference with regard to the prediction of a future record value and test for spuriously of the current record values are also developed.

Asymmetrical models such as the Gumbel, logistic, Weibull and generalized extreme value distributions have been extensively utilized for modeling various random phenomena encountered for instance in the course of certain survival, financial or reliability studies.

The Extremal Types Theorem [see Haan (2006)] characterizes the limit cdf G as of the type of one of three classes. The three types are often called the Gumbel, Frechet and Weibull types, respectively gathered in the following family:

$$G_\xi(x; \mu, \beta, \xi) = \begin{cases} \exp\{-(1+\xi(\frac{x-\mu}{\beta}))^{\frac{-1}{\xi}}\}, \xi \neq 0, \\ \exp\{-\exp(-\frac{x-\mu}{\beta})\}, \xi \to 0, \end{cases} \quad (3)$$

and

$$g_\xi(x; \mu, \sigma, \xi) = \begin{cases} \frac{1}{\beta}\exp\{-(1+\xi(\frac{x-\mu}{\beta}))^{\frac{-1}{\xi}}\} \\ \times(1+\xi(\frac{x-\mu}{\beta}))^{\frac{-1}{\xi}-1}, \xi \neq 0, \\ \frac{1}{\beta}\exp\{-\exp(-(\frac{x-\mu}{\beta}))\} \\ \times\exp(-(\frac{x-\mu}{\beta})), \xi \to 0, \end{cases} \quad (4)$$

where μ is a location parameter, β is a positive scale parameter and ξ is the shape parameter. The support of the distribution is

$$x \in \begin{cases} (\mu - \frac{\beta}{\xi}, \infty), & \xi > 0, \\ (-\infty, \infty), & \xi \to 0, \\ (-\infty, \mu - \frac{\beta}{\xi}), & \xi < 0. \end{cases}$$

The distribution in (4) is known as the generalized extreme value distribution under linear normalization. We denote it by $GEVL(x; \mu, \sigma, \xi)$. The Gumbel probability distribution in (3) and (4) as $\xi \to 0$ is used to analyse and model the behaviour of random phenomena in engineering, business, biology, management, sports (Mbah and Tsokos, (2007)), and economics, among other fields. For more information, see Luo and Zhu (2005), Coles, (2001), Gumbel, (1958), Hosking et al. (1985), Kotz et al.(2000), for examples of how the Gumbel probability distribution, also known as the double exponential probability distribution.

Provost et al. (2018) hereby introduced q-analogues of the generalized extreme value and Gumbel distributions, the additional parameter q allowing for increased modeling flexibility. They introduced the cdf and pdf of the q-GEVL and q-Gumbel (obtained by letting $\xi \to 0$ in the q-GEVL model) distributions are respectively given by

$$F(x;\mu,\sigma,\xi,q)=\begin{cases}[1+q(\xi(sx-m)+1)^{-\frac{1}{\xi}}]^{-\frac{1}{q}}, & \xi\neq 0, q\neq 0\\ (1+qe^{-(sx-m)})^{-\frac{1}{q}}, & \xi\to 0,\ \ q\neq 0\end{cases}$$

and

$$f(x;\mu,\sigma,\xi,q)=\begin{cases}s(1+\xi(sx-m))^{\frac{-1}{\xi}-1} & \\ \times[1+q(\xi(sx-m)+1)^{-\frac{1}{\xi}}]^{-\frac{1}{q}-1}, & \xi\neq 0, q\neq 0\\ (1+qe^{-(sx-m)})^{-\frac{1}{q}-1}se^{-(sx-m)}, & \xi\to 0,\neq 0,\end{cases} \quad (5)$$

where $s=\frac{1}{\beta}$ and $m=\frac{\mu}{\beta}$. The support of q- Gumbel distribution is

$$x\in\begin{cases}(-\infty,\infty), & \xi\to 0, q>0,\\ (\frac{m+\ln(-q)}{\beta},\infty), & \xi\to 0, q<0.\end{cases}$$

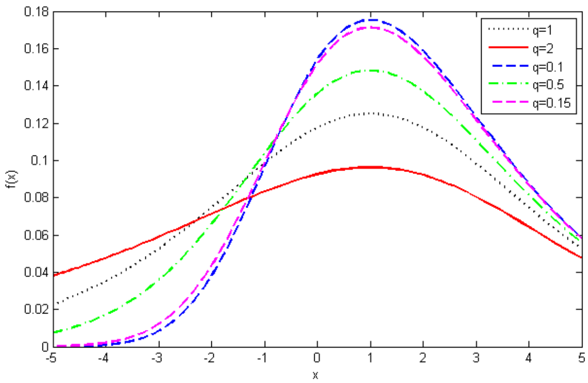

Figure 1 The PDF of q-Gumbel ($q>0$)

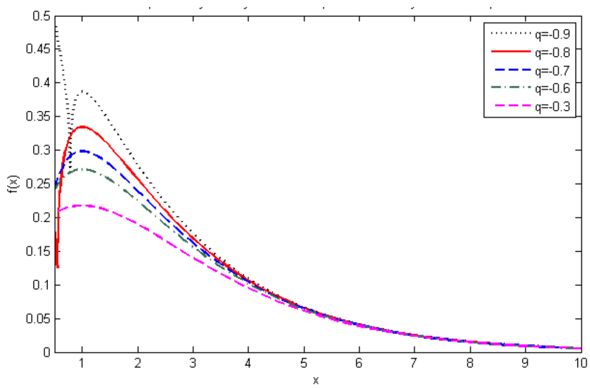

Figure 2 The PDF of q-Gumbel ($q<0$)

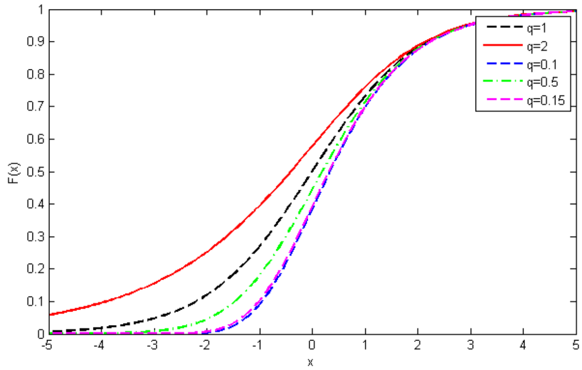

Figure 3 The CDF of q-Gumbel ($q > 0$)

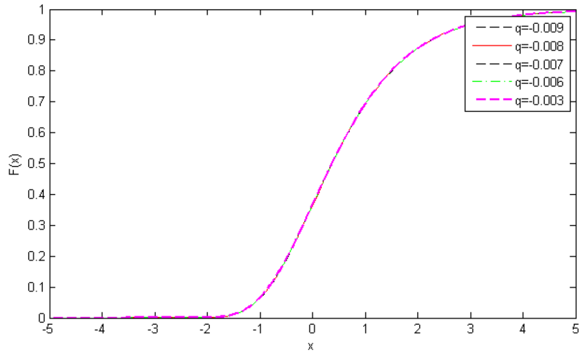

Figure 4 The CDF of q-Gumbel ($q < 0$)

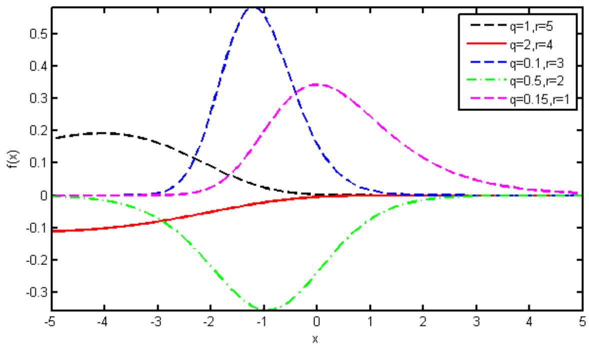

Figure 5 The q-Gumbel PDF of the lower record values($q > 0$)

The effects of the parameter q on the shape of the distributions are illustrated graphically in Figures 1 to 8. Plots of the q-Gumbel PDF and CDF of X are displayed in Figures 1 to 4 for certain parameter values. Plots of the q-Gumbel PDF and CDF of lower record values are displayed in Figures 5 to 8 for certain parameter values. These plots illustrate the impressive versatility of the proposed models.

In this paper, we consider the analysis of record breaking data sets, where only observations that exceed, or only those that fall below, the current extreme value are recorded. Example of application areas include industrial stress testing, meteorological analysis, sporting and athletic events, and oil

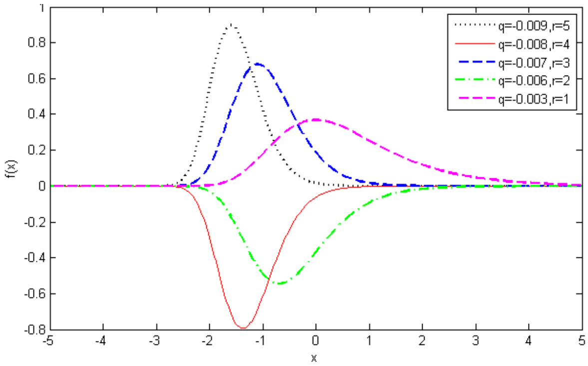

Figure 6 The q-Gumbel PDF of the lower record values($q < 0$)

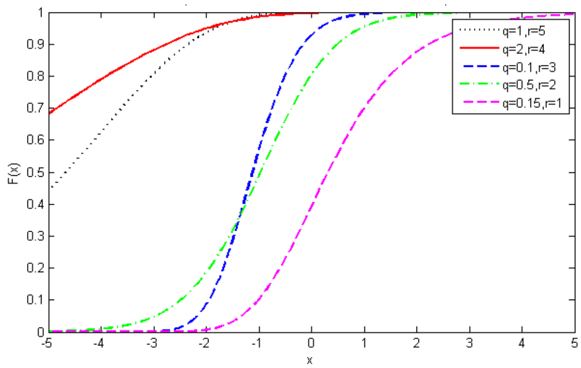

Figure 7 The q-Gumbel CDF of the lower record values($q > 0$)

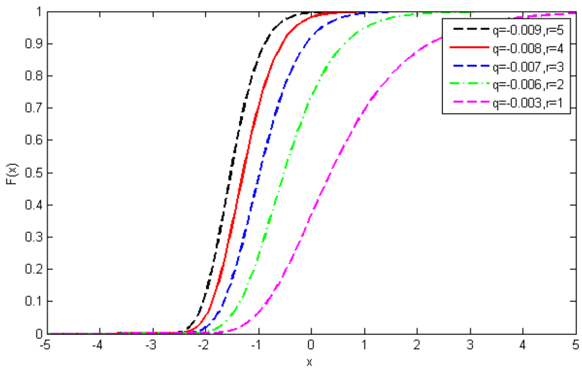

Figure 8 The q-Gumbel CDF of the lower record values($q < 0$)

and mining surveys.

The paper is organized as follows. Section 2 contains estimation of the parameters of Gumbel probability distribution used best linear unbiased estimates (BLUE) and best linear invariant estimators (BLIE) methods based on lower record values. The prediction of the Future Record and simulation study are developed in this section. In Section 3, we used maximum likelihood estimator (MLE) method to estimate the parameters of the Gumbel distribution based on inter record times sequence and complete sample. The rest of this section, we used some goodness of fit tests to compare between the parameters in the inter record times and complete sample. Section 4 contains estimation of the parameters of q-Gumbel probability distribution BLUE and BLIE methods based on lower record values. The prediction of the Future Record and simulation study are developed in this section. In Section 5, we used MLE method to estimate the parameters of the q-Gumbel distribution based on inter record times sequence and complete sample. The rest of this section, we used some goodness of fit tests to compare between the parameters in the inter record times and complete sample.

2. Inference Based on Lower Record Values for Gumbel Distribution

In this section, we estimate the parameters of the Gumbel probability distribution using lower record values. The BLUE based on r lower record values from the Gumbel probability distribution are obtained in Subsection 2.1. The BLIE based on r lower record values from the Gumbel probability distribution are derived in Subsection 2.2. The prediction of the future record are developed in Subsection 2.3. In Subsection 2.4, we presented simulation study.

2.1. Best Linear Unbaised Estimates (BLUEs)

Applying Eqn. (4) and letting $\mu = 0$ and $\beta = 1$, the n^{th} moment of $X_{L(r)}$ from the Gumbel function probability distribution is given by:

$$E(X_{L(r)})^n = \frac{1}{\Gamma(r)} \int_{-\infty}^{\infty} x^n [e^{-x}]^{r-1} e^{-x} e^{-e^{-x}} dx.$$

Let $u = e^{-x}$, then

$$\begin{aligned} E(X_{L(r)})^n &= \frac{1}{\Gamma(r)} \int_0^{\infty} [-\ln u]^n u^{r-1} e^{-u} du \\ &= \frac{(-1)^n}{\Gamma(r)} \frac{\partial^n}{\partial r^n} [\Gamma(r)], \quad n \geq 1. \end{aligned} \tag{6}$$

For $n = 1$, we get

$$E(X_{L(r)}) = -\Psi(r) = b_r, \tag{7}$$

where $\Psi(r)$ known as the digamma function, is the logarithmic derivative of the gamma function. For $n = 2$, we get

$$E(X_{L(r)})^2 = \Psi(r)^2 + \Psi(1, r), \tag{8}$$

where $\Psi(1, r)$ is the trigamma function. Using Eqns. (7) and (8), we get the variance of $X_L(r)$ as

$$Var(X_{L(r)}) = \Psi(1, r) = a_r b_r.$$

Also, using Eqn. (1) and (2), for $s > r, \ \ x_s < x_r$ we have the covariance of $X_{L(r)}$ and $X_{L(s)}$ is given by:

$$Cov(X_{L(r)}, X_{L(s)}) = E(X_{L(r)} X_{L(s)}) - E(X_{L(r)}) E(X_{L(s)}) = b_s.a_r. \tag{9}$$

To estimate the parameters of Gumbel probability distribution using lower record values, by using the following theorem:

Theorem 1 *Let $x_1, x_2, ..., x_r$ be r record values from the Gumbel probability distribution (4). Then the best linear unbiased estimates (BLUE) , $\hat{\mu}$, $\hat{\beta}$ for μ and β are respectively:*

$$\hat{\mu} = \frac{\alpha' V^{-1} (\alpha 1' - 1\alpha') V^{-1} h}{\Delta}$$

and

$$\hat{\beta} = \frac{1' V^{-1} (1\alpha' - \alpha 1') V^{-1} h}{\Delta},$$

where $h' = (x_1, x_2, ..., x_r), \alpha' = (b_1, b_2, ..., b_r), V = (v_i j), \ \ v_i j = a_i b_j, \ \ 1 \leq i, \ j \leq r.$

Proof:

$$h' = (x_1, x_2, ..., x_r),$$

then
$$E(h') = \mu 1 + \beta^2 \alpha,$$
$$Var(h') = \beta^2 V$$

where from Eqns. (6) and (9),

$$1' = (1, 1, ..., 1),$$
$$\alpha' = (b_1, b_2, ..., b_r)$$
$$V = (v_i j),\ \ v_{ij} = a_i b_j,\ \ 1 \le i,\ j \le r,$$
$$V^{-1} = (V^{ij}),\ \ 1 \le i < j \le r.$$

Then the entries of V^{-1} are given by:

$$V^{ii} = \frac{a_{i+1}b_{i-1} - a_{i-1}b_{i+1}}{(a_{i+1}b_{i-1} - a_{i-1}b_i)(a_{i+1}b_i - a_i b_{i+1})},\ \ i = 1, ..., r-1,$$
$$V^{ij} = V^{ji} = \frac{-1}{a_{i+1}b_i - a_i b_{i+1}}$$
$$V^{rr} = \frac{-b_{r-1}}{b_r(a_r b_{r-1} - a_{r-1} b_r)},$$
$$\Delta = (\alpha' V^{-1} \alpha)(1' V^{-1} 1) - (\alpha' V^{-1} 1)^2.$$

Applying the method introduced by Lioyd(1952), the best linear unbiased estimates (BLUE), $\hat{\mu},\ \hat{\beta}$ for μ and β based on r lower record values from the Gumbel distribution are given by:

$$\hat{\mu} = \frac{\alpha' V^{-1}(\alpha 1' - 1\alpha') V^{-1} h}{\Delta}$$

and
$$\hat{\beta} = \frac{1' V^{-1}(1\alpha' - \alpha 1') V^{-1} h}{\Delta}$$

The variance and covariance of $\hat{\mu},\ \hat{\beta}$ are given by:

$$Var(\hat{\mu}) = \frac{\alpha' V^{-1} \alpha}{\Delta} \beta^2,$$
$$Var(\hat{\beta}) = \frac{1' V^{-1} 1}{\Delta} \beta^2,$$
$$Cov(\hat{\mu}, \hat{\beta}) = \frac{\alpha' V^{-1} 1}{\Delta} \beta^2,$$

By using Matlab program(ver 2018), coefficients of the BLUES for $\hat{\mu}, \hat{\beta}$ and variance covariance for μ and β are given in Tables 1 and 2, respectively.

Table 1 Coefficients for the BLUE of μ and β

n	r	Coefficient for the BLUE of μ	Coefficient for the BLUE of β
2	1	0.4228	1
2	2	0.5772	-1
3	1	0.40599	1.029
3	2	0.6276	-1.087
3	3	-0.0336	0.058
4	1	0.397	1.044
4	2	0.614	-1.064
4	3	0.0488	-0.085
4	4	-0.05996	0.104
5	1	0.391	1.055
5	2	0.6044	-1.047
5	3	0.048	-0.08
5	4	0.0345	-0.0598
5	5	-0.078	0.135
7	1	0.3821	1.07
7	2	0.591	-1.023
7	3	0.047	-0.08
7	4	0.0337	-0.0585
7	5	0.0285	-0.0495
7	6	0.0256	-0.0444
7	7	-0.1076	0.1864
8	1	0.379	1.077
8	2	0.585	-1.014
8	3	0.0465	-0.0805
8	4	0.0334	-0.0579
8	5	0.028	-0.0489
8	6	0.025	-0.04398
8	7	0.023	-0.041
8	8	-0.121	0.209
9	1	0.375	1.082
9	2	0.58	-1.005
9	3	0.046	-0.0799
9	4	0.033	-0.057
9	5	0.028	-0.0486
9	6	0.025	-0.044
9	7	0.023	-0.04
9	8	0.0219	-0.038
9	9	-0.134	0.23
15	1	0.361	1.1066
15	2	0.5584	-0.9673
15	3	0.0444	-0.0768
15	4	0.0319	-0.0553
15	5	0.02699	-0.04675
15	6	0.02422	-0.04196
15	7	0.0224	-0.0388
15	8	0.0211	-0.03652
15	9	0.0201	-0.03477
15	10	0.0193	-0.033377
15	11	0.0186	-0.0322
15	12	0.018	-0.03127
15	13	0.0176	-0.0304
15	14	0.0172	-0.02973

Table 2 Coefficient for variance covariance of the BLUE of μ and β in terms of β^2

	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$	$r=10$
1	0.079	0.076	0.074	0.073	0.0723	0.0715	0.0709	0.0703	0.0698
2	1.09	1.1009	1.1058	1.11	1.112	1.114	1.116	1.12	1.1196
3	-0.137	-0.132	-0.1288	-0.127	-0.125	-0.124	-0.1228	-0.122	-0.1208

2.2. Best Linear Invariant Estimates (BLIEs)

The best linear invariant (in terms of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIE) $\tilde{\mu}, \tilde{\beta}$ of μ and β are:

$$\tilde{\mu} = \hat{\mu} - \hat{\beta}(\frac{E_{12}}{1+E_{22}}) \tag{10}$$

and

$$\tilde{\beta} = \frac{\hat{\beta}}{1+E_{22}}, \tag{11}$$

where $\hat{\mu}$ and $\hat{\beta}$ are BLUE of μ and β, and

$$\begin{pmatrix} Var(\hat{\mu}) & Cov(\hat{\mu}, \hat{\beta}) \\ Cov(\hat{\mu}, \hat{\beta}) & Var(\hat{\beta}) \end{pmatrix} = \hat{\beta}^2 \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}.$$

The mean square errors of these estimators are:

$$MSE(\tilde{\mu}) = \hat{\beta}^2[E_{11} - \frac{E_{12}^2}{1+E_{22}}]$$

and

$$MSE(\tilde{\beta}) = \hat{\beta}^2[\frac{E_{22}^2}{1+E_{22}}].$$

2.3. Prediction of the future record

Finally, the notions of records for a specific phenomena that is probabilistically defined by the Gumbel function probability distribution have been introduced in this paper. We generated some lower record value distributional features and achieved certain attributes that are important to this distribution. To predict future observations, we can accomplish this by using return levels

$$F(x_s) = 1/s, \qquad s > r$$

which gives

$$x_s = \hat{\mu} - \hat{\beta} \ln \ln s. \tag{12}$$

2.4. Simulation study

A simulation study is used to demonstrate the performance of the estimators produced in the preceding section. We used the Gumbel probability distribution with $\mu = 10$ and $\beta = 2$ to simulate a small random sample of size n = 12:

```
10.7330, 11.0289, 9.3467, 9.6287, 8.7193, 11.6843
12.0619, 12.4918, 9.9028, 9.0482, 7.8056, 13.6339
```

From the given random sample, four record values can be derived, namely,

```
10.7330, 9.3467, 8.7193, 7.8056
```

By using the BLUE and BLIE methods we obtained the estimate parameters of μ and β, for $r = 1, 2, 3, 4$. The standard error in each case are calculated. Finally, the prediction 5^{th} future observation by applying (12) is obtained in each case. All of these results are given below in Table 3:

The simulation results indicate that the estimates for μ and β are quite close to the true values. The S.E. by BLIE method is smaller than that of the BLUE. This means that the BLIE method is the best.

Table 3 The result of the simulation

	BLUE ($\hat{\Theta}$)	S.E ($\hat{\Theta}$)	BLIE ($\hat{\Theta}$)	S.E ($\hat{\Theta}$)
μ	9.957	0.36	10.038	0.162
β	1.33	1.4	0.63	0.453
Prediction 5^{th} observation	9.3241		9.7382	

3. Maximum Likelihood Method

Let $x_1, x_2, ..., x_n$ represent a complete random sample from the Gumbel probability distribution function (5). The records required for this investigation are obtained as follow: The first record, $X_{L(1)}$, is x_1, so the first observation, $X_{L(1)} = x_1$. Observing the independent and identically distributed random variables $X_i^{'s}$ sequentially from $x_2, ...x_n$ yields the second record value, $X_{L(2)}$. Let the next observation that is less than $X_{L(1)}$ needs number of trials to acquire $X_{L(2)}$ equal K_1. For example, let the next observation that is less than $X_{L(1)}$ be X_7, so the number of trials to get $X_{L(2)}$ be $K_1 = 6$.

$X_{L(1)} = x_1, K_1 = k_1, X_{L(2)} = x_2, K_2 = k_2..., X_{L(r)} = x_r, K_r = k_r$, where $\{X_{L(i)}, 1 \leq i \leq r\}$ is the record value sequence and $\{K_i, i > 0\}$ and $k_r = 1$ is the inter record time sequence. Note that The number of records acquired (r) will be smaller than n, the size of the whole random data sample, if this approach is used. It's worth noting that the record numbers that don't include the inter-record times are referred to as the lower record values.

We may express the likelihood function as

$$L(x, \mu, \beta) = \prod_{i=1}^{r} f(x_i)[1 - F(x_i)]^{(k_i - 1)}$$

for the record-breaking samples $X_{L(1)} = x_1, K_1 = k_1, X_{L(2)} = x_2, K_2 = k_2..., X_{L(r)} = x_r, K_r = k_r$. where $f(x_i)$ and $F(x_i)$ is the pdf and cdf of the random variable from which the record observations are obtained.

Applying likelihood function for record observations are obtained from Gumbel distribution we get

$$L_1(x, \mu, \beta) = \prod_{i=1}^{r} \frac{1}{\beta} z_i e^{-z_i} [1 - e^{-z_i}]^{(k_i - 1)}, \tag{13}$$

where $z_i = e^{-\frac{x_i - \mu}{\beta}}$.

The log of likelihood function is

$$\log L_1(x, \mu, \beta) = \sum_{i=1}^{r} \{\log(\frac{1}{\beta}) + \log(z_i) - z_i + (k_i - 1) \log [1 - e^{-z_i}]\}. \tag{14}$$

We have by taking the partial derivative of (14) with regard to μ and β the following equations

$$\frac{\partial \log L_1(x, \mu, \beta)}{\partial \mu} = \frac{1}{\beta} \sum_{i=1}^{r} \{1 - z_i + (k_i - 1) \frac{z_i e^{-z_i}}{1 - e^{-z_i}}\}, \tag{15}$$

$$\frac{\partial \log L_1(x, \mu, \beta)}{\partial \beta} = \frac{1}{\beta} \sum_{i=1}^{r} \{-1 - log(z_i) + z_i \log(z_i) - (k_i - 1) \frac{z_i e^{-z_i} \log(z_i)}{1 - e^{-z_i}}\}. \tag{16}$$

The maximum likelihood estimators for μ and β for the record samples by setting Eqns. (15) and (16) to zero.

The estimates of the parameters that are inherent in Eqns. (13) and (14) are obtained as follows for the complete sample $X_1, X_2, ..., X_n$. We can write the log-likelihood from the Gumbel probability density function given by Eqn. (4) as follows

$$\log(L_2(x,\mu,\beta)) = \sum_{i=1}^{n}\{\log(\frac{1}{\beta}) + \log(z_i) - z_i\}. \tag{17}$$

We have by taking the partial derivative of (17) with regard to μ and β the following equations

$$\frac{\partial \log L_2(x,\mu,\beta)}{\partial\mu} = \frac{1}{\beta}\sum_{i=1}^{n}\{1 - z_i\} \tag{18}$$

$$\frac{\partial \log L_2(x,\mu,\beta)}{\partial\beta} = \frac{1}{\beta}\sum_{i=1}^{n}\{-1 - \log(z_i) + z_i\log(z_i)\}. \tag{19}$$

The maximum likelihood estimators for μ and β for the complete samples by setting Eqns. (18) and (19) to zero.

3.1. Application

Here, we will apply the result in this section on the following example: We used the Gumbel function probability distribution with $\mu = 10$ and $\beta = 2$ to simulate a small random sample of size $n = 15$:

```
12.999, 11.34, 10.1748, 10.733, 11.0289, 9.9028, 9.9287, 9.0482
10.45, 10.174, 8.7193, 13.6339, 12.0619, 8.3319, 7.8056
```

First, using the entire data, we have the following record values and inter record times. $x_i = 12.999, 11.34, 10.1748, 9.9028, 9.0482, 8.7193, 8.3319, 7.8056$ and $k_i = 1, 1, 3, 2, 3, 3, 1, 1$.

The maximum likelihood estimates with respect to the complete sample and the inter record times for μ and β are given below in Table 4:

Table 4 The maximum likelihood estimates

	$(\hat{\mu})$	$(\hat{\beta})$
Complete sample	9.6906	1.3718
Inter record times	9.6852	1.4856

3.2. Goodness-of-fit tests

A statistical model's goodness of fit defines how well it fits a collection of data. The disparity between actual values and predicted values under the model in issue is often summarised by goodness of fit measures. Such measurements can be used in statistical hypothesis testing, for example, to check for residual normality, to see whether two samples are taken from the same distribution. By applying Akaike's information criterion (AIC) (see Akaike(1974)) and corrected Akaike information criterion (CAIC) statistics using the previous example where the two statistics can be evaluated from:

$$AIC = -2L + 2K \quad \text{and} \quad AICC = AIC + \frac{2K(K+1)}{n-K-1}, \tag{20}$$

where L is the likelihood of the function K is the number of parameters which estimated and n is the sample size which used for estimation. In the same direction by applying Cram$\acute{e}$-von Mises statistic which given by:

$$T = \frac{1}{12} + \sum_{i=1}^{n}(F(x_i) - \frac{2i-1}{2n})^2, \tag{21}$$

$F(x_i)$ is the cumulative function for ordered observations.

By applying (20), (21) we obtained the following result in Table 5:

Table 5 Statistical models for goodness of fit

	AIC	AICC	T
Complete sample	63.6285	64.5516	0.195956498
Inter record times	40.0106	42.4106	0.100196816

Table 5 provided that the value of AIC in the case of inter record times is smaller than the value in the case of complete sample. This means that the using of inter record times is the best.

4. Inference Based on Lower Record Values for q-Gumbel Distribution

In this section, we estimate the parameters of the q-Gumbel probability distribution using lower record values. The BLUE based on r lower record values from the q-Gumbel probability distribution are obtained in Subsection 5.1. The BLIE based on r lower record values from the q-Gumbel probability distribution are derived in Subsection 5.2. The prediction of the future record are developed in Subsection 5.3. In Subsection 5.4, we presented simulation study.

4.1. Best Linear Unbiased Estimates (BLUEs)

Applying Eqn. (5) and letting $\mu = 0$ and $\beta = 1$, the n^{th} moment of $X_L(r)$ from the q-Gumbel function probability distribution is given by:

$$E(X_{L(r)})^n = \frac{1}{\Gamma(r)} \int_{-\infty}^{\infty} x^n [-\ln(1+qe^{-x})^{-}\frac{1}{q}]^{r-1} e^{-x}[1+qe^{-x}]^{(-\frac{1}{q}-1)}dx$$

For $n = 1$

$$E(X_{L(r)}) = \frac{1}{\Gamma(r)} \int_{-\infty}^{\infty} x[-\ln(1+qe^{-x})^{-}\frac{1}{q}]^{r-1} e^{-x}[1+qe^{-x}]^{(-\frac{1}{q}-1)}dx$$

Let

$$1 + qe^{-x} = \frac{1}{1-t} \tag{22}$$

then
$$\begin{aligned} E(X_{L(r)}) &= \frac{q^{1-r}}{\Gamma(r)} \int_0^1 [-\ln \frac{t}{q(1-t)}][\ln \frac{1}{(1-t)}]^{(r-1)}[1-t]^{(\frac{1}{q}-1)}dt \\ &= \frac{q^{-r}}{\Gamma(r)} \int_0^1 [-\ln t + \ln(1-t) + \ln q][-\ln(1-t)]^{(r-1)}[1-t]^{(\frac{1}{q}-1)}dt \\ &= \frac{q^{-r}}{\Gamma(r)}[I_1 + I_2 + I_3] = bq_r, \end{aligned} \tag{23}$$

where
$$I_1=\sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty}...\sum_{i_{r-1}=0}^{\infty}\int_0^1\frac{1}{j.i_1....i_{r-1}}t^{(i_1+...+i_{r-1})}(1-t)^{(j+\frac{1}{q}-1)}dt$$
$$=\sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty}...\sum_{i_{r-1}=0}^{\infty}\frac{1}{j.i_1....i_{r-1}}\beta(i_1+...+i_{r-1}+1,j+\frac{1}{q})$$
$$I_2=-\sum_{i_1=0}^{\infty}...\sum_{i_r=0}^{\infty}\int_0^1\frac{1}{j.i_1....i_r}t^{(i_1+...+i_r)}(1-t)^{(\frac{1}{q}-1)}dt$$
$$=-\sum_{i_1=0}^{\infty}...\sum_{i_r=0}^{\infty}\frac{1}{i_1....i_r}\beta(i_1+...+i_r+1,\frac{1}{q}),$$
$$I_3=\ln q\sum_{i_1=0}^{\infty}...\sum_{i_{r-1}=0}^{\infty}\int_0^1\frac{1}{i_1....i_{r-1}}t^{(i_1+...+i_{r-1})}(1-t)^{(\frac{1}{q}-1)}dt$$
$$=\ln q\sum_{i_1=0}^{\infty}...\sum_{i_{r-1}=0}^{\infty}\frac{1}{i_1....i_{r-1}}\beta(i_1+...+i_{r-1}+1,\frac{1}{q}).$$

As the same, for $n=2$:

$$E(X_{L(r)})^2=\frac{1}{\Gamma(r)}\int_{-\infty}^{\infty}x^2[-\ln(1+qe^{-x})^-\frac{1}{q}]^{r-1}e^{-x}[1+qe^{-x}]^(-\frac{1}{q}-1)dx.$$

From (22) then

$$E(X_{L(r)})^2=\frac{q^{1-r}}{\Gamma(r)}\int_0^1[-\ln\frac{t}{q(1-t)}]^2[\ln\frac{1}{(1-t)}]^{(r-1)}[1-t]^{(\frac{1}{q}-1)}dt$$
$$=\frac{q^{-r}}{\Gamma(r)}\int_0^1[-\ln t+\ln(1-t)+\ln q]^2[-\ln(1-t)]^{(r-1)}[1-t]^{(\frac{1}{q}-1)}dt$$
$$=\frac{q^{-r}}{\Gamma(r)}[I_1+I_2+I_3+I_4+I_5+I_6] \tag{24}$$

where
$$I_1=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty}...\sum_{i_{r-1}=0}^{\infty}\int_0^1\frac{1}{i.j.i_1....i_{r-1}}t^{(i_1+...+i_{r-1})}(1-t)^{(i+j+\frac{1}{q}-1)}dt$$
$$=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty}...\sum_{i_{r-1}=0}^{\infty}\frac{1}{i.j.i_1....i_{r-1}}\beta(i_1+...+i_{r-1}+1,i+j+\frac{1}{q}),$$
$$I_2=\sum_{i_1=0}^{\infty}...\sum_{i_{r+1}=0}^{\infty}\int_0^1\frac{1}{i_1....i_{r+1}}t^{(i_1+...+i_{r+1})}(1-t)^{(\frac{1}{q}-1)}dt$$
$$=\sum_{i_1=0}^{\infty}...\sum_{i_r=0}^{\infty}\frac{1}{i_1....i_{r+1}}\beta(i_1+...+i_{r+1}+1,\frac{1}{q}),$$

$$I_3 = (\ln q)^2 \sum_{i_1=0}^{\infty} ... \sum_{i_{r-1}=0}^{\infty} \int_0^1 \frac{1}{i_1....i_{r-1}} t^{(i_1+...+i_{r-1})}(1-t)^{(\frac{1}{q}-1)}dt$$
$$= \ln q \sum_{i_1=0}^{\infty} ... \sum_{i_{r-1}=0}^{\infty} \frac{1}{i_1....i_{r-1}} \beta(i_1+...+i_{r-1}+1, \frac{1}{q}),$$
$$I_4 = -2\sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty} ... \sum_{i_r=0}^{\infty} \int_0^1 \frac{1}{j.i_1....i_r} t^{(i_1+...+i_r)}(1-t)^{(j+\frac{1}{q}-1)}dt$$
$$= \sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty} ... \sum_{i_r=0}^{\infty} \frac{1}{j.i_1....i_r} \beta(i_1+...+i_r+1, j+\frac{1}{q}),$$
$$I_5 = 2\ln q \sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty} ... \sum_{i_{r-1}=0}^{\infty} \int_0^1 \frac{1}{j.i_1....i_r - 1} t^{(i_1+...+i_{r-1})}(1-t)^{(j+\frac{1}{q}-1)}dt$$
$$= 2\ln q \sum_{j=0}^{\infty}\sum_{i_1=0}^{\infty} ... \sum_{i_{r-1}=0}^{\infty} \frac{1}{j.i_1....i_r - 1} \beta(i_1+...+i_{r-1}+1, j+\frac{1}{q}),$$
$$I_6 = 2\ln q \sum_{i_1=0}^{\infty} ... \sum_{i_r=0}^{\infty} \int_0^1 \frac{1}{i_1....i_r - 1} t^{(i_1+...+i_r)}(1-t)^{(\frac{1}{q}-1)}dt$$
$$= 2\ln q \sum_{i_1=0}^{\infty} ... \sum_{i_r=0}^{\infty} \frac{1}{i_1....i_r} \beta(i_1+...+i_r+1, \frac{1}{q}).$$

From (23) and (24) we can compute

$$Var(X_{L(r)}) = E(X_{L(r)})^2 - (E(X_{L(r)}))^2 = aq_r.bq_r, \quad (25)$$
$$Cov(X_{L(r)}, X_{L(s)}) = E(X_{L(r)}X_{L(s)}) - E(X_{L(r)})E(X_{L(s)}) = bq_s.aq_r \quad (26)$$

By applying Theorem 1 (Subsection 2.1) and the method introduced by Lioyd (1952) using Eqns. (24), (25) and (26), we have coefficients of the BLUES for mu and beta. Also, the variance covariance for μ, β are given in Tables 6 and 7, respectively according different values of q.

4.2. Best Linear Invariant Estimates (BLIEs)

The best linear invariant (in terms of minimum mean squared error and invariance with respect to the location parameter μ) estimators (BLIE) $\tilde{\mu}, \tilde{\beta}$ of μ and β are also computed by applying (10) and (11) using Eqns. (23), (24) and (26) (see Section 2.2).

4.3. Prediction of the Future Record

Finally, the notions of records for a specific phenomena that is probabilistically defined by the q-Gumbel function probability distribution have been introduced in this paper. We generated some lower record values distributional features and achieved certain attributes that are important to this distribution. To predict future observations, we can accomplish this by using return levels:

$$F(x_s) = 1/s, \qquad s > r.$$

Then,

$$x_s = \hat{\mu} - \hat{\beta}\ln\frac{s^q - 1}{q}.$$

Table 6 Coefficients for the BLUE of μ and β

n	r	Coefficients for the BLUE of μ and β for q = 0.5		Coefficients for the BLUE of μ and β for q = 1		Coefficients for the BLUE of μ and β for q = 1.5	
2	1	0.6778	-0.3241	0.8471	-0.3333	0.8931	-0.2901
2	2	0.3222	0.3241	0.1529	0.3333	0.1069	0.2901
3	1	1.0323	0.0325	0.9889	-0.0242	2.2969	3.5193
3	2	0.2491	0.2506	0.0918	0.2001	-0.6068	-1.6466
3	3	-0.2814	-0.2831	-0.0807	-0.176	-0.6901	-1.8728
4	1	0.7912	-0.2101	0.8948	-0.2294	1.0047	0.0128
4	2	0.1533	0.1542	0.0831	0.1811	-0.2894	-0.7853
4	3	-0.0634	-0.0638	-0.0159	-0.0347	0.0357	0.097
4	4	0.119	0.1197	0.0381	0.0831	0.249	0.6756
5	1	0.9133	-0.0873	0.949	-0.1111	2.5651	4.2472
5	2	0.1423	0.1431	0.0881	0.1921	-0.7773	-2.1094
5	3	-0.0457	-0.046	-0.0169	-0.0369	0.0837	0.2272
5	4	0.0828	0.0833	0.0071	0.0155	-0.1974	-0.5356
5	5	-0.0926	-0.0932	-0.0274	-0.0596	-0.6742	-1.8295

Table 7 Coefficient for the BLUE of μ and β

	r=2	r=3	r=4	r=5
	0.8473	1.0844	0.7522	0.8227
q=0.5	1.0089	0.471	0.6582	0.5124
	-0.8524	-1.0909	-0.7567	-0.8276
	0.6596	0.7701	0.6968	0.739
q=1	1.2337	0.709	1.0573	0.8565
	-1.4378	-1.6785	-1.5187	-1.6108
	3.5982	16.3757	11.8358	48.113
q=1.5	6.7231	61.804	9.9756	199.6442
	-9.7644	-44.4381	-32.1184	-130.562

4.4. Simulation study

To show the performance of the estimators developed in the preceding section, a simulation study is employed.We used the q-Gumbel function probability distribution with $r = 5$, $q = 0.5$, $\mu = 5$ and $\beta = 0.1$ to simulate a small random sample of size $n = 15$:

```
5.1173,5.2224,5.094,5.2957,5.1776,5.0361,5.444,4.9498
5.0733,5.0188,4.985,4.8536,4,8848,4.9677,5.0019
```

From the given random samples, five record values can be derived, namely, $5.1173, 5.094, 5.0361, 4.9498, 4.8536$.

By using the BLUE and BLIE methods we obtained the estimate parameters of μ and β for $r = 1, 2, 3, 4, 5$. The standard error in each case are calculated. Finally, the prediction 6^{th} future observation by applying (25) is obtained in each case.

All of these results are given below in Table 8.

Table 8 The result of the simulation

	BLUE ($\hat{\Theta}$)	S.E ($\hat{\Theta}$)	BLIE ($\hat{\Theta}$)	S.E ($\hat{\Theta}$)
μ	5.1283	0.0224	5.134	0.00444
β	0.011	0.0079	0.0073	0.00425
Prediction 6^{th} observation	5.1166		5.12622	

The simulation results indicate that the estimates for μ and β are quite close to the true values. The S.E. by BLIE method is smaller than that of the BLUE. This means that the BLIE method is the best.

5. Maximum Likelihood Method

Let $X_1, X_2, ..., X_n$ represent a full random sample from the q-Gumbel probability distribution function (5):
The records required for this investigation are obtained as follow:
$X_{L(1)} = x_1, K_1 = k_1, X_{L(2)} = x_2, K_2 = k_2, ..., X_{L(r)} = x_r, K_r = k_r$, where $\{X_L(i), 1 \leq i \leq r\}$ is the record value sequence and $\{K_i, 1 > 0\}$ is the inter record time sequence (see section 3). The likelihood function expressed as:

$$L(x, \mu, \beta, q) = \prod_{i=1}^{r} f(x_i)[1 - F(x_i)]^{(k_i - 1)}$$

for the record-breaking samples $X_{L(1)} = x_1, K_1 = k_1, X_{L(2)} = x_2, K_2 = k_2, ..., X_{L(r)} = x_r, K_r = k_r$. where $f(x_i)$ and $F(x_i)$ is the pdf and cdf of the random variable from which the record observations are obtained from q-Gumbel distribution.

Applying likelihood function for record observations are obtained from q-Gumbel distibution we get:

$$L_1(x, \mu, \beta, q) = \prod_{i=1}^{r} \frac{1}{\beta} z_i [1 + q z_i]^{-(\frac{1}{q}+1)} [1 - (1 + q z_i)^{\frac{-1}{q}}]^{(k_i - 1)},$$

where $z_i = e^{-(\frac{x_i - \mu}{\beta})}$
the log of likelihood function is:

$$\log L_1(x, \mu, \beta, q) = \sum_{i=1}^{r} \{\log(\frac{1}{\beta}) + \log(z_i) - (\frac{1}{q} + 1) \log[1 + q z_i] + (k_i - 1) \log[1 - (1 + q z_i)^{\frac{-1}{q}}]\} \quad (27)$$

By taking the partial derivative of (27) with regard to μ and β the following equations:

$$\frac{\partial \log L_1(x, \mu, \beta, q)}{\partial \mu} = \frac{1}{\beta} \sum_{i=1}^{r} \{1 - \frac{q(\frac{1}{q} + 1) z_i}{[1 + q z_i]} + (k_i - 1) \frac{z_i [1 + z_i]^{\frac{-1}{q} - 1}}{[1 - (1 + q z_i)^{\frac{-1}{q}}]}\}, \quad (28)$$

$$\frac{\partial \log L_1(x, \mu, \beta, q)}{\partial \beta} = \frac{1}{\beta} \sum_{i=1}^{r} \{-1 - \log(z_i) + \frac{q(\frac{1}{q} + 1) z_i \log(z_i)}{[1 + q z_i]} - (k_i - 1) \frac{z_i \log(z_i) [1 + q z_i]^{\frac{-1}{q} - 1}}{[1 - (1 + q e^{z_i})^{\frac{-1}{q}}]}\} \quad (29)$$

and

$$\frac{\partial \log L_1(x, \mu, \beta, q)}{\partial q} = \sum_{i=1}^{r} \{\frac{-(\frac{1}{q} + 1) z_i}{[1 + q z_i]} + \frac{\log[1 + q z_i]}{q^2} + \frac{(k_i - 1) z_i}{[1 + q e^{z_i}]^{\frac{1}{q} + 1}} - \frac{(k_i - 1) \log[1 + q z_i]}{q^2 [1 + q z_i]^{1/q}}\} \quad (30)$$

The maximum likelihood estimators for μ , β and q for the record samples by setting Eqns. (28), (29) and (30) to zero.

The estimates of the parameters that are inherent in Eqns. (28), (29) and (30) are obtained as follows for the complete sample $X_1, X_2, ..., X_n$.

We can write the log-likelihood from the q-Gumbel probability density function given by Eqn. (5) as follows:

$$\log(L_2(x, \mu, \beta, q)) = \sum_{i=1}^{n} \{\log(\frac{1}{\beta}) + \log(z) - (\frac{1}{q} + 1) \log[1 + q z]\} \quad (31)$$

We have by taking the partial derivative of (31) with regard to μ, β and q the following equations:

$$\frac{\partial \log L_2(x,\mu,\beta,q)}{\partial \mu} = \frac{1}{\beta}\sum_{i=1}^{n}\{1-\frac{q(\frac{1}{q}+1)z_i}{[1+qz_i]}\}, \tag{32}$$

$$\frac{\partial \log L_2(x,\mu,\beta,q)}{\partial \beta} = \frac{1}{\beta}\sum_{i=1}^{n}\{-1-\log(z_i)+\frac{q(\frac{1}{q}+1)z_i\log(z_i)}{[1+qz_i]}\} \tag{33}$$

The maximum likelihood estimators for μ, β and q for the complete samples by setting Eqns. (31), (32) and (33) to zero.

5.1. Application

Here, we will apply the result in this section on the following example:

We used the q-Gumbel function probability distribution with $\mu = 5$, $\beta = 2$ and $q = 0.01$ to simulate a small random sample of size $n = 15$:

```
7.9976, 7.4889, 5.7261, 8.6323, 6.6800, 5.4420, 5.1657, 4.8923
3.7003, 4.3329, 3.3089, 2.7756 ,7.0583, 4.0321, 6.0229
```

First, using the entire data , we have the following record values and inter record times. $x_i = 7.9976, 7.4889, 5.7261, 5.442, 5.1657, 4.8923, 3.7003, 3.3089, 2.7756$ and $k_i = 1, 1, 3, 1, 1, 1, 2, 1, 1$

The maximum likelihood estimates with respect to the complete sample and the inter record times for μ, β and q are given below in Table 9.

Table 9 The maximum likelihood estimates

	$(\hat{\mu})$	$(\hat{\beta})$	$(\hat{q})$
Complete sample	5.4999	1.0273	0.98
Inter record times	5.46522	1.0273	0.8802

5.1.1 Goodness-of-fit

By applying Akaike's information criterion (AIC), corrected Akaike information criterion (CAIC) and Cramé-von Mises statistics using data from the q-Gumbel function probability distribution, since the two statistics can be evaluated from:

$$AIC = -2L + 2K \quad \text{and} \quad AICC = AIC + \frac{2K(K+1)}{n-K-1} \tag{34}$$

and

$$T = \frac{1}{12} + \sum_{i=1}^{n}(F(x_i) - \frac{2i-1}{2n})^2, \tag{35}$$

where L is the likelihood of the function, K is the number of parameters which estimated and n is the sample size which used for estimation. By applying (34) and (35) we obtained the following results which given below in Table 10:

Table 10 Statistical models for goodness of fit

	AIC	AICC	T
Complete sample	65.684	67.8663	0.049865511
Inter record times	53.76456	458.56456	0.036495806

Table 10, provided that the value of AIC in the case of inter record times is smaller than the value in the case of complete sample. This means that the using of inter record times is the best.

6. Conclusions

In the present study, we have introduced the concepts of Records for a given phenomenon that is probabilistically characterized by the q-Gumbel pdf. Coefficients of the best linear unbiased estimates have been obtained. In addition, a method for predicting future observations given based on the current data. We have developed the analytical structure of the records, along with their maximum likelihood estimates. We have illustrated the usefulness of our analytical developments in two interesting classical applications. Finally, the estimates of our analysis using inter record times are better than previous results.

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