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One Sample Empirical Likelihood Ratio Test for Coefficient of Variation

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Abstract

Coefficient of Variation (CV) is widely used as a measure of variation by researchers in applied disciplines like chemistry, engineering, climatology, finance, agriculture and biological sciences. CV is a better measure for analysing health science data as the units of measurement of the index of different organs are often different. To assess precision in immunoassays and morphological measurements, CV is used. The present study aims to propose an empirical likelihood ratio (ELR) test for testing CV. The asymptotic null distribution of the proposed test statistic is obtained as Chi-square distribution with 1 degree of freedom. Simulation is carried out to check the adequacy of Chi-square approximation for finite samples. The proposed test is compared to Wald, bootstrap tests and ELR test constructed by Wang et al. ($ELRT_2$) using real data sets and also simulated data sets. The study indicates that the proposed empirical likelihood ratio test possesses higher power compared to Wald, bootstrap and $ELRT_2$ tests when the underlying distributions considered are normal, lognormal, gamma and Weibull.

Keywords: Chi-square approximation, simulation, Wald test, bootstrap test, type 1 error rate, power of the test, confidence interval.

1. Introduction

CV is widely used as a measure of variation in applied disciplines by researchers. CV is invariant to the number of replications, therefore it is an ideal index of certainty of measurements when the number of replications varies across samples. Mean Residual Life (MRL) improves with increasing replications. Comparison of MRL of two or more populations is meaningful when the same number of samples are studied, otherwise, CV is a better option. CV is a popular measure for describing the amount of repeat variability present in ECG measurements from one recording to another. Here, the aim is to assess repeat variation (reclassification) in computer-measured ECG criteria, i.e. positive to negative or vice versa, and compare this with the coefficient of variability [McLaughlin et al. (1998)]. CV is better measure for analysing health science data as the units of measurement of the index of different organs are often different. CV is used for assessing precision in immunoassays and morphological measurements. It is the best measure of variability of population size over time if there are zeros in the data. CV is better indicator of relative risk among different levels of risk for different securities. It is also used to study investment volatility.

Rao and Bhatt (1989) proposed tests for CV(s) of one (two) population(s) and derived Edgeworth expansion for distribution function of sample CV. The asymptotic robustness of these tests is discussed by Rao and Vidya (1992). Rao and Bhatt (1995) further proposed tests based on jackknifing and bootstrapping techniques for one and two sample cases. Singh (1993) proposed tests based on inverse sample CV. Banik et al. (2012) proposed a bootstrap test for testing population CV and compared it with existing methods. Kalkur and Rao (2014) have proposed six tests for testing equality of CVs of bivariate normal distribution.

Empirical likelihood is the nonparametric analogue of parametric likelihood. The first paper on empirical likelihood is with reference to survival analysis, wherein Thomas and Grunkemeier (1975) have addressed the issue of censored observations. Owen (1988) and Owen (1990) introduced ELR for testing specified value of mean of a continuous distribution. Qin and Lawless (1994) extended ELR test for testing specified values of quantiles of continuous distribution. ELR test shared the same property with the parametric likelihood ratio test, i.e. ELR test satisfies Wilks' theorem as in the case of parametric likelihood ratio test. The asymptotic null distribution of ELR test statistic is central chi-square distribution with 1 degree of freedom. Naik-Nimbalkar and Rajrshi (1997) used this idea and developed a test for equality of median survival times for censored data. Adimari (1997) addressed the issue of censored observation with reference to empirical likelihood as defined by Owen (1988) and has proposed a simple method to obtain empirical likelihood type confidence interval for the mean under random censorship. Shen and He (2007) used smoothed empirical likelihood method to investigate the difference of quantiles under censoring. Further, they extended these techniques for the construction of confidence intervals for hazard and density functions when observations are right censored. Yu et al. (2011) proposed four ELR tests for testing the equality of medians of two populations. Wang et al. (2018) proposed two nonparametric methods to construct confidence intervals for the coefficient of variation using empirical likelihood method after transforming the original data and using modified jackknife empirical likelihood method.

An ELR test for testing CV is proposed in Section 2. Type I error rates of proposed ELR, Wald, bootstrap and $ELRT_2$ tests are estimated in Section 3 through simulation. Performance of the proposed test with respect to three competent tests is analysed in the same section using power comparisons. Section 4 considers the illustration of analysis of proposed test through real data set.

2. Test Statistics

2.1. Empirical likelihood ratio test

Let X be a random variable and $x_1, x_2, ..., x_n$ denote a sample of size n from a distribution with cumulative density function (cdf) $F(x)$, mean $\mu(\mu \neq 0)$ and standard deviation σ. For testing $H_0 : \theta = c_0$, where θ denotes the CV of the distribution and c_0 is the specified value of CV, the test statistic is

$$ELRT_1 = -2\log\left(L_1(c_0)\right),$$

where empirical likelihood ratio $L_1(c_0)$ is given by

$$L_1(c_0) = \max_{w_1,\dots,w_n}\left\{\prod_{i=1}^{n} nw_i \,\middle|\, w_i \geq 0, \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} w_i[(x_i-\mu)^2 - c_0^2\mu^2] = 0\right\}. \tag{1}$$

The asymptotic null distribution of $(ELRT_1)$ test statistic is central χ^2 with 1 degree of freedom. Alternatively one can construct $(1-\alpha)100\%$ empirical likelihood confidence interval for θ as

$$R_\theta = \{\theta : ELRT_1 \leq k\}$$

where k is $(1-\alpha)$ quantile of χ^2_1.

Theorem 1 *Under regularity conditions, the asymptotic null distribution of $-2\log\left(L_1(c_0)\right)$ is central χ^2 with 1 degree of freedom.*

Proof: Since it is similar to the proof as given in Owen (1988)), the sketch of the proof by omitting finer details is presented.

Let $z_i = (x_i - \widehat{\mu})^2 - c_0^2\widehat{\mu}^2$, such that $Ez_i = 0$.

$$G = \sum_{i=1}^{n} n\log(w_i) - \gamma\left(1 - \sum_{i=1}^{n} w_i\right) + n\lambda\left(0 - \sum_{i=1}^{n} w_i z_i\right)$$

$$\text{Equating} \sum_{i=1}^{n} w_i \frac{\delta G}{\delta w_i} = 0 \text{ gives } \gamma = -n$$

$$\text{and hence, } w_i = [n(1+\lambda z_i)]^{-1}$$

$$\text{where } \lambda \text{ is a solution of } \sum_{i=1}^{n} \frac{z_i}{[n(1+\lambda z_i)]} = 0.$$

Using Taylor series expansion of $\sum_{i=1}^{n} \frac{z_i}{[n(1+\lambda z_i)]}$ around $\lambda = 0$, λ is obtained as

$$\lambda = \frac{\sum_{i=1}^{n} z_i}{\sum_{i=1}^{n} {z_i}^2} + O_p\left(n^{-\frac{1}{2}}\right).$$

Using Taylor series expansion of $log(1+x)$, omitting the term $nlog(n)$ which eventually gets cancelled, results in

$$\begin{aligned} -2\sum_{\mathrm{i=1}}^{\mathrm{n}} \log(w_i) &= 2\sum_{i=1}^{n} \log\ [1+\lambda z_i] \\ &= 2\lambda\sum_{i=1}^{n} z_i - \lambda^2\sum_{i=1}^{n} z_i^2 + O_p(1) \end{aligned}$$

which is simplified as $-2\sum_{\mathrm{i=1}}^{\mathrm{n}} log(w_i) = \frac{\mathrm{n}(\overline{z})^2}{\sigma_z^2}$. where $\frac{1}{\mathrm{n}}\sum_{i=1}^{n} z_i = \overline{z}$ and $\frac{1}{\mathrm{n}}\sum_{i=1}^{n} z_i^2 = \sigma_z^2$.

Hence the proof.
The regularity condition $\int |X|^3 dF(X)$ ensures that the reminder term is $O_p(1)$, for further details Owen (1988) can be referred.

2.2. Wald test

The Wald test statistic as proposed by Rao and Bhatt (1989) for testing $H_0 : \theta = c_0$ is

$$W = \frac{(c - c_0)^2}{Var(c)} \tag{2}$$

where, c is sample CV, $E(c) = \theta + O\left(n^{-2}\right)$,

$$Var(c) = \frac{\theta^2}{n}\left(\frac{\mu_4 - \mu_2^2}{4\mu_2^2} - \frac{\mu_3}{\mu_2\mu_1^{'}} + \frac{\mu_2}{(\mu_1^{'})^2}\right) + O\left(n^{-2}\right)$$

and $\mu_r = E(X-\mu)^r$, $\mu_r^{'} = EX^r, EX^4 < \infty$.

The asymptotic null distribution of W is central χ^2 with 1 degree of freedom. The distribution of sample CV is difficult to derive when the underlying distribution is skewed [Acharna (2012)].

2.3. Bootstrap test

One can construct a test for testing $H_0 : \theta = c_0$ using bootstrap technique. The test statistic is

$$BT = \frac{(c - c_0)^2}{Var\,(c_1)} \tag{3}$$

where, $Var(c_1)$ is Variance of bootstrap sample's CV. The asymptotic null distribution of BT is central χ^2 with 1 degree of freedom.

2.4. Empirical likelihood ratio test due to Wang et al.

Wang et al. (2018) proposed interval estimator for a single CV via the empirical likelihood method by transforming the original variable. Let $m = \lfloor n/2 \rfloor$ denote the integer part of $n/2$. Define $y_i = \frac{1}{2}(x_i - x_{m+i})^2$ and $z_i = \frac{1}{2}(x_i^2 - x_{m+i}^2)$ for $i = 1, 2, \ldots .m$. Then $\tau = \left(\frac{Ey_i}{Ez_i - Ey_i}\right)^{\frac{1}{2}}$ is estimated CV and $E\left(y_i - \tau^2\left(z_i - y_i\right)\right) = 0$ is the estimating equation. For testing $H_0 : \theta = c_0$, the test statistics is

$$ELRT_2 = -2\log\left(L_2(c_0)\right),$$

where empirical likelihood ratio $L_2\left(c_0\right)$ is given by,

$$L_2(c_0) = \max_{w_1,\ldots,w_n}\left\{\prod_{i=1}^{n} nw_i \middle| w_i \geq 0, \sum_{i=1}^{n} w_i = 1, \sum_{i=1}^{n} w_i[(x_i - c_0^2(z_i - y_i)] = 0\right\}. \tag{4}$$

Asymptotic null distribution of $ELRT_2$ is χ^2 with 1 degree of freedom and it can be proved from Owen (1988) similarly as Theorem 1.

3. Type I Error Rates and Power Comparisons when Samples are Drawn from Distributions

ELR test does not have any distributional assumption. The test is distribution-free asymptotically. Therefore type I error rates can be estimated for generated observations from various probability distributions through simulation.

3.1. Simulation configurations

For simulation experiment, normal distribution is taken as the reference distribution. For example, under normality, if the parameters considered are $\mu = 100$ and $\sigma = 10$, the sample is generated with the CV of 10%. For other distributions that are to be considered, the parameters are adjusted so as to give same CV with respective combination of mean and variance as in the case of normal distribution under consideration. The simulation configurations are

Level of significance $\alpha = 0.05$.
$c_0 = 0.1,\ 0.2,\ 0.3$.
Sample size $n = 20,\ 40,\ 60, 100$.
Number of simulations = 2000.
Distributions considered: Normal, Lognormal, Gamma, and Weibull.

The simulations are carried out by using the "emplik" package of R Programming developed by Zhou and Yang (2012).

3.2. Estimated type I error rates

In this study, type I error rates are estimated using two methods. One method is by considering χ^2 critical value and another one is by constructing the frequency distribution of test statistics under the null hypothesis. Table 1 to Table 4 show the estimated type I error rates by considering the χ^2 distribution for normal, lognormal, gamma, and Weibull distributions respectively.

Table 1 Estimated type I error rates for normal distribution

CV	n	Type I error rate			
		$ELRT_1$	Wald	BT	$ELRT_2$
0.1	20	0.144	0.096	0.120	0.172
	40	0.095	0.081	0.093	0.106
	60	0.082	0.066	0.081	0.081
	100	0.066	0.066	0.069	0.068
0.2	20	0.138	0.106	0.125	0.179
	40	0.091	0.079	0.095	0.112
	60	0.085	0.074	0.084	0.079
	100	0.067	0.056	0.072	0.070
0.3	20	0.154	0.103	0.120	0.155
	40	0.111	0.078	0.075	0.108
	60	0.094	0.065	0.067	0.078
	100	0.084	0.062	0.071	0.060

Table 2 Estimated type I error rates for lognormal distribution

CV	n	Type I error rate			
		$ELRT_1$	Wald	BT	$ELRT_2$
0.10	20	0.150	0.104	0.108	0.183
	40	0.090	0.076	0.096	0.109
	60	0.072	0.072	0.082	0.084
	100	0.071	0.059	0.064	0.071
0.20	20	0.124	0.094	0.110	0.172
	40	0.081	0.095	0.081	0.116
	60	0.067	0.060	0.072	0.094
	100	0.053	0.058	0.064	0.068
0.30	20	0.135	0.093	0.115	0.191
	40	0.074	0.077	0.088	0.123
	60	0.063	0.070	0.072	0.102
	100	0.055	0.050	0.064	0.082

Table 3 Estimated type I error rates for gamma distribution

CV	n	Type I error rate		
		$ELRT_1$	BT	$ELRT_2$
0.1	20	0.128	0.121	0.162
	40	0.097	0.087	0.116
	60	0.073	0.083	0.082
	100	0.066	0.062	0.078
0.2	20	0.129	0.119	0.159
	40	0.089	0.088	0.096
	60	0.065	0.078	0.090
	100	0.048	0.073	0.071
0.3	20	0.136	0.102	0.171
	40	0.082	0.090	0.115
	60	0.078	0.078	0.089
	100	0.052	0.066	0.067

Table 4 Estimated type I error rates for Weibull distribution

CV	n	Type I error rate		
		ELRT_1	BT	ELRT_2
0.1	20	0.209	0.152	0.170
	40	0.119	0.111	0.124
	60	0.102	0.108	0.100
	100	0.103	0.080	0.077
0.2	20	0.178	0.104	0.156
	40	0.115	0.092	0.102
	60	0.091	0.080	0.070
	100	0.097	0.080	0.068
0.3	20	0.153	0.100	0.146
	40	0.122	0.072	0.094
	60	0.084	0.074	0.085
	100	0.092	0.049	0.069

A test is said to maintain type I error rate when the attained level of significance $\widehat{\alpha} \in [0.04, 0.06]$. A similar criterion is also used in the past by several researchers (see Nairy and Rao (2003)). For $ELRT_1$, Wald, Bootstrap and $ELRT_2$ tests the chi-squared approximation does not maintain type I error rates when the underlying distribution is normal lognormal , gamma and Weibull because these tests are asymptotically χ^2 distribution under null hypothesis. Type 1 error rate of proposed ELR test $(ELRT_1)$ tends to 0.05 for $c_0 = 0.2,\ 0.3$ and $n = 100$ under lognormal and gamma distributions. Here the aim is to check the performance of $\hat{\alpha}$ by using χ^2 critical value. On the other hand, all the estimated type I error values constructed by using empirical distribution of the test statistics are less than 0.05, which means all the tests maintain the type I error rate. In terms of Type 1 error rate, all the considered tests exhibit similar behavior.

3.3. Power comparison

Simulations are also carried out to compute the power of empirical likelihood ratio test. The estimated α^{th} percentile values are used for power computation so that all the tests have the same size α. Simulation configurations are similar to the one used for estimation of type I error rate. Computed power is compared with Wald, bootstrap, and *ELRT*$_2$ tests. The power functions of the proposed test along with its competitors are given in Figure 1 to Figure 4 for normal, lognormal, gamma, and Weibull distributions respectively when n =20, 40, 60 and 100, c_0= $0.1,\ 0.2 \text{ and } 0.3,\ \alpha = 0.05.$

From Figure 1 to Figure 4, it is clear that the proposed ELR test has greater power compared to Wald, bootstrap, and *ELRT*$_2$ tests in all considered scenarios. The power function is observed to be increasing with an increasing shift in CV. However, the power is even better for the smaller difference in observed and assumed CV whenever the latter is small. It is also noted that for larger sample sizes the power function approaches 1 much faster than for smaller sample sizes.

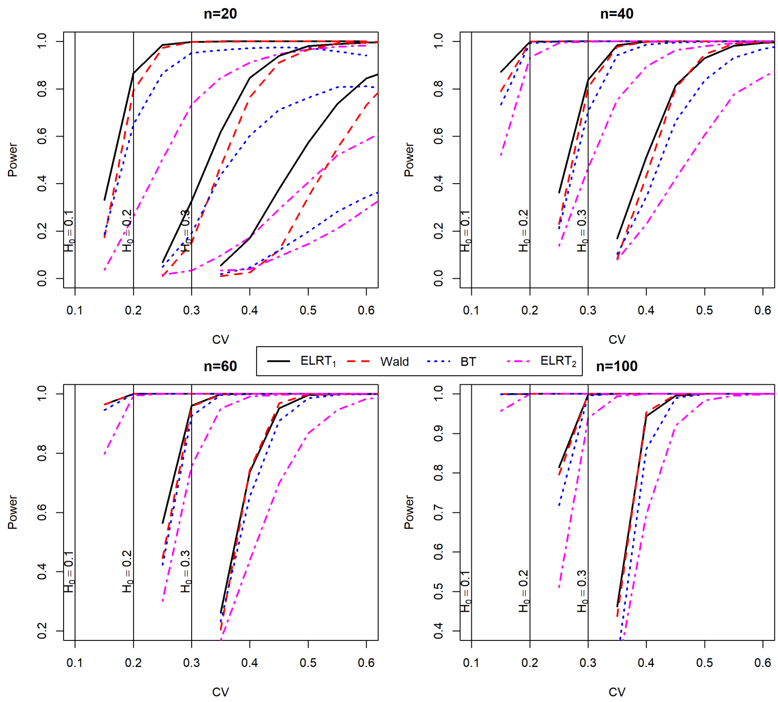

Figure 1 Estimated power functions of normal distribution

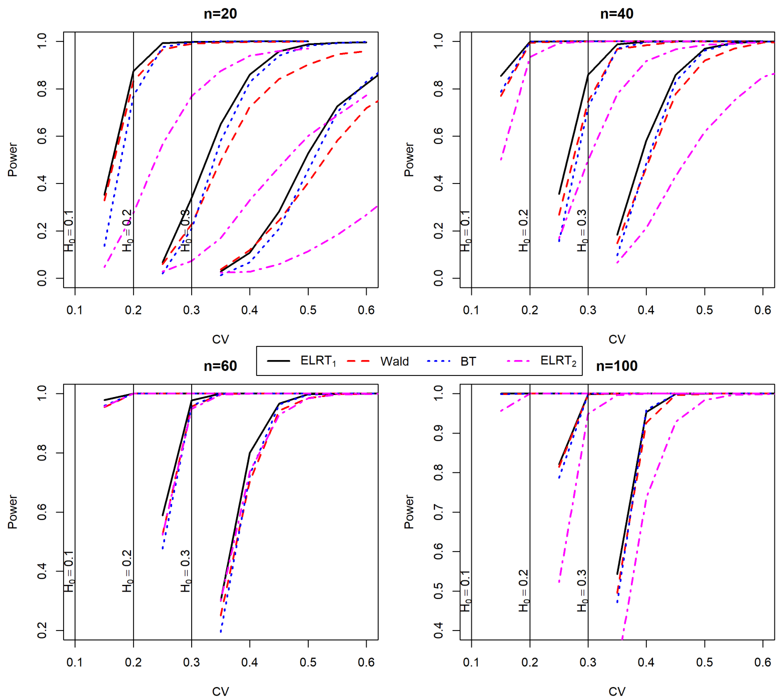

Figure 2 Estimated power functions of lognormal distribution

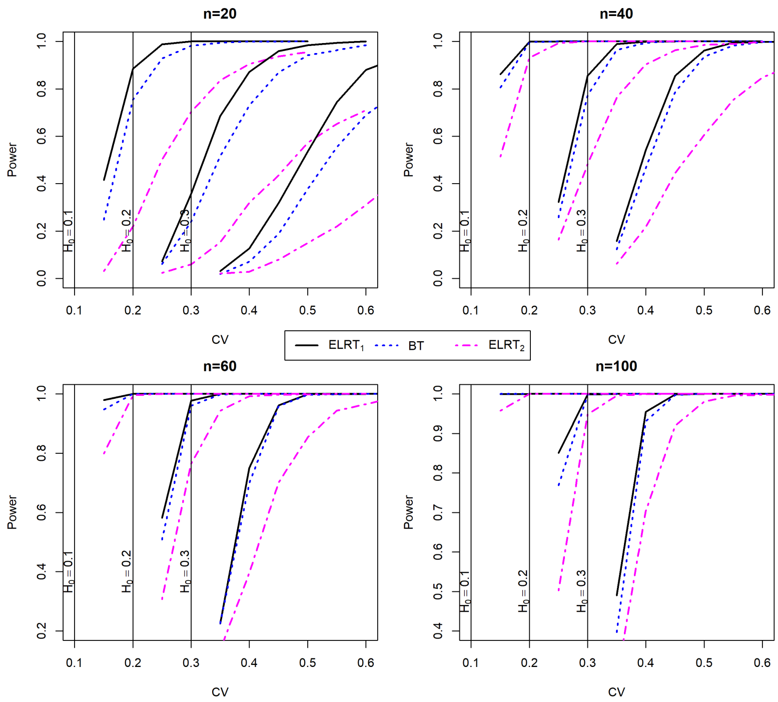

Figure 3 Estimated power functions of gamma distribution

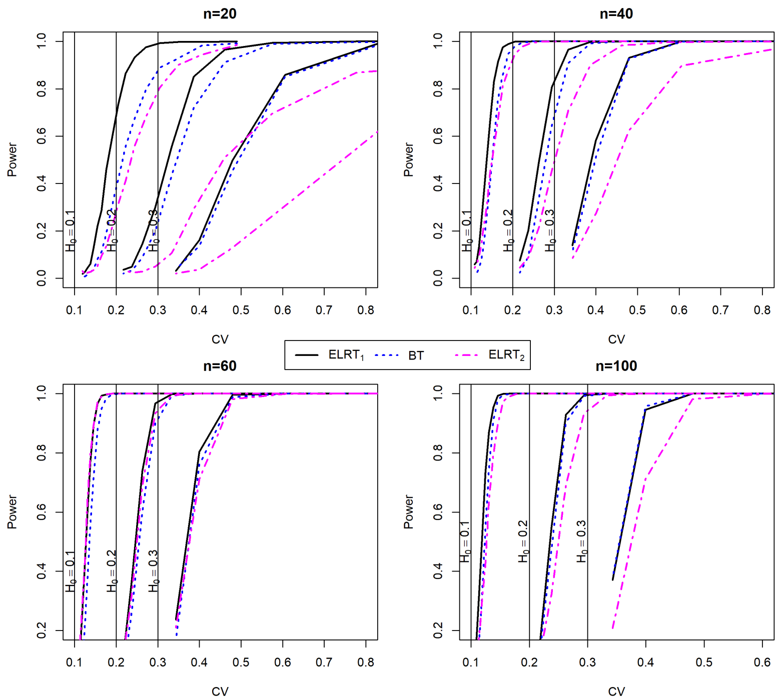

Figure 4 Estimated power functions of Weibull distribution

4. Illustrations

4.1. Testing the significance of CV

CV is interpreted as volatility per mean return and inverse CV is referred to as a sharp ratio in stock market analysis and relative risk in the area of finance. The volatility in finance is a measure of change in the price of a financial instrument over time. It is generally determined by the standard deviation of prices and returns of financial assets observed. Higher the price or return difference, higher the standard deviation, which in turn is linked with higher risk. Volatility of financial assets based on historical values, over the stated duration, with the most recent observation, and the most recent price. For illustration purpose, Hindustan Unilever Ltd. (HUL) stock prices of the Bombay Stock Exchange (BSE) is considered and the proposed test is applied along with three competent tests to check whether volatility of HUL stock is significantly different from zero. Hence, $H_0 : \theta = 0$. The data pertains to a period from 01.01.2020 to 17.12.2020 which yields a sample size (n) as 243 and CV of sample is 5.1%. The test statistic values of $ELRT_1$, Wald, Bootstrap, and $ELRT_2$ tests are 4514.3, 483.5, 394.3 and 3988.753 respectively. All the tests indicate rejection of H_0, since volatility of HUL stock is differ significantly from zero. The proposed test gathers more evidence against the null hypothesis as compared to other three.

4.2. Interval estimation

To measure the reproducibility of serological tests, CV has been widely used. A dataset of the measurements of the antibody titers on 30 distinct days of a single serum specimen from a serological test from Wang et al. (2018) is considered. Confidence intervals for various tests also obtained in Wang et al. (2018). The P-value 0.414 of Shapiro and Wilk (1965) normality test suggests that the data may come from a normal distribution. The sample CV is 0.327. The 95% confidence intervals (CI) from proposed and considered three tests are provided in Table 5.

Table 5 95% Confidence interval estimates

Test	95% CI	
ELRT_1	0.252	0.423
Wald	0.220	0.434
Bootstrap	0.223	0.431
ELRT_2	0.231	0.487

From the table, it is clear that all proposed estimators cover the true CV value 0.327. It is observed that $ELRT_1$ has the narrowest width followed by Wald test. The $ELRT_2$ has the widest width compared with the rest.

5. Conclusion

The objective of the present study is to propose an empirical likelihood ratio test to test one sample CV without transforming the original data. The asymptotic null distribution of the proposed test statistics is obtained as a Chi-square distribution with 1 degree of freedom. Using simulated data and also real data sets, the proposed test is compared to Wald, bootstrap and $ELRT_2$ tests. Simulation is carried out to compute type I error rate and power. Type I error rates of all considered tests are similar whereas the power of $ELRT_1$ is higher than the other three tests when the underlying distributions are normal, lognormal, gamma and Weibull. Two data sets are used to illustrate the proposed test. $ELRT_1$ gathers more evidence against the null hypothesis than other three tests in testing the significance of CV and $ELRT_1$ produces a comparably shorter length of confidence interval from Wald, bootstrap and $ELRT_2$ tests in the estimation of confidence intervals. Thus one can prefer the proposed test ($ELRT_1$) to construct confidence interval of CV and to test the significance of CV over other three.

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