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The Exponentiated Half-Logistic-Generalized Marshall-Olkin-G Family of Distributions with Properties and Applications

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Abstract

In this article, we present a new generalized family of distributions called the Exponentiated Half-Logistic-Generalized Marshall-Olkin-G (EHL-GMO-G) distribution. Some of the useful mathematical and statistical properties for this new family of distributions such as the hazard rate function, quantile function, moments and moment generating functions, Rényi entropy, order statistics and stochastic order are derived. The method of maximum likelihood estimation is used for estimating the model parameters. Simulation experiments are conducted to illustrate consistency of the maximum likelihood estimates for model parameters and furthermore we apply one special case of this new family to real life data sets to demonstrate its flexibility in modelling various types of real life data.

Keywords: Consistency, flexibility, model parameters, maximum likelihood estimation, simulations.

1. Introduction

More recently, various statisticians developed new distributions having the ability to capture special features critical in data modelling and analysis. The widely used and most common approach is the use of the so called "generators of distributions" which improves the flexibility of the newly developed models using the existing distributions. The development of these generators uses the existing distributions with one or more parameters and well established structural properties. Some of the well-known families are the Marshall-Olkin-G by Marshall and Olkin (1997), the beta-G by Eugene et al. (2002), odd log-logistic-G by Alizadeh et al. (2017), the transmuted-G by Shaw and Buckley (2009), the gamma-G by Zografos and Balakhrishnan (2009), the Kumaraswamy-G by Cordeiro and de Castro (2011), the logistic-G by Torabi and Montazeri (2014), exponentiated generalized-G by Cordeiro et al. (2013), T-X family by Alzaatreh et al. (2013), the Weibull-G by Bourguignon et al. (2014), the exponentiated half-logistic generated family by Cordeiro et al. (2014) and the beta odd log-logistic generalized by Cordeiro et al. (2016) to mention just a few. Burr (1942) developed a system of cumulative distribution functions which have been widely extended by many researchers to generate more flexible and useful distributions. Afify et al. (2017) developed a new flexible family of distributions called the Odd exponentiated half logistic-G (OEHL-G) family of distributions using the Half-Logistic (HL) distribution as the generator. The use of these new generators of continuous distributions have attracted the attention of various authors in recent times.

Cordeiro et al. (2014) introduced another family of distributions called the exponentiated half-logistic (EHL) family from the gamma-generator by Zografos and Balakhrishnan (2009). The cumulative distribution function (cdf) of the EHL-G family of distributions is given by

$$F(x;\alpha,\beta,\underline{\xi}) = \left\{\frac{1-[1-G(x;\underline{\xi})]^{\alpha}}{1+[1-G(x;\underline{\xi})]^{\alpha}}\right\}^{\beta}, \tag{1}$$

where $\alpha,\beta > 0$ are additional shape parameters with $\underline{\xi}$ as the vector of parameters. Marshall and Olkin (see Marshall and Olkin (1997), for details) introduced the family of distributions called Marshall-Olkin-G (MO-G). Various authors have done more work on the generalization of Marshall-Olkin-G family of distributions, some of which include; Marshall–Olkin-Kumaraswamy-G family by Handique and Chakraborty (2015), Kumaraswamy Marshall–Olkin-G family by Alizadeh et al. (2015), generalized Marshall–Olkin Kumaraswamy-G family by Chakraborty and Handique (2017) and beta generalized Marshall–Olkin-G family by Handique and Chakraborty (2016) as some of the few among others. The cdf of Marshall-Olkin-G (MO-G) family of distributions is given by

$$F_{MO-G}(x;\delta,\psi) = 1-\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}, \tag{2}$$

where δ is the tilt parameter, $\overline{\delta}=1-\delta$, $\delta>0$, $\overline{G}(x;\psi)=1-G(x;\psi)$ and $G(x;\psi)$ is the baseline cdf with parameter vector ψ. Note that when $\delta=1$, we have the baseline distribution function, $F_{MO}(x;\psi)=F(x;\psi)$. The generalized-G (G-G) family has a cdf given by

$$F_{G-G}(x;\alpha,\psi) = 1-\left[\overline{G}(x;\psi)\right]^{\alpha}, \qquad x\in\mathbb{R}, \tag{3}$$

and the corresponding probability density function (pdf) is given by

$$f_{G-G}(x;\alpha,\psi) = \alpha g(x;\psi)\left[\overline{G}(x;\psi)\right]^{\alpha-1}, \qquad x\in\mathbb{R}, \tag{4}$$

where $\alpha>0$, $\overline{G}(x;\psi)=1-G(x;\psi)$ and ψ is the vector of parameters.

Inserting Eqn. (3) into (2), we obtain the Marshall-Olkin generalized-G (MOG-G) family of distributions with the cdf given by

$$F_{MOG-G}(x;\alpha,\delta,\psi) = 1-\frac{\delta\left[\overline{G}(x;\psi)\right]^{\alpha}}{1-\overline{\delta}\left[\overline{G}(x;\psi)\right]^{\alpha}}, \tag{5}$$

for $\alpha,\delta>0$ and the parameter vector ψ, (see Yousof et al. (2018), for additional details).

We can therefore write the cdf of generalized Marshall-Olkin-G (GMO-G) family of distributions as

$$F_{GMOG-G}(x;\alpha,\delta,\psi) = 1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}, \tag{6}$$

for $\alpha,\delta>0$ and the parameter vector ψ.

The main aim of this paper is to propose an alternative family of distributions using the existing generators having great advantages in data analysis with more flexible kurtosis and skewness in comparison to the baseline distribution. The proposed family can generate distributions with symmetric, left-skewed, right-skewed, reversed-J shaped with hazard rate function (hrf) plots revealing decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

This paper is organized as follows: Section 2, defines the proposed family of distributions and also presents its hazard rate and quantile functions. The linear representation of the density function and some of the important structural properties for this newly generated family of distributions, namely; moments and moment generating functions, Rényi entropy, order statistics and stochastic order are presented under Section 3. Section 5 contains the maximum likelihood estimates and some of the special cases for this family are discussed under Section 6. Simulation studies are performed under Section 7 and Section 8 presents model applications. We finally give concluding remarks under Section 9.

2. Developing the New Model & Some Properties

In this section, we present the cdf, pdf, hazard rate function (hrf) and quantile function (qf) for the exponentiated half-logistic-generalized Marshall-Olkin (EHL-GMO-G) family of distributions. Note that by using the generator given under Eqn. (1) with Eqn. (6), we can write the cdf and pdf for the new family of distributions as

$$F(x;\alpha,\delta,\beta,\psi)=\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}\right]^{\beta}, \tag{7}$$

and

$$\begin{aligned} f(x;\alpha,\delta,\beta,\psi) &= 2\alpha\beta\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}\right]^{\beta-1}\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha-1} \\ &\times \frac{\delta g(x;\psi)}{\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{2}\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}\right)^{2}}, \end{aligned} \tag{8}$$

respectively, for $\alpha,\delta,\beta>0$, $\overline{\delta}=1-\delta$ and parameter vector ψ.

2.1. The hazard rate and quantile functions

Let a random variable X follow the underlying family of distribution given by Eqn.(7). The hrf for this new family of distribution is given by

$$\begin{aligned} h_F(x;\alpha,\delta,\beta,\psi) &= 2\alpha\beta\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}\right]^{\beta-1}\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha-1} \\ &\times \frac{\delta g(x;\psi)}{\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{2}\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}\right)^{2}} \\ &\times \left(1-\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}\right]^{\beta}\right)^{-1}, \end{aligned}$$

for $\alpha,\delta,\beta>0$, $\overline{\delta}=1-\delta$ and parameter vector ψ. The corresponding qf for the EHL-GMO-G family of distributions is derived by solving the non-linear equation

$$F(x;\alpha,\delta,\beta,\psi)=\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}\right]^{\beta}=u,$$

for $0\le u\le 1$. That is,

$$G(x;\psi) = 1-\delta^{-1}\left(\left(\left(\frac{\left(1-u^{\frac{1}{\beta}}\right)}{\left(1+u^{\frac{1}{\beta}}\right)}\right)^{\frac{1}{\alpha}}\right)^{-1}+\frac{\overline{\delta}}{\delta}\right)^{-1}.$$

We can finally write the quantile function for this new family of distributions as

$$Q(u) = G^{-1}\left[1-\delta^{-1}\left(\left(\left(\frac{\left(1-u^{\frac{1}{\beta}}\right)}{\left(1+u^{\frac{1}{\beta}}\right)}\right)^{\frac{1}{\alpha}}\right)^{-1}+\frac{\overline{\delta}}{\delta}\right)^{-1}\right]. \tag{9}$$

3. Linear Representation of Density Function

This section presents series expansion for the pdf of the EHL-GMO-G family of distributions. The linear representation of the pdf further allows for the study of important mathematical and statistical properties for this new family of distributions. To derive the series expansion for the pdf of this family, we apply the generalized binomial expansion given by

$$(1-z)^{-b} = \sum_{w=0}^{\infty}\frac{\Gamma(b+w)}{\Gamma(b)w!}z^{w}, \quad \text{for } |z|<1 \text{ and } b>0.$$

The pdf of EHL-GMO-G family of distributions can be written as

$$\begin{aligned}
f(x;\alpha,\delta,\beta,\psi) &= 2\alpha\beta\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}}\right]^{\beta-1}\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha-1}\\
&\times \frac{\delta g(x;\psi)}{\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{2}\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}\right)^{2}}\\
&= 2\alpha\beta\left[1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}\right]^{\beta-1}\left(\delta\overline{G}(x;\psi)\right)^{\alpha-1}\\
&\times \left[1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha}\right]^{-(\beta+1)}\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{-(\alpha+1)}\delta g(x;\psi)\\
&= 2\alpha\beta\sum_{k,i=0}^{\infty}(-1)^{k}\binom{\beta-1}{k}\frac{\Gamma(\beta+1+i)}{\Gamma(\beta+1)i!}\left(\delta\overline{G}(x;\psi)\right)^{\alpha(k+i+1)-1}\\
&\times \left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{-\alpha(k+i+1)-1}\delta g(x;\psi)\\
&= 2\alpha\beta\sum_{k,i,q=0}^{\infty}\delta^{\alpha(k+i+1)}\overline{\delta}^{q}(-1)^{k+q}\binom{\beta-1}{k}\frac{\Gamma(\beta+1+i)}{\Gamma(\beta+1)i!}\\
&\times \binom{-\alpha(k+i+1)-1}{q}\left(\overline{G}(x;\psi)\right)^{\alpha(k+i+1)-1+q}g(x;\psi)\\
&= 2\alpha\beta\sum_{k,i,q,p=0}^{\infty}\delta^{\alpha(k+i+1)}\overline{\delta}^{q}(-1)^{k+q+p}\binom{\beta-1}{k}\frac{\Gamma(\beta+1+i)}{\Gamma(\beta+1)i!}\\
&\times \binom{-\alpha(k+i+1)-1}{q}\binom{\alpha(k+i+1)-1+q}{p}\left(G(x;\psi)\right)^{p}g(x;\psi)
\end{aligned}$$

$$
\begin{aligned}
&= 2\alpha\beta \sum_{k,i,q,p=0}^{\infty} \delta^{\alpha(k+i+1)}\overline{\delta}^{q}(-1)^{k+q+p}\binom{\beta-1}{k}\frac{\Gamma(\beta+1+i)}{\Gamma(\beta+1)i!}\left(\frac{p+1}{p+1}\right)\\
&\times \binom{-\alpha(k+i+1)-1}{q}\binom{\alpha(k+i+1)-1+q}{p}\left(G(x;\psi)\right)^{p} g(x;\psi)\\
&= \sum_{p=0}^{\infty} w_{p+1} g_{p+1}(x;\psi), \qquad (10)
\end{aligned}
$$

where $g_{p+1}(x;\psi) = (p+1)\left(G(x;\psi)\right)^{p} g(x;\psi)$ is the exponentiated-G (E-G) distribution with power parameter $p+1>0$ having parameter vector ψ and

$$
\begin{aligned}
w_{p+1} &= 2\alpha\beta \sum_{k,i,q,p}^{\infty} \delta^{\alpha(k+i+1)}\overline{\delta}^{q}(-1)^{k+q+p}\binom{\beta-1}{k}\frac{\Gamma(\beta+1+i)}{\Gamma(\beta+1)i!}\left(\frac{1}{p+1}\right)\\
&\times \binom{-\alpha(k+i+1)-1}{q}\binom{\alpha(k+i+1)-1+q}{p}. \qquad (11)
\end{aligned}
$$

The mathematical and statistical properties of the EHL-GMO-G family of distributions follows directly from those of the exponentiated-G (E-G) distribution.

3.1. Moments and generating function

Let $Y_{p+1} \sim E-G(p+1)$, then using Eqn. (10) we derive the r^{th} raw moment, μ_r' of the EHL-GMO-G family of distributions as

$$
\mu_r' = E(X^r) = \int_{-\infty}^{\infty} x^r f(x;\alpha,\delta,\beta,\psi)dx = \sum_{p=0}^{\infty} w_{p+1}E(Y_{p+1}^r),
$$

where $E(Y_{p+1}^r)$ is the r^{th} raw moment of exponentiated-G (E-G) distribuiton with parameter $p+1$ and w_{p+1} is given by Eqn. (11). The moment generating function (MGF) $M(t) = E(e^{tX})$ is given by:

$$
M(t) = \sum_{p=0}^{\infty} w_{p+1}M_{p+1}(t),
$$

where $M_{p+1}(t)$ is the MGF of Y_{p+1} and w_{p+1} is given by Eqn. (11). We present the first five moments with the standard deviation (SD or σ), coefficient of variation (CV), coefficient of skewness (CS) and coefficient of kurtosis (CK) for the exponentiated half-logistic-generalized Marshall-Olkin-exponential (EHL-GMO-E) distribution for some parameter values (see Table 1, for details). The quantiles for the selected parameter values are also given under Table 2. Figure 1 presents the 3D plots of skewness and kurtosis for the EHL-GMO-Exponential (EHL-GMO-E) distribution for some selected parameter values. The degree of decay for skewness and kurtosis measures depend on the values of the shape parameters.

It can be noted that the skewness and kurtosis measures for the new family of distributions are highly influenced by the choice of parameters.

Table 1 Moments of the EHL-GMO-E distribution for some parameter values

	(1.5,0.5,0.5,1)	(0.5,1.5,1,1.5)	(0.5,1.5,1,1.5)	(1,1.5,1,0.5)	(1.3,1.5,1.5,0.5)
$E(X)$	0.21647	0.12609	0.12609	0.07205	0.06673
$E(X^2)$	0.11029	0.08889	0.08889	0.05116	0.04793
$E(X^3)$	0.07156	0.06851	0.06851	0.03963	0.03739
$E(X^4)$	0.05229	0.05568	0.05568	0.03233	0.03065
$E(X^5)$	0.04096	0.04687	0.04687	0.02729	0.02597
SD	0.25186	0.27017	0.27017	0.21442	0.20851
CV	1.16350	2.14262	2.14262	2.97566	3.12473
CS	1.26552	1.97237	1.97237	2.97447	3.13211
CK	3.66481	5.41474	5.41474	10.60725	11.58278

Table 2 Quantiles for selected parameters of the EHL-GMO-E distribution

u	(1,0.5,0.5,1.5)	(0.5,1,0.5,2.5)	(1.5,0.5,1.5,1)	(1.5,1.5,1,1.5)	(1,1.5,1,0.5)
0.1	0.02720	0.19420	0.07901	0.31675	1.42540
0.3	0.06287	0.31982	0.17546	0.42877	1.92946
0.4	0.11623	0.46804	0.30758	0.54065	2.43295
0.5	0.19178	0.64377	0.47535	0.65897	2.96538
0.6	0.29754	0.85602	0.68276	0.79107	3.55987
0.7	0.44889	1.12268	0.94215	0.94855	4.26849
0.8	0.68110	1.48543	1.28705	1.15571	5.20078
0.9	1.10714	2.07939	1.83110	1.48861	6.69878

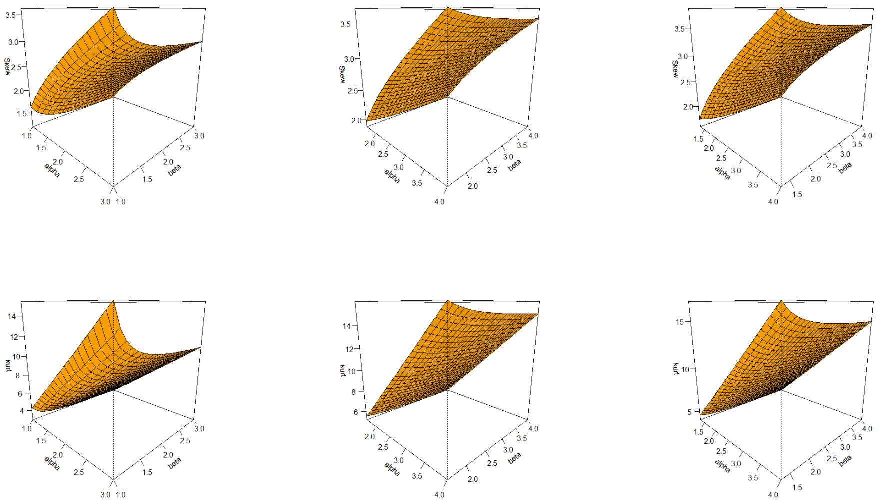

Figure 1 3D plots of the skewness (above) and kurtosis (below) measures for the EHL-GMO-E distribution for selected parameter values

4. Order Statistics

Consider a random sample $X_1, X_2, ..., X_n$ from the EHL-GMO-G family of distributions. Using the binomial series expansion

$$(1-F(x))^{n-w} = \sum_{z=0}^{n-w} \binom{n-w}{z} (-1)^z [F(x)]^z,$$

we can express the pdf of the w^{th} order statistic as

$$\begin{aligned} f_{w:n}(x) &= \frac{n! f(x)}{(w-1)!(n-w)!} [F(x)]^{w-1} [1-F(x)]^{n-w} \\ &= \frac{n! f(x)}{(w-1)!(n-w)!} \sum_{z=0}^{n-w} (-1)^z \binom{n-w}{z} [F(x)]^{z+w-1}. \end{aligned} \tag{12}$$

Using Eqns. (7) and (8), we can write $f(x)[F(x)]^{z+w-1}$ as:

$$\begin{aligned} f(x)F(x)^{z+w-1} &= 2\alpha\beta \left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\delta\overline{G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\delta\overline{G}(x;\psi)}\right)^{\alpha}} \right]^{\beta(z+w)-1} \left(\frac{\delta\overline{G}(x;\psi)}{1-\delta\overline{G}(x;\psi)}\right)^{\alpha-1} \\ &\times \frac{\delta g(x;\psi)}{\left(1-\delta\overline{G}(x;\psi)\right)^2 \left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\delta\overline{G}(x;\psi)}\right)^{\alpha}\right)^2}. \end{aligned} \tag{13}$$

Following the same procedure used to establish Eqn. (10), we get

$$f(x)F(x)^{z+w-1} = \sum_{p=0}^{\infty} w^{*}_{p+1} g_{p+1}(x;\psi), \tag{14}$$

where $g_{p+1}(x;\psi) = (p+1)\,(G(x;\psi))^p\, g(x;\psi)$ is the exponentiated-G (E-G) distribution with power parameter $p+1>0$ and parameter vector ψ, and

$$\begin{aligned} w^{*}_{p+1} &= 2\alpha\beta \sum_{k,i,q}^{\infty} \delta^{\alpha(k+i+1)} \overline{\delta}^{q} (-1)^{k+q+p} \binom{\beta(z+w)-1}{k} \frac{\Gamma(\beta+1+i)}{\Gamma(\beta+1)i!} \left(\frac{1}{p+1}\right) \\ &\times \binom{-\alpha(k+i+1)-1}{q} \binom{\alpha(k+i+1)-1+q}{p}. \end{aligned} \tag{15}$$

By substituting Eqn. (14) into (12), we obtain

$$f_{w:n}(x) = \frac{n!}{(w-1)!(n-w)!} \sum_{p=0}^{\infty} \sum_{z=0}^{n-w} (-1)^z \binom{n-w}{z} w^{*}_{p+1} g_{p+1}(x;\psi), \tag{16}$$

where $g_{p+1}(x;\psi) = (p+1)[G(x;\psi)]^p g(x;\psi)$ is the exponentiated-G (E-G) pdf with the power parameter $p+1>0$ and parameter vector ψ. The density function of the EHL-GMO-G order statistics is therefore a linear combination of E-G densities.

4.1. Rényi Entropy

Consider the two commonly known entropies used a measure of variation for the random variable X, being Shannon entropy (Shannon , 1951) and Rényi entropy (Rényi , 1961). Rényi entropy is defined to be

$$I_R(v)=\frac{1}{1-v}\log\left(\int_0^\infty [f(x;\alpha,\delta,\beta,\psi)]^v dx\right), v\neq 1, v>0.$$

Note that

$$\begin{aligned}
[f(x;\alpha,\delta,\beta,\psi)]^v &= (2\alpha\beta)^v\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}\right]^{v(\beta-1)}\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{v(\alpha-1)}\\
&\times \frac{\delta^v g^v(x;\psi)}{\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{2v}\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha\right)^{2v}}\\
&= (2\alpha\beta)^v\left[1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha\right]^{v(\beta-1)}\left(\delta\overline{G}(x;\psi)\right)^{v(\alpha-1)}\\
&\times \left[1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha\right]^{-v(\beta+1)}\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{-v(\alpha+1)}\delta^v g^v(x;\psi)\\
&= (2\alpha\beta)^v\sum_{k,i=0}^\infty(-1)^k\binom{v(\beta-1)}{k}\frac{\Gamma(v(\beta+1+i))}{\Gamma(v(\beta+1))i!}\left(\delta\overline{G}(x;\psi)\right)^{\alpha(k+i+v)-v}\\
&\times \left(1-\overline{\delta}\overline{G}(x;\psi)\right)^{-\alpha(k+i+v)-v}\delta^v g^v(x;\psi)\\
&= (2\alpha\beta)^v\sum_{k,i,q=0}^\infty \delta^{\alpha(k+i+v)}\overline{\delta}^q(-1)^{k+q}\binom{v(\beta-1)}{k}\frac{\Gamma(v(\beta+1+i))}{\Gamma(v(\beta+1))i!}\\
&\times \binom{-\alpha(k+i+v)-v}{q}\left(\overline{G}(x;\psi)\right)^{\alpha(k+i+v)-v+q}g^v(x;\psi)\\
&= (2\alpha\beta)^v\sum_{k,i,q,p=0}^\infty \delta^{\alpha(k+i+v)}\overline{\delta}^q(-1)^{k+q+p}\binom{v(\beta-1)}{k}\frac{\Gamma(v(\beta+1+i))}{\Gamma(v(\beta+1))i!}\\
&\times \binom{-\alpha(k+i+v)-v}{q}\binom{\alpha(k+i+v)-v+q}{p}\left(G(x;\psi)\right)^p g^v(x;\psi).
\end{aligned}$$

Finally, the Rényi entropy for the EHL-GMO-G family of distributions is given by

$$\begin{aligned}
I_R(v) &= \frac{1}{1-v}\log\left[(2\alpha\beta)^v\sum_{k,i,q,p}^\infty \delta^{\alpha(k+i+v)}\overline{\delta}^q(-1)^{k+q+p}\binom{v(\beta-1)}{k}\frac{\Gamma(v(\beta+1+i))}{\Gamma(v(\beta+1))i!}\right.\\
&\times \binom{-\alpha(k+i+v)-v}{q}\binom{\alpha(k+i+v)-v+q}{p}\frac{1}{\left[\frac{p}{v}+1\right]^v}\\
&\times \left.\int_0^\infty\left(\left[\frac{p}{v}+1\right](G(x;\psi))^{\frac{p}{v}}g(x;\psi)\right)^v dx\right]\\
&= \frac{1}{1-v}\log\left[\sum_{p=0}^\infty \tau_p\exp((1-v)I_{REG})\right],
\end{aligned}$$

for $v>0$, $v\neq 1$, where $I_{REG}=\frac{1}{1-v}\log\left[\int_0^\infty\left(\left[\frac{p}{v}+1\right](G(x;\psi))^{\frac{p}{v}}g(x;\psi)\right)^v dx\right]$ is the Rényi entropy of E-G distribution with power parameter $\frac{q}{v}+1$, and

$$\tau_p = (2\alpha\beta)^v \sum_{k,i,q}^{\infty} \delta^{\alpha(k+i+v)}\overline{\delta}^q(-1)^{k+q+p}\binom{v(\beta-1)}{k}\frac{\Gamma(v(\beta+1+i))}{\Gamma(v(\beta+1))i!}$$
$$\times \binom{-\alpha(k+i+v)-v}{q}\binom{\alpha(k+i+v)-v+q}{p}\frac{1}{\left[\frac{p}{v}+1\right]^v}.$$

4.2. Stochastic Ordering

In this section, we present the three commonly applied orders for the EHL-GMO-G family of distributions being; the usual stochastic order, the hazard rate order and the likelihood ratio order (Shaked and Shanthikumar , 2007).

Let X and Y be the two random variables with cdfs $F_X(t)$ and $F_Y(t)$, respectively, with the survival function given by $\overline{F}_X(t) = 1 - F_X(t)$. A random variable, say X is stochastically smaller than the random variable Y if $\overline{F}_X(t) \leq \overline{F}_Y(t)$ for all t or $F_X(t) \geq F_Y(t)$ for all t. This is denoted by $X <_{st} Y$. The hazard rate order and likelihood ratio order are stronger and are given by $X <_{hr} Y$ if $h_X(t) \geq h_Y(t)$ for all t, and $X <_{\ell_r} Y$ if $\frac{f_X(t)}{f_Y(t)}$ is decreasing in t. It holds that $X <_{\ell_r} Y \implies X <_{hr} Y \implies X <_{st} Y$.

Suppose X_1 and X_2 are two independent random variables following $EHL{-}GMO{-}G(\alpha,\delta,\beta_1,\psi)$ and $EHL-GMO-G(\alpha,\delta,\beta_2,\psi)$ distributions, then the pdfs of X_1 and X_2 are

$$f_1(x) = 2\alpha\beta_1\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}\right]^{\beta_1-1}\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha-1}$$
$$\times \frac{\delta g(x;\psi)}{\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^2\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha\right)^2}$$

and

$$f_2(x) = 2\alpha\beta_2\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}\right]^{\beta_2-1}\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^{\alpha-1}$$
$$\times \frac{\delta g(x;\psi)}{\left(1-\overline{\delta}\overline{G}(x;\psi)\right)^2\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha\right)^2}.$$

The ratio, $\frac{f_1(x)}{f_2(x)}$ takes the form

$$\frac{f_1(x)}{f_2(x)} = \frac{\beta_1}{\beta_2}\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta}\overline{G}(x;\psi)}\right)^\alpha}\right]^{\beta_1-\beta_2}, \tag{17}$$

and differentiating Eqn. (17) with respect to x, we obtain

$$\begin{aligned}\frac{d}{dx}\left(\frac{f_1(x)}{f_2(x)}\right) &= \frac{\beta_1}{\beta_2}(\beta_1-\beta_2)\left[\frac{1-\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta G}(x;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta G}(x;\psi)}\right)^{\alpha}}\right]^{\beta_1-\beta_2-1}\\ &\times \left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta G}(x;\psi)}\right)^{\alpha-1}\\ &\times \frac{\delta g(x;\psi)}{\left(1-\overline{\delta G}(x;\psi)\right)^2\left(1+\left(\frac{\delta\overline{G}(x;\psi)}{1-\overline{\delta G}(x;\psi)}\right)^{\alpha}\right)^2}.\end{aligned}$$

Consequently, if $\beta_1 \leq \beta_2$, then $\left(\frac{f_1(x)}{f_2(x)}\right)$ is decreasing and therefore, the likelihood ratio order $X_1 <_{\ell_r} X_2$ exists. As a result, the random variables X_1 and X_2 are stochastically ordered.

5. Maximum Likelihood Estimation

In this section, the maximum likelihood estimation technique is used to estimate the parameters for the EHL-GMO-G family of distributions. If we let $X \sim EHL-GMO-G(\alpha,\delta,\beta,\psi)$ and $\mathbf{\Delta}=(\alpha,\delta,\beta,\psi)^T$ be the vector of model parameters, then the log-likelihood function $\ell_n=\ell_n(\mathbf{\Delta})$ based on a random sample of size n from the EHL-GMO-G family of distributions is given by

$$\begin{aligned}\ell_n(\mathbf{\Delta}) &= n\ln(2\alpha\beta)+(\beta-1)\sum_{i=1}^{n}\ln\left[\frac{1-\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\overline{\delta G}(x_i;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\overline{\delta G}(x_i;\psi)}\right)^{\alpha}}\right]+\delta\sum_{i=1}^{n}\ln\left(g(x_i;\psi)\right)\\ &+ (\alpha-1)\sum_{i=1}^{n}\ln\left(\delta\overline{G}(x_i;\psi)\right)-(\alpha-1)\sum_{i=1}^{n}\ln\left(1-\overline{\delta G}(x_i;\psi)\right)\\ &- 2\sum_{i=1}^{n}\ln\left(1-\overline{\delta G}(x_i;\psi)\right)-2\sum_{i=1}^{n}\ln\left(1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\overline{\delta G}(x_i;\psi)}\right)^{\alpha}\right).\end{aligned}$$

We obtain the maximum likelihood estimates (MLEs) by differenting the log-likelihood function with respect to α, δ, β, ψ, and solving the nonlinear equation $(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\delta},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\psi_k})^T=\mathbf{0}$. The corresponding elements of a score vector $U(\mathbf{\Delta})=(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\delta},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\psi_k})^T$ are given under **appendix** 9. Since these system of non-linear equations has no closed form, numerical methods such as Newton-Raphson procedure can be used obtain the maximum likelihood estimates for the parameters for specified baseline cdf $G(x_i;\psi)$. The multivariate normal distribution $N_{q+3}(\underline{0},J(\hat{\mathbf{\Delta}})^{-1})$, where the mean vector $\underline{\mathbf{0}}=(0,0,0,\underline{0})^T$ and $J(\hat{\mathbf{\Delta}})^{-1}$ is the observed Fisher information matrix evaluated at $\hat{\mathbf{\Delta}}$, is critical in the construction of confidence intervals and regions for the model parameters.

6. Some Special Cases

We consider some special cases by changing the baseline distribution function $G(x;\psi)$ to flexible distributions. The parameter vector space is limited to atmost 2 component vector to avoid over parametrization and redundancy.

6.1. EHL-GMO-Exponential (EHL-GMO-E) distribution

Let an exponential distribution be the baseline distribution with paramter $a>0$ having cdf and pdf given by $G(x;a)=1-\mathrm{e}^{-ax}$ and $g(x;a)=a\mathrm{e}^{-ax}$, respectively. If we let $t_1=\left(\frac{\delta\mathrm{e}^{-ax}}{1-(1-\delta)\mathrm{e}^{-ax}}\right)$, then the cdf and pdf of EHL-GMO-E distribution are given by

$$F(x;\alpha,\delta,\beta,a)=\left[\frac{1-t_1^{\alpha}}{1+t_1^{\alpha}}\right]^{\beta}, \tag{18}$$

and

$$f(x;\alpha,\delta,\beta,a) \quad = \quad 2\alpha\beta\left[\frac{1-t_1^{\alpha}}{1+t_1^{\alpha}}\right]^{\beta-1} t_1^{\alpha-1}\frac{\delta a \mathrm{e}^{-ax}}{(1-(1-\delta)\mathrm{e}^{-ax})^2\,(1+t_1^{\alpha})^2}, \tag{19}$$

respectively, for $\alpha,\delta,\beta,a>0$. The corresponding hrf is given by

$$\begin{aligned} h_F(x;\alpha,\delta,\beta,a) &= 2\alpha\beta\left[\frac{1-t_1^{\alpha}}{1+t_1^{\alpha}}\right]^{\beta-1} t_1^{\alpha-1}\frac{\delta a \mathrm{e}^{-ax}}{(1-(1-\delta)\mathrm{e}^{-ax})^2\,(1+t_1^{\alpha})^2} \\ &\times \left(1-\left[\frac{1-t_1^{\alpha}}{1+t_1^{\alpha}}\right]^{\beta}\right)^{-1}, \end{aligned}$$

for $\alpha,\delta,\beta,a>0$.

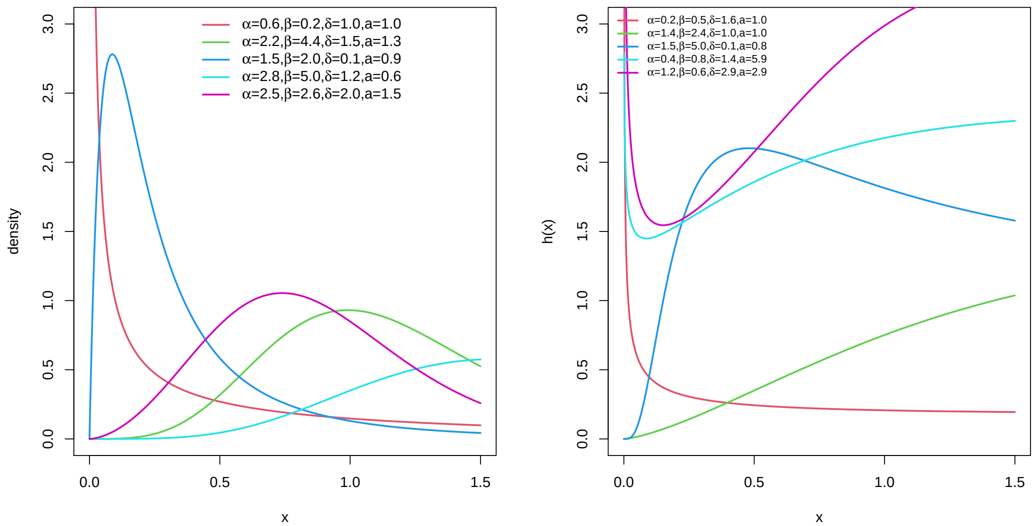

Figure 2 Plots of the pdf and hrf for the EHL-GMO-E distribution

Figure 2 demonstrates the flexibility of the EHL-GMO-E distribution for selected parameter values. The pdfs of the EHL-GMO-E distribution can adopt various shapes that include reverse-J, uni-modal, left or right skewed shapes. Furthermore, the EHL-GMO-E distribution hrf plots reveal decreasing, increasing, bathtub and upside down bathtub shapes.

6.2. EHL-GMO-Log-Logistic (EHL-GMO-LLoG) distribution

Consider the log-logistic distribution as the baseline distribution with parameter $c>0$ having cdf and pdf $G(x;c)=1-(1+x^c)^{-1}$ and $g(x;c)=cx^{c-1}(1+x^c)^{-2}$, respectively. Let $t_2=\left(\frac{\delta(1+x^c)^{-1}}{1-(1-\delta)(1+x^c)^{-1}}\right)$, then the cdf, pdf and hrf of EHL-GMO-LLoG distribution are given by

$$F(x;\alpha,\delta,\beta,c)=\left[\frac{1-t_2^{\alpha}}{1+t_2^{\alpha}}\right]^{\beta}, \tag{20}$$

$$f(x;\alpha,\delta,\beta,c) \quad = \quad 2\alpha\beta\left[\frac{1-t_2^{\alpha}}{1+t_2^{\alpha}}\right]^{\beta-1} t_2^{\alpha-1}\frac{\delta c x^{c-1}(1+x^c)^{-2}}{\left(1-(1-\delta)(1+x^c)^{-1}\right)^2(1+t_2^{\alpha})^2} \tag{21}$$

and

$$\begin{aligned} h_F(x;\alpha,\delta,\beta,c) \quad &= \quad 2\alpha\beta\left[\frac{1-t_2^{\alpha}}{1+t_2^{\alpha}}\right]^{\beta-1} t_2^{\alpha-1}\frac{\delta c x^{c-1}(1+x^c)^{-2}}{\left(1-(1-\delta)(1+x^c)^{-1}\right)^2(1+t_2^{\alpha})^2} \\ &\times \left(1-\left[\frac{1-t_2^{\alpha}}{1+t_2^{\alpha}}\right]^{\beta}\right)^{-1}, \end{aligned}$$

respectively, for $\alpha,\delta,\beta,c>0$.

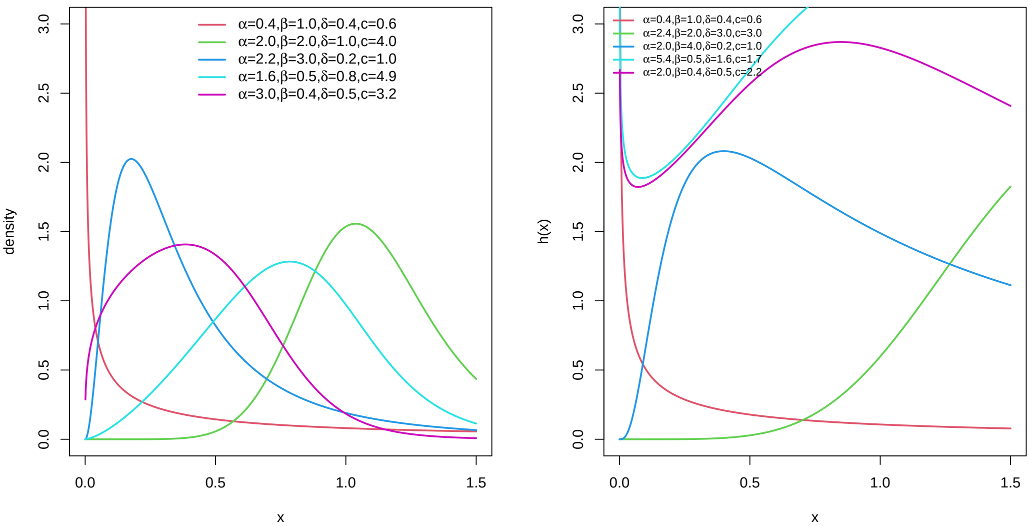

Figure 3 Plots of the pdf and hrf for the EHL-GMO-LLoG distribution

Figure 3 illustrates the efficacy and flexibility of the EHL-GMO-LLoG distribution for selected parameter values. The pdfs of the EHL-GMO-LLoG distribution exhibits various shapes that include reverse-J, uni-modal, left or right skewed shapes. Additionally, the EHL-GMO-LLoG distribution hrf plots give decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

6.3. EHL-GMO-Kumaraswamy (EHL-GMO-K) distribution

Let the Kumaraswamy distribution be the baseline distribution with paramters $a,\lambda>0$ having cdf and pdf $G(x;a,\lambda)=1-(1-x^a)^{\lambda}, 0\leq x\leq 1$, and $g(x;a,\lambda)=a\lambda x^{a-1}(1-x^a)^{\lambda-1}$, respectively. If we let $t_3 = \left(\frac{\delta(1-x^a)^\lambda}{1-(1-\delta)(1-x^a)^\lambda}\right)$, then the cdf and pdf of EHL-GMO-K distribution are given by

$$F(x;\alpha,\delta,\beta,a,\lambda) = \left[\frac{1-t_3^\alpha}{1+t_3^\alpha}\right]^\beta, \tag{22}$$

and

$$f(x;\alpha,\delta,\beta,a,\lambda) = 2\alpha\beta\left[\frac{1-t_3^\alpha}{1+t_3^\alpha}\right]^{\beta-1} t_3^{\alpha-1}\frac{\delta a\lambda x^{a-1}(1-x^a)^{\lambda-1}}{\left(1-(1-\delta)(1-x^a)^\lambda\right)^2(1+t_3^\alpha)^2}, \tag{23}$$

respectively, for $\alpha,\delta,\beta,a,\lambda>0$, and the corresponding hrf is given by

$$\begin{aligned} h_F(x;\alpha,\delta,\beta,a,\lambda) &= 2\alpha\beta\left[\frac{1-t_3^\alpha}{1+t_3^\alpha}\right]^{\beta-1} t_3^{\alpha-1}\frac{\delta a\lambda x^{a-1}(1-x^a)^{\lambda-1}}{\left(1-(1-\delta)(1-x^a)^\lambda\right)^2(1+t_3^\alpha)^2} \\ &\times\left(1-\left[\frac{1-t_3^\alpha}{1+t_3^\alpha}\right]^\beta\right)^{-1}, \end{aligned}$$

for $\alpha,\delta,\beta,a,\lambda>0$.

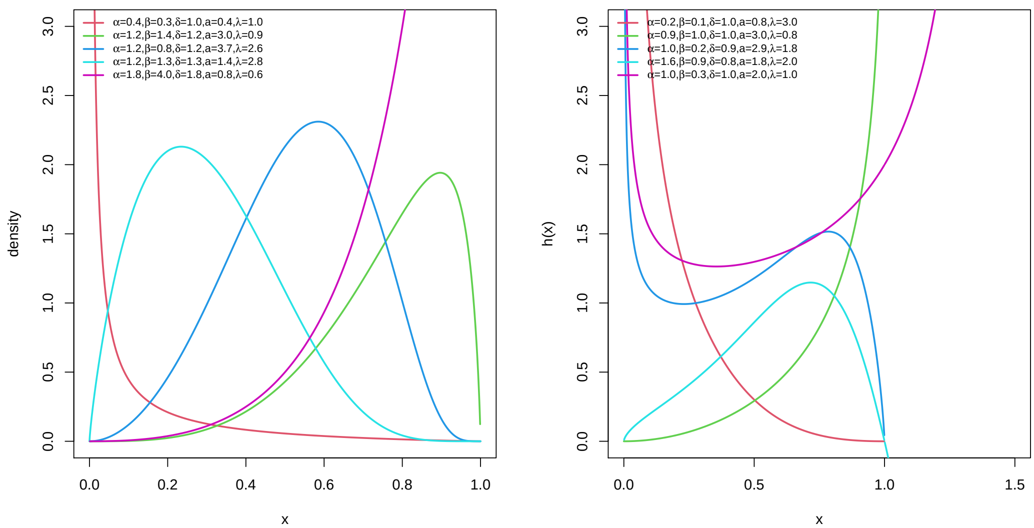

Figure 4 Plots of the pdf and hrf for the EHL-GMO-K distribution

From Figure 4, the flexible nature of the EHL-GMO-K distribution for selected parameter values can be noted clearly. The pdfs of the EHL-GMO-K distribution can take different shapes namely; reverse-J, uni-modal, left or right skewed shapes. Also, the hrf plots for the EHL-GMO-K distribution yields decreasing, increasing, bathtub, upside down bathtub and bathtub followed by upside down bathtub shapes.

6.4. EHL-GMO-Weibull (EHL-GMO-W) distribution

Let the one parameter Weibull distribution be the baseline distribution with pdf and cdf given by $g(x;\lambda) = \lambda x^{\lambda-1}\exp(-x^{\lambda})$ and $G(x;\lambda) = 1-\exp(-x^{\lambda})$, for $\lambda > 0$, respectively. Let $t_4 = \left(\frac{\delta \mathrm{e}^{-x^{\lambda}}}{1-(1-\delta)\mathrm{e}^{-x^{\lambda}}}\right)$, then the cdf, pdf and hrf of the EHL-GMO-W distribution are given by

$$F(x;\alpha,\delta,\beta,\lambda) = \left[\frac{1-t_4^{\alpha}}{1+t_4^{\alpha}}\right]^{\beta}, \tag{24}$$

$$f(x;\alpha,\delta,\beta,\lambda) = 2\alpha\beta\left[\frac{1-t_4^{\alpha}}{1+t_4^{\alpha}}\right]^{\beta-1} t_4^{\alpha-1}\frac{\delta\lambda x^{\lambda-1}\exp(-x^{\lambda})}{\left(1-(1-\delta)\mathrm{e}^{-x^{\lambda}}\right)^2(1+t_4^{\alpha})^2} \tag{25}$$

and

$$\begin{aligned} h_F(x;\alpha,\delta,\beta,\lambda) &= 2\alpha\beta\left[\frac{1-t_4^{\alpha}}{1+t_4^{\alpha}}\right]^{\beta-1} t_4^{\alpha-1}\frac{\delta\lambda x^{\lambda-1}\exp(-x^{\lambda})}{\left(1-(1-\delta)\mathrm{e}^{-x^{\lambda}}\right)^2(1+t_4^{\alpha})^2} \\ &\times \left(1-\left[\frac{1-t_4^{\alpha}}{1+t_4^{\alpha}}\right]^{\beta}\right)^{-1}, \end{aligned}$$

respectively, for $\alpha,\delta,\beta,\lambda > 0$.

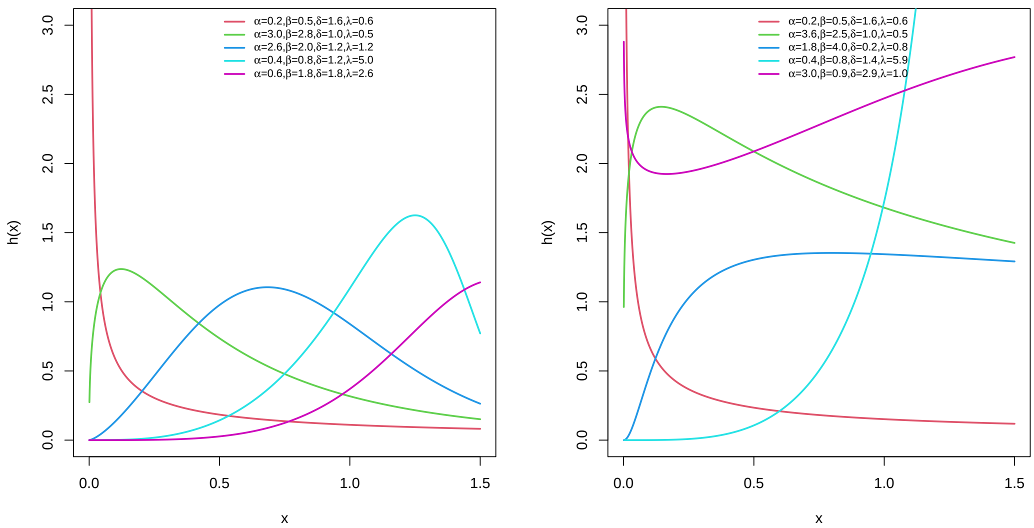

Figure 5 Plots of the pdf and hrf for the EHL-GMO-W distribution

Figure 5 depicts the efficacy of the EHL-GMO-W distribution for selected parameter values. The pdfs of the EHL-GMO-W distribution easily adopt diiferent shapes which are; reverse-J, uni-modal, left or right skewed shapes. The EHL-GMO-W distribution hrf plots in addition show decreasing, increasing, bathtub and upside down bathtub shapes.

7. Monte Carlo Simulations

In this section, we conduct a simulation study to evaluate consistency of the maximum likelihood estimators for the EHL-GMO-E distribution using different parameter values. The simulation study is repeated for N = 1000 times with sample size n = 25, 50, 100, 200, 400, 800 and 1000. The simulation results (mean of the MLEs, average bias and root mean square error (RMSE)) are presented under Tables 3 and 4. The results shows that as the sample size n increases, the mean estimates of the parameters tend to be closer to the true parameter values, with Average Bias and RMSEs converging towards zero. The Average Bias and RMSEs are given by:

$$\text{Bias}(\hat{\theta}) = \frac{\sum_{i=1}^{n}\hat{\theta}_i}{n} - \theta \quad \text{and} \quad \text{RMSE}(\hat{\theta}) = \sqrt{\frac{\sum_{i=1}^{n}(\hat{\theta}_i - \theta)^2}{n}},$$

respectively.

Table 3 Monte Carlo simulation results for EHL-GMO-E distribution; mean, average bias and RMSE

Parameter	n	Set I: α=0.2, δ=0.9 , β=0.2 , a=0.1			Set II: α=0.2, δ=0.9, β=0.5, a=0.1		
		Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.66625	5.23912	0.46625	0.44175	0.48371	0.24175
	50	0.29894	0.24941	0.09894	0.40053	0.40940	0.20054
	100	0.27965	0.23442	0.07965	0.36965	0.39124	0.16965
	200	0.23655	0.14971	0.03655	0.27968	0.24713	0.07968
	400	0.22035	0.08213	0.02035	0.24927	0.17159	0.04927
	800	0.21167	0.05458	0.01167	0.23074	0.13250	0.03074
	1000	0.20774	0.04394	0.00774	0.22152	0.11644	0.02152
δ	25	1.74884	1.35600	0.84884	1.85934	1.97100	0.95934
	50	1.45707	1.01096	0.55707	1.46470	1.24845	0.56470
	100	1.17498	0.66699	0.27498	1.22148	0.85211	0.32148
	200	1.10518	0.51033	0.20518	1.07599	0.51057	0.17599
	400	1.04635	0.40970	0.14635	0.99248	0.31528	0.09248
	800	0.96037	0.23090	0.06037	0.93752	0.16406	0.03752
	1000	0.95550	0.20945	0.05550	0.94218	0.17327	0.04218
β	25	5.31205	9.36909	5.11205	2.26455	10.70787	1.76455
	50	0.74386	5.00047	0.54386	2.17964	15.04179	1.67964
	100	0.31318	0.57434	0.11318	0.86686	2.76489	0.36686
	200	0.23054	0.23680	0.03054	0.54116	0.65306	0.04116
	400	0.22952	0.21147	0.02952	0.53115	0.50329	0.03115
	800	0.22064	0.14027	0.02064	0.51896	0.30809	0.01896
	1000	0.21204	0.11559	0.01204	0.49852	0.23610	-0.00147
a	25	0.13541	0.17772	0.03541	0.14360	0.21317	0.04360
	50	0.12945	0.15372	0.02945	0.13654	0.23489	0.03654
	100	0.12891	0.14580	0.02891	0.11799	0.14876	0.01799
	200	0.12062	0.10274	0.02062	0.13712	0.13829	0.03712
	400	0.10623	0.05578	0.00623	0.12589	0.11360	0.02589
	800	0.10152	0.02906	0.00152	0.11970	0.08388	0.01970
	1000	0.10166	0.02453	0.00166	0.11786	0.07927	0.01786

Table 4 Monte Carlo Simulation Results for EHL-GMO-E Distribution; Mean, Average Bias and RMSE

Parameter	n	Set III: α=0.2, δ=0.5 , β=0.2 , a=0.1			Set IV: α=0.2, δ=0.5 , β=0.5 , a=0.1		
		Mean	RMSE	Bias	Mean	RMSE	Bias
α	25	0.29669	0.25762	0.09669	0.36223	0.43629	0.16223
	50	0.28085	0.22814	0.08085	0.32617	0.34017	0.12617
	100	0.27038	0.20741	0.07038	0.31511	0.31597	0.11511
	200	0.23797	0.14967	0.03797	0.30242	0.54929	0.10242
	400	0.21594	0.07806	0.01594	0.24079	0.17368	0.04079
	800	0.20632	0.04605	0.00632	0.22525	0.15190	0.02525
	1000	0.20935	0.03910	0.00935	0.23022	0.12253	0.03022
δ	25	0.69097	0.45776	0.19097	0.69165	0.49501	0.19165
	50	0.60502	0.26614	0.10502	0.61064	0.33128	0.11064
	100	0.57487	0.18832	0.07487	0.58058	0.21031	0.08058
	200	0.56144	0.147247	0.06144	0.55771	0.14698	0.05771
	400	0.54455	0.09381	0.04455	0.54418	0.09736	0.04418
	800	0.52862	0.05939	0.02862	0.52777	0.05730	0.02777
	1000	0.52901	0.05705	0.02901	0.52512	0.05167	0.02512
β	25	8.48860	8.41746	8.28860	5.38985	5.42071	4.88985
	50	0.35967	1.05267	0.15967	1.29991	3.35774	0.79991
	100	0.22805	0.27960	0.02805	0.79437	2.06739	0.29437
	200	0.19922	0.14511	-0.00077	0.54331	1.23260	0.04331
	400	0.18764	0.12563	-0.01235	0.44746	0.29238	-0.05253
	800	0.18106	0.07412	-0.01893	0.44194	0.16817	-0.05805
	1000	0.18212	0.07145	-0.01787	0.45742	0.17024	-0.04257
a	25	0.14926	0.18698	0.04926	0.13954	0.16824	0.03954
	50	0.13620	0.18577	0.03620	0.15183	0.18996	0.05183
	100	0.11534	0.10377	0.01534	0.12788	0.12769	0.02788
	200	0.11234	0.07653	0.01234	0.13412	0.13227	0.03412
	400	0.10667	0.07117	0.00667	0.12245	0.09951	0.02244
	800	0.10320	0.02525	0.00320	0.11956	0.08799	0.01956
	1000	0.10019	0.02114	0.00019	0.10895	0.06019	0.00895

8. Applications

This section presents some model fittings and applications using two real life data sets. We compare the performance of the applied special case distribution with other existing equal parameter non-nested models namely; the New Modified Weibull (NMW) distribution introduced by Doostmoradi et al. (2014), the Exponentiated Modified Weibull (EMW) distribution by Elbatal (2011), the Weibull-Lomax (WLx) distribution by Tahir et al. (2015), the Gamma Exponentiated Lindley log-logistic (GELLLoG) distribution by Oluyede et al. (2020) and the Odd log-logistic exponential Weibull (OLLEW) distribution by Afify et al. (2017). We used several goodness-of-fit statistics to compare the model performances and these include; -2 log-likelihood $(-2\ln(L))$, Akaike Information Criterion $(AIC = 2p - 2\ln(L))$, Bayesian Information Criterion $(BIC = p\ln(n) - 2\ln(L))$ and Consistent Akaike Information Criterion $\left(AICC = AIC + 2\frac{p(p+1)}{n-p-1}\right)$, where $L = L(\hat{\boldsymbol{\Delta}})$ is the value of the likelihood function evaluated at the parameter estimates, n is the number of observations, and p is the number of estimated parameters. We also obtained results on the Crameŕ-von Mises (W^*) and Anderson-Darling Statistics (A^*) (Chen and Balakrishnan , 1995), as well as Kolmogorov-Smirnov (K-S) statistic and the associated p-values. The pdfs of the non-nested models used in applicatiions are;

Gamma Exponentiated Lindley log-logistic (GELLLoG) distribution

$$
\begin{aligned}
f(x;\lambda,c,\alpha,\delta) &= \frac{1}{\Gamma(\delta)}\left[-\log\left(1-\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha}\right)\right]^{\delta-1}\\
&\times\ \alpha\left[1-\frac{1+\lambda+\lambda x}{1+\lambda}\frac{e^{-\lambda x}}{(1+x^c)}\right]^{\alpha-1}\\
&\times\ \frac{(1+x^c)^{-1}}{1+\lambda}e^{-\lambda x}\left[\lambda^2(1+x)+\frac{(1+\lambda+\lambda x)cx^{c-1}}{1+x^c}\right],
\end{aligned}
$$

for $\lambda, c, \alpha, \delta > 0$ and $x > 0$.

Exponentiated Modified Weibull (EMW) distribution

$$
f(x;\gamma,\delta,\lambda,\theta)=\gamma\left[\delta+\lambda\theta^{\alpha}x^{\lambda-1}\right]e^{-(\delta x+(\theta x^{\lambda})}\left[1-e^{-(\delta x+(\theta x)^{\lambda})}\right]^{\delta-1},
$$

for $\gamma, \delta, \lambda, \theta > 0$ and $x > 0$.

New Modified Weibull (NMW) distribution

$$
f(x;\alpha,\gamma,\lambda,\beta)=\left(\alpha\gamma x^{\gamma-1}e^{\alpha x^{\gamma}}+\lambda\beta x^{\lambda-1}e^{-\beta x^{\lambda}}\right)e^{-e^{\alpha x^{\gamma}}+e^{-\beta x^{\lambda}}},
$$

for $\alpha, \gamma, \lambda, \beta > 0$ and $x > 0$.

Weibull-Lomax (WLx) distribution

$$
\begin{aligned}
f(x;a,b,\alpha,\beta) &= \frac{ab\alpha}{\beta}\left[1+\left(\frac{x}{\beta}\right)\right]^{b\alpha-1}\left\{1-\left[1+\left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b-1}\\
&\times\ \exp\left\{-a\left\{1+\left(\frac{x}{\beta}\right)^{\alpha}-1\right\}^{b}\right\},
\end{aligned}
$$

for $a, b, \alpha, \beta > 0$ and $x > 0$

and

Odd log-logistic exponential Weibull (OLLEW) distribution

$$
\begin{aligned}
f(x;\alpha,\beta,\gamma,\theta) &= \frac{\theta\beta\gamma x^{\beta-1}\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\left[1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right]^{\gamma\theta-1}}{\alpha^{\beta}\left\{\left[1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right]^{\gamma\theta}+\left\{1-\left[1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right]\right\}^{\gamma}\right\}^{2}}\\
&\times\ \left\{1-\left[1-\exp\left\{-\left(\frac{x}{\alpha}\right)^{\beta}\right\}\right]\right\}^{\theta-1},
\end{aligned}
$$

for $\alpha, \beta, \gamma, \theta > 0$ and $x > 0$.

8.1. Aircraft windshield failure times data

The first real data set represent the failure times of 84 aircraft windshields. This data was also analyzed by El-Bassiouny et al. (2015). The data points are as follows:

```
3.70, 2.74, 2.73, 2.50, 3.60, 3.11, 3.27, 2.87, 1.47, 3.11, 4.42, 2.41,
3.19, 3.22, 1.69, 3.28, 3.09, 1.87, 3.15, 4.90, 3.75, 2.43, 2.95, 2.97,
3.39, 2.96, 2.53, 2.67, 2.93, 3.22, 3.39, 2.81, 4.20, 3.33, 2.55, 3.31,
3.31, 2.85, 2.56, 3.56, 3.15, 2.35, 2.55, 2.59, 2.38, 2.81, 2.77, 2.17,
2.83, 1.92, 1.41, 3.68, 2.97, 1.36, 0.98, 2.76, 4.91, 3.68, 1.84, 1.59,
3.19, 1.57, 0.81, 5.56, 1.73, 1.59, 2.00, 1.22, 1.12, 1.71, 2.17, 1.17,
5.08, 2.48, 1.18, 3.51, 2.17, 1.69, 1.25, 4.38, 1.84, 0.39, 3.68, 2.48,
0.85, 1.61, 2.79, 4.70, 2.03, 1.80, 1.57, 1.08, 2.03, 1.61, 2.12, 1.89,
2.88, 2.82, 2.05, 3.65
```

The parameter estimates for the EHL-GMO-E distribution (with standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (KS) and its p-value are given in Table 5. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 6.

Table 5 The models estimates and goodness-of-fit statistics for failure times data

	Estimates				Statistics							
Model	α	β	δ	a	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
EHL-GMO-E	42.796	2.660	23.550	0.299	282.5	290.5	290.9	300.9	0.0673	0.3970	0.0636	0.81
	(6.032)	(0.734)	(12.510)	(0.111)								
	a	b	α	β								
WLx	0.125	1.296	9.56×10^4	1.64×10^5	293.8	301.8	302.3	312.3	0.1350	1.0808	0.0910	0.34
	(0.027)	(0.093)	(7.73×10^{-7})	(4.51×10^{-7})								
	γ	δ	λ	θ								
EMW	7.788	1.013	1.701	1.11×10^{-4}	292.4	300.4	300.8	310.8	0.2267	1.1860	0.1078	0.20
	(1.496)	(0.087)	(6.49×10^{-16})	(1.10×10^{-10})								
	α	γ	λ	β								
NMW	0.005	3.314	3.477	0.023	318.2	326.2	326.2	336.6	0.1766	0.8980	0.1771	0.004
	(0.006)	(0.843)	(0.451)	(0.012)								
	λ	c	α	δ								
GELLLoG	3.664	0.008	0.047	12.287	283.1	291.1	291.6	301.6	0.9634	5.4083	1.0100	2.2×10^{-16}
	(1.357)	(0.156)	(0.250)	(8.801)								
	α	β	γ	θ								
OLLEW	0.298	0.101	2.022	20.179	291.9	299.9	300.3	310.3	0.2303	1.1921	0.0904	0.39
	(0.432)	(0.012)	(0.373)	(0.003)								

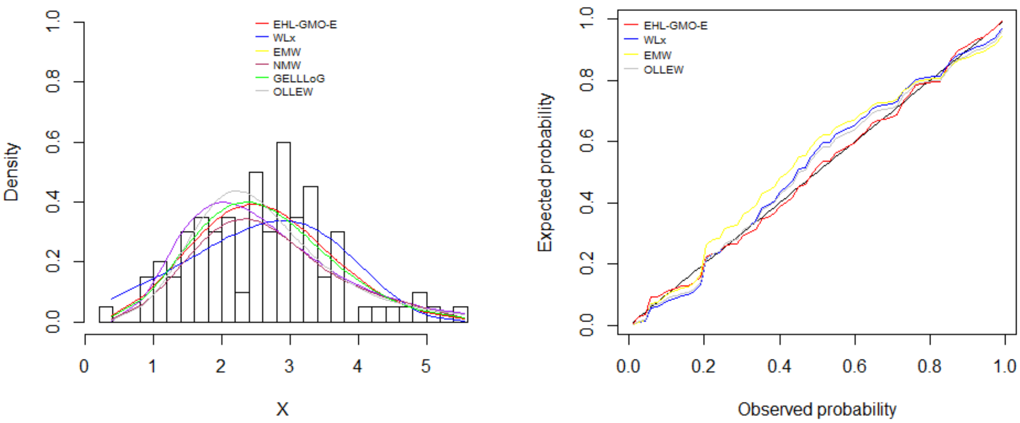

Figure 6 Fitted Densities and Probability Plots for the failure times data

From Figure 6, it is evident that the proposed model provides better fits compared to other equal parameter models used for comparison. The fitted density of the EHL-GMO-E distribution remain closer to the sample histogram and similarly the fitted probability plot remains closer to the diagonal line. This shows that the model is consistent in providing a better fit for both skewed and symmetric data than other competing models used for comparison. From the results presented under Table 5, it is also shown that the EHL-GMO-E distribution provides a better fit than the competing models since it has the smallest values for the goodness-of-fit statistics: W^*, A^*, $K - S$ with a higher P-value compared to other models.

8.2. COVID-19 data

The second data set relates to the number of deaths due to COVID-19 in China. This data is reported in (https://www.worldometers.info/coronavirus/country/china/) which projects daily deaths due to COVID-19 in China from 23 January to 28 March. The data set is also analyzed by Sindhu et al. (2020) and is given as:

```
8, 16, 15, 24, 26, 26, 38, 43, 46, 45, 57, 64, 65, 73, 73, 86, 89, 97,
108, 97, 146, 121, 143, 142, 105, 98, 136, 114, 118, 109, 97, 150,
71, 52, 29, 44, 47, 35, 42, 31, 38, 31, 30, 28, 27, 22, 17, 22, 11, 7,
13, 10, 14, 13, 11, 8, 3, 7, 6, 9, 7, 4, 6, 5, 3, 5
```

Estimates of the parameters for the EHL-GMO-E distribution (with standard error in parentheses), AIC, AICC, BIC, and the goodness-of-fit statistics W*, A*, Kolmogorov-Smirnov (K-S) and its p-value are given in Table 6. Plots of the fitted densities and the histogram, observed probability vs predicted probability are given in Figure 7.

Table 6 The models estimates and goodness-of-fit statistics for Covid-19 data

	Estimates				Statistics							
Model	α	β	δ	a	$-2\log L$	AIC	$AICC$	BIC	W^*	A^*	$K-S$	P-value
EHL-GMO-E	1.33×10^{-2}	0.6320	7.09×10^{2}	1.6853	639.81	647.81	648.47	656.57	0.0489	0.3903	0.0820	0.77
	(5.58×10^{-2})	(0.1479)	(3.65×10^{-4})	(0.7843)								
	a	b	α	β								
WLx	0.0381	2.3844	0.3539	0.5477	645.87	653.87	654.52	662.62	0.0873	0.6799	0.0912	0.64
	(0.1392)	(2.4473)	(0.5006)	(1.8881)								
	γ	δ	λ	θ								
EMW	6.0897	0.0305	0.0011	1.0128	664.81	672.81	673.47	681.57	0.2299	1.5623	0.1166	0.33
	(0.8440)	(0.0033)	(0.0065)	(0.0024)								
	α	γ	λ	β								
NMW	2.46×10^{-2}	0.8245	1.96×10^{-2}	0.0112	646.78	654.78	655.43	663.54	0.1348	0.9143	0.0952	0.59
	(1.03×10^{-2})	(8.81×10^{-2})	(2.4420)	(3.33×10^{-10})								
	λ	c	α	δ								
GELLLoG	0.0344	1.2651	0.8159	5.4240	640.91	648.91	649.57	657.67	0.4143	2.4947	0.9794	2.2×10^{-16}
	(0.0131)	(0.8121)	(2.3723)	(5.8394)								
	α	β	γ	θ								
OLLEW	5.5700	0.9713	33.4296	0.1526	643.36	651.36	652.02	660.12	0.0705	0.6942	0.1618	0.06
	(2.4157)	(0.2936)	(0.2209)	(0.0846)								

Figure 7 shows that the model fits the data well compared to other competing non-nested models. The goodness-of-fit statistics W^*, A^* and K-S presented under table 6 demonstrates that the EHL-GMO-E distribution has better fits than other equal parameter models. Furthermore, the values of AIC and BIC also show that the EHL-GMO-E distribution is better than other non-nested models used in applications.

9. Concluding Remarks

In this study, we developed and studied in detail a new generalized family of distributions called exponentiated half-logistic-generalized Marshall-Olkin-G (EHL-GMO-G). Some of the structural properties of this new family of distributions have been extensively derived and studied. We discussed some of its special cases and applied EHL-GMO-Exponential to two real life data examples with other existing non-nested models for comparison. From the application results, the model provides better fits and performs better than the other non-nested models in fitting real life data. Furthermore, based on the simulation results, the consistency of the model estimators is indicated by

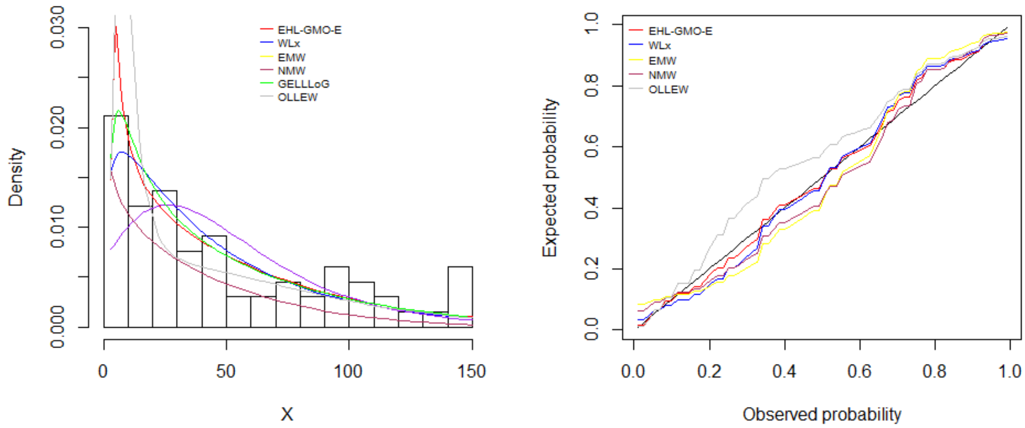

Figure 7 Fitted Densities and Probability Plots for the Covid-19 data

bias and RMSE coverging towards zero as the sample size increases which conforms to the theoretical convergence properties of estimators. The newly generated family of distributions can fit various forms of real life datasets as evident from its ability to adopt various shapes based on its pdf and hrf plots.

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Appendix

A. Elements of the score vector

The elements of a score vector $U(\boldsymbol{\Delta})=(\frac{\partial\ell_n}{\partial\alpha},\frac{\partial\ell_n}{\partial\delta},\frac{\partial\ell_n}{\partial\beta},\frac{\partial\ell_n}{\partial\psi_k})^T=\mathbf{0}$ are given by

$$\begin{aligned}\frac{\partial\ell_n}{\partial\alpha} &= \frac{n}{\alpha}-(\beta-1)\sum_{i=1}^{n}\frac{2\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\ln\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)}{\left[\frac{1-\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}\right]\left[1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right]^2}\\
&+ \sum_{i=1}^{n}\ln\left(\delta\overline{G}(x_i;\psi)\right)-\sum_{i=1}^{n}\ln\left(1-\delta\overline{G}(x_i;\psi)\right)\\
&- 2\sum_{i=1}^{n}\frac{\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\ln\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)}{\left(1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right)},\end{aligned}$$

$$\begin{aligned}\frac{\partial\ell_n}{\partial\delta} &= -\alpha(\beta-1)\sum_{i=1}^{n}\frac{2\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha-1}}{\left[\frac{1-\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}\right]\left[1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right]^2}+\sum_{i=1}^{n}\ln\left(g(x_i;\psi)\right)\\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{\overline{G}(x_i;\psi)}{\delta\overline{G}(x_i;\psi)}-(\alpha-1)\sum_{i=1}^{n}\frac{\overline{G}(x_i;\psi)}{\left(1-\delta\overline{G}(x_i;\psi)\right)}\\
&- 2\sum_{i=1}^{n}\frac{\overline{G}(x_i;\psi)}{\left(1-\delta\overline{G}(x_i;\psi)\right)}-2\sum_{i=1}^{n}\frac{\overline{G}(x_i;\psi)\left[\left(1-\delta\overline{G}(x_i;\psi)\right)-\delta\overline{G}(x_i;\psi)\right]}{\left(1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right)\left(1-\delta\overline{G}(x_i;\psi)\right)^2},\end{aligned}$$

$$\frac{\partial\ell_n}{\partial\beta} = \frac{n}{\beta}+\sum_{i=1}^{n}\ln\left[\frac{1-\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}\right],$$

$$\begin{aligned}\frac{\partial\ell_n}{\partial\psi_k} &= (\beta-1)\sum_{i=1}^{n}\frac{\delta\frac{\partial\overline{G}(x_i;\psi)}{\partial\psi_k}\left[1-\delta\overline{G}(x_i;\psi)\right]+\overline{\delta}\frac{\partial\overline{G}(x_i;\psi)}{\partial\psi_k}\left[\delta\overline{G}(x_i;\psi)\right]}{\left[\frac{1-\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}{1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}}\right]\left(1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right)^2}\\
&\times \frac{\left[\left(1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right)-\left(1-\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right)\right]}{\left(1-\delta\overline{G}(x_i;\psi)\right)^2}+\delta\sum_{i=1}^{n}\frac{\frac{\partial g(x_i;\psi)}{\partial\psi_k}}{\left(g(x_i;\psi)\right)}\\
&+ (\alpha-1)\sum_{i=1}^{n}\frac{\frac{\partial\overline{G}(x_i;\psi)}{\partial\psi_k}}{\left(\overline{G}(x_i;\psi)\right)}+(\alpha-1)\sum_{i=1}^{n}\frac{\overline{\delta}\frac{\partial\overline{G}(x_i;\psi)}{\partial\psi_k}}{\left(1-\delta\overline{G}(x_i;\psi)\right)}\\
&+ 2\sum_{i=1}^{n}\frac{\frac{\overline{\delta}\partial\overline{G}(x_i;\psi)}{\partial\psi_k}}{\left(1-\delta\overline{G}(x_i;\psi)\right)}-2\alpha\sum_{i=1}^{n}\frac{\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha-1}}{\left(1+\left(\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right)^{\alpha}\right)}\\
&\times \frac{\left[\left(\delta\frac{\partial\overline{G}(x_i;\psi)}{\partial\psi_k}\right)\left[1-\delta\overline{G}(x_i;\psi)\right]-\left[\overline{\delta}\frac{\partial\overline{G}(x_i;\psi)}{\partial\psi_k}\right]\left[\frac{\delta\overline{G}(x_i;\psi)}{1-\delta\overline{G}(x_i;\psi)}\right]\right]}{\left(1-\delta\overline{G}(x_i;\psi)\right)^2}.\end{aligned}$$

B. R-Code for Applications

```
main_LL<- function(alpha,beta,delta,a) {
 -sum(log(2*alpha*beta*(((1-((delta*exp(-a*x) )/
 (1-(1-delta)*exp(-a*x) ))^alpha)/(1+((delta*
 exp(-a*x))/(1-(1-delta)*exp(-a*x)))^alpha))^
 (beta-1))*(((delta*exp(-a*x))/(1-(1-delta)*exp(-a*x)))
 ^(alpha-1))*((1+((delta*exp(-a*x) )
 /(1-(1-delta)*exp(-a*x) ))^alpha)^(-2))*((1-(1-delta)*
 exp(-a*x) )^(-2))*(delta*  a*exp(-a*x))))
}
main.result<-mle2(main_LL,hessian = NULL,start=list
(alpha=2.18118,beta=0.0000090,delta=0.0001,a=0.091),
optimizer="nlminb",lower=0)summary(main.result)

###############################################Weibull Lomax

WL<- function(a,b,alpha,beta) {
 -sum(log(((a*b*alpha)/beta)*((1+(x/beta))^(alpha*b-1))*
 ((1-(1+(x/beta))^(-alpha))^(b-1))*exp((-a)*(((1+(x/beta))
 ^alpha)-1)^b)))}
WL.result<-mle2(WL,hessian = NULL,start=list(a=0.61519,
b=0.000000002756,alpha=0.2771,beta=0.96774),
optimizer="nlminb",lower=0)summary(WL.result)

###############################Exponentiated Modifed Weibull

BW<- function(gamma,delta,lambda,theta) {
 -sum(log(gamma*(delta+lambda*(theta^lambda)*(x^lambda-1))
 *exp(-(delta*x+(theta*x)^lambda))*(1-exp(-(delta*x+(theta*x)^
 lambda)))^(gamma-1)))}
BW.result<-mle2(BW,hessian=NULL,start=list(gamma=20.11,
delta=10.001,lambda=17.001,theta=0.0001),optimizer="nlminb"
,lower=0)summary(BW.result)

###############################################NMW

NMW <- function(alpha,gamma,lambda,beta){
 -sum(log(alpha*gamma*x^(gamma-1)+lambda*beta*(x^(lambda-1))
 *exp(-beta*x^lambda))-exp(alpha*x^gamma)+exp(-beta*x^lambda)
 )}NMW.result<-mle2(NMW,hessian=NULL,start=
list(alpha=1.09,gamma=1.0987,lambda=1.0,beta=1.0),
optimizer="nlminb",lower=0)
summary(NMW.result)

#################################################GELLLoG

main_LL<- function(lambda,c,alpha,delta) {
 -sum(log((1/gamma(delta))*((-log(1-(1-
 (((1+lambda+lambda*x)*(exp(-lambda*x)))/((1+lambda)
 *(1+x^c))))^alpha))^(delta-1))*alpha*((1-
```

```
 (((1+lambda+lambda*x)*(exp(-lambda*x)))/((1+lambda)
 *(1+x^c))))^(alpha-1))*(((1+x^c)^-1)*(exp(-lambda*x))
 /(1+lambda))*((lambda^2)*(1+x)+((1+lambda+lambda*x)*
 (c*x^(c-1))/(1+x^c))))))}
main.result<-mle2(main_LL,hessian = NULL,start=list
(lambda=0.18118,c=1.0090,alpha=1.001,delta=1.091),
optimizer="nlminb",lower=0)
summary(main.result)

################################################OLLEW

OLLEW_LL<-function(alpha,beta,gamma,theta){-sum(log
 ((theta*beta*gamma*(x^(beta-1))*(exp(-(x/alpha)^beta))
 *((1-exp(-(x/alpha)^beta))^(gamma*theta-1))*
 ((1-(1-exp(-(x/alpha)^beta))^gamma)^(theta-1)))/
 ((alpha^beta)*(((1-exp(-(x/alpha)^beta))^(gamma*theta))
 +(1-(1-exp(-(x/alpha)^beta))^gamma)^theta)^2))))}
mle.result<-mle2(OLLEW_LL,hessian = NULL,
start=list(alpha=0.006,beta=0.000000021,gamma=0.6644,
theta=0.9454),optimizer="nlminb",lower=0)
summary(mle.result)
```