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Partial Bayes Estimation in a Normal Family

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Abstract

Areas of Bayesian analysis became vast throughout several decades. Bayes estimation, empirical and hierarchical Bayes estimation are important areas among them. In multi-parameter case, notion of 'Partial Bayes (PB) Estimation' is introduced and in $N(\theta, \sigma^2)$, where parameters being unknown, 'PB Estimation' of θ is done by putting the estimator of σ^2 obtained by some other methods. When we do not have enough information regarding the joint parameters of the model of the variable and when we are estimating one parameter in presence of others, such method may be used instead of empirical and hierarchical methods when the information about their parameters are not sufficient. Integrated Risk of the PB estimator and Bayes estimator derived under squared error loss function are compared along with the classical estimator MLE through simulation technique. From the results, it is found that PB estimator has almost same risk values as that of the Bayes estimator, whereas the risk of MLE is higher compared to both of the estimator. For illustrative purpose, a real dataset is used to apply this PB estimation technique. PB is appropriate name to describe. However, partial Bayes term has been used in different context of Bayesian hierarchical model and meta analysis.

Keywords: Bayesian inference, integrated Bayes risk, empirical Bayes.

1. Introduction

Bayes procedure has application in almost all areas starting from Industrial statistics to Epidemiology in Biomedical research (Greenland, 2009; Maclehose and Hamra, 2014). A good account of theoretical works are included in Berger (2013) and Ferguson (2014). Ramos et al. (2017) discussed Bayesian analysis of Generalized Gamma distribution using Non-informative Prior. In case of Skewed Normal, works have been done in Kim (2005) and Rikhtehgaran and Kazemi (2013). Under balanced loss function and exponential family, Bayes procedure is carried out by Aggarwal (2006). Parameter estimation of Markov transition matrix in Bayesian context are considered by Seal and Hossain (2013). Under Log-normal distribution, Bayes procedure is carried out by Fabrizi and Trivisano (2012). By considering two parameter Gamma model, Banerjee and Seal (2021) described the PB estimation procedure.

Sometimes, form of the prior belief is guessed but not in full form i.e. some of the parameters of the prior remain unknown and those parameters are estimated from data. This is known as Empirical Bayes procedure or Hierarchical Bayes procedure. These are actually step by step understanding of beliefs.

Here, a new approach is addressed as 'Partial Bayes Estimation' (PBE) and is different from earlier procedures. Suppose, we have two parameters and our interest is to estimate one of them only. Then according to the procedure to be described in the next section, we take a prior of that parameter

of interest only and for other, we estimate by some other method if necessary, i.e. we assume prior for parameter of interest and take estimator of that in Bayesian way. Thus, we estimate partially parameter of interest and that is why it is to be called as PB estimation. During the progress of the work it was seen that PB estimation is available in different context for example Qiu et al. (2018) and Xie et al. (2013), where meta analysis and different approaches are taken. However, it is a semi Bayes.

In Section 2, the method is described in case of $N(\theta, \sigma^2)$, where θ is parameter of interest and σ^2 is nuisance parameter. Prior distribution for θ is assumed and Bayes estimate involved σ^2 , and it is replaced by $\hat{\sigma}^2$ obtained by method of 'Maximum Likelihood Estimation'. Some other estimates may be used. Thus an estimate of θ is obtained. In Section 3, integrated risks of PB estimator and MLE are obtained. In next Section 4, integrated Bayes risk of the usual Bayes estimator is studied. In Section 5, numerical studies are given for understanding the theoretical results. Application to a real dataset is shown in Section 6. Finally, in Section 7, we conclude the paper. It is seen that PB estimator performs well.

2. PB Estimation of Mean

Let us take,

$X|\theta \sim N(\theta, \sigma^2)$ and prior distribution of $\theta \sim N(\mu, \tau^2)$, where σ^2 is unknown.

Then, the marginal distribution of X becomes,

$$m(x) = \frac{1}{\sqrt{2\pi(\tau^2 + \sigma^2)}} \exp \left\{ -\frac{(x - \mu)^2}{2(\tau^2 + \sigma^2)} \right\} \quad (1)$$

i.e., $x \overset{\text{marginally}}{\sim} N(\mu, \tau^2 + \sigma^2)$.

Now, if we take $x_1, x_2, \dots, x_n \sim N(\theta, \sigma^2)$, with the nuisance parameter σ^2 , then as in Eqn. (1) the sufficient statistic

$$\bar{x} \sim N\left(\theta, \frac{\sigma^2}{n}\right),$$

and marginal distribution of $\bar{x}$ becomes

$$f(\bar{x}; \mu, \tau^2, \sigma^2) = \frac{1}{\sqrt{2\pi(\tau^2 + \frac{\sigma^2}{n})}} \exp \left\{ -\frac{(\bar{x} - \mu)^2}{2(\tau^2 + \frac{\sigma^2}{n})} \right\}, \quad (2)$$

and the Posterior distribution becomes,

$$\begin{aligned} \Pi(\theta|\bar{x}) &\propto f(\bar{x}|\theta)\pi(\theta) \\ &\propto e^{-\frac{1}{2}\left[\frac{n}{\sigma^2}(\bar{x}-\theta)^2 + \frac{1}{\tau^2}(\theta-\mu)^2\right]} \\ &\propto e^{-\frac{1}{2}\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)\left[\theta - \frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}\right]^2}. \end{aligned}$$

Therefore, using Eqn. (2) we have

$$\theta|\bar{x} \propto N\left(\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \frac{1}{\left(\frac{n}{\sigma^2} + \frac{1}{\tau^2}\right)}\right).$$

So, the Bayes estimator of θ under squared error loss function becomes,

$$\frac{\frac{n\bar{x}}{\sigma^2} + \frac{\mu}{\tau^2}}{\frac{n}{\sigma^2} + \frac{1}{\tau^2}}, \quad \text{where, } \sigma^2 \text{ is unknown.} \quad (3)$$

From this we find PB estimate of θ , i.e. plugging the estimate of σ^2 , obtained from some other ways into the expression of the Bayes estimator.

We have M.L.E of σ^2 as,

$$\hat{\sigma}^2 = \sum_{i=1}^n \frac{(x_i - \bar{x})^2}{n}.$$

Now, putting $\hat{\sigma}^2$ in place of σ^2 in Eqn. (3), the Bayes estimate becomes PB estimate as,

$$\begin{aligned} PB &= \frac{\frac{n\bar{x}}{\hat{\sigma}^2} + \frac{\mu}{\tau^2}}{\frac{n}{\hat{\sigma}^2} + \frac{1}{\tau^2}} \\ &= \frac{\frac{n^2\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} + \frac{\mu}{\tau^2}}{\frac{n^2}{\sum_{i=1}^n (x_i - \bar{x})^2} + \frac{1}{\tau^2}} \\ &= \frac{\tau^2 \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \left[\frac{n^2\bar{x}}{\sum_{i=1}^n (x_i - \bar{x})^2} + \frac{\mu}{\tau^2} \right] \\ &= \frac{n^2\bar{x}\tau^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} + \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \\ &= \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2 + n^2\bar{x}\tau^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2}. \end{aligned} \quad (4)$$

3. Integrated Risks of PB Estimator and MLE

As, $\theta \sim N(\mu, \tau^2)$ we have,

$$E(\theta^2) = \tau^2 + \mu^2 \quad \text{and} \quad E(\theta) = \mu \quad (5)$$

Now, the risk function of PB estimator using (4) is

$$\begin{aligned} &E \left[\frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2 + n^2\bar{x}\tau^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} - \theta \right]^2 \\ &= E_{\sum_{i=1}^n (x_i - \bar{x})^2} E_{\bar{x}} \left[\frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2 + n^2\bar{x}\tau^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} - \theta \right]^2 \\ &= E_{\sum_{i=1}^n (x_i - \bar{x})^2} E_{\bar{x}} \left[\frac{n^2\bar{x}\tau^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} - \theta + \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \right]^2 \\ &= E_{\sum_{i=1}^n (x_i - \bar{x})^2} E_{\bar{x}} \left[(c\bar{x} - \theta) + \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \right]^2 \end{aligned}$$

where $c = \frac{n^2\tau^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2}$

$$\begin{aligned} &= E_{\sum_{i=1}^n (x_i - \bar{x})^2} E_{\bar{x}} \left[\{c(\bar{x} - \theta) + (c-1)\theta\} + \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \right]^2 \\ &= E_{\sum_{i=1}^n (x_i - \bar{x})^2} \left[c^2 \frac{\sigma^2}{n} + (c-1)^2 \theta^2 + \left\{ \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \right\}^2 + 2(c-1)\theta \right. \\ &\quad \left. \frac{\mu \sum_{i=1}^n (x_i - \bar{x})^2}{n^2\tau^2 + \sum_{i=1}^n (x_i - \bar{x})^2} \right] \end{aligned}$$

$$= E_{\sigma^2, \chi_{n-1}^2} \left[\frac{n^3 \tau^4 \sigma^2}{(n^2 \tau^2 + \sigma^2 \chi_{n-1}^2)^2} + \left(\frac{\sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right)^2 \theta^2 + \left(\frac{\mu \sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right)^2 \right. \\ \left. - 2\theta \left(\frac{\sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right) \left(\frac{\mu \sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right) \right].$$

Using (5), the integrated risk

$$= E_{\sigma^2, \chi_{n-1}^2} \left[\frac{n^3 \tau^4 \sigma^2}{(n^2 \tau^2 + \sigma^2 \chi_{n-1}^2)^2} + \left(\frac{\sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right)^2 (\tau^2 + \mu^2) + \left(\frac{\mu \sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right)^2 \right. \\ \left. - 2 \left(\frac{\sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right) \left(\frac{\mu^2 \sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right) \right] \\ = E_{\sigma^2, \chi_{n-1}^2} \left[\frac{n^3 \tau^4 \sigma^2}{(n^2 \tau^2 + \sigma^2 \chi_{n-1}^2)^2} + \left(\frac{\sigma^2 \chi_{n-1}^2}{n^2 \tau^2 + \sigma^2 \chi_{n-1}^2} \right)^2 \tau^2 \right]. \quad (6)$$

Now, as we see that the above expression is difficult to solve analytically, we proceed calculation using simulation and the results are discussed in numerical Section 5.

Now, the maximum likelihood estimator of θ is $\hat{\theta}_{MLE} = \bar{x}$.

Hence, the integrated risk of $\hat{\theta}_{MLE}$ is,

$$E[\hat{\theta} - \theta]^2 = E(\bar{x} - \theta)^2 \\ = E(\bar{x}^2) - 2\theta E(\bar{x}) + \theta^2 \\ = \frac{\sigma^2}{n} + \theta^2 - 2\theta^2 + \theta^2 \\ = \frac{\sigma^2}{n}. \quad (7)$$

4. Integrated Risk of Bayes Estimator of Mean

If we have, $X_i \sim N(\theta, \sigma^2)$; $i=1,2,\dots,n$ and $\theta \sim N(\mu, \tau^2)$.

Then the Bayes Estimate of θ is

$$= \frac{\mu \frac{\sigma^2}{n} + \bar{x} \tau^2}{\tau^2 + \frac{\sigma^2}{n}} \quad (\text{where } \sigma^2 \text{ is known}) \quad (8) \\ = \frac{\mu(\tau^2 + \frac{\sigma^2}{n}) + \tau^2(\bar{x} - \mu)}{\tau^2 + \frac{\sigma^2}{n}} \\ = \mu + \frac{\tau^2(\bar{x} - \mu)}{\tau^2 + \frac{\sigma^2}{n}}.$$

Now the Risk function of this Bayes estimator is

$$E \left[\left(\mu + \frac{\tau^2(\bar{x} - \mu)}{\tau^2 + \frac{\sigma^2}{n}} - \theta \right) \right]^2 = \frac{2n\mu(\tau^2 - \mu + \theta)}{n\tau^2 + \sigma^2} + \frac{n\tau^2}{\sigma^2} \left(2 + \frac{n\tau^2}{\sigma^2} \right) + \frac{2(\theta - \mu)(n\sigma^2 + n\tau^2)}{\sigma^2(n\tau^2 + \sigma^2)} \\ + \frac{n^2(\sigma^2 + \mu^2) + n^2\theta(\theta - 2\mu)}{n^2\tau^4 + \sigma^4 - 2n^2\tau^2\sigma^2} - \frac{2(\mu\theta\sigma^2 + n\tau^2\theta^2)}{n\tau^2 + \sigma^2} + \theta^2 + \mu^2 + 1$$

So, the Integrated Bayes Risk is

$$\begin{aligned}
 \int riskfun(\theta)\pi(\theta)d\theta &= \int var(\theta|\bar{x})m(\bar{x})dx \\
 &= \int \frac{\tau^2 \frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}} m(x)dx \\
 &= \frac{\tau^2 \frac{\sigma^2}{n}}{\tau^2 + \frac{\sigma^2}{n}}.
 \end{aligned} \tag{9}$$

5. Numerical comparison between PB estimator and Bayes estimator through integrated Bayes risk

In this section, performance of the proposed PB estimator is studied numerically and compared with the Bayes Estimator and the Maximum likelihood estimator also in terms of Integrated risk values. As in the Eqn. (6), we see that the risk is not in a closed form. So, in order to calculate the integrated risk of PB estimator, the following algorithm is constructed.

- *Step 1:* Initialize the nuisance parameter $\sigma = 0.3, 0.5, 1, 1.5, 2$ and the known hyperparameter $\tau = 0.05, 0.25, 0.50, 1$.
- *Step 2:* Select the sample size $n = 25, 50, 75, 100$ to represent small, moderate and large size of sample respectively.
- *Step 3:* First consider the parameter choice $\sigma = 0.3$ and $\tau = 0.05$.
- *Step 4:* Simulate one realization from χ_{n-1}^2 using sample size, $n = 25$.
- *Step 5:* Calculate the expression from (6).
- *Step 6:* Repeat *Step 5* for $K = 5000$ times.
- *Step 7:* Calculate the average of that K times computed expressions to obtain the integrated risk of PBE & by using Strong Law of Large number it is approximately the expectation over χ_{n-1}^2 in (6).
- *Step 8:* Similarly compute the integrated risk for the remaining parameter choices and the mentioned sample sizes.

To obtain the integrated risk of the MLE and Bayes estimator numerically, we calculate it by using the expressions (7) and (9) respectively for the same set of parameter values. To compare the performance of the PB estimator, we tabulate the integrated risk values in Tables 1 - 4 with an increment of sample sizes.

Table 1 Integrated risks of Partial Bayes (PB), Bayes Estimator (BE) and MLE when $\tau = 0.05$

Parameter choice	Sample size (n)	Integrated Bayes risk of PBE	Integrated Bayes risk of BE	Integrated Bayes risk of MLE
$\sigma = 0.3$	25	0.001510	0.001475	0.003600
	50	0.001057	0.001047	0.001800
	75	0.000816	0.000811	0.001200
	100	0.000664	0.000662	0.000900
	25	0.002037	0.002000	0.010000

Table 1 (Continued)

$\sigma = 0.5$	50	0.001684	0.001667	0.005000
	75	0.001439	0.001429	0.003333
	100	0.001257	0.00125	0.002500
$\sigma = 1.0$	25	0.002371	0.002353	0.040000
	50	0.002234	0.002222	0.020000
	75	0.002114	0.002105	0.013333
	100	0.002007	0.002000	0.010000
$\sigma = 1.5$	25	0.002442	0.002432	0.090000
	50	0.002375	0.002368	0.045000
	75	0.002313	0.002308	0.030000
	100	0.002255	0.002250	0.022500
$\sigma = 2.0$	25	0.002467	0.002462	0.160000
	50	0.002428	0.002424	0.080000
	75	0.002391	0.002388	0.053333
	100	0.002356	0.002353	0.040000

Table 2 Integrated risks of Partial Bayes (PB), Bayes Estimator (BE) and MLE when $\tau = 0.25$

Parameter choice	Sample size (n)	Integrated Bayes risk of PBE	Integrated Bayes risk of BE	Integrated Bayes risk of MLE
$\sigma = 0.3$	25	0.003418	0.003404	0.003600
	50	0.001752	0.001750	0.001800
	75	0.001178	0.001177	0.001200
	100	0.000887	0.000887	0.000900
$\sigma = 0.5$	25	0.008704	0.008621	0.010000
	50	0.004643	0.004630	0.005000
	75	0.003169	0.003165	0.003333
	100	0.002406	0.002404	0.002500
$\sigma = 1.0$	25	0.024892	0.024390	0.040000
	50	0.015267	0.015152	0.020000
	75	0.011033	0.010989	0.013333
	100	0.008642	0.008621	0.010000
$\sigma = 1.5$	25	0.037749	0.036885	0.090000
	50	0.026435	0.026163	0.045000
	75	0.020393	0.020270	0.030000
	100	0.016611	0.016544	0.022500
$\sigma = 2.0$	25	0.045919	0.044944	0.160000
	50	0.035473	0.035088	0.080000
	75	0.028979	0.028777	0.053333
	100	0.024511	0.024390	0.040000

Table 3 Integrated risks of Partial Bayes (PB), Bayes Estimator (BE) and MLE when $\tau = 0.50$

Parameter choice	Sample size (n)	Integrated Bayes risk of PBE	Integrated Bayes risk of BE	Integrated Bayes risk of MLE
$\sigma = 0.3$	25	0.003553	0.003549	0.003600
	50	0.001788	0.001787	0.001800
	75	0.001194	0.001177	0.001200
	100	0.000897	0.000897	0.000900
$\sigma = 0.5$	25	0.009644	0.009615	0.010000
	50	0.004906	0.004902	0.005000
	75	0.003291	0.003289	0.003333
	100	0.002476	0.002475	0.002500
$\sigma = 1.0$	25	0.034815	0.034483	0.040000
	50	0.018571	0.018519	0.020000
	75	0.012675	0.012658	0.013333
	100	0.009623	0.009615	0.010000
$\sigma = 1.5$	25	0.067244	0.066176	0.090000
	50	0.038338	0.038136	0.045000
	75	0.026856	0.026786	0.030000
	100	0.02067	0.020642	0.022500
$\sigma = 2.0$	25	0.099570	0.097561	0.160000
	50	0.061067	0.060606	0.080000
	75	0.044130	0.043956	0.053333
	100	0.034567	0.034483	0.040000

Table 4 Integrated risks of Partial Bayes (PB), Bayes Estimator (BE) and MLE when $\tau = 1$

Parameter choice	Sample size (n)	Integrated Bayes risk of PBE	Integrated Bayes risk of BE	Integrated Bayes risk of MLE
$\sigma = 0.3$	25	0.003588	0.003587	0.003600
	50	0.001797	0.001797	0.001800
	75	0.001199	0.001199	0.001200
	100	0.000899	0.000899	0.000900
$\sigma = 0.5$	25	0.009909	0.009901	0.010000
	50	0.004976	0.004975	0.005000
	75	0.003323	0.003322	0.003333
	100	0.002494	0.002494	0.002500
$\sigma = 1.0$	25	0.038577	0.038462	0.040000
	50	0.019623	0.019608	0.020000
	75	0.013163	0.013158	0.013333
	100	0.009903	0.009901	0.010000
$\sigma = 1.5$	25	0.083074	0.082569	0.090000
	50	0.043135	0.043062	0.045000
	75	0.029149	0.029126	0.030000
	100	0.022015	0.022005	0.022500
$\sigma = 2.0$	25	0.139260	0.137931	0.160000
	50	0.074282	0.074074	0.080000
	75	0.050699	0.050633	0.053333
	100	0.038491	0.038462	0.040000

From Tables 1 - 4, it is seen that, as we increase the standard deviation σ of the model, the integrated risk of PBE, BE and MLE increases. Also, with the increment of the hyperparameter τ , integrated risks of PBE, BE and MLE increase. So, large variance may lead to the larger value of integrated risk. In case of prior distribution, the small value of τ , provides more information about the prior as the prior becomes more concentrated. Also increase in sample sizes, leads to the decreases in integrated risk values for all the three estimators. These are also expected to our common sense.

After comparing our proposed PB estimator with the Bayes estimator and the MLE in terms of the integrated risk values, it is seen that BE performs the best among the three estimates and MLE has the largest value of integrated risk compared with the others estimators. Though the integrated risk value of PBE is larger than that of the BE, having the difference very small. So, from the simulation study, we see that the PB estimate behaves like the Bayes estimate. Moreover, our proposed PB estimate performs very well for small choices of σ and τ i.e., the standard deviation of the model and the prior respectively. Thus, it is a very useful method when we have partial information of parameters.

6. Real data application

From application point of view, a real dataset has been considered in this section to illustrate the estimation procedure. Data given in the Table 5 represents the measurements (in millimeters) of the wing lengths of 100 houseflies. This dataset was primarily used by Sokal and Hunter (1955) in a research project on the genetics and ethology of DDT-resistance in *Drosophila* and houseflies.

Table 5 Length of the wings of 100 houseflies (x.1mm)

36	37	38	38	39	39	40	40	40	40	41	41	41	41	41	41
42	42	42	42	42	42	42	43	43	43	43	43	43	43	43	44
44	44	44	44	44	44	44	44	45	45	45	45	45	45	45	45
45	45	46	46	46	46	46	46	46	46	46	46	47	47	47	47
47	47	47	47	47	48	48	48	48	48	48	48	48	49	49	49
49	49	49	49	50	50	50	50	50	50	51	51	51	51	52	52
53	53	54	55												

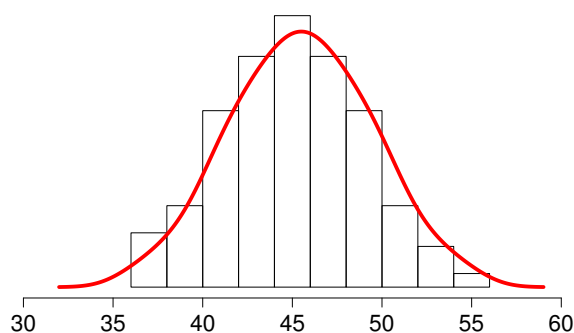
The summary of the dataset and a graphical overview are given in Table 6 and Figure 1 respectively. From Table 6, it is observed that the Mean, Median and Mode coincide and also the data form a bell shaped curve in Figure 1. To check the normality, the skewness and kurtosis tests are performed with the respective values 0 and 2.7 approximately. Therefore, the value of skewness justifies the symmetry of the data and the kurtosis value i.e. peakedness of the data is also near about the value 3. So, this dataset is believed to be normal and used to find the Bayes estimate and PB estimate here. The sample mean and sample standard deviation for the real data set are 45.5 and 3.9 respectively. The Maximum likelihood estimates of the parameters θ and σ are obtained as $\hat{\theta}_{MLE} = 45.5$ and $\hat{\sigma}_{MLE} = 3.9$ for this dataset. We computed the Kolomogorov–Smirnov distances between the empirical distribution function and the fitted distribution functions, and the associated p values (reported within brackets) for the dataset and it is found as 0.051 (0.9571).

Here, the intuition is to show whether the PB estimate is consistent or not. For this purpose we compare the proposed PB estimate with the original Bayes estimate and with Maximum likelihood estimator. In Table 7, we calculate the PB estimate and Bayes estimate of θ based on the real data set by using the expressions (4) and (8) respectively. However, in (8), we plug the MLE of σ^2 for unknown σ^2 . Also, we divide the real data set equally into two groups with sample size $n=50$ and for each of these groups, Bayes estimate and PB estimate are calculated. The idea is to see how the proposed PB estimate performs in real life situations.

From Table 7, it is clearly seen that while considering the whole data set, the PB estimate is same as the Bayes estimate for all combinations of parameter choice and while splitting the data set into

Table 6 Summary statistics

Length (n)	100
Mean	45.5
Min	36.0
Max	55.0
1st quartile (Q_1)	43.0
Median	45.5
3rd quartile (Q_3)	48.0
Standard Deviation	3.9
Mode	45

**Figure 1** Histogram of the data

two groups, the PB estimate does not differ too much from the MLE as well as the Bayes estimate of θ . It is also observed that, for larger value of the standard deviation (τ) of the prior distribution, the PB estimate and the Bayes estimate are very close to the MLE of θ . Therefore, for a larger value of τ , the Partial Bayes, Bayes and the classical estimate, MLE of θ are close enough to estimate the mean parameter of normal distribution.

Table 7 Partial Bayes(PB) estimate and Bayes Estimate (BE) using real data for different parametric values

Parameter choice	Groups	$\tau = 2$		$\tau = 2.5$		$\tau = 3$	
		PBE	BE	PBE	BE	PBE	BE
$\mu = -3$	Total	43.7233	43.7233	44.3477	44.3477	44.6940	44.6940
	Group 1	41.1858	39.1356	41.5945	40.2356	41.8197	40.8576
	Group 2	47.3449	45.0089	47.8105	46.2623	48.0671	46.9710
$\mu = -2$	Total	43.7600	43.7600	44.3715	44.3715	44.7106	44.7106
	Group 1	41.2112	39.2063	41.6109	40.2820	41.8311	40.8903
	Group 2	47.3703	45.0796	47.8270	46.3087	48.0786	47.0037
$\mu = -1$	Total	43.7966	43.7966	44.3953	44.3953	44.7272	44.7272
	Group 1	41.2367	39.2769	41.6274	40.3285	41.8426	40.9230
	Group 2	47.3958	45.1503	47.8434	46.3551	48.0901	47.0364
$\mu = 0$	Total	43.8332	43.8332	44.4190	44.4190	44.7438	44.7438
	Group 1	41.2622	39.3476	41.6438	40.3749	41.8541	40.9557
	Group 2	47.4213	45.2209	47.8599	46.4015	48.1016	47.0691
$\mu = 1$	Total	43.8699	43.8699	44.4428	44.4428	44.7604	44.7604
	Group 1	41.2876	39.4183	41.6602	40.4213	41.8656	40.9884
	Group 2	47.4467	45.2916	47.8763	46.4480	48.1130	47.1018
$\mu = 2$	Total	43.9065	43.9065	44.4665	44.4665	44.7771	44.7771
	Group 1	41.3131	39.4890	41.6767	40.4677	41.8770	41.0211
	Group 2	47.4722	45.3623	47.8928	46.4944	48.1245	47.1345
$\mu = 3$	Total	43.9431	43.9431	44.4903	44.4903	44.7937	44.7937
	Group 1	41.3385	39.5596	41.6931	40.5141	41.8885	41.0538
	Group 2	47.4976	45.4330	47.9092	46.5408	48.1360	47.1672

7. Conclusions

In this article, a new variant Partial Bayes (PB) of Bayes estimate, is introduced. Bayes estimation can be done simultaneously for many parameters and also there is empirical Bayes estimation and that has a different context. But in case when there is more than one parameter and we do not have prior knowledge of the nuisance parameter, then we should use the PB estimation. Performance of proposed PB estimator with the conventional Bayes estimator and the traditional MLE have been compared using integrated risk values.

Tables 1 - 4 reveal that the PB estimate performs well for small values of the parameters σ and τ . So, smaller values of the standard deviations of the data and the prior, increase the performance of our proposed PB estimator.

To support the idea with the real life situation, a real data set about the measurements (in millimetres) of the wing lengths of 100 houseflies is considered and illustrates the performance of the PB estimator along with the original Bayes estimator and the MLE. From Table 7, we say that our PB estimator exactly behaves like the Bayes estimator. Also, the hyperparameter τ has an impact on the PB estimator, as for the higher values of τ , the values of the PB estimate is very close to the Bayes estimator and the MLE. Therefore, there are enough scopes where the PB estimation procedure may be used for other distributions also.

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