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Discrete Gompertz-Lomax Distribution and Its Applications

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Abstract

A new distribution, called the discrete Gompertz-Lomax distribution, is proposed. It has been developed by combining the properties of two existing distributions, namely discrete Gompertz-G family of distributions and Lomax distribution. Its probability mass function is characterized by a flexible probability function that can exhibit unimodality and reverse J-shape (decreasing). It is interesting that some statistical properties of the discrete Gompertz-Lomax distribution have been discussed, including the quantile function, moments, probability generating function, discrete hazard function and discrete reversed hazard function. The maximum likelihood estimation has been formulated to estimate the unknown parameters of the discrete Gompertz-Lomax distribution. A simulation study and applications of this distribution have been illustrated. The development of the discrete Gompertz-Lomax distribution seems to be a valuable contribution to the field of probability theory and statistics. It has potential applications which data with skewed or decreasing patterns may be encountered.

Keywords: Discrete Gompertz-G family of distributions, Lomax distribution, maximum likelihood estimation, quantile function, discretization method.

1. Introduction

Over the past two decades or so, many researchers have recognized the need for alternative distributions for modeling lifetime data that is collected in discrete analogue form. There are many situations where collecting lifetime data in discrete form is more convenient, such as measuring the life of a gaslighter by the number of shots (Roy and Gupta, 1999), the degree of temperature shown on screen, the number of blood pressure shown on the blood pressure monitor (Liu and Abeyratne, 2019), the number of voltage fluctuations which an electronic item can withstand before its failure (Krishna and Pundir, 2009), measuring the survival times of guinea pigs in days (Bjerkedal, 1960) and measuring the number of weeks a cancer patient survives after treatment (Hussain et al., 2016). In such cases, traditional continuous distributions like the exponential and Weibull distributions may not be appropriate, and new discrete distributions are needed. The discretization method is one interesting approach to developing new distributions for discrete analogue data. This method involves discretizing a continuous distribution to obtain a discrete distribution that can better model the data.

The discretization method is a technique used to derive new discrete analogues of continuous distributions. It has gained popularity in recent years due to its theoretical appeal and practical applications in both discrete data and count data analysis (Roy and Gupta, 1992; Roy, 2004). Different types of discretization methods are based on various functions, such as survival function, probability density function (pdf), cumulative distribution function (cdf) and hazard rate function (Chakraborty, 2015). Roy's method is a widely used discretization technique that constructs the probability mass function (pmf) of a new distribution by computing the difference of survival function values at nearly two non-negative integer values. Roy first introduced this method, demonstrating how the geometric distribution can be derived from the exponential distribution. (Roy, 2004). Several distributions have been derived using Roy's method, such as the discrete Rayleigh distribution (Roy, 2004), the discrete Burr Type XII distribution (Krishna and Pundir, 2009), the discrete Lindley distribution (Gómez-Déniz and Calderín-Ojeda, 2011), the discrete Burr Type III distribution (Al-Huniti and AL-Dayian, 2012), the discrete weighted exponential distribution (Khongthip et al., 2018), the discrete generalized odd Lindley-Weibull distribution (Aryuyuen et al., 2020) and the discrete moment exponential distribution (Afify et al., 2022).

The discrete Gompertz-G family of distributions is a new discrete generator that was proposed in 2020 (Eliwa et al., 2020). It is based on discretizing the Gompertz-G family of distributions using Roy's method. The resulting distribution has more flexibility than the baseline distribution and is suitable for modeling discrete analogue lifetime data and count data. The generator has been used to derive several new discrete distributions, including the discrete Gompertz-exponential (DGz-Exp), the discrete Gompertz-Weibull (DGz-Wei), and the discrete Gompertz-inverse Weibull (DGz-Inv Wei) distributions (Eliwa et al., 2020).

The Lomax distribution (also known as Pareto type II) was introduced in 1954. (Lomax, 1954). It is a modification of the Pareto distribution (Arnold, 2014). It is categorized as a heavy-tailed distribution, that means a density function going to zero rapidly less than an exponential function (Bryson, 1974). Its adaptability has made it an appropriate choice for modeling various types of data. Many researchers have proposed different modifications and extensions to the Lomax distribution to make it even more flexible and suitable for various applications, such as the inverse Lomax distribution (Kleiber and Kotz, 2003), the transmuted Lomax distribution (Ashour and Eltehiwy, 2013), the Poisson-Lomax distribution (Al-Zahrani and Sagor, 2014), the gamma-Lomax distribution (Cordeiro et al., 2015), the Weibull-Lomax distribution (Tahir et al., 2015), the power Lomax distribution (Rady et al., 2016), the weighted Lomax distribution (Kilany, 2016), the Lomax-exponential distribution (Hami Golzar et al., 2017), the Nadarajah-Haghighi Lomax distribution (Nagarjuna et al., 2022), the Maxwell-Lomax distribution (Abiodun and Ishaq, 2022) and the X-gamma Lomax distribution (Almetwally et al., 2022).

In this article, a new alternative distribution for count data and discrete analogue data, called the discrete Gompertz-Lomax (DGz-Lomax) distribution, is proposed. In Section 2, material and method for development of the proposed distribution are provided. In Section 3, some properties of the proposed distribution are discussed. In Section 4, the model parameter estimation is estimated by using the maximum likelihood estimation (MLE). In Section 5, simulation study of the proposed distribution is illustrated. In Section 6, applications with two real data sets of the proposed distribution are presented. Finally, in Section 7, the conclusions are presented.

2. Material and Method for Development

In this section, a description of discretization method is provided. Moreover, the properties of Gompertz-G family of distributions, discrete Gompertz-G family of distributions and Lomax distribution are introduced to derive a new distribution in this article.

2.1. Discretization method

One of the popular discretization methods is Roy's method. This method is simplify for generating the pmf of a new discrete distribution (Roy, 2004). This method is chosen for developing the Gompertz-G family of distributions (Eliwa et al., 2020).

Let Y be a random variable on $[0,\infty)$. The random variable X construct a discrete counterpart supported on the set of integers $\mathbb{N}_0 = \{0,1,2,3,\ldots\}$ or $X = \lfloor Y \rfloor$, where $\lfloor\cdot\rfloor$ is the floor function, means the largest integer less than or equal to Y, then its pmf is given by

$$f_X(x) = P(X = x) = P(x \le Y < x+1) = S_Y(x) - S_Y(x+1) \tag{1}$$

where $S_Y(x)$ is the survival function of Y, $S_Y(x) = P(Y > x) = 1 - G_Y(x)$ and $G_Y(x)$ is cdf of Y.

Roy's method is a versatile technique that can be applied not only to derive discrete analogues of continuous distributions, but also to develop discrete generators like the discrete Gompertz-G family of distributions (Eliwa et al., 2020).

2.2. Gompertz-G family of distributions

The Gompertz-G family of distributions is developed from the Gompertz distribution by transformed-transformer method (Alizadeh et al., 2017).

If Y be a Gompertz-G random variable, then its cdf and pdf, respectively are

$$G_Y(y) = \int_0^{-\log(1-J_Y(y;\psi))} \gamma e^{ct-\frac{\gamma}{c}(e^{ct}-1)}dt = 1 - e^{-\frac{\gamma}{c}\{[1-J_Y(y;\psi)]^{-c}-1\}}, \tag{2}$$

$$g_Y(y) = \gamma j_Y(y;\psi)[1-J_Y(y;\psi)]^{-(c+1)}e^{-\frac{\gamma}{c}\{[1-J_Y(y;\psi)]^{-c}-1\}}, \quad y > 0, \tag{3}$$

where c is the scale parameter, $c > 0, \gamma > 0$ and ψ is a vector of parameters $(1\times m; m = 1,2,3,\ldots)$. $J_Y(y;\psi)$ and $j_Y(y;\psi)$ are the cdf and pdf of baseline distribution, respectively.

2.3. Discrete Gompertz-G family of distributions

The discrete Gompertz (DGz)-G family of distributions, derived from discretizing the Gompertz-G family of distributions, offers both flexibility and improved goodness of fit test compared to the baseline distribution. This suggests that the discretization process enhances the fit of the Gompertz-G family to discrete data, making it a valuable tool for modeling various types of discrete data sets. The improved fit of the DGz-G family of distributions indicates its potential for accurately representing real-world phenomena and capturing the underlying patterns present in discrete data (Eliwa et al., 2020).

If X be a DGz-G random variable, then its cdf, survival function and pmf, respectively are

$$F_X(x) = 1 - p^{\frac{1}{c}\{[1-J_X(x+1;\psi)]^{-c}-1\}}, \tag{4}$$

$$S_X(x) = p^{\frac{1}{c}\{[1-J_X(x+1;\psi)]^{-c}-1\}}, \tag{5}$$

$$f_X(x) = p^{-\frac{1}{c}}[p^{\frac{1}{c}\{[1-J_X(x;\psi)]^{-c}\}} - p^{\frac{1}{c}\{[1-J_X(x+1;\psi)]^{-c}\}}], \quad x \in \mathbb{N}_0, \tag{6}$$

where p is the shape parameter and $p = e^{-\gamma}$.

2.4. Lomax distribution

Let Y be a random variable distributed as the Lomax distribution with parameters α and θ, then its cdf and pdf, respectively are

$$J_Y(y) = 1 - \left\{\frac{\alpha}{y+\alpha}\right\}^{\theta}, \tag{7}$$

$$j_Y(y) = \frac{\theta}{\alpha}\left(\frac{\alpha}{y+\alpha}\right)^{\theta+1}, \quad y > 0, \tag{8}$$

where α is the shape parameter, $\alpha > 0$ and θ is the scale parameter, $\theta > 0$.

Some plots of the Lomax pdf are shown in Figure 1.

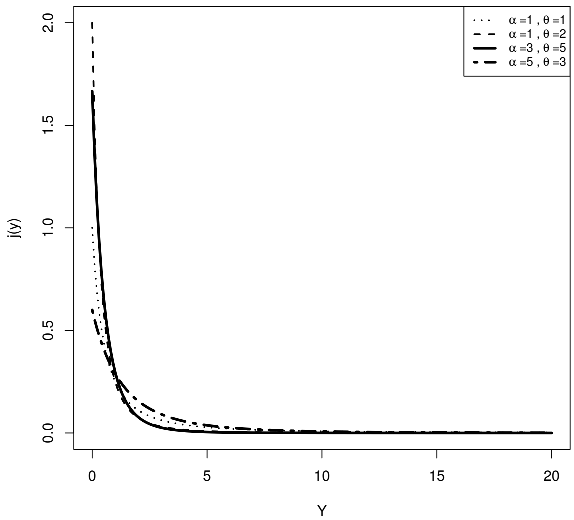

Figure 1 Plots of the Lomax pdf with the specified parameters of α and θ

3. Results

In this section, the DGz-Lomax distribution is presented. The procedure for generating this distribution is a Lomax distribution be chosen to be baseline distribution of DGz-G family of distributions. Moreover, some statistical properties of this distribution, including cdf, survival function, pmf, quantile function, moments, mean, variance, probability generating function (pgf), moment generating function (mgf), characteristic function, index of dispersion (ID), discrete hazard function and discrete reversed hazard function are presented.

3.1. The discrete Gompertz-Lomax distribution

Theorem 1 *Let X be a random variable distributed as the DGz-Lomax distribution with parameters c, p, α and θ, denoted by $X \sim$ DGz-Lomax(c, p, α, θ). The cdf of DGz-Lomax distribution is*

$$F_X(x) = 1 - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}. \tag{9}$$

Proof: By replacing the Lomax cdf into Eqn. (4), the cdf of the DGz-Lomax distribution is obtained as in Eqn. (9) that can be shown as following;

$$F_X(x) = 1 - p^{\frac{1}{c}\left\{\left[1-\left(1-\left(\frac{\alpha}{(x+1)+\alpha}\right)^{\theta}\right)\right]^{-c}-1\right\}} = 1 - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}.$$

Some cdf plots of X are shown in Figure 2.

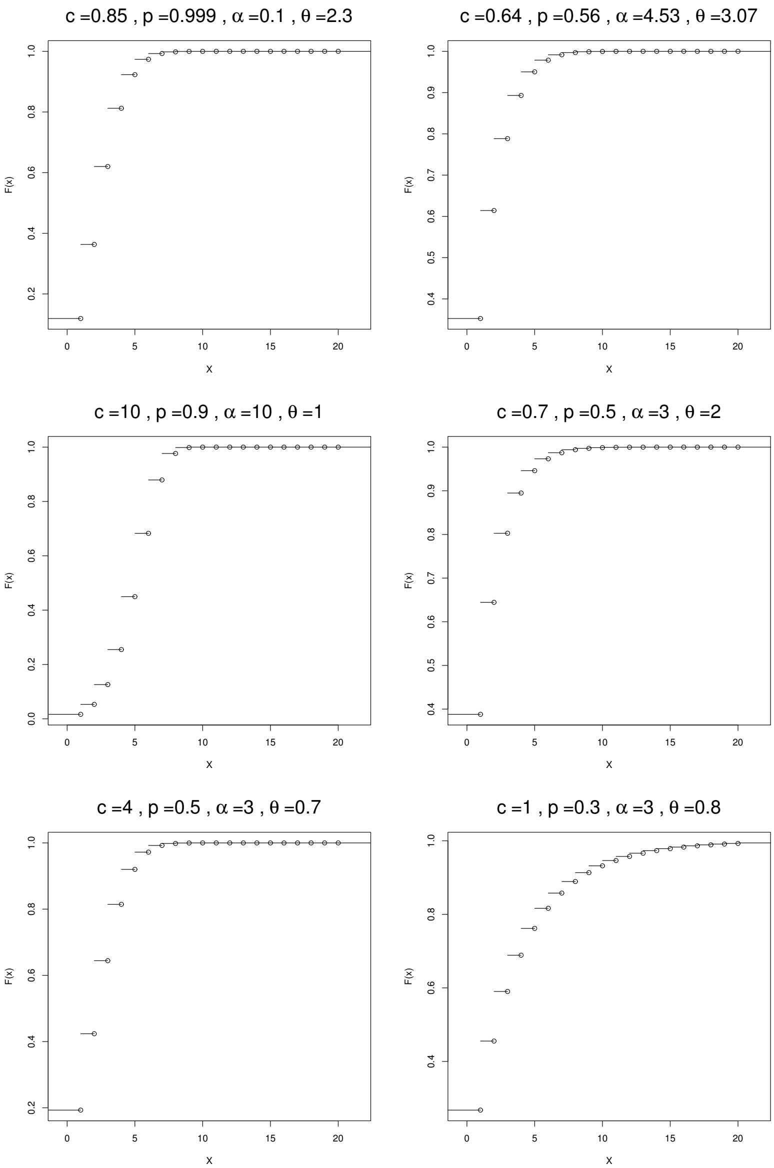

Figure 2 Plots of the DGz-Lomax cdf with the specified parameters c, p, α and θ

3.2. The survival function

Theorem 2 *Let $X \sim$ DGz-Lomax(c, p, α, θ), then the survival function of DGz-Lomax distribution is*

$$S_X(x) = p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}. \tag{10}$$

Proof: Since the survival function is a probability that a subject survives longer than time x, then the survival function of X is defined as $S_X(x) = P(X > x) = 1 - F_X(x)$. Immediately,

$$S_X(x) = 1 - \left\{1 - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}\right\} = p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}.$$

3.3. The probability mass function

Theorem 3 *Let $X \sim$ DGz-Lomax(c, p, α, θ), then its pmf of DGz-Lomax distribution is*

$$f_X(x) = p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}} - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}. \quad (11)$$

Proof: According to Roy's method and utilizing Eqn. (10), the pmf of DGz-Lomax distribution can be obtained as

$$\begin{aligned} f_X(x) &= S_X(x) - S_X(x+1) \\ &= p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}} - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}. \end{aligned}$$

Some pmf plots of X are shown in Figure 3.

Figure 3 illustrates some pmf plots with different values of the parameters c, p, α, and θ. Note that the DGz-Lomax distribution can exhibit different behaviors, such as unimodality and reverse J-shape (decreasing). This indicates that this distribution is flexible in modeling count data with different patterns.

3.4. The quantile function

Theorem 4 *Let $X \sim$ DGz-Lomax(c, p, α, θ), then its quantile function is*

$$Q_F(u) = \left\lfloor \alpha \left\{ \left(1 + c\left[\frac{\log(1-u)}{\log(p)}\right]\right)^{\frac{1}{c\theta}} - 1 \right\} - 1 \right\rfloor, \quad (12)$$

when $u \sim U[0, 1]$.

Proof: Given the cdf of the DGz-Lomax distribution as in Eqn. (9), we can derive a quantile function of the DGz-Lomax distribution, denoted as $Q_F(u)$, by inverting its cdf. Hence,

$$Q_F(u) = F^{-1}(u) = \left\lfloor \alpha \left\{ \left(1 + c\left[\frac{\log(1-u)}{\log(p)}\right]\right)^{\frac{1}{c\theta}} - 1 \right\} - 1 \right\rfloor.$$

3.5. Moments

Theorem 5 *Let $X \sim$ DGz-Lomax(c, p, α, θ), then the rth moment of X is*

$$\mu_r'(x) = p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ \left\{ x^r - (x-1)^r \right\} p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\}. \quad (13)$$

Proof: Let $f_X(x)$ be a pmf of the DGz-Lomax distribution, the rth moment of X can be obtained as

$$\begin{aligned} \mu_r'(x) &= E(X^r) = \sum_{x=0}^{\infty} x^r \cdot f_X(x) \\ &= \sum_{x=0}^{\infty} \left[x^r \left\{ p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}} - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}} \right\} \right] \\ &= p^{-\frac{1}{c}} \left[\left\{ \sum_{x=0}^{\infty} x^r p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}\right\}} \right\} - \left\{ \sum_{x=0}^{\infty} x^r p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}\right\}} \right\} \right] \\ &= p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ \left\{ x^r - (x-1)^r \right\} p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\}. \end{aligned}$$

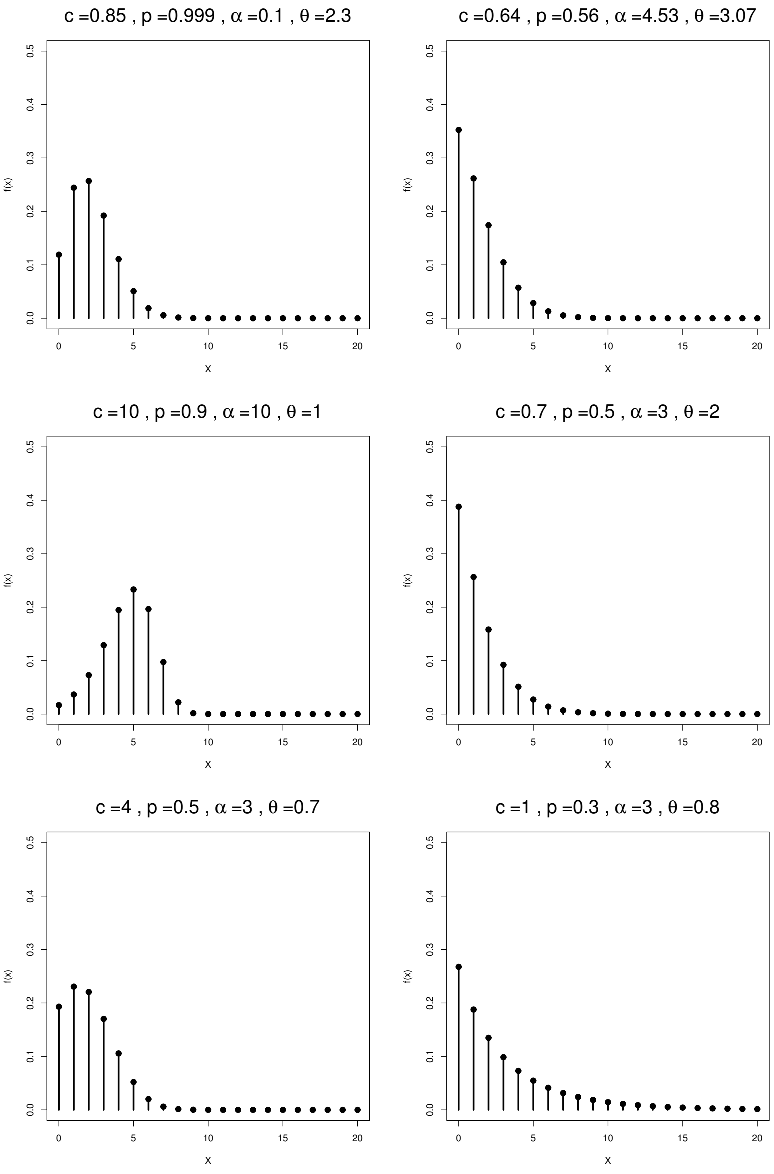

Figure 3 Plots of the DGz-Lomax pmf with the specified parameters of c, p, α and θ

From Eqn. (13), the mean and variance of DGz-Lomax distribution, respectively are

$$\mu_1'(x) = E(X) = p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\}, \tag{14}$$

$$Var(X) = \left[p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ 2x-1 \right\} \left\{ p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\} \right] - \left[\mu_1'(x) \right]^2. \tag{15}$$

3.6. Probability generating function

Theorem 6 *Let $X \sim$ DGz-Lomax(c, p, α, θ), then the pgf of X is*

$$P_X(t) = 1 + \left[p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ \left\{ t^x - t^{(x-1)} \right\} p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\} \right], \tag{16}$$

where t is a real number.

Proof: The pgf of X can be obtained as

$$\begin{aligned} P_X(t) &= E(t^x) = \sum_{x=0}^{\infty} t^x \cdot f_X(x) \\ &= \sum_{x=0}^{\infty} \left[t^x \left\{ p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}} - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}} \right\} \right] \\ &= p^{-\frac{1}{c}} \left[\left\{ \sum_{x=0}^{\infty} t^x p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}\right\}} \right\} - \left\{ \sum_{x=0}^{\infty} t^x p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}\right\}} \right\} \right] \\ &= 1 + \left[p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ \left\{ t^x - t^{(x-1)} \right\} p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\} \right]. \end{aligned}$$

If t in the Eqn. (16) is replaced by e^t, then the mgf of the DGz-Lomax distribution is

$$M_X(t) = 1 + \left[p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ \left\{ e^{tx} - e^{t(x-1)} \right\} p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\} \right].$$

In addition, if t in Eqn. (16) is replaced by e^{it}, then the characteristic function of the DGz-Lomax distribution is

$$\phi_X(t) = 1 + \left[p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ \left\{ e^{itx} - e^{it(x-1)} \right\} p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\} \right],$$

where $i = \sqrt{-1}$.

3.7. Index of dispersion

The ID is measure for determining whether the any distribution is over or under dispersed. The ID is defined as the ratio between varaiance and mean. If ID > 1, the distribution is over-dispersed, on the other hand, if ID < 1, the dispersed is under-dispersed (Cox and Lewis, 1966). The ID of the DGz-Lomax distribution is

$$\mathrm{ID}_X(x) = \frac{\left[p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ 2x - 1 \right\} \left\{ p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\} \right] - \left[\mu_1'(x) \right]^2}{p^{-\frac{1}{c}} \sum_{x=1}^{\infty} \left\{ p^{\frac{1}{c}\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}} \right\}}. \tag{17}$$

3.8. Discrete hazard and discrete reversed hazard functions

Theorem 7 *If $X \sim$ DGz-Lomax(c, p, α, θ), then the discrete hazard and discrete reversed hazard functions, respectively are*

$$h(x) = 1 - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta} - \left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}\right\}}, \tag{18}$$

and

$$\lambda(x)=\frac{\left[p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}}-p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}\right]}{\left[1-p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}\right]}. \tag{19}$$

Proof: From the pmf and the survival function of the DGz-Lomax distribution in Eqn. (11) and Eqn. (10) respectively, the discrete hazard function of the DGz-Lomax distribution (Gupta, 2015) is

$$\begin{aligned}
h(x) &= \frac{f_X(x)}{S_X(x-1)}\\
&= \frac{\left[p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}}-p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}\right]}{\left[p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}}\right]}\\
&= 1-\frac{p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}}{p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}}}\\
&= 1-p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}\right\}}.
\end{aligned}$$

Also, its discrete reversed hazard function can be obtained as

$$\lambda(x)=\frac{f_X(x)}{F_X(x)}=\frac{\left[p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha}\right)^{-c\theta}-1\right\}}-p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}\right]}{\left[1-p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x+\alpha+1}\right)^{-c\theta}-1\right\}}\right]}.$$

Some plots of DGz-Lomax hazard function are shown in Figure 4.

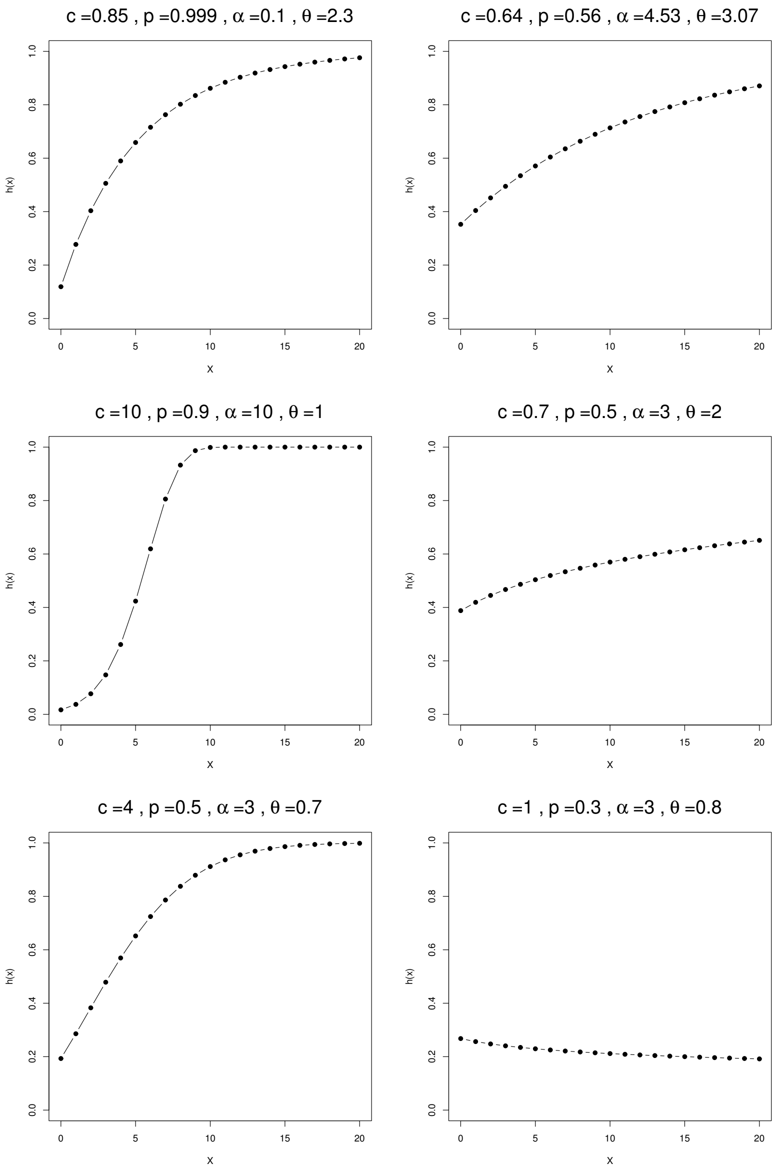

Figure 4 Plots of the DGz-Lomax hazard function with the specified parameters of c, p, α and θ

Figure 4 illustrates how the discrete hazard function of the DGz-Lomax distribution changes with different values of the parameters c, p, α, and θ. Note that the DGz-Lomax distribution can exhibit different behaviors, such as increasing and decreasing.

In addition, some plots of DGz-Lomax reversed hazard function are shown in Figure 5.

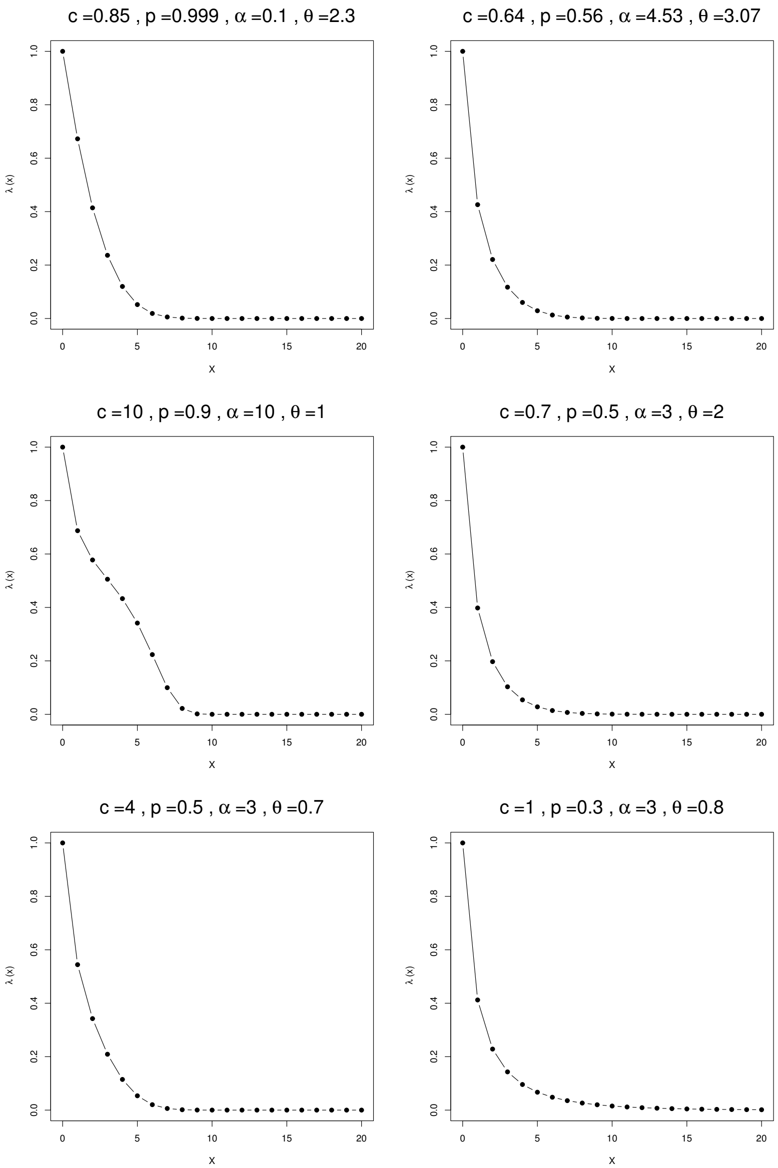

Figure 5 Plots of the DGz-Lomax reversed hazard function with the specified parameters of c, p, α and θ

Figure 5 illustrates how the discrete reversed hazard function of the DGz-Lomax distribution changes with different values of the parameters c, p, α, and θ. Note that all of these cases of parameters of the DGz-Lomax distribution can exhibit decreasing behaviors.

4. Parameter Estimation

One of popular estimation methods is the MLE. It is a frequentist approach. In this article, it is chosen to estimate parameters.

Let $X = (X_1, X_2, \ldots, X_n)$ be an independent and identically distributed (iid) the DGz-Lomax distribution with a parameter vector of $\Theta = (c, p, \alpha, \theta)$. If $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a random sample, then its log-likelihood function is

$$\begin{aligned}\ell(\Theta|\mathbf{x}) &= \log\prod_{i=1}^{n} f_X(x_i|\Theta) = \sum_{i=1}^{n}\log\Big[f_X(x_i|\Theta)\Big] \\ &= \sum_{i=1}^{n}\log\left[p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x_i+\alpha}\right)^{-c\theta}-1\right\}} - p^{\frac{1}{c}\left\{\left(\frac{\alpha}{x_i+\alpha+1}\right)^{-c\theta}-1\right\}}\right].\end{aligned}$$

To estimate the unknown parameters c, p, α and θ, the partial derivatives of the log-likelihood function are taken respectively with the unknown parameters. These nonlinear equations are equated to zero,

$$\frac{\partial\ell(\Theta|\mathbf{x})}{\partial c} = 0, \quad \frac{\partial\ell(\Theta|\mathbf{x})}{\partial p} = 0, \quad \frac{\partial\ell(\Theta|\mathbf{x})}{\partial \alpha} = 0, \quad \frac{\partial\ell(\Theta|\mathbf{x})}{\partial \theta} = 0.$$

To obtain the maximum likelihood estimates, including $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, from the above system of nonlinear equations is complicated and difficult, so Nelder-Mead method utilizing the `optimr` package (Nash, 2014, 2022) in the R programming language is used to solve these nonlinear equations (R Core Team, 2021).

5. Simulation Study

In this section, simulation study for parameter estimation of the DGz-Lomax parameters is carried. The simulations are described as below:

(i) The sample sizes (n) are taken as $n = 20, 30, 50, 100, 200$ and 500.

(ii) The data are generated from

$$X_i = \left\lfloor \alpha\left\{\left(1 + c\left[\frac{\log(1-u_i)}{\log(p)}\right]\right)^{\frac{1}{c\theta}} - 1\right\} - 1\right\rfloor \tag{20}$$

where u_i is the value of a uniform random variable on interval $[0, 1]$. The parameter values are set as four cases,

(a) $c = 0.85, p = 0.999, \alpha = 0.1$ and $\theta = 2.3$;
(b) $c = 0.64, p = 0.56, \alpha = 4.53$ and $\theta = 3.07$;
(c) $c = 10, p = 0.9, \alpha = 10$ and $\theta = 1$;
(d) $c = 0.7, p = 0.5, \alpha = 3$ and $\theta = 2$.

(iii) Each sample size is replicated 1000 times.

(iv) Formulas used for calculating estimates (means), bias and mean square error (MSE) of $\hat{\Theta}$ are given by
$\bar{\hat{\Theta}} = \frac{1}{1000}\sum_{i=1}^{1000}\hat{\Theta}_i$, $Bias(\hat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}_i - \Theta)$ and $MSE(\hat{\Theta}) = \frac{1}{1000}\sum_{i=1}^{1000}(\hat{\Theta}_i - \Theta)^2$.

(v) Step 5 is also repeated for the other parameters.

The results of simulation studies are provided in Tables 1 - 4. It shows that the estimated value of each maximum likelihood estimators is closed to the true value of the parameters of the DGz-Lomax distribution. The MSE values of each maximum likelihood estimators trends to decreasing when the sample sizes is increasing. Since the biasedness generated from maximum likelihood estimators in all cases are small, it can concludes that MLE is an alternative for estimating parameters.

Table 1 The estimates and MSE of the maximum likelihood estimators of the DGz-Lomax distribution with parameters $c = 0.85$, $p = 0.999$, $\alpha = 0.1$ and $\theta = 2.3$

Sample sizes (n)		Parameters c	p	α	θ
20	Estimates	4.2206	0.6595	3.4863	1.6094
	Bias	3.3706	-0.3395	3.3863	-0.6906
	MSE	41.5421	0.2907	40.5662	5.1184
30	Estimates	4.1295	0.6621	3.1203	1.5023
	Bias	3.2795	-0.3369	3.0203	-0.7977
	MSE	39.0304	0.2882	31.3279	3.6871
50	Estimates	3.6514	0.6879	2.8619	1.5389
	Bias	2.8014	-0.3111	2.7619	-0.7611
	MSE	30.0468	0.2586	25.9419	3.4178
100	Estimates	3.1361	0.6807	2.6409	1.5879
	Bias	2.2861	-0.3183	2.5409	-0.7121
	MSE	18.2407	0.2542	19.7034	3.4397
200	Estimates	2.6816	0.6944	2.4283	1.5174
	Bias	1.8316	-0.3046	2.3283	-0.7826
	MSE	11.6996	0.2225	15.7558	2.5615
500	Estimates	2.0395	0.7374	1.9513	1.5222
	Bias	1.1895	-0.2616	1.8513	-0.7778
	MSE	4.7985	0.1576	9.5487	1.8474

Table 2 The estimates and MSE of the maximum likelihood estimators of the DGz-Lomax distribution with parameters $c = 0.64$, $p = 0.56$, $\alpha = 4.53$ and $\theta = 3.07$

Sample sizes (n)		Parameters c	p	α	θ
20	Estimates	3.2302	0.4751	6.2270	2.4310
	Bias	2.5902	-0.0849	1.6970	-0.6390
	MSE	27.0804	0.1794	66.8527	13.8948
30	Estimates	2.7356	0.4536	5.6037	2.4321
	Bias	2.0956	-0.1064	1.0737	-0.6379
	MSE	15.3721	0.1886	43.6879	13.2617
50	Estimates	2.2389	0.4533	5.1918	2.3490
	Bias	1.5989	-0.1067	0.6618	-0.7210
	MSE	8.3660	0.1721	35.9831	10.6432
100	Estimates	1.8164	0.4438	4.5449	2.3073
	Bias	1.1764	-0.1162	0.0149	-0.7627
	MSE	4.3597	0.1508	25.3820	10.0874
200	Estimates	1.5556	0.4382	4.3180	2.2764
	Bias	0.9156	-0.1218	-0.2120	-0.7936
	MSE	2.5874	0.1393	22.1851	7.8179
500	Estimates	1.3722	0.4178	4.1709	2.0769
	Bias	0.7322	-0.1422	-0.3591	-0.9931
	MSE	1.5502	0.1171	16.9881	4.7215

Table 3 The estimates and MSE of the maximum likelihood estimators of the DGz-Lomax distribution with parameters $c = 10$, $p = 0.9$, $\alpha = 10$ and $\theta = 1$

Sample sizes		Parameters			
(n)		c	p	α	θ
20	Estimates	10.7985	0.8838	8.6248	1.4325
	Bias	0.7985	-0.0162	-1.3752	0.4325
	MSE	43.5517	0.0287	92.1335	2.6953
30	Estimates	10.9005	0.8548	9.6601	1.3998
	Bias	0.9005	-0.0452	-0.3399	0.3998
	MSE	44.0064	0.0302	97.9064	2.3363
50	Estimates	10.6804	0.8474	9.7569	1.3359
	Bias	0.6804	-0.0526	-0.2431	0.3359
	MSE	39.7134	0.0223	82.6633	1.6315
100	Estimates	10.3426	0.8294	9.9986	1.2532
	Bias	0.3426	-0.0707	-0.0014	0.2532
	MSE	31.3951	0.0222	67.7894	1.1014
200	Estimates	10.0913	0.8048	11.0121	1.2659
	Bias	0.0913	-0.0952	1.0121	0.2659
	MSE	27.4523	0.0242	61.5658	0.8774
500	Estimates	10.2711	0.7869	10.9215	1.1384
	Bias	0.2711	-0.1131	0.9215	0.1384
	MSE	18.6610	0.0237	43.3738	0.5201

Table 4 The estimates and MSE of the maximum likelihood estimators of the DGz-Lomax distribution with parameters $c = 0.7$, $p = 0.5$, $\alpha = 3$ and $\theta = 2$

Sample sizes		Parameters			
(n)		c	p	α	θ
20	Estimates	3.2760	0.5075	4.1075	1.5991
	Bias	2.5760	0.0075	1.1075	-0.4009
	MSE	23.8538	0.1777	29.4363	4.4887
30	Estimates	2.5415	0.5204	3.6486	1.6400
	Bias	1.8415	0.0204	0.6486	-0.3600
	MSE	12.0539	0.1742	22.9466	4.6514
50	Estimates	2.1565	0.5052	3.5492	1.5663
	Bias	1.4565	0.0052	0.5492	-0.4337
	MSE	7.1778	0.1601	21.2967	4.4231
100	Estimates	1.7698	0.4970	3.5604	1.6834
	Bias	1.0698	-0.0030	0.5604	-0.3166
	MSE	3.6682	0.1494	19.1863	4.1677
200	Estimates	1.4597	0.4554	3.6149	1.6927
	Bias	0.7597	-0.0446	0.6149	-0.3073
	MSE	1.7931	0.1323	18.1863	4.0195
500	Estimates	1.2545	0.4681	3.4092	1.7227
	Bias	0.5545	-0.0319	0.4092	-0.2773
	MSE	1.0491	0.1155	16.3775	2.7459

Figures 6 - 9 are shown how the biases vary with respect to sample sizes. The broken line in Figures 6 - 9 corresponds to the biases being zero. Figures 10 - 13 are shown trending of MSE values in each case when the sample sizes is increasing.

From Figure 6, when $c = 0.85$ and $\alpha = 0.1$, the biases are positive. On the other hand, when $p = 0.999$ and $\theta = 2.3$, the biases are negative. For $c = 0.85$, $p = 0.999$ and $\alpha = 0.1$, the magnitude of biases are decreasing when the sample sizes is increasing. Nevertheless, for $\theta = 2.3$, the magnitude of biases is increasing when the sample sizes is increasing.

From Figure 7, when $c = 0.64$, the biases are positive. When $p = 0.56$ and $\theta = 3.07$, the biases are negative. When $\alpha = 4.53$, the biases are both positive and negative. For $c = 0.64$, the magnitude of biases are decreasing to zero when the sample sizes is increasing. For $p = 0.56$ and $\theta = 3.07$, the magnitude of biases is increasing when the sample sizes is increasing. For $\alpha = 4.53$, the magnitude of biases are decreasing to zero when $n = 20, 30, 50$ and 100, nevertheless, the magnitude of biases is increasing when $n = 200, 500$.

From Figure 8, when $c = 10$ and $\theta = 1$, the biases are positive. When $p = 0.9$, the biases are negative. When $\alpha = 10$, the biases are both positive and negative. For $c = 10$, the magnitude of biases are decreasing to zero when $n = 20, 30, 50, 100$ and 200, nevertheless, the magnitude of biases is increasing when $n = 500$. For $p = 0.9$, the magnitude of biases is increasing when the sample sizes is increasing. For $\alpha = 10$, the magnitude of biases are decreasing to zero when $n = 20, 30, 50$ and 100, nevertheless, the magnitude of biases is increasing when $n = 200$, finally, the magnitude of biases trends to decreasing when $n = 500$. For $\theta = 1$, the magnitude of biases are decreasing to zero when the sample sizes is increasing.

From Figure 9, when $c = 0.7$ and $\alpha = 3$, the biases are positive. When $p = 0.5$, the biases are both positive and negative. When $\theta = 2$, the biases are negative. For $c = 0.7$ and $\alpha = 3$, the magnitude of biases are decreasing to zero when the sample sizes is increasing. For $p = 0.5$, the magnitude of biases trends to increasing when the sample sizes is increasing. For $\theta = 2$, the magnitude of biases trends to decreasing when the sample sizes is increasing.

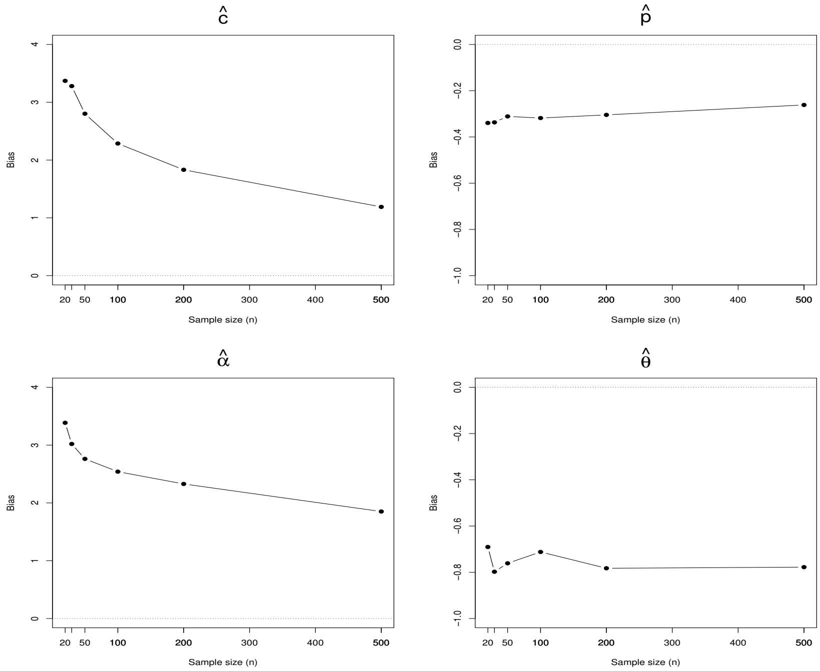

Figure 6 Bias of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for X $\sim$ DGz-Lomax$(0.85, 0.999, 0.1, 2.3)$ with various sample sizes

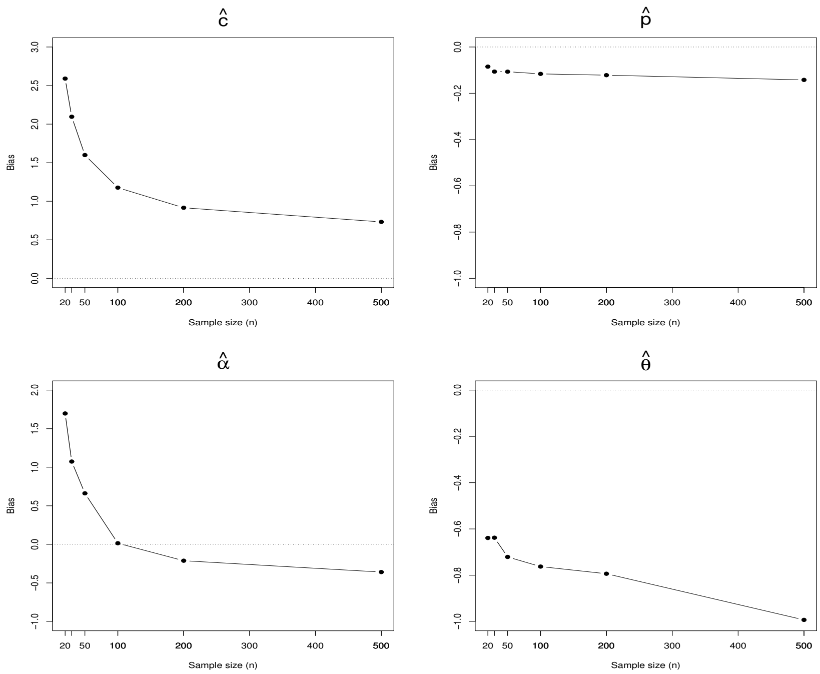

Figure 7 Bias of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for X $\sim$DGz-Lomax$(0.64, 0.56, 4.53, 3.07)$ with various sample sizes

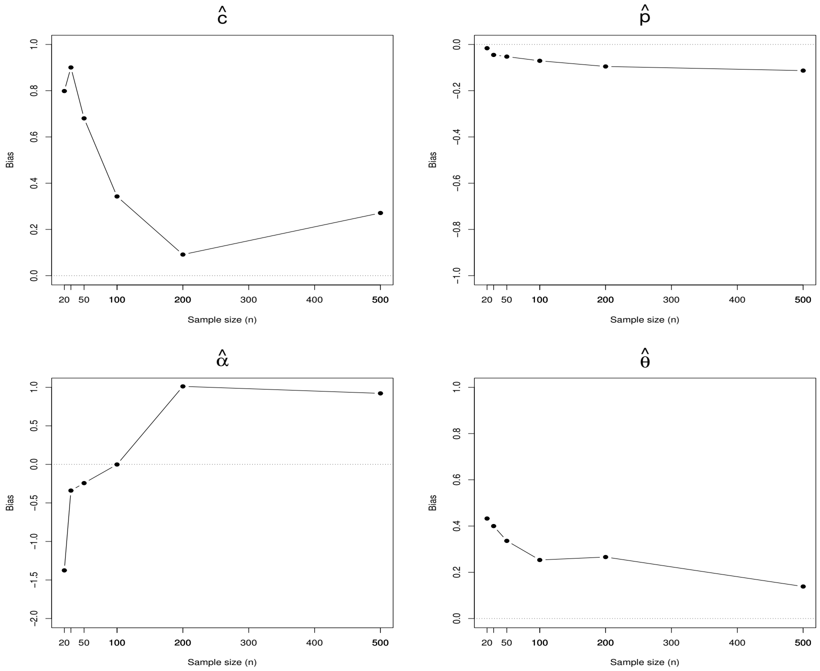

Figure 8 Bias of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for $X \sim$ DGz-Lomax$(10, 0.9, 10, 1)$ with various sample sizes

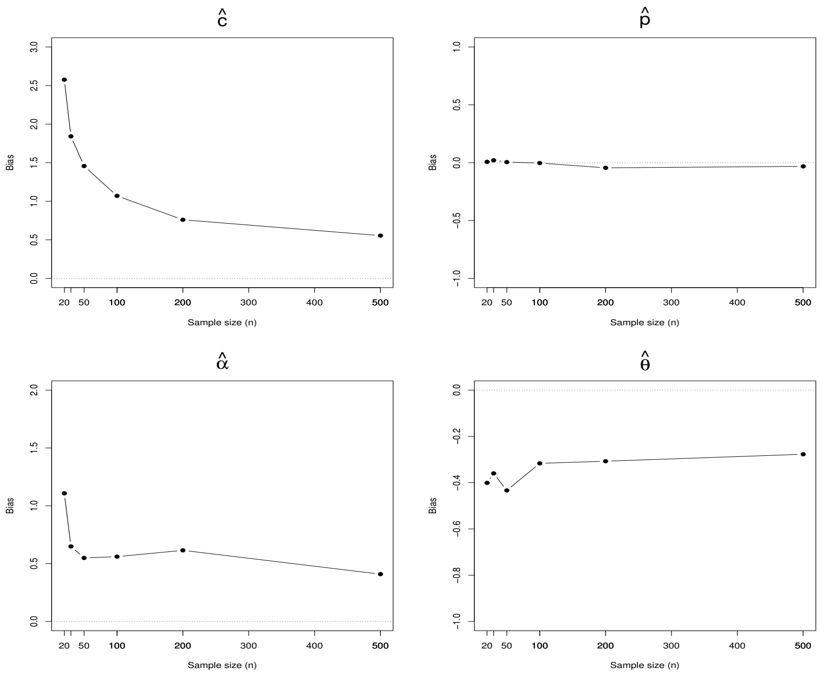

Figure 9 Bias of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for $X \sim$ DGz-Lomax$(0.7, 0.5, 3, 2)$ with various sample sizes

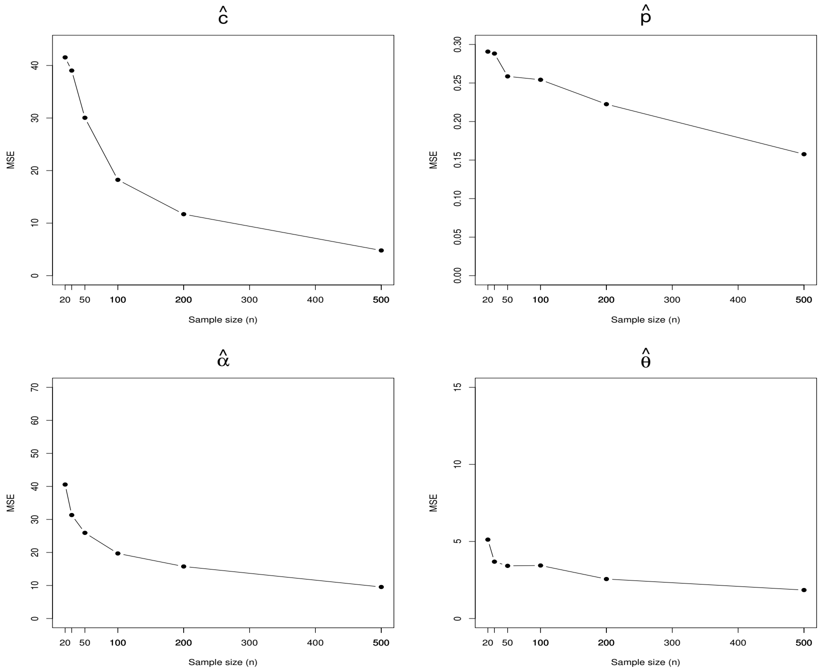

Figure 10 MSE of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for $X \sim$ DGz-Lomax$(0.85, 0.999, 0.1, 2.3)$ with various sample sizes

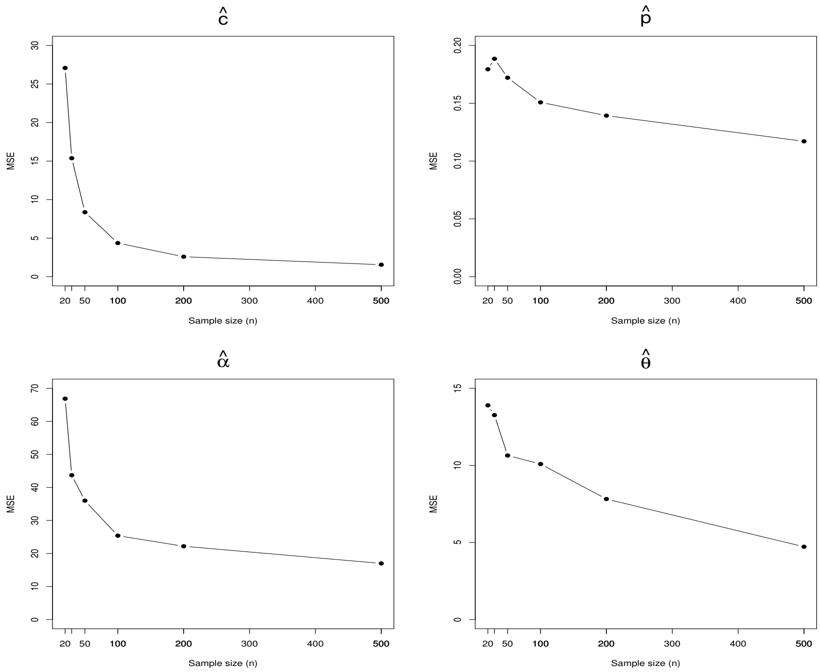

Figure 11 MSE of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for $X \sim$ DGz-Lomax$(0.64, 0.56, 4.53, 3.07)$ with various sample sizes

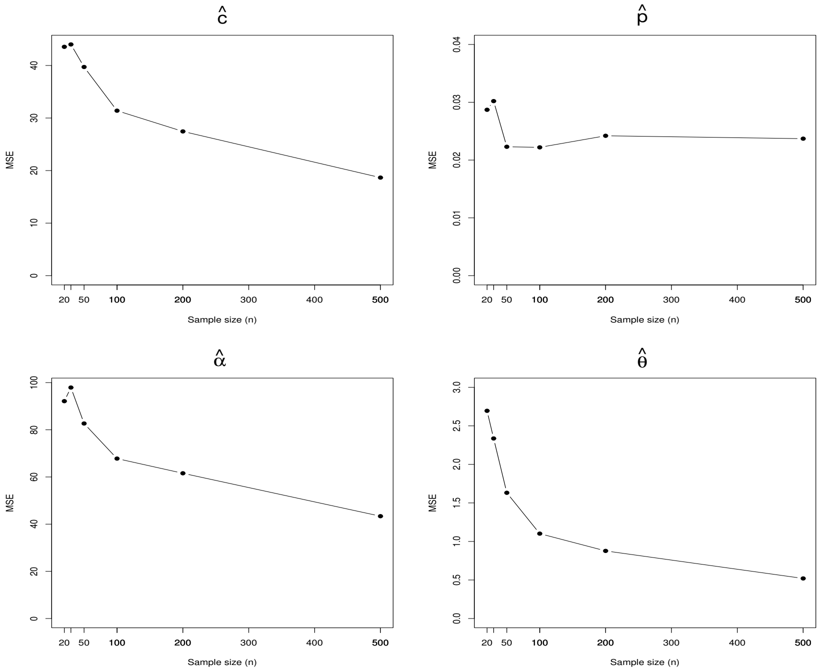

Figure 12 MSE of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for $X \sim$ DGz-Lomax$(10, 0.9, 10, 2)$ with various sample sizes

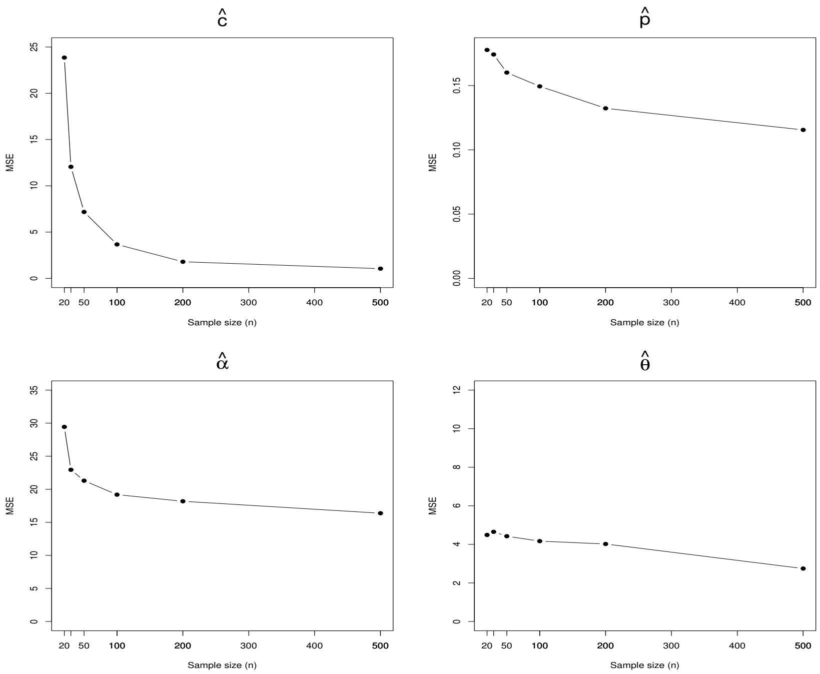

Figure 13 MSE of $\hat{c}$, $\hat{p}$, $\hat{\alpha}$ and $\hat{\theta}$, resulted from maximum likelihood estimators for $X \sim$ DGz-Lomax$(0.7, 0.5, 3, 2)$ with various sample sizes

6. Applications

In this section, the DGz-Lomax distribution is applied to two real data sets. The efficiency for fitted models of four competitive distributions which the DGz-Exp, the DGz-Wei and the DGz-Inv Wei distributions are compared with the proposed distribution. Minus log-likelihood (-LL) and discrete Anderson-Darling (AD) test are criteria.

The first data set (Data1) represents the number of goals per match scored by individual teams in the National Hockey League 1966-67 (Reep et al., 1971). The mean, variance, and ID values of Data1 are 2.9786, 3.5389 and 1.1881 respectively (420 observations). The behavior of Data1 is unimodality.

The second data set (Data2) represents the number of unit areas of larvae in some experiment (beall, 1940). The mean, variance, and ID values of Data2 are 1.400, 2.3272 and 1.6623 respectively (325 observations). The behavior of Data2 is reverse J-shape (decreasing).

The third data set (Data3) represents the life lengths (in days) of 35 conventional mice in some experiment. These mice were given a dose of 300 rads of radiation at 5-6 weeks of age and followed to death. The data are as follow: 163, 179, 206, 222, 228, 249, 252, 282, 324, 333, 341, 366, 385, 407, 420, 431, 441, 461, 462, 482, 517, 517, 524, 564, 567, 586, 619, 620, 621, 622, 647, 651, 686, 761, 763 (Hoel and Walburg , 1972). The mean and variance of Data3 are 454.2571 and 29015.6700 respectively.

Three selected distributions, including, negative binomial (NB) (Greenwood and Yule, 1920), discrete exponential (DExp) (Roy, 2004), discrete gamma (DGam) (Chakraborty and Chakravarty , 2012), DGz-Exp, DGz-Wei and DGz-Inv Wei distributions are compared to conclude their performances with the DGz-Lomax distribution. The result shows that in the Data2 and Data3, the DGz-Lomax distribution has a smallest minus log-likelihood and smallest AD test statistic value. We can conclude that the DGz-Lomax distribution is the best model to fit these data sets.

In the Data1, although the DGz-Lomax distribution has minus log-likelihood more than the NB distribution, the DGz-Lomax distribution has AD value less than the NB distribution. Moreover, the DGz-Lomax has a smallest AD value when comparing with NB, DExp, DGam, DGz-Exp, DGz-Wei and DGz-Inv Wei distributions. In the Data2 and Data3, the DGz-Lomax distribution has a smallest minus log-likelihood and smallest AD test statistic value. We can conclude that the DGz-Lomax distribution is the alternative distribution to fit these data sets.

From the results in Table 5, the NB distribution gives the smallest minus log-likelihood but the DGz-Lomax distribution gives the smallest discrete AD test statistic. From the results in Table 6 and Table 7, the DGz-Lomax distribution gives the smallest value of both minus log-likelihood and discrete AD test statistic. Figures 14, 15 and 16 are supported the results shown in Tables 5, 6 and 7, respectively.

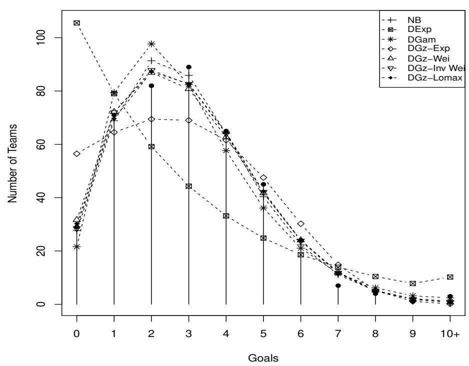

Figure 14 The empirical and fitted distribution plots for Data1

Table 5 Observed and expected frequencies for the number of goals per match scored from the National Hockey League 1966-67

Goals	Number of Teams (Observed Frequency)	Expected Frequency						
		NB	DExp	DGam	DGz-Exp	DGz-Wei	DGz-Inv Wei	DGz-Lomax
0	29	27.6630	105.5250	21.5883	56.4820	31.6573	28.4595	30.2574
1	71	68.8457	79.0118	79.4553	64.5169	72.1603	71.9654	69.6249
2	82	91.3555	59.1601	97.6678	69.4485	86.9105	87.8109	87.2000
3	89	85.8471	44.2961	82.3531	69.0094	80.8190	82.5365	82.7223
4	65	64.0484	33.1667	57.6495	61.5711	62.7010	63.9997	64.3310
5	45	40.3440	24.8336	36.1270	47.5305	41.7110	42.0839	42.3351
6	24	22.2879	18.5942	21.0376	30.2166	24.0382	23.7975	23.9566
7	7	11.0799	13.9224	11.6274	14.8168	12.0267	11.6657	11.7655
8	4	5.0483	10.4244	6.1803	5.1402	5.2093	4.9822	5.0442
9	1	2.1373	7.8053	3.1870	1.1263	1.9411	1.8599	1.8953
10+	3	1.1695	10.2200	2.3958	0.1416	0.7814	0.7827	0.8079
Estimated Parameters		$\hat{r} = 15.0659$ $\hat{p} = 0.8348$	$\hat{\theta} = 0.7488$	$\hat{k} = 3.1444$ $\hat{\theta} = 0.9025$	$\hat{a} = 0.5633$ $\hat{c} = 0.5364$ $\hat{p} = 0.8028$	$\hat{a} = 0.0052$ $\hat{b} = 1.8513$ $\hat{c} = 0.6126$ $\hat{p} = 0.0000$	$\hat{a} = 0.8164$ $\hat{b} = 1.1583$ $\hat{c} = 1.7464$ $\hat{p} = 0.9331$	$\hat{c} = 0.8543$ $\hat{p} = 0.9994$ $\hat{\alpha} = 0.1046$ $\hat{\theta} = 2.3300$
-LL		839.8084	942.4152	844.5137	859.5346	840.7894	840.2656	840.2186
Discrete AD Statistic		0.1997	36.9990	1.3433	4.3319	0.3492	0.1828	0.1780
p-value		0.9575	< 0.05	0.2166	0.0056	0.8412	0.9669	0.9690

Table 6 Observed and expected frequencies for the number of larvae in each unit areas

Number of Larvae	Number of Unit Areas (Observed Frequency)	Expected Frequency						
		NB	DExp	DGam	DGz-Exp	DGz-Wei	DGz-Inv Wei	DGz-Lomax
0	117	113.6753	135.3950	114.1515	118.8692	116.6843	111.1111	116.5626
1	87	91.8655	78.9894	93.5993	83.7843	86.3576	100.2615	85.7960
2	50	56.5425	46.0824	55.0907	55.0439	55.7420	53.6773	56.4191
3	38	31.1706	26.8845	30.0260	33.3671	32.9931	28.1365	33.3981
4	21	16.1828	15.6844	15.7819	18.4491	17.9257	14.8300	17.8857
5	7	8.0899	9.1503	8.1205	9.1825	8.8907	7.8742	8.6949
6	2	3.9403	5.3383	4.1203	4.0528	3.9916	4.2065	3.8467
7	2	1.8830	3.1144	2.0700	1.5592	1.6054	2.2576	1.5516
8	0	0	0	0	0	0	0	0
9	1	0.4130	1.0600	0.5118	0.1410	0.1775	0.6562	0.1924
Estimated Parameters		$\hat{r} = 1.9112$ $\hat{p} = 0.5772$	$\hat{\theta} = 0.5834$	$\hat{k} = 1.4051$ $\hat{\theta} = 0.7469$	$\hat{a} = 0.1079$ $\hat{c} = 1.2605$ $\hat{p} = 0.0195$	$\hat{a} = 0.1297$ $\hat{b} = 1.1072$ $\hat{c} = 0.6463$ $\hat{p} = 0.0374$	$\hat{a} = 3.1739$ $\hat{b} = 0.4207$ $\hat{c} = 3.1714$ $\hat{p} = 0.0001$	$\hat{c} = 0.6441$ $\hat{p} = 0.5535$ $\hat{\alpha} = 4.5310$ $\hat{\theta} = 3.0711$
-LL		524.0024	529.7708	524.6161	523.2982	523.2069	526.6207	523.0943
Discrete AD Statistic		0.2814	2.1789	0.3578	0.0998	0.1008	0.7474	0.0994
p-value		0.7904	0.0545	0.7064	0.9715	0.9707	0.3958	0.9717

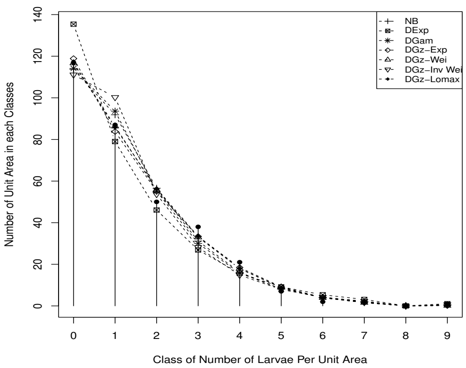

Figure 15 The empirical and fitted distribution plots for Data2

Table 7 Fitted estimates for life lengths of conventional mice caused radiation

	Distribution						
	NB	DExp	DGam	DGz-Exp	DGz-Wei	DGz-Inv Wei	DGz-Lomax
Estimated Parameters	$\hat{r} = 6.4884$ $\hat{p} = 0.0141$	$\hat{\theta} = 0.9978$	$\hat{k} = 6.3978$ $\hat{\theta} = 0.0141$	$\hat{a} = 0.0061$ $\hat{c} = 0.9637$ $\hat{p} = 0.9561$	$\hat{a} = 0.0025$ $\hat{b} = 1.1349$ $\hat{c} = 0.8613$ $\hat{p} = 0.9381$	$\hat{a} = 0.2572$ $\hat{b} = 0.3298$ $\hat{c} = 5.2141$ $\hat{p} = 0.9999$	$\hat{c} = 5.2771$ $\hat{p} = 0.9999$ $\hat{\alpha} = 3.0521$ $\hat{\theta} = 0.5384$
-LL	229.4718	249.1920	229.4784	228.9609	228.9137	235.4819	228.4635
Discrete AD Statistic	0.4931	4.9280	0.4940	0.2738	0.2768	1.7572	0.2347
p-value	0.6903	0.0014	0.6891	0.9561	0.9381	0.0930	0.9487

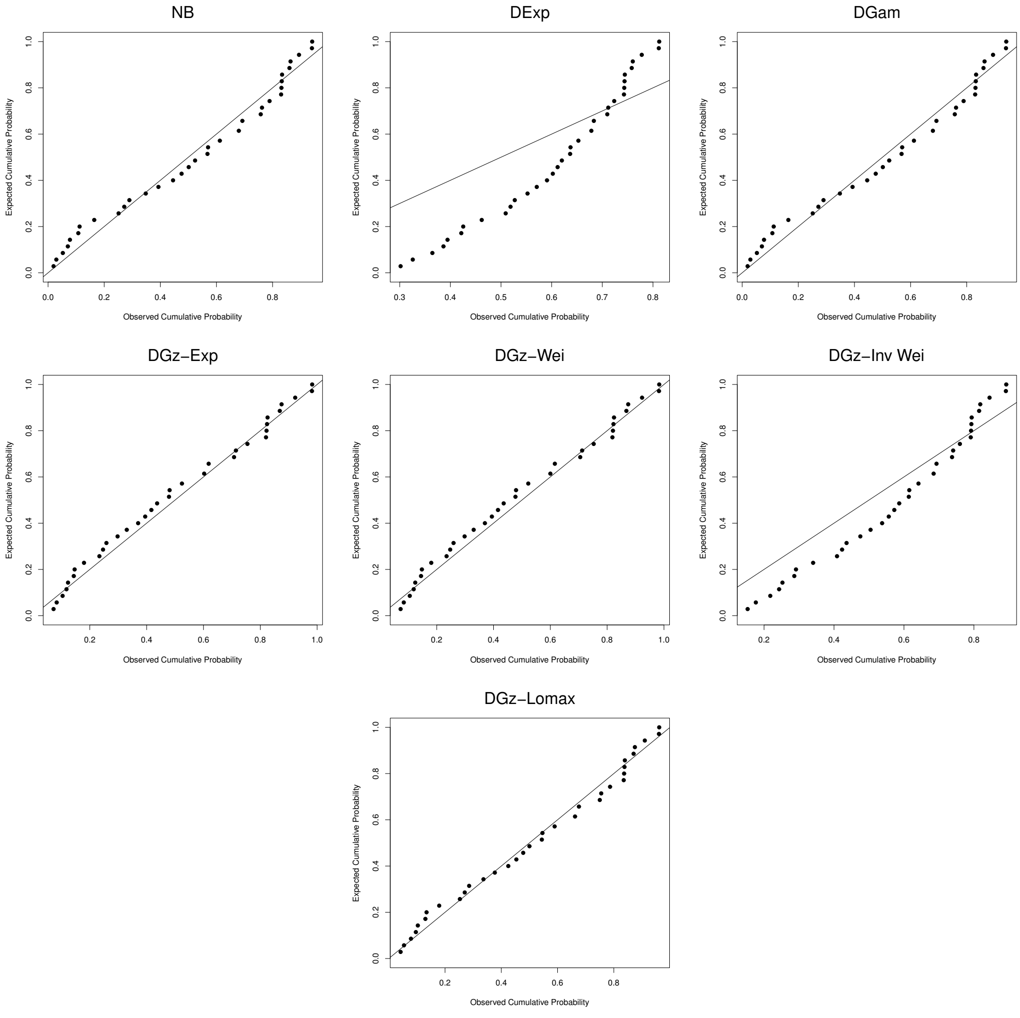

Figure 16 PP-Plots for all fitted distributions of Data3

7. Conclusions and Discussion

The new alternative distribution, called the discrete Gompertz-Lomax distribution, is proposed. It is interesting to note that Roy's method has been used to generate the probability mass function of this distribution. The probability mass function of the discrete Gompertz-Lomax distribution can exhibit two shapes, namely unimodality and reverse J-shape (decreasing). Some properties of this distribution have been discussed which can help in understanding its behavior and making statistical inference. The maximum likelihood estimation has been used to estimate the unknown parameters of the distribution. It is good to know that the maximum likelihood estimators have high efficiency when the sample size is large, as shown in the simulation study results. It is also interesting to see that in three real data sets, the discrete Gompertz-Lomax distribution has shown better fit compared to other discrete analogue distributions, such as the discrete exponential, the discrete gamma, the discrete Gompertz-exponential, the discrete Gompertz-Weibull, and the discrete Gompertz-inverse Weibull distributions. In the other hands, the discrete Gompertz-Lomax distribution has shown better fit compared to the negative binomial distribution in some real data sets. However, in other real data sets, its efficiency of fitting may be less than the other distributions. This indicates that the performance of the discrete Gompertz-Lomax distribution may vary depending on the data being modeled. It is good to see that future work is planned to analyze the goodness of fit test of the data by using other distributions to compare the efficiency of the discrete Gompertz-Lomax distribution. This will help in assessing the performance of the distribution and its usefulness in different applications.

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