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A New Generalized Exponentiated Weibull Distribution: Properties and Applications

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Abstract

The Weibull distribution is the most important statistical distribution to study the data especially from reliability theory. We propose a new generalized exponentiated Weibull distribution which is developed by using the exponentiated Weibull distribution and exponentiated generalized Weibull distribution. Various well-known lifetime distributions are particular case of the proposed distribution. The failure rate of the newly constructed distribution is monotone and non-monotone such as bathtub, unimodal, increasing and decreasing. Thus, the proposed model seems more flexible. Some important mathematical properties of the proposed distribution are studied and simple expressions for the generating function, moments, mean deviations, entropy, and order statistics density are provided. Some important aspects of the distribution are also discussed numerically and graphically. Parameters are estimated by using popular technique of maximum likelihood. We apply newly developed model to two real data sets and make comparison with some well-known sub-models. We explore that the proposed model is more flexible and useful as compared to the particular sub-models for modeling lifetime, skewed and survival time data.

Keywords: Weibull model, lifetime distributions, exponentiated generalized distribution, moments, maximum likelihood estimation.

1. Introduction

Problems and models are intertwined and various models have been developed to solve these problems. A class of such models developed by using Weibull distribution are often called Weibull models. Though, these models can be used to study problems in reliabiliy, biology, engineering, medical and other life sciences. Therefore, Weibull distribution is consider as a fundamental lifetime distribution.

Firstly, Fisher and Tippett (1928) studied the Weibull distribution as limiting distribution of extreme minimum in a sample. Weibull (1939) introduced the Weibull model for studying material's strength. Then, Weihull (1951) used the Weibull distribution to study the problems from various disciplines and illustrated its importance. The Weibull distribution study only monotone hazard rate functions (hrfs). There are many systems in reliability and survival theory that follow the hrfs that are not monotone like unimodal and bathtub shaped. To study such complex systems with non-monotone hrfs, various authors generalized the Weibull model.

Gupta et al. (1998) proposed prominent exponentiated exponential (EE) distribution, which is well-known as proportional reversed hazard rate model (PRHRM), with adding one extra and additional shape pararmeter to the exponential model. Gupta and Kundu (2001) discussed some important characteristics of the EE distribution and noted that it has many properties similar to those of the Weibull and gamma models. These three models are extension of the exponential model by distinct approaches. Mudholkar and Srivastava (1993) developed the exponentiated Weibull (EW) distribution by

$$H_1(x) = [1 - \mathrm{e}^{-(\lambda x)^{\delta}}]^{\alpha}. \tag{1}$$

The EE, Weibull and exponential models are particular examples of the EW distribution. Mudholkar et al. (1995) mentioned some generalized distributions developed by some authors to study non-monotone hrfs. Those distributions were not parsimonious in parameters and experienced complications to study censored data. The hrf of the EW examined by dividing parameter space into four parts where the hrf is increasing, bathtub, decreasing and unimodal and noted that the EW distribution gives sometimes remarkable results. It is noted that the EW distribution is suitable for approximating the extreme value distributions when samples size is moderate and data is censored.

Some well-known authors including Jiang and Murthy (1999), Nassar and Eissa (2003), Pal et al. (2006), Lai et al. (2011) and Nadarajah et al. (2013) carefully studied important characteristics of the EW model. Famoye et al. (2005) generalized the Weibull model by adopting the beta generalized distribution. Lee et al. (2007) discussed some important characteristics of this distribution. Cordeiro et al. (2018) introduced the Lindley Weibull distribution. kumar and Nair (2018) worked on the log-inverse Weibull distribution and discussed its important properties.

Nadarajah and Kotz (2006) proposed another type of exponentiated distribution $H_2(x) = [1 - (1-F(x))^{\beta}]$ to generalize prominent extreme value Gumbel and Fréchet distributions. Cordeiro et al. (2013) developed the exponentiated generalized (EG) distribution as $H_3(x) = \{H_2(x)\}^{\gamma}$, here both β and γ are non-negative real and shape parameters of the EG distribution. Elbatal and Muhammed (2014) developed the EG inverse Weibull (EGIW) model and carefully studied its properties. Oguntunde et al. (2015) considered the EG Weibull (EGW) distribution $H_3(x) = \left[1 - \mathrm{e}^{-\beta(\frac{x}{\theta})^{\delta}}\right]^{\gamma}$. By considering the parameter $\lambda = \frac{1}{\theta}$, the EGW distribution is defined by

$$H_3(x) = [1 - \mathrm{e}^{-\beta(\lambda x)^{\delta}}]^{\gamma}. \tag{2}$$

Abbas et al. (2022) generalized the weighted exponential distribution by using the EG distribution and discussed its important characteristics.

The rest of the article is organized in following way. In Section 2, we define a new generalized exponentiated Weibull distribution, which is generalization of the Weibull distribution (which is extensively studied), by adopting a new generalized class of distributions (Ashraf et al., 2022). In Section 3, some basic properties such as the mode, density function shapes, hazard rate function (hrf) shapes, limiting behavior of the density function and hrf, reversed hazard rate function (rhrf) and survival function are discussed. A useful expansion of distribution and density function is provided in Section 4. We provide explicit expressions of some important mathematical features including the moments, mean deviations, Lorenz and Bonferroni curve, generating function, entropy measures and order statistics density in Section 5. The maximum likelihood (ML) technique is discussed in Section 6. In Section 7, two real data sets are studied to illustrate the usefulness and appropriateness of proposed distribution for modeling the data. Lastly, final remarks are given in Section 8.

2. A New Generalized Exponentiated Weibull Distribution

By using (1) and (2), cumulative distribution function (cdf) of new generalized exponentiated Weibull (NGEW) distribution is

$$F(x) = [1 - \mathrm{e}^{-(\lambda x)^{\delta}}]^{\alpha}[1 - \mathrm{e}^{-\beta(\lambda x)^{\delta}}]^{\gamma}. \tag{3}$$

Hereafter, we suppose that $H(x) = [1 - \mathrm{e}^{-(\lambda x)^{\delta}}]$ and $H(x;\beta) = [1 - \mathrm{e}^{-\beta(\lambda x)^{\delta}}]$. Thus, Eqn.(3) becomes

$$F(x) = \{H(x)\}^{\alpha}\{H(x;\beta)\}^{\gamma}, \tag{4}$$

Then, the NGEW probability density function (pdf) is

$$f(x) = \alpha h(x)\{H(x)\}^{\alpha-1}\{H(x;\beta)\}^{\gamma} + \gamma h(x;\beta)\{H(x)\}^{\alpha}\{H(x;\beta)\}^{\gamma-1}, \tag{5}$$

where $x, \lambda, \delta, \alpha, \beta, \gamma > 0$. A random variable X follows (5) can simply be specified by $X \sim NGEW(\alpha, \beta, \gamma, \lambda, \delta)$. It is noted that the NGEW is the product of the multiplicative Weibull model $H_1(x)$ and the multiplicative Weibull competing risk model $H_2(x)$. These models are among various generalizations of the Weibull model by Lai et al. (2011) and Murthy et al. (2004). The NGEW model is quite flexible because some popular lifetime distributions are special cases of (5). These sub-cases are given by Table 1.

Table 1 Important sub-cases of the NGEW distribution

λ	δ	α	β	γ	Distribution	Reference
1	1	α	β	γ	New Generalized Exponential	Ashraf et al. (2022)
λ	1	0	β	γ	Exponentiated Generalized Exponential	Cordeiro et al. (2013)
λ	δ	0	1	γ	Exponentiated Weibull	Mudholkar and Srivastava (1993)
λ	2	0	1	γ	Burr type X	Ahmad Sartawi and Abu-Salih (1991)
λ	1	0	1	γ	Exponentiated Exponential	Gupta et al. (1998)
λ	δ	0	1	1	Weibull	Weibull (1939)
λ	2	0	1	1	Rayleigh	F.R.S. and L.R. (1880)
λ	1	0	1	1	Exponential	Johnson et al. (1994)

3. Basic Properties

3.1. The mode

Differentiate the Eqn.(5) and equate the result to zero, the mode of the X is given as

$$\begin{aligned}
&\alpha\{H(x;\beta)\}^{2}\Big[(\delta-1)(H(x)) - \delta\lambda^{\delta}x^{\delta}(1-\alpha\mathrm{e}^{-(\lambda x)^{\delta}})\Big] \\
&+ \beta\gamma\mathrm{e}^{-(\beta-1)(\lambda x)^{\delta}}\{H(x)\}^{2}\Big[(\delta-1)H(x;\beta) - \beta\delta\lambda^{\delta}x^{\delta}(1-\gamma\mathrm{e}^{-\beta(\lambda x)^{\delta}})\Big] \\
&+ 2\alpha\beta\gamma\delta\lambda^{\delta}x^{\delta}\mathrm{e}^{-\beta(\lambda x)^{\delta}}H(x)H(x;\beta) = 0.
\end{aligned} \tag{6}$$

The Eqn. (6) provides the mode for different parameter values. To obtain the maximum, minimum and inflection points of (5), we will examine the sign of $f^{''}(x)$ at value of the x which solves the Eqn. (6). For $\gamma = 0$, the mode of the X reduces

$$(\delta-1)x^{-1} - \delta\lambda^{\delta}x^{\delta-1} + (\alpha-1)\delta\lambda^{\delta}x^{\delta-1}\{H(x)\}^{-1} = 0. \tag{7}$$

Eqn. (7) provides the mode of (1) which basically derived by Nassar and Eissa (2003). The approximate expressions for mode are derived from (6) for some sub-cases by following the techniques used by Nassar and Eissa (2003) are given in Table 2.

Table 2 The mode of sub-models when $\lambda = 1$

$(\alpha, \beta, \gamma, \delta)$	Sub-Model	Mode
$(0, 1, \gamma, \delta)$	Exponentiated Weibull	$\left[\frac{2(\delta\gamma-1)}{\delta(\gamma+1)}\right]^{\frac{1}{\delta}}$
$(0, 1, \gamma, 2)$	Burr	$\left[\frac{2\gamma-1}{\gamma+1}\right]^{\frac{1}{2}}$
$(0, 1, \gamma, 1)$	Exponentiated exponential	$\frac{2(\gamma-1)}{\gamma+1}$
$(0, 1, 1, \delta)$	Weibull	$\left[\frac{\delta-1}{\delta}\right]^{\frac{1}{\delta}}$
$(0, 1, 1, 2)$	Rayleigh	$\frac{1}{\sqrt{2}}$
$(0, 1, 1, 1)$	Exponential	0

3.2. Shapes of density

i. For $\delta(\alpha + \gamma) \leq 1$ and $\beta > 0$, the NGEW density is reversed J-shaped. It is monotonically decreasing with heavy and light right tails.

ii. For $\delta(\alpha + \gamma) > 1$ and $\beta > 0$, the NGEW density is unimodal, positively skewed with heavier and lighter right tails and skewness increases when $\beta \to \infty$.

iii. For $\delta = \beta = 1, \alpha \leq 1, \gamma = 0$ or $\delta = \beta = 1, \alpha = 0, \gamma \leq 1$ or $\delta = \beta = 1, \alpha + \gamma \leq 1$, the NGEW density is strictly decreasing.

iv. For $\delta > 2, \alpha > 1, \gamma = 0, \beta = 1$ or $\delta = \beta = 1, \alpha = 0, \gamma > 1$ or $\delta = \beta = 1, \alpha + \gamma > 1$, the NGEW density is unimodal and positive skewed.

v. For $\beta \to 0, \delta > 2$ and $\alpha > \gamma$, the NGEW density is bimodal and it reduces to unimodal and symmetric as difference between α and γ decreases.

The visual representation of (5) with selective parameter values is presented by Figure 1. The density of the X takes various forms for different parameter values. Thus, the NGEW model is more appropriate and useful than its special sub-models.

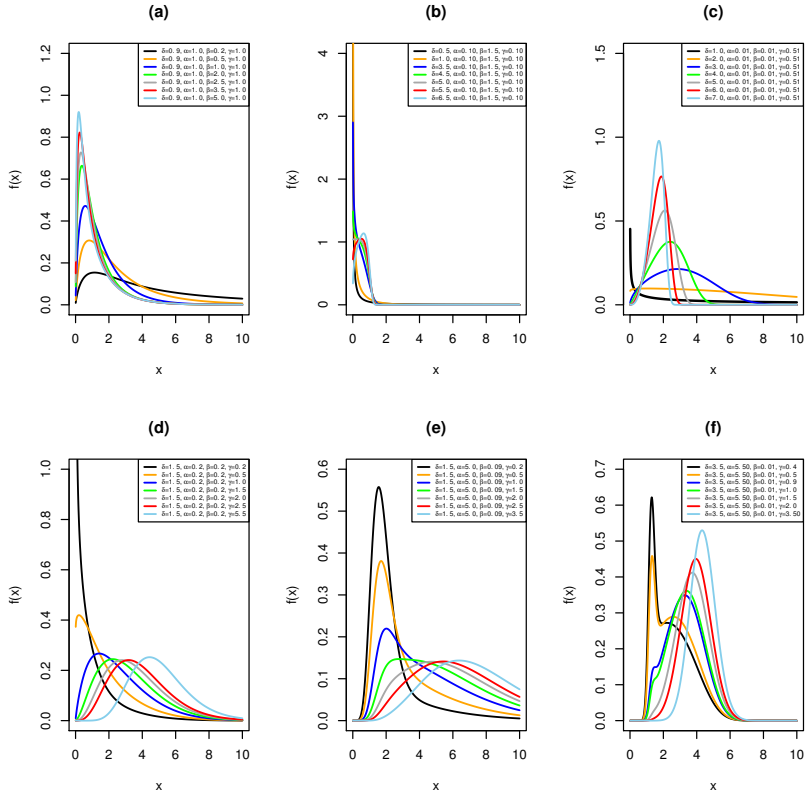


Figure 1 Plots of NGEW pdf (5) with specified parameters values.

3.3. Hazard rate function

This function has vital role in reliability and survival theory. The hrf of (5) is

$$h(x) = \left[\frac{\{H(x)\}^{\alpha}\{H(x;\beta)\}^{\gamma}}{1-\{H(x)\}^{\alpha}\{H(x;\beta)\}^{\gamma}}\right]\left[\frac{\alpha h(x)}{H(x)} + \frac{\gamma h(x;\beta)}{H(x;\beta)}\right]. \tag{8}$$

The graphs for the hrf of the X are represented in Figure 2, which illustrate that the hrf of the X is monotone as well as non-monotone, for instance increasing, decreasing, bathtub shaped and unimodal. Some detail about the shapes of the hrf of the X is given in Subsection 3.4.

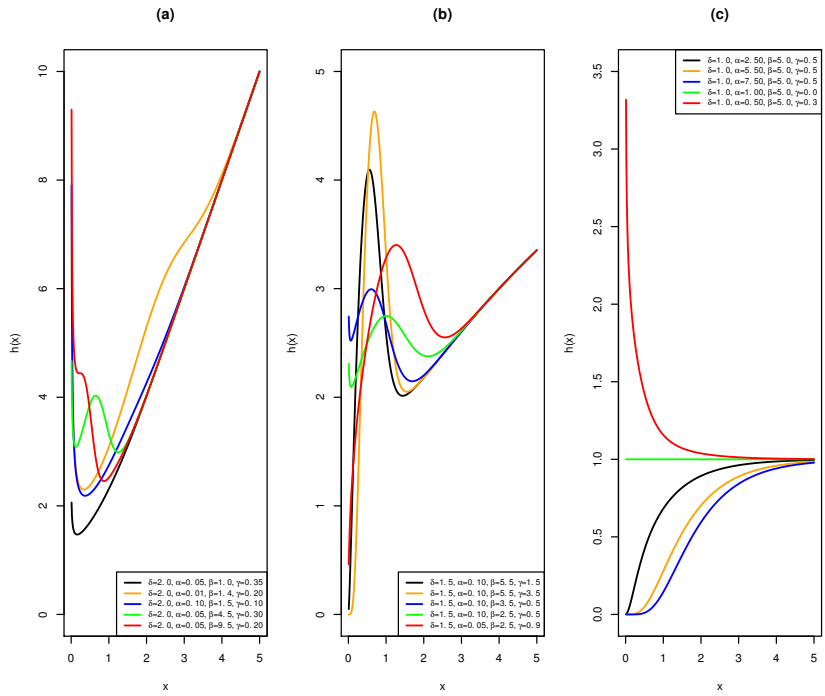


Figure 2 The NGEW hrf with selective parameter values.

3.4. Shapes of the hrf

i. For $\delta(\alpha+\gamma) > 1$ and $\beta > 0$, the hrf of the X is uni-model with unique change point.

ii. For $\delta(\alpha+\gamma) \leq 1$ and $\beta > 0$, (8) is monotonically decreasing.

iii. For $\delta > 1 : \delta\alpha < 1, \gamma = 0$ or $\alpha = 0, \beta = 1, \delta\gamma < 1$ or $\beta = 1, \delta(\alpha+\gamma) < 1$, the hrf of the X is bathtub shapes with point of change is unique.

iv. For $\delta < 1$: $\delta\alpha > 1, \gamma = 0$ or $\alpha = 0, \beta = 1, \delta\gamma > 1$ or $\beta = 1, \delta(\alpha+\gamma) > 1$, the hrf of the X is uni-modal with point of change is unique.

v. For $\delta > 1$: $\delta\alpha > 1, \gamma = 0$ or $\alpha = 0, \beta = 1, \delta\gamma > 1$ or $\beta = 1, \delta(\alpha+\gamma) > 1$, the hrf of the X is monotonically increasing.

vi. For $\delta < 1$: $\delta\alpha < 1, \gamma = 0$ or $\alpha = 0, \beta = 1, \delta\gamma < 1$ or $\beta = 1, \delta(\alpha+\gamma) < 1$, the hrf of the X is monotonically decreasing.

vii. For $\delta = 1$: $\alpha \leq 1, \gamma = 0$ or $\beta = 1, \alpha = 0, \gamma \leq 1$ or $\alpha+\gamma \leq 1, \beta = 1$, the hrf of the X is decreasing from ∞ to λ

viii. For $\delta = 1$: $\alpha > 1, \beta = 1, \gamma = 0$ or $\alpha = 0, \gamma > 1, \beta = 1$ or $\alpha+\gamma > 1, \beta = 1$, the hrf of the X is increasing from 0 to λ.

ix. For $\delta = \beta = \gamma = 1$ and $\alpha = 0$, the hrf of the X is constant λ.

3.5. The rhrf

The rhrf of the X is

$$r(x) = \frac{d}{dx}[\ln\{F(x)\}] = \frac{\alpha h(x)}{H(x)} + \frac{\gamma h(x;\beta)}{H(x;\beta)}. \tag{9}$$

The rhrf (9) is sum of the EW and EGW rhrfs. For $\gamma = 0$, the rhrf (9) is proportionally equal to the Weibull hrf.

3.6. Survival function

The survival function (sf) is likelihood of surviving after some definite time and is monotonically decreasing continuous function of time. The sf can be employed for comparison of the survival distributions and to obtain the survival time's percentiles. Thus, the sf for the NGEW distribution is

$$\begin{aligned} S(x) &= P(X > x) = \int_x^{\infty} f(t)d(t) \\ &= \left[1 - \{H(x)\}^{\alpha}\{H(x;\beta)\}^{\gamma}\right]. \end{aligned}$$

4. Linear Presentation

We use the power series any real θ as

$$(1-z)^{\theta} = \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\theta)}{\Gamma(\theta - i)i!} z^i, \quad \forall\, |z| < 1. \tag{10}$$

By using (10) in (4), The NGEW df is

$$F(x) = \sum_{j=0}^{\infty} \nu_j W_{\alpha+j}(x), \tag{11}$$

where $\nu_j = \frac{(-1)^j \Gamma(\gamma+1)}{j!} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(k\beta+1)}{\Gamma(k\beta-j+1)\Gamma(\gamma-k+1)k!}$ and $W_{\alpha+j}(x) = \{H(x)\}^{\alpha+j}$ is the EW cdf and $\alpha + j$ is power parameter. By differentiating Eqn.(11), we obtain

$$f(x) = \sum_{j=0}^{\infty} \nu_j w_{\alpha+j}(x), \tag{12}$$

where $w_{\alpha+j}(x)$ is the EW density and $\alpha + j$ is power parameter. Eqn.(12) demonstrates that the NGEW pdf is a simply linear mixture of the EW densities. Hence, various important statistical characteristics of (5) can easily be determined from those characteristics of the EW distribution.

5. Mathematical and Statistical Properties

Present section contains extensive mathematical and statistical properties of the X, like, moments, generating function, mean deviations, Bonferroni and Lorenz curves, Rényi entropy and density function of the *r*-th order statistic.

5.1. The moments

Moments usually use to explore very significant features of the X including average, dispersion, skewness and kurtosis. By substituting the (10) in $E(x^r) = \int_{-\infty}^{\infty} x^r f(x)dx$, set $(i+1)(\lambda x)^{\delta} = u$ and after some mathematics, we obtain

$$E(X^r) = \lambda^{-r}\Gamma(\frac{r}{\delta} + 1) \sum_{j=0}^{\infty} \nu_j(\alpha + j) \sum_{i=0}^{\infty} \frac{(-1)^i \Gamma(\alpha + j)}{i!(i+1)^{\frac{r+\delta}{\delta}} \Gamma(\alpha + j - i)}.$$

For any $r > -\delta$. By following the relation $(1-\alpha)_i = \frac{(-1)^i \Gamma(\alpha)}{\Gamma(\alpha-i)}$ in last equation, we have

$$E(X^r) = \lambda^{-r}\Gamma(\frac{r}{\delta}+1)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{\nu_j(\alpha+j)(1-\alpha-j)_i}{i!(i+1)^{\frac{r+\delta}{\delta}}}. \tag{13}$$

For $r > -\delta$. If $\frac{r}{\delta} = p \geq 1$ is any integer, another useful expression for the moments of the X is given as

$$E(X^r) = \lambda^{-r}(-1)^p \sum_{j=0}^{\infty}\nu_j(\alpha+j)\frac{\partial^p}{\partial y^p}\{B(\alpha+j,y)\}\Big|_{y=1}. \tag{14}$$

The values of first two moments, median, mode, standard deviation, skewness and kurtosis of (5) with particular parameter values are provided in Table 3. We illustrate the shape parameters impact on these measures in Figures 3 and 4.

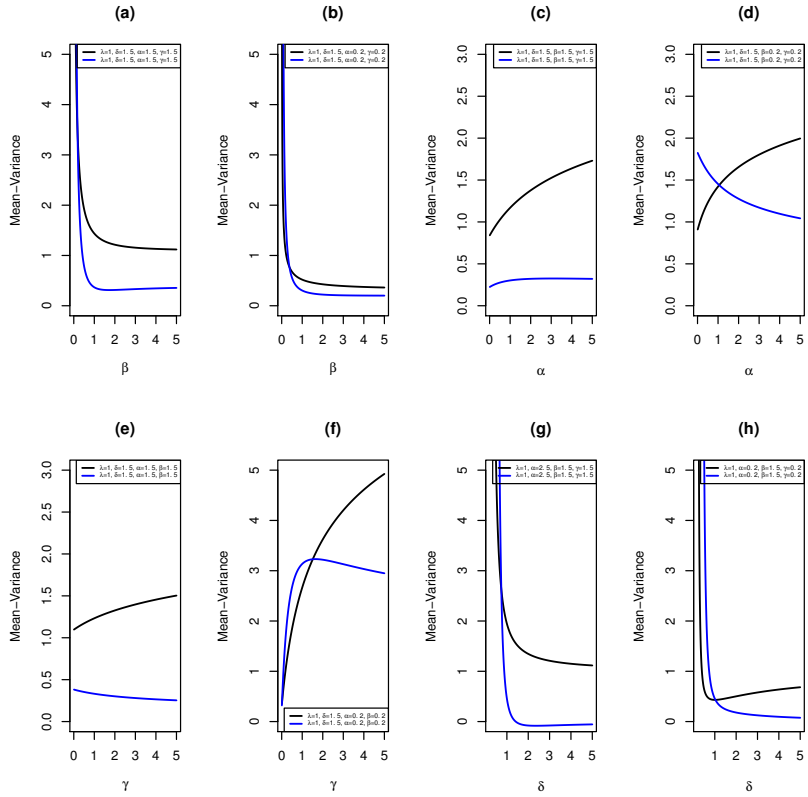


Figure 3 The NGEW Mean and Variance with selected parameter values. Here, black line represents the NGEW mean and blue lines describes the Variation.

Table 3 The numerical observations of first two moments, standard deviation, median, mode, skewness (Sk) and kurtosis (Kur) of (5) with selective shape parameter values when scale parameter λ is unity

$(\delta, \alpha, \beta, \gamma)$	$E(X)$	$E(X^2)$	*SD*	*Median*	*Mode*	*Sk*	*Kur*
(1.5,1.5,0.3,1.5)	2.6	8.2	2.30	1.79	1.30	1.10	4.6
(1.5,1.5,0.7,1.5)	1.6	3.2	1.54	1.34	0.71	0.89	4.3
(1.5,1.5,1.0,1.5)	1.4	2.4	1.36	1.19	0.61	0.80	4.0
(1.5,1.5,1.5,1.5)	1.3	2.0	1.20	1.04	0.56	0.92	4.3
(1.5,1.5,3.5,1.5)	1.1	1.6	1.03	0.79	0.58	1.10	4.6
(1.5,1.5,5.5,1.5)	1.1	1.6	1.00	0.70	0.60	1.00	4.4
(1.5,5.5,0.3,5.5)	3.9	17	3.71	3.34	1.30	0.84	4.1
(1.5,5.5,0.7,5.5)	2.4	6.1	2.28	2.11	0.67	0.86	4.3
(1.5,5.5,1.0,5.5)	2.1	4.5	1.98	1.86	0.56	0.77	4.0
(1.5,5.5,1.5,5.5)	1.9	3.7	1.78	1.65	0.53	0.87	4.3
(1.5,5.5,3.5,5.5)	1.7	3.4	1.66	1.50	0.57	0.83	4.0
(1.5,5.5,5.5,5.5)	1.7	3.3	1.66	1.52	0.58	0.79	4.0
(1.5,0.2,0.3,0.2)	0.86	1.8	0.51	1.7×10^{-7}	1.0	2.10	9.2
(1.5,0.2,0.7,0.2)	0.59	0.75	0.38	1.6×10^{-7}	0.63	1.70	6.7
(1.5,0.2,1.0,0.2)	0.52	0.57	0.33	1.6×10^{-7}	0.55	1.60	6.2
(1.5,0.2,1.5,0.2)	0.46	0.45	0.29	1.6×10^{-7}	0.49	1.70	6.8
(1.5,0.2,3.5,0.2)	0.38	0.35	0.22	1.6×10^{-7}	0.45	2.20	9.3
(1.5,0.2,5.5,0.2)	0.36	0.33	0.20	1.6×10^{-7}	0.45	2.30	10.0
(2.5,1.5,0.3,1.5)	1.7	3.2	1.65	1.52	0.52	0.53	3.2
(2.5,1.5,0.7,1.5)	1.3	1.9	1.30	1.26	0.35	0.37	3.2
(2.5,1.5,1.0,1.5)	1.2	1.6	1.20	1.17	0.31	0.30	3.1
(2.5,1.5,1.5,1.5)	1.1	1.4	1.12	1.08	0.30	0.38	3.2
(2.5,1.5,3.5,1.5)	1.1	1.2	1.02	0.93	0.32	0.53	3.2
(2.5,1.5,5.5,1.5)	1.0	1.2	1.00	0.90	0.33	0.48	3.0
(2.5,5.5,0.3,5.5)	2.2	5.2	2.19	2.12	0.44	0.46	3.2
(2.5,5.5,0.7,5.5)	1.7	2.8	1.64	1.75	0.28	0.50	3.4
(2.5,5.5,1.0,5.5)	1.5	2.4	1.51	1.47	0.25	0.43	3.3
(2.5,5.5,1.5,5.5)	1.4	2.1	1.41	1.37	0.24	0.51	3.4
(2.5,5.5,3.5,5.5)	1.4	2.0	1.36	1.31	0.27	0.45	3.2
(2.5,5.5,5.5,5.5)	1.4	2.0	1.35	1.32	0.28	0.39	3.2
(3.5,1.5,0.3,1.5)	1.4	2.2	1.43	1.38	0.32	0.29	2.9
(3.5,1.5,0.7,1.5)	1.2	1.5	1.20	1.19	0.23	0.14	3.0
(3.5,1.5,1.0,1.5)	1.1	1.3	1.14	1.13	0.21	0.09	3.0
(3.5,1.5,1.5,1.5)	1.1	1.2	1.08	1.07	0.21	0.16	3.0
(3.5,1.5,3.5,1.5)	1.0	1.1	1.01	0.97	0.23	0.29	2.9
(3.5,1.5,5.5,1.5)	1.0	1.1	1.00	0.97	0.24	0.25	2.8

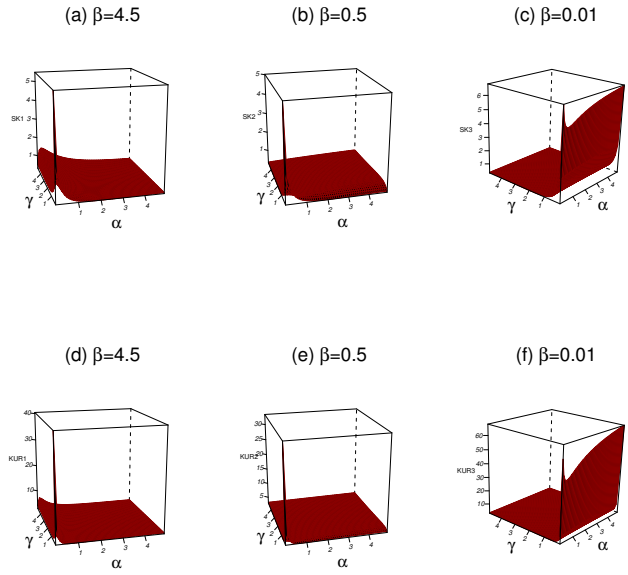


Figure 4 Plots of skewness (a,b,c) and kurtosis (d,e,f) for the NGEW distribution when $\lambda = 1$ and $\delta = 2.5$.

5.2. Moment generating function (mgf)

The mgf uses for characterization and generation of the moments of a distribution.

By using (14) in the formula $M(t) = \sum_{r=0}^{\infty} \frac{\mu_r'}{r!} t^r$, the mgf of the X is

$$M(t) = \sum_{r=0}^{\infty} \frac{(-1)^m}{r!} (\frac{t}{\lambda})^r \sum_{j=0}^{\infty} \nu_j(\alpha + j) \frac{\partial^m}{\partial y^m} \{B(\alpha + j, y)\} \Big|_{y=1}.$$

5.3. Quantile function

We can find the quantile function (qf) of the NGEW distribution numerically by following equation

$$\left\{[H(x)]^{\alpha}[H(x;\beta)]^{\gamma}\right\} = q. \tag{15}$$

The first three quartiles of the NGEW distribution can be determined by using $q = 0.25, 0.50$ and 0.75.

5.4. The mean deviation

Variation or mean deviations of a distribution can be computed by total deviation from median and mean and can be determined by using the expressions given by

$$\zeta_1 = 2\mu_1 F(\mu_1) - 2I(\mu_1) \text{ and } \zeta_2 = \mu_1 - 2I(M), \tag{16}$$

where the value of first moment (mean) of the X is $E(X) = \mu_1$ which can be determined from (13) or (14) when $r = 1$ and the median (M) of (5) can directly be determined from (15). The general

formula to determine the first incomplete moment is $I(z) = \int_0^z xf(x)dx$. Now, using the (10) and (12), $I(z)$ reduces to

$$I(z) = \lambda^{-1}\sum_{j=0}^{\infty}(\alpha+j)\nu_j\sum_{p=0}^{\infty}\binom{\alpha+j}{p}\int_0^{(\lambda z)^{\delta}} u^{\frac{1}{\delta}}\mathrm{e}^{-(p+1)u}du.$$

By setting $(p+1)u = t$ and using $\Gamma(b,x) = \int_0^x z^{b-1}\mathrm{e}^{-z}dz$, It reduces to

$$I(z) = \lambda^{-1}\sum_{j=0}^{\infty}(\alpha+j)\nu_j\sum_{p=0}^{\infty}\binom{\alpha+j}{p}\frac{(-1)^p\Gamma(\frac{1}{\delta}+1,(p+1)(\lambda z)^{\delta})}{(p+1)^{\frac{1}{\delta}+1}}. \tag{17}$$

by using (17) in (16), we determine these dispersion measures (mean deviations) of the X. We compute the Bonferroni and Lorenz curves by using (17). These curves are extensively used in reliability theory and economics. For a probability given by f, these important curves are $B(f) = I(q)/(f\mu_1)$ and $L(f) = I(q)/\mu_1$. Here $q = Q(f)$ is calculated by (15) and $\mu_1 = E(X)$.

5.5. Rényi entropy

The entropy measures are frequently used to compute uncertain variation of the X. There are several measures which are comprehensively studied in the literature. These entropy measures are very popular in field of engineering and applied sciences. Amidst the Rényi entropy is most famous entropy measure which is commonly formulated as

$$I_R(\zeta) = (1-\zeta)^{-1}\log\left(\int_0^{\infty} f(x)^{\zeta}dx\right), \tag{18}$$

where $\zeta > 0$ and $\zeta \neq 1$, using (10) thrice in Eqn.(5), $f(x)^{\zeta}$ can be rewritten in simple form as

$$f(x)^{\zeta} = \alpha^{\zeta}\Big[\frac{\beta\gamma}{\alpha}\Big]^i\sum_{k=0}^{\infty}\epsilon_k G(x)^{\zeta(\alpha-1)+k-i}g(x)^{\zeta}, \tag{19}$$

where the coefficient ϵ_k is $\sum_{i,j=0}^{\infty}(-1)^{j+k}\binom{\zeta}{i}\binom{\zeta\gamma}{j}\binom{(i+j)\beta}{k}$. Now, substitute $G(x)$ and $g(x)$ with $H(x)$ and $h(x)$, respectively, in Eqn.(19) and put the result in (18). we obtain

$$I_R(\zeta) = \frac{1}{(1-\zeta)}\log\left\{(\alpha\lambda^{\delta}\delta)^{\eta}\Big[\frac{\beta\gamma}{\alpha}\Big]^i\sum_{k=0}^{\infty}\epsilon_k\int_0^{\infty}x^{\zeta(\delta-1)}\{h(x)\}^{\zeta(\alpha-1)+k-i}\mathrm{e}^{-\zeta(\lambda x)^{\delta}}dx\right\}.$$

Further, by using (10) and after simple mathematics, we obtain

$$I_R(\zeta) = \frac{1}{(1-\zeta)}\log\left\{(\alpha\lambda)^{\zeta}\delta^{\zeta-1}\Big[\frac{\beta\gamma}{\alpha}\Big]^i\sum_{k=0}^{\infty}\sum_{r=0}^{\infty}\epsilon_k\binom{\zeta(\alpha-1)+k-i}{r}\frac{\Gamma(\varrho)}{(\zeta+r)^{\varrho}}\right\},$$

where $\varrho = \frac{\zeta(\delta-1)+1}{\delta}$. The Shannon entropy of the X can be obtain as particular case of (18) when $\zeta \uparrow 1$.

5.6. Order statistics

The *r*-th order statistic density $f_{r:n}(x)$, for $r = 1,\ldots,n$, from independently and identically (i.i.d) distributed random variables $X_1,\ldots,X_n$ is defined by

$$f_{r:n}(x) = C_{r,n}[F(x)]^{r-1}[\overline{F}(x)]^{n-r}f(x), \tag{20}$$

where $\{C_{r,n} = 1/B(r, n-r+1)\}$. By employing the general binomial expansion, (20) is expressed as

$$f_{r:n}(x) = C_{r,n} \sum_{k=0}^{n-r} (-1)^k \binom{n-r}{k} f(x)\{F(x)\}^{r+k-1}. \tag{21}$$

By substituting the (4) and (5) in (21) and applying the generalized binomial expansion (10). For simplicity of expression set $\xi = (r+k)\alpha + j$, we have

$$f_{r:n}(x) = C_{r,n} \sum_{p=0}^{\infty} \upsilon_p h(x)\{H(x)\}^{\xi+p-1}, \tag{22}$$

where the coefficient $\upsilon_p = \upsilon_p(\alpha, \beta, \gamma, r, n) = \sum_{j=0}^{\infty} \sum_{k=0}^{n-r} \sum_{l=0}^{\infty} (\alpha+j)\tau_j \, (-1)^{k+l+p} \binom{n-r}{k} \binom{(r+k-1)\gamma}{l} \binom{l\beta}{p}$. By simple mathematical manipulation, last equation reduces to following simple expression

$$f_{r:n}(x) = C_{r,n} \sum_{p=0}^{\infty} \upsilon_p^* h_{\xi+p}(x), \tag{23}$$

where the $\upsilon_p^* = \frac{\upsilon_p}{\xi+p}$ and $h_{\xi+p}(x) = (\xi+p)h(x)\{H(x)\}^{\xi+p-1}$ is the EW density. Eqn.(23) is a linear mixture of the EW densities. Hence, important and significant characteristics of the order statistics of the X, for instance the moments and the mgf, can simply be achieved from those characteristics of the EW model.

6. Estimation

There are various techniques in literature to determine the point estimates of unknown parameters but the method of maximum likelihood is most popular among these techniques. The maximum likelihood estimators (MLEs) are generally good estimators with optimal properties, which can be employed to determine the confidence intervals and important test statistics. So, We adopted the method of ML to estimates the unknown parameters of (5) from complete sample. Assume x_i be a random sample for $i = 1, \ldots, n$ taken from the X and vector $\Theta = (\lambda, \delta, \alpha, \beta, \gamma)^T$ of parameters of the X. Thus, the log-likelihood , say $\ell(\Theta)$ is

$$\begin{aligned} \ell(\Theta) = {} & n\log\delta + n\delta\log\lambda + (\delta-1)\sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}(\lambda x_i)^{\delta} + \sum_{i=1}^{n}\log[H(x_i)] \\ & + \gamma\sum_{i=1}^{n}\log[H(x_i;\beta)] + \sum_{i=1}^{n}\log\Big[\frac{\alpha H(x_i;\beta) + \beta\gamma \mathrm{e}^{-(\beta-1)(\lambda x_i)^{\delta}} H(x_i)}{H(x_i;\beta)}\Big]. \end{aligned} \tag{24}$$

We can maximized the Eqn.(24) by adopting the R function *optim*, a general-purpose optimization based on different numerical optimization techniques such as Nelder and Mead (NM), Broyden, Fletcher, Goldfarb and Shanno (BFGS), and simulated annealing (SANN) or by solving nonlinear equations which determined by differentiating (24). The score elements are

$$\begin{aligned} \frac{\partial\ell(\Theta)}{\partial\delta} = {} & \frac{n}{\delta} + \sum_{i=1}^{n}\log x_i - \sum_{i=1}^{n}(\lambda x_i)^{\delta}\log(\lambda x_i) + n\log\lambda + (\alpha-1)\sum_{i=1}^{n}\frac{(\lambda x_i)^{\delta}\log(\lambda x_i)\mathrm{e}^{-(\lambda x_i)^{\delta}}}{H(x_i)} \\ & + \beta\sum_{i=1}^{n}\mathrm{e}^{-\beta(\lambda x_i)^{\delta}}(\lambda x_i)^{\delta}\log(\lambda x_i)\left[\frac{(\alpha+\gamma) - \gamma(\beta-1)\mathrm{e}^{(\lambda x_i)^{\delta}}}{\nu(x_i)} + \frac{(\gamma-1)}{H(x_i;\beta)}\right], \end{aligned}$$

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{n\delta}{\lambda} - \delta \sum_{i=1}^{n} \lambda^{\delta-1} x_i^{\delta} + (\alpha-1) \sum_{i=1}^{n} \frac{\lambda^{\delta-1} \mathrm{e}^{-(\lambda x_i)^{\delta}}}{H(x_i)} + \beta \sum_{i=1}^{n} \delta \lambda^{\delta-1} x_i^{\delta} \mathrm{e}^{-\beta(\lambda x_i)^{\delta}}$$
$$\left[\frac{(\gamma-1)}{H(x_i;\beta)} + \frac{(\alpha+\gamma) + \gamma(\beta-1)(1-\mathrm{e}^{(\lambda x_i)^{\delta}})}{\nu(x_i)} \right],$$
$$\frac{\partial \ell(\Theta)}{\partial \alpha} = \sum_{i=1}^{n} \log[H(x_i)] + \sum_{i=1}^{n} \frac{H(x_i;\beta)}{\nu(x_i)},$$
$$\frac{\partial \ell(\Theta)}{\partial \beta} = (\gamma-1) \sum_{i=1}^{n} \frac{\mathrm{e}^{-\beta(\lambda x_i)^{\delta}} (\lambda x_i)^{\delta}}{H(x_i;\beta)} + \sum_{i=1}^{n} \frac{\alpha \mathrm{e}^{-\beta(\lambda x_i)^{\delta}} (\lambda x_i)^{\delta} + \gamma \mathrm{e}^{-(\beta-1)(\lambda x_i)^{\delta}} (H(x_i))(1-\beta(\lambda x_i)^{\delta})}{\nu(x_i)},$$
$$\frac{\partial \ell(\Theta)}{\partial \gamma} = \sum_{i=1}^{n} \log(H(x_i;\beta)) + \sum_{i=1}^{n} \frac{\beta \mathrm{e}^{-(\beta-1)(\lambda x_i)^{\delta}} H(x_i)}{\nu(x_i)},$$

where $\nu(x_i) = \alpha H(x_i;\beta) + \beta\gamma \mathrm{e}^{-(\beta-1)(\lambda x_i)^{\delta}} H(x_i)$. These equations are difficult to solve analytically. Therefore, we can determine the MLEs of the unknown parameters numerically. A 5×5 observed information matrix is used for set estimation and hypothesis testing of the unknown quantities (parameters) λ, δ, α, β and γ of (5). Which is numerically computed by using some maximization procedure, provided by $J(\Theta) = \{-U_{pq}\}$ where $U_{pq} = \partial^2 \ell(\Theta)/\partial p \partial q$ for $p, q \in \{\lambda, \delta, \alpha, \beta, \gamma\}$. By following the regularity conditions which are true for the parameters in the interior of parameter space, the asymptotic distribution of $\sqrt{n}(\hat{\Theta} - \Theta)$ is multivariate $N_5(0, J(\hat{\Theta})^{-1})$. Here $J(\hat{\Theta})$ is the information estimated at $\Theta = \hat{\Theta}$.

The NGEW distribution can be compared with particular sub-models by using the likelihood ratio (LR) statistic. For purpose of testing particular sub-models of (5), the LR statistics are determined by computing the maximum values of the unrestricted and restricted log-likelihoods (LL). For instance, the test hypothesis $H_0 : EGW$ versus $H_1 : H_0$ *is false* is comparison of the NGEW and EGW distributions and the LR statistic becomes

$$d = 2\{\ell(\hat{\lambda}, \hat{\delta}, \hat{\alpha}, \hat{\beta}, \hat{\gamma}) - \ell(\tilde{\lambda}, \tilde{\delta}, \tilde{\beta}, \tilde{\gamma})\},$$

where $\hat{\lambda}, \hat{\delta}, \hat{\alpha}, \hat{\beta}$ $\hat{\gamma}$ are the MLEs under H_1 and $\tilde{\lambda}, \tilde{\delta}, \tilde{\beta}$ and $\tilde{\gamma}$ are the null hypothesis H_0.

7. Simulation

The performance of the maximum likelihood estimation technique is evaluated by simulation study. We used the inversion method numerically in *R* for simulating from (5). The simulation study is replicated 1,000 times each for sample size $n = 50, 100, 200, 300, 500, 1000$ with different values of parameters $\lambda, \delta, \alpha, \beta$ and γ and computed the mean and the root mean square error (RMSE). The MLEs are obtained using the routine *optim* in *R*. Table 4 gives the results of simulation study. The simulation results show that estimates are close to the true parameters and RMSE decreases as the sample size increases.

Table 4 The means and the RMSE (in parenthesis) of the NGEW distribution for parameter combination (I) $\lambda = 1.5, \delta = 1.5, \alpha = 2.0, \beta = 2.5, \gamma = 3.0$ and (II) $\lambda = 1.0, \delta = 1.5, \alpha = 3.0, \beta = 2.5, \gamma = 2.0$

Parameter Combination	n	$\hat{\lambda}$	$\hat{\delta}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\gamma}$
I	100	1.387	1.289	2.935	3.495	4.160
		(0.253)	(0.367)	(1.215)	(1.312)	(1.731)
	200	1.401	1.377	2.716	3.286	3.893
		(0.185)	(0.293)	(1.021)	(1.047)	(1.361)
	300	1.432	1.412	2.541	3.152	3.721
		(0.132)	(0.223)	(0.976)	(0.877)	(1.025)
	400	1.456	1.448	2.376	2.957	3.568
		(0.112)	(0.183)	(0.753)	(0.675)	(0.843)
	500	1.463	1.457	2.114	2.718	3.304
		(0.098)	(0.153)	(0.614)	(0.583)	(0.531)
	1000	1.468	1.476	2.013	2.507	3.142
		(0.063)	(0.116)	(0.317)	(0.124)	(0.303)
II	100	0.557	1.173	5.879	3.124	2.905
		(0.056)	(0.461)	(1.932)	(1.232)	(1.505)
	200	0.638	1.257	4.718	2.853	2.776
		(0.047)	(0.395)	(1.413)	(1.045)	(1.217)
	300	0.717	1.388	4.176	2.735	2.578
		(0.038)	(0.251)	(1.132)	(0.863)	(0.854)
	400	0.803	1.410	3.977	2.635	2.354
		(0.024)	(0.176)	(0.847)	(0.671)	(0.547)
	500	0.901	1.478	3.355	2.587	2.116
		(0.011)	(0.151)	(0.547)	(0.423)	(0.247)
	1000	1.057	1.512	3.034	2.507	2.005
		(0.008)	(0.124)	(0.267)	(0.212)	(0.107)

8. Applications

Here, we apply the NGEW model to two sets of data from real world to reveal capability of the proposed model. The NGEW distribution is analyzed with four sub-models like, the exponentiated generalized exponential (EGE), EGW, EW and Weibull distribution. All study is conducted in a statistical computing language R developed and maintained by R Core Team (2018). First, we give essential summary statistics of two data sets associated with real life problems in Table 5. Then, for each data, the MLEs of fitted models (unknown) parameters are determined along with standard errors values (provided in parentheses) using the R package *AdequacyModel* (Diniz Marinho et al., 2016) and results are given in Tables 6 and 7. The suggested model is compared by other models by using well-established criterion AIC (Akaike Information Criterion) and goodness-of-fit statistics including Anderson-Darling (A*), Kolmogrov-Smirnov (K-S) and Cramér-von Mises (W*). These criteria and statistics are presented with Tables 8 and 9. Generally, the statistics with smaller values, reveal the fit is better. The proposed model is compared with its sub-models by employing LR statistics in Tables 10 and 11. The significant p-values in these tables indicate that the NGEW model is more useful and flexible as compare to other four sub-models. Aarset (1987) introduced the empirical total time on test (TTT) transform for obtaining critical information about the hrf's shape. Since the TTT transform plot with diagonal, convex, concave, concave-convex and convex-concave shape describes that the hrf is constant, decreasing, increasing, unimodal and bathtub shaped. So we can say that on the basis of the TTT transform plots, the proposed model is capable for analyzing nonmonotone hazard rate functions. we display the TTT transform plots, histograms, estimated densities and estimated cdfs in Figure 5. Finally the probability-probability (P-P) plots, to check the validity of the proposed

models, are provided in Figures 6 and 7. They compare the empirical cdf of the sample data against the proposed theoretical cdf. These plots indicate that the proposed model is superior to the other models in terms of model fitting.

Breaking stress data: This real life data contains 100 values about (breaking) stress of carbon fibres (in Gba) is provided by Birnbaum and Saunders (1969).

3.70	2.74	2.73	2.50	3.60	3.11	3.27	2.87	1.47	3.11	4.42	2.41	3.19	3.22	1.69
3.28	3.09	1.87	3.15	4.90	3.75	2.43	2.95	2.97	3.39	2.96	2.53	2.67	2.93	3.22
3.39	2.81	4.20	3.33	2.55	3.31	3.31	2.85	2.56	3.56	3.15	2.35	2.55	2.59	2.38
2.81	2.77	2.17	2.83	1.92	1.41	3.68	2.97	1.36	0.98	2.76	4.91	3.68	1.84	1.59
3.19	1.57	0.81	5.56	1.73	1.59	2.00	1.22	1.12	1.71	2.17	1.17	5.08	2.48	1.18
3.51	2.17	1.69	1.25	4.38	1.84	0.39	3.68	2.48	0.85	1.61	2.79	4.70	2.03	1.80
1.57	1.08	2.03	1.61	2.12	1.89	2.88	2.82	2.05	3.65					

Sale price data: The data is obtained by Shafiei et al. (2016) which list the sale price of 31 wooden toys made for children in Suffolk craft for April, 1991 and used by Peng (2018).

4.2	1.12	1.39	2.00	3.99	2.15	1.74	5.81	1.70	2.85	0.50
0.99	11.50	5.12	0.90	1.99	6.24	2.60	3.00	12.20	7.36	
4.75	11.59	8.69	9.80	1.85	1.99	1.35	10.00	0.65	1.45	

Table 5 Critical descriptive statistics for both data sets

Statistic	Breaking stress data	Sale price data
n	100	31
Mean	2.62	4.24
Median	2.7	2.60
Mode	2.75	1.00
Variance	1.03	13.27
Skewness	0.37	0.99
Kurtosis	0.10	-0.38

Table 6 MLEs for models (NGEW, EGW, EGE, EW and W) parameters and corresponding standard error (SE) values presented (in parentheses) to breaking stress data

Model	λ	δ	α	β	γ
NGEW	0.2922	11.8806	0.1694	0.0058	0.0230
	(0.0077)	(0.1137)	(0.0239)	(0.0041)	(0.0092)
EGW	20.2286	0.6081	-	0.5992	21.8606
	(20.2286)	(0.6081)		(0.5992)	(21.8606)
EGE	4.6983	-	-	7.7891	0.2156
	(21.0234)			(1.4965)	(0.9649)
EW	0.3731	2.4058	1.3198	-	-
	(0.0627)	(0.6047)	(0.5984)		
W	0.3397	2.7929	-	-	-
	(0.0128)	(0.2141)			

Table 7 MLEs for models (NGEW, EGW, EGE, EW and W) parameters and corresponding standard error (SE) values presented (in parentheses) to sale price data

Model	λ	δ	α	β	γ
NGEW	0.5712	3.2798	0.6257	0.0027	0.1456
	(0.1496)	(0.6442)	(0.3886)	(0.0021)	(0.0553)
EGW	1.5922	0.3793	-	19.8228	1.8947
	(133.7682)	(0.3107)		(58.2662)	(60.4173)
EGE	0.5566	-	-	0.1.5602	0.5566
	(23.0815)			(0.4050)	(23.0815)
EW	7.1463	0.3916	17.8742	-	-
	(23.2372)	(0.2184)	(35.5195)		
W	0.2195	1.2279	-	-	-
	(0.0340)	(0.1700)			

Table 8 The log-likelihood, formal goodness-of-fit criteria and statistics for fitted distributions (W, EW, EGE, EGW and NGEW) to breaking stress data

Model	-LL	AIC	CAIC	W*	A*	K-S	p-value (K-S)
W	141.5293	287.0586	287.1823	0.0623	0.4158	0.0605	0.8578
EW	141.3320	288.6641	288.9141	0.0705	0.4134	0.0646	0.7987
EGE	146.1823	298.3646	298.6146	0.2267	1.1860	0.1077	0.1963
EGW	141.3320	290.6641	291.0851	0.0704	0.4131	0.0644	0.8007
NGEW	138.2095	286.4190	287.0573	0.0362	0.2475	0.0565	0.9069

Table 9 The log-likelihood, formal goodness-of-fit criteria and statistics for fitted distributions (W, EW, EGE, EGW and NGEW) to sales price data

Model	-LL	AIC	CAIC	W*	A*	K-S	p-value (K-S)
W	74.7887	153.5774	154.0060	0.1274	0.7672	0.0605	0.8578
EW	73.6003	153.2005	154.0894	0.0600	0.3861	0.0646	0.7987
EGE	74.3968	154.7935	155.6824	0.1110	0.6697	0.1077	0.1963
EGW	73.5987	155.1975	156.7360	0.0590	0.3819	0.0644	0.8007
NGEW	70.3607	150.7214	153.1214	0.0174	0.1488	0.0565	0.9069

Table 10 LR tests for breaking stress data

Hypotheses	LR Statistic (d)	p-value
H_0: EGW vs $H_1 : NGEW(H_0$ is false)	6.2	0.01278
H_0: EGE vs $H_1 : NGEW(H_0$ is false)	15.9	0.00007
H_0: EW vs $H_1 : NGEW(H_0$ is false)	6.2	0.01278
H_0: Weibull vs $H_1 : NGEW(H_0$ is false)	6.6	0.01020

Table 11 LR tests for price data

Hypotheses	LR Statistic (d)	p-value
H_0: EGW vs H_1 : $NGEW(H_0$ is false)	6.5	0.01079
H_0: EGE vs H_1 : $NGEW(H_0$ is false)	8.1	0.00443
H_0: EW vs H_1 : $NGEW(H_0$ is false)	6.5	0.01079
H_0: Weibull vs H_1 : $NGEW(H_0$ is false)	8.9	0.00285

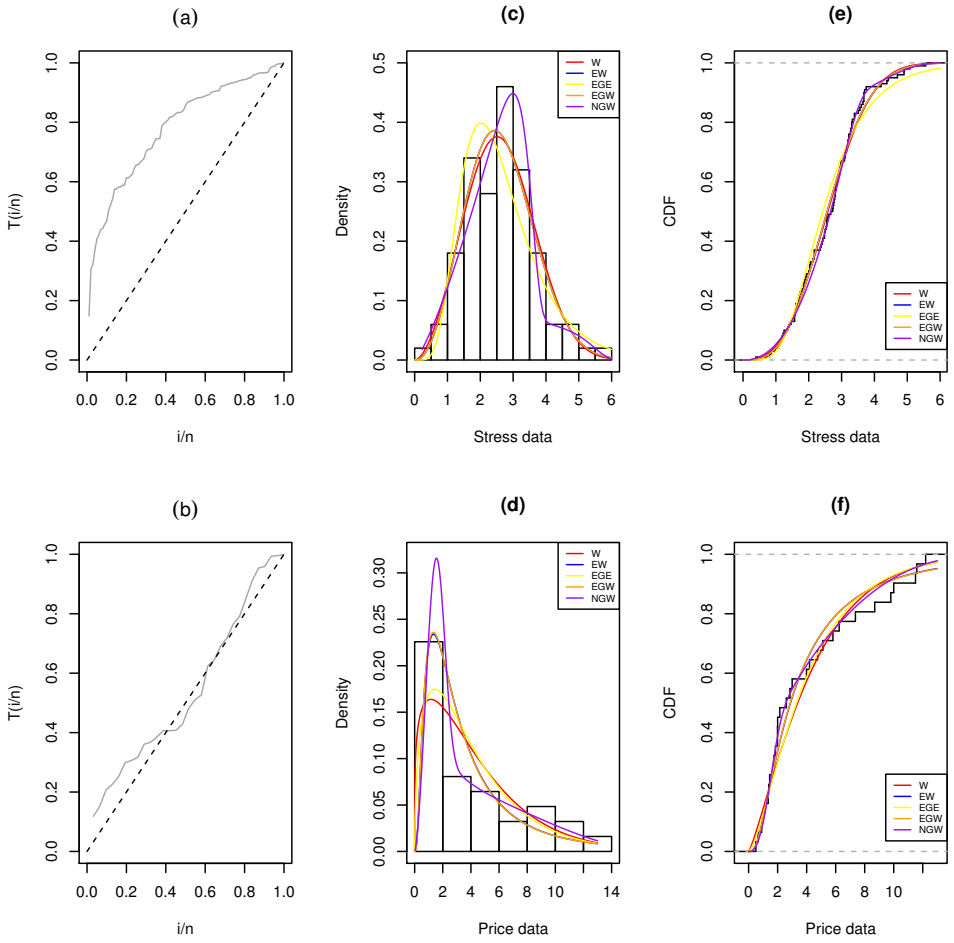


Figure 5 (a,b) TTT plot of the stress and price data (c,d,e,f) Plots of the fitted (W, EW, EGE, EGW and NGEW) densities and cdfs for the stress and price data.

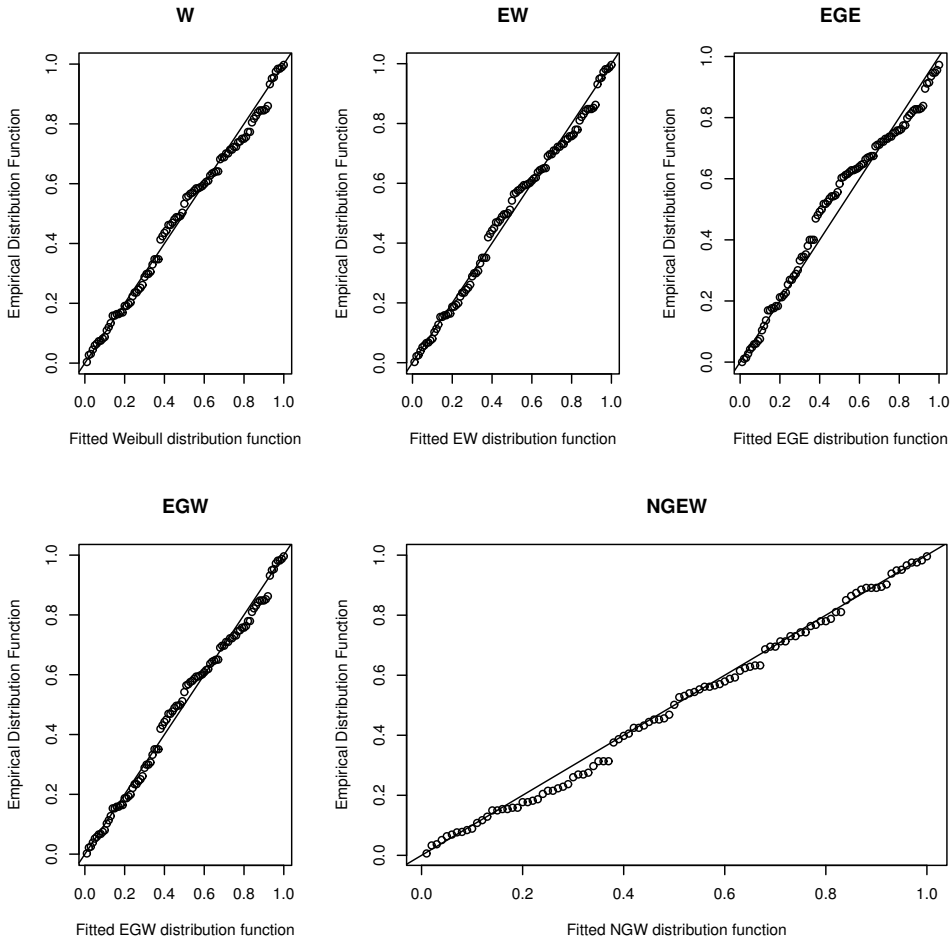


Figure 6 Fitted and empirical distribution functions for breaking stress data.

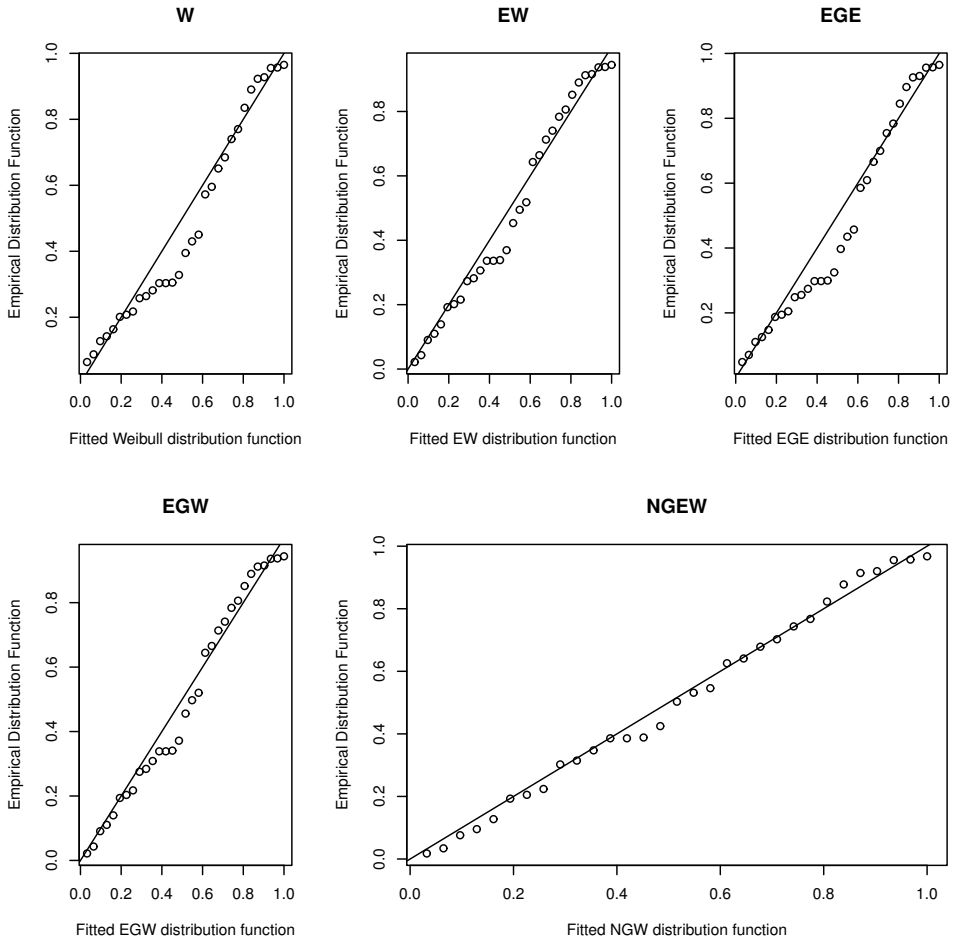


Figure 7 Fitted and empirical distribution functions for sales price data.

9. Conclusions

In this article, we introduce a new generalized exponentiated Weibull (NGEW) distribution by generalizing well-known Weibull distribution. The Weibull distribution is generalized by using a new generalized (NG) class of distribution introduced by Ashraf et al. (2022). We study comprehensively some basic properties of the NGEW distribution. We give simple presentation for the cdf and pdf of the X which is quite effective to derive the explicit expressions of several mathematical properties of the proposed distribution. Then, we provide the explicit expressions for some mathematical properties including moments, generating function, quartile function, Rényi entropy, mean deviations and Bonferroni and Lorenz curves. The density of the order statistics is defined as infinite sum of the EW densities. The parameters of the proposed distribution are estimated by the ML approach. The applications to two sets of real data reveals that the NGEW distribution performs better than the EGW, EGE, EW and Weibull distributions. We are quite confident that the NGEW distribution is a useful addition in the family of the continuous probability distributions for modeling the real data especially lifetime and skewed data.

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