Thailand Statistician
October 2024; 22(4): 909-925
http://statassoc.or.th
Contributed paper

Precise Average Run Length of an Exponentially Weighted Moving Average Control Chart for Time Series Model

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Received: 26 January 2024
Revised: 18 April 2024
Accepted: 7 May 2024

Abstract

This study aims to develop a precise formula for calculating the average run length in the context of an exponentially weighted moving average (EWMA) control chart, specifically in the presence of a seasonal autoregressive with exogenous variable ($SARX(P,r)_L$) model. The research also introduces a novel method for estimating the average run length using numerical integral equations, facilitating a comparison between the outcomes derived from the formula and those obtained through the numerical integral equation method. Additionally, control charts are applied to real-world data across diverse domains. The explicit formula is evaluated based on the absolute percentage difference and CPU time. The results show that the average run length calculated using the proposed method precisely corresponds to the findings from the numerical integral equation method. In addition, it's important to mention that the explicit formulas demonstrated a significant improvement in computational efficiency, requiring much fewer computations than the NIE approach.

Keywords: Average run length, SARX model, explicit formulas, numerical integral equation.

1. Introduction

Statistical process control is a widely used method of monitoring processes, which involves the use of control charts. The Shewhart control chart was introduced by Shewhart in 1931 (Shewhart 1931). This was followed by the Cumulative Sum (CUSUM) control chart, which was initially proposed by Page in 1954 (Page 1954). Another type of control chart is the exponentially weighted moving average (EWMA) control chart, which was first reported by Roberts in 1959 (Roberts 1959). The Shewhart control chart is most suitable for detecting large shifts in the mean or variance of a process when the observations follow a normal distribution. On the other hand, CUSUM and EWMA control charts are more effective for detecting small shifts in the statistic parameter, as well as complex patterns. In a recent study, Areepong and Sunthornwat (2021) utilized an EWMA control chart to monitor COVID-19 outbreaks in Thailand, Singapore, Vietnam, and Hong Kong. They employed a quantitative and probabilistic approach and compared two methods for estimating the expected value and variance of cases. The study established alert levels with the help of the EWMA control chart.

The goal of this research is to manage process variability through the use of statistical quality control tools. In general, the efficiency of control chart studies often assumes an initial agreement that data follows a normal distribution. However, in many real-world situations, data tends to exhibit a time-series pattern. Therefore, the selection of an appropriate control chart is crucial for effective monitoring of process changes. EWMA control chart can be applied to correlated random events in hospitals, stock prices, or daily rainfall volumes. As a result, the researchers are interested in developing an explicit formula and estimation method for the average run length (ARL) of EWMA control charts.

The ARL is a significant metric used to assess the effectiveness of control charts when it comes to detecting shifts in process mean. Displays average subgroups needed before control chart indicates out-of-control process. ARL comprises two vital components: the in-control ARL (ARL_0) and the out-of-control ARL (ARL_1). ARL_0 represents the average number of samples taken from a stable process before a false out-of-control signal is generated. On the other hand, an alarm indicating that the process is out of control is signaled by an average number of samples falling within control limits, or ARL_1.

Methods for estimating ARL include Monte Carlo simulations (MC), Markov Chain approach (MCA), and Integral Equation approach (IE). The MC technique may require considerable computing time, despite being useful for validating analytical findings. Roberts (1959) employed the MC technique to compute the ARL for the EWMA control chart. Crowder (1987) presents a numerical procedure for calculating run length in EWMA control chart based on normal distribution and extends to non-normal cases and one-sided EWMA control chart. Lucas and Saccucci (1990) studied the EWMA control scheme for monitoring the mean of a process. A design MCA procedure is provided, with parameter values as well for small shifts. However, due to the limitations of the MCA and MC methods, researchers have started to investigate the integral equation method. In the EWMA control chart for exponential distribution observations, Areepong (2009) suggested analytical solutions for the average delay (AD) and ARL. Recently, Mititelu et al. (2010) use the Fredholm integral equations method to derive explicit formulas for the ARL in special control charts, including CUSUM and EWMA control charts, which require fewer computations.

Control charts are typically designed with the assumption of independent and identically distributed observations. However, in cases where processes exhibit autocorrelation, specialized control charts are required. Integral Equation methods have been introduced for evaluating ARL for control charts when processes are serially correlated. When there is an AR(1) model with extra random error, Vanbrackle and Reynolds (1997) suggest using IE techniques for finding the ARL of EWMA and CUSUM control charts. Subsequently, Busaba et al. (2012) provided analytical ARL solutions for CUSUM control chart in the context of stationary AR(1) models. Additionally, Petcharat et al. (2013) developed explicit ARL formulas for EWMA and CUSUM control charts using a moving average (MA) model. Sunthornwat et al. (2018) compared analytical and numerical EWMA ARL, and analytical CUSUM ARL. They proposed a method to estimate optimal parameters for EWMA and AFRIMA processes. Results showed analytical EWMA ARL is an alternative to measure chart efficiency due to good performance. Phanthuna et al. (2021) introduced a technique to compute the ARL of a modified EWMA control chart under trend AR(1) mode. They compared the NIE method with an explicit formula, finding that the latter is more accurate and faster. They also found that the modified EWMA chart is more effective than the conventional scheme in detecting small to moderate shifts. Phanthuna and Areepong (2021) proposed an explicit formula for calculating the ARL on a modified EWMA control chart for observations generated by a $SAR(p)_L$ with exponential residuals. They validated the accuracy of the explicit formulas, which are applicable to various real-world datasets. They also compared the modified EWMA control chart with the conventional EWMA scheme and concluded that the former is more effective in detecting small shifts. Silpakob et al. (2021) have developed explicit formulas for the ARL with a modified EWMA control chart based on an ARX(p,r) to detect changes in the process mean. They also conducted a comparative analysis of the performance between the modified EWMA control chart and EWMA control charts, utilizing the Relative Mean Index (RMI). Their findings indicate that the explicit formulas for the ARL of the modified EWMA control chart outperformed those of the EWMA control chart in monitoring process mean shifts. Later, Phanyaem (2022) developed the explicit formula and numerical integral equation (NIE) of the ARL for the CUSUM control chart based on the $\mathrm{SARX(P,r)_L}$ model. The Fredholm integral equation was employed, and numerical methods like the midpoint rule, the trapezoidal rule, Simpson's rule, and the Gaussian rule were used to approximate the ARL. Petcharat (2022) constructs the ARL for a CUSUM control chart using the Fredholm integral equation approach and Banach's Fixed Point theorem to ensure the solution's existence and uniqueness based on $\mathrm{SAR(P)_L}$ with the trend process. Furthermore, Peerajit (2023) compares analytical integral equations (ARL) derived from Banach's fixed-point theorem to the numerical integral equation (NIE) method for a fractionally integrated moving average with exogenous variables (FIMAX) model with underlying exponential white noise.

In this paper, an explicit formula for the ARL of the EWMA control chart under the seasonal autoregressive with exogenous variable; the $\mathrm{SARX(P,r)_L}$ model is introduced. This is a novel contribution that has not been explored previously. The ARL obtained from the proposed method is compared with numerical integral equation approaches. The paper is organized as follows: Section 2 describes the materials utilized; Sections 3 and 4 describe the methods used; Section 5 presents the results of the proposed method; and Section 6 provides concluding views.

2. Characteristics of the SARX Model and the EWMA Control Chart

In this section, we describe the characteristics of an SARX model featuring exponential white noise. We define the SARX model employed on the EWMA control chart for efficient monitoring of process mean shifts. The final subsection delves into ARL features integral to the assessment of control chart performance.

2.1. The seasonal autoregressive with exogenous variable model

The $\mathrm{SARX(P,r)_L}$ model is a time series model that combines autoregressive components with seasonality and exogenous variables. P represents the autoregressive order, while r represents the exogenous variable order in the model. The $\mathrm{SARX(P,r)_L}$ model can be generalized as

$$Y_t = \sum_{j=1}^{r} \beta_j X_{jt} + \mu + \phi_1 Y_{t-L} + \ldots + \phi_P Y_{t-PL} + \varepsilon_t ,$$

where μ is a constant, the initial values of Y_t are represented by $\{Y_{t-L}, Y_{t-2L}, \ldots, Y_{t-PL}\}$, ϕ_i, $i = 1, 2, \ldots, P$, refers to the autoregressive coefficient parameters, X_{jt} are exogenous variables of Y_t, β_j are exogenous coefficient parameters, ε_t is a white noise process assumed to be exponentially distributed.

2.2. EWMA control chart characteristics

The EWMA control chart was initially introduced by Robert (1959). It is widely accepted that the EWMA control chart outperforms the Shewhart control chart when it comes to detecting small-to-medium shift sizes in the process mean.

The prevalent form of the EWMA control chart relies on the sequence

$$E_t = (1-\lambda)E_{t-1} + \lambda Y_t; \; t = 1,2,....,$$

where E_t is the EWMA statistic, E_{t-1} denotes the previous value of the EWMA statistic (E_0 is set to u and Y_t is the sequence of the SARX(P,r)$_L$ model with exponential white noise), and λ is an exponential smoothing parameter with of EWMA control chart with $0 < \lambda < 1.$ EWMA control chart stopping time definition:

$$\tau_b = \inf\{t > 0; E_t > b\},$$

where b is a constant parameter representing the upper control limit. The ARL for SARX(P,r)$_L$ model with an initial value $E_0 = u,$ the expectation under density function $f(x,\alpha)$ that the change-point occurs at point $\theta,$ where $\theta < \infty,$ denoted by $\mathbb{E}_\infty(.).$

$$ARL = H(u) = \mathbb{E}_\infty(\tau_b) < \infty.$$

3. Explicit Formulas for ARL of EWMA Control Chart Based on SARX(P,r)$_L$ model

In this section, explicit formulas are employed to calculate the average run length (ARL) of the EWMA control chart for a seasonal autoregressive model with an exogenous variable SARX(P,r)$_L$ Specifically, we derive analytical formulas for ARL by utilizing the Fredholm Integral Equation of the second kind. The lower and upper control limits are assumed to be zero and $b,$ respectively, and the function $H(u)$ is defined as the ARL of the EWMA chart for the SARX(P,r)$_L$ model. Let $\mathbb{P}_E$ represent the probability measure and $\mathbb{E}_E$ represent the expectation corresponding to initial value $E_0 = u.$ Finally, we extend the function into the Fredholm Integral Equations of the second kind.

$$H(u) = 1 + \mathbb{E}_{\mathrm{E}}\left[I\{0 < E_1 < b\}L(E_1)\right] + \mathbb{P}_{\mathrm{E}}\{E_1 = 0\}L(0).$$

Thus, E_1 represents an in-control state if $0 \le E_1 \le b,$ it can be written as

$$0 \le (1-\lambda)E_0 + \lambda\mu + \lambda\phi_1 Y_{t-L} + ... + \lambda\phi_P Y_{t-PL} + \lambda\sum_{j=1}^{r}\beta_j X_{jt} \le b. \tag{1}$$

If Y_1 gives the out-of-control state for $E_1,$ it can be written as

$$(1-\lambda)E_0 + \lambda\mu + \lambda\phi_1 Y_{t-L} + ... + \lambda\phi_P Y_{t-PL} + \lambda\sum_{j=1}^{r}\beta_j X_{jt} > b,$$

or

$$(1-\lambda)E_0 + \lambda\mu + \lambda\phi_1 Y_{t-L} + ... + \lambda\phi_P Y_{t-PL} + \lambda\sum_{i=1}^{r}\beta_i X_{it} < 0.$$

After assigning the initial value of $E_0 = u,$ (1) can be represented as

$$0 \le (1-\lambda)u + \lambda\mu + \lambda\phi_1 Y_{t-L} + ... + \lambda\phi_P Y_{t-PL} + \lambda\sum_{j=1}^{r}\beta_j X_{jt} \le b.$$

Following Champ and Rigdon's method (1991), the initial value of the EWMA statistics is set to $E_0 = u,$ with $\varepsilon_t \sim Exp(\alpha).$ Then, the function $H(u)$ can be rewritten

$$H(u) = 1 + \int_0^b H(E_1) f(\varepsilon_1) d\varepsilon_1.$$

To obtain the function $H(u),$ ε_t is substituted with $z.$

$$H(u) = 1 + \int_0^b H\left\{(1-\lambda)u + \lambda\mu + \lambda\phi_1 Y_{t-L} + ... + \lambda\phi_P Y_{t-PL} + \lambda\sum_{j=1}^{r}\beta_j X_{jt}\right\} f(z) dz.$$

Let $v=(1-\lambda)u+\lambda\mu+\lambda\phi_1 Y_{t-L}+...+\lambda\phi_P Y_{t-PL}+\lambda\sum_{j=1}^{r}\beta_j X_{jt}$. To clarify, the function $H(u)$ can be expressed by changing the integration variable.

$$H(u)=1+\frac{1}{\lambda}\int_0^b H(v)f\left(\frac{v-(1-\lambda)u}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right)dv. \tag{2}$$

As a result, the following integral equation is obtained:

$$H(u)=\ 1+\frac{1}{\lambda\alpha}\int_0^b H(v)e^{-\frac{v}{\lambda\alpha}}.e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}dv. \tag{3}$$

In this section, Banach's Fixed Point Theorem will be presented. It guarantees the existence and uniqueness of the results of an integral equation. The theorem applies to a metric space consisting of continuous functions on a closed interval $(C(I),\|\ \|_\infty)$ where I denote the compact interval. The norm $\|H\|_\infty=Sup_{u\in I}|H(u)|$ and the operator T are defined on this space. If there exists a number $0\le q<1$ such that the operator T is a contraction, then the theorem holds true

$$\|T(H_1)-T(H_2)\|_\infty\le q\|H_1-H_2\| \text{ for all } H_1,H_2\in I.$$

Let $C(I_1)$ as a continuous function over a range $I_1=[0,b]$ and define the operator T as

$$T(H(u))=\ 1+\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}}{\lambda\alpha}\int_0^b H(v)e^{-\frac{v}{\lambda\alpha}}dv. \tag{4}$$

According to the Banach Fixed Point Theorem, if the operator T is a contraction, then fixed point equations $T(H(u))=H(u)$ have a unique solution. So, in this case, if T is a contraction, then the integral equation can be written as $T(H(u))=H(u),$ and it will have a unique solution.

Theorem 1. (Banach's Fixed-point Theorem) *In the complete metric space* (X,d) *where* $T:X\to X$ *is a mapping satisfying the criteria of a contraction mapping with contraction constant* $q<1$ *such that* $\|T(H_1)-T(H_2)\|_\infty\le q\|H_1-H_2\|$, *there is a unique function* $H(\cdot)\in X$ *for which* $T(H(u))=H(u)$ *has a unique fixed point in* X.

Proof: For any given $u\in I$ and $H_1,H_2\in C(I)$, we have the inequality $\|T(H_1)-T(H_2)\|_\infty\le q\|H_1-H_2\|$ where $q<1$. According to (4), we obtain

$$\begin{aligned}\|T(H_1)-T(H_2)\|_\infty &= Sup_{u\in[0,b]}|H_1(v)-H_2(v)|\\
&= Sup_{u\in[0,b]}\left|\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}}{\lambda\alpha}\int_0^b\big(H_1(v)-H_2(v)\big)e^{-\frac{v}{\lambda\alpha}}dv\right|\\
&\le \underset{u\in[0,b]}{Sup}\left\{\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}}{\lambda\alpha}\left|\int_0^b e^{-\frac{v}{\lambda\alpha}}dv\right|\right\}\|H_1-H_2\|_\infty\end{aligned}$$

$$\leq \sup_{u\in[0,b]}\left[e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}(1-e^{-\frac{b}{\lambda\alpha}})\right]\left\|H_1-H_2\right\|_\infty$$

$$\leq q\left\|H_1-H_2\right\|_\infty ,$$

where $q=\sup_{u\in[0,b]}\left[e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}(1-e^{-\frac{b}{\lambda\alpha}})\right]<1.$ Thus, the function $H(u)$ can be defined as an existence and unique solution.

The explicit formula of the ARL for the EWMA control chart based on the SARX(P,r)$_L$ model was derived using the Fredholm integral equation. First, let's take into consideration

$$H(u)=1+\frac{e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}}{\lambda\alpha}\int_0^b H(v)e^{-\frac{v}{\lambda\alpha}}dv.$$

Let $C(u)=e^{\frac{(1-\lambda)u}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}$, then we can rewrite the (3) as follows:

$$H(u)=1+\frac{C(u)}{\lambda\alpha}\int_0^b H(v)e^{-\frac{v}{\lambda\alpha}}dv,\quad 0\leq u\leq b.$$

Let $k=\int_0^b H(v)e^{-\frac{v}{\lambda\alpha}}dv.$ Consequently, we obtain

$$H(u)=1+\frac{C(u)}{\lambda\alpha}k. \tag{5}$$

The next step is to find the value of k as follows:

$$\begin{aligned}
k &= \int_0^b H(v)e^{-\frac{v}{\lambda\alpha}}dv,\\
&= \int_0^b\left(1+\frac{C(v)}{\lambda\alpha}k\right)e^{-\frac{v}{\lambda\alpha}}dv,\\
&= \int_0^b e^{-\frac{v}{\lambda\alpha}}dv+\int_0^b\frac{C(v)}{\lambda\alpha}ke^{-\frac{v}{\lambda\alpha}}dv,\\
&= \int_0^b e^{-\frac{v}{\lambda\alpha}}dv+\frac{k}{\lambda\alpha}\int_0^b e^{\frac{(1-\lambda)v}{\lambda\alpha}+\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}.e^{-\frac{v}{\lambda\alpha}}dv,\\
&= \int_0^b e^{-\frac{v}{\lambda\alpha}}dv+\frac{k}{\lambda\alpha}.e^{\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}\int_0^b e^{\frac{(1-\lambda)v}{\lambda\alpha}-\frac{v}{\lambda\alpha}}dv,\\
&= -\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)+\frac{k}{\lambda\alpha}.e^{\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}\int_0^b e^{-\frac{v}{\alpha}}dv,
\end{aligned}$$

$$= -\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)-\frac{k}{\lambda}.e^{\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}.(e^{\frac{-b}{\alpha}}-1).$$

Consequently, the following formula can be used to find a constant k :

$$k = \frac{-\lambda\alpha(e^{-\frac{b}{\lambda\alpha}}-1)}{1+\frac{1}{\lambda}.e^{\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{j=1}^{r}\beta_j X_{jt}}{\alpha}}.(e^{-\frac{b}{\alpha}}-1)}.$$

When we substitute a constant k into (5), we can obtain the function $H(u)$. This solution of $H(u)$ is the explicit formulas for the ARL of EWMA control chart for SARX(P,r)$_L$ model,

$$H(u) = 1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha}}(e^{-\frac{b}{\lambda\alpha}}-1)}{\lambda e^{-\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{i=1}^{r}\beta_j X_{jt}}{\alpha}}+(e^{-\frac{b}{\alpha}}-1)}.$$

The explicit formula for ARL$_0$ of EWMA control chart for SARX(P,r)$_L$ model in the in-control state with an exponential parameter $\alpha=\alpha_0$ is

$$ARL_0 = 1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha_0}}(e^{-\frac{b}{\lambda\alpha_0}}-1)}{\lambda e^{-\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{i=1}^{r}\beta_j X_{jt}}{\alpha_0}}+(e^{-\frac{b}{\alpha_0}}-1)}.$$

On the other hand, the explicit formula for ARL$_1$ of EWMA control chart for SARX(P,r)$_L$ model in the out-of-control state with an exponential parameter $\alpha_1=\alpha_0(1+\delta)$ is

$$ARL_1 = 1-\frac{\lambda e^{\frac{(1-\lambda)u}{\lambda\alpha_1}}(e^{-\frac{b}{\lambda\alpha_1}}-1)}{\lambda e^{-\frac{\mu+\phi_1 Y_{t-L}+...+\phi_P Y_{t-PL}+\sum_{i=1}^{r}\beta_j X_{jt}}{\alpha_1}}+(e^{-\frac{b}{\alpha_1}}-1)},$$

where α is a parameter of exponential white noise, b is upper control limit, Y_{t-L} are the initial values of SARX model, $\phi_i; i=1,2,...,P$ is an autoregressive coefficient; $0\le\phi_i\le 1$.

4. Numerical Integration of ARL of EWMA Control Chart for SARX(P,r)$_L$

In this section, we will explain how to calculate the numerical integration of the ARL of the EWMA control chart for the SARX(P,r)$_L$ model, considering exponential distribution for the white noise processes. To solve the integral equation of ARL, we will be using Gauss-Legendre Quadrature as an approximation technique for integration. As a result, the integral equation in (2) can be expressed in the following $\tilde{H}(u)$.

$$\tilde{H}(u) = 1+\frac{1}{\lambda}\int_0^b H(v)f\left(\frac{v-(1-\lambda)u}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right)dv.$$

The quadrature rule is the foundation for numerical integration of integral equations, allowing for the estimation of integrals with finite sums. The approximation for an integral has the following form:

$$\int_0^b W(z)f(z)dz \approx \sum_{j=1}^{m} w_j f(a_j),$$

where $w_j = \dfrac{b}{m}$ and $a_j = \dfrac{b}{m}\left(j-\dfrac{1}{2}\right); j = 1,2,...,m.$ The integral equation is approximated numerically as $\tilde{H}(a_i),$ and its solution can be found using a method of solving linear algebraic equations.

$$\tilde{H}(a_i) \approx 1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right).$$

Thus,

$$\tilde{H}(a_1) = 1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_1}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right),$$

$$\tilde{H}(a_2) = 1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_2}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right),$$

$$\vdots$$

$$\tilde{H}(a_m) = 1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)a_m}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right),$$

or as a matrix $\mathbf{H}_{m\times 1} = \mathbf{1}_{m\times 1}+\mathbf{R}_{m\times m}\mathbf{H}_{m\times 1},$

where $\mathbf{H}_{m\times 1} = \begin{pmatrix}\tilde{H}(a_1)\\ \tilde{H}(a_2)\\ \vdots \\ \tilde{H}(a_m)\end{pmatrix},\ \mathbf{1}_{m\times 1} = \begin{pmatrix}1\\1\\ \vdots\\1\end{pmatrix},$

$$[\mathbf{R}]_{ij} \approx \frac{1}{\lambda} w_j f\left(\frac{a_j-(1-\lambda)a_i}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right),$$

and $\mathbf{I}_m = \operatorname{diag}(1,1,...,1).$ If $(\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}$ there exist

$$\mathbf{H}_{m\times 1} = (\mathbf{I}_m-\mathbf{R}_{m\times m})^{-1}\mathbf{1}_{m\times 1}.$$

The integral equation in (2) can be roughly represented by the following (6), where $\tilde{H}(u)$ indicates the numerical integration solution of $H(u).$

$$\tilde{H}(u) \approx 1+\frac{1}{\lambda}\sum_{j=1}^{m} w_j \tilde{H}(a_j) f\left(\frac{a_j-(1-\lambda)u}{\lambda}-\mu-\phi_1 Y_{t-L}-...-\phi_P Y_{t-PL}-\sum_{j=1}^{r}\beta_j X_{jt}\right). \tag{6}$$

5. Numerical Results

The study compared the performance of explicit formulas and NIE method in detecting changes in the process mean on an EWMA control chart for the SARX(P,r)$_L$ model. The precision and accuracy of ARL values that are obtained from explicit formulas are examined, and the results are compared with the results that are obtained by numerical integration. This assessment will specifically consider the absolute percentage difference; *Diff*(%) between the exact solution represented by $H(u)$ using explicit formulas and the numerical integration solution denoted as $\tilde{H}(u).$ The absolute percentage difference of ARL can be calculated by

$$Diff\,(\%) \;=\; \frac{\left|H(u)-\tilde{H}(u)\right|}{H(u)}\times 100.$$

For the numerical result, Table 1 presents the ARL of the $SARX(1,1)_{12}$, $SARX(2,1)_{12}$, $SARX(1,2)_{12}$, and $SARX(2,2)_{12}$ models with different values of the autoregressive coefficient parameter ($\phi_1 = 0.10, 0.20$ and $\phi_2 = 0.10,\ 0.20$) and exogenous coefficient parameter ($\beta_1 = 0.10, 0.50$ and $\beta_2 = 0.60$). Exponential parameter of white noise process (α_0) is set to 1 in the in-control state. In the out-of-control state, α_1 can take values of 1.01, 1.03, 1.05, 1.10, 1.20, 1.30, and 1.40, respectively. The ARL_1 values of the EWMA control chart can be calculated using two methods: the explicit formula and the NIE method, both with the initial upper control limits b and the exponential smoothing parameter λ set to 0.10 and ARL_0 = 370. The results of both methods indicate that their ARL solutions are similar, with an absolute percentage difference *Diff*(%) of less than 0.001. However, the explicit formula approach outperforms the NIE method in terms of CPU time.

According to Table 2 shows the ARL of four different SARX models: $SARX(1,1)_{12}$, $SARX(2,1)_{12}$, $SARX(1,2)_{12}$, and $SARX(2,2)_{12}$. The models have different values of the autoregressive coefficient parameter ($\phi_1 = 0.10, 0.20$ and $\phi_2 = 0.10,\ 0.20$), the exogenous coefficient parameter ($\beta_1 = 0.10, 0.50$ and $\beta_2 = 0.60$). The ARL_1 of the EWMA control chart for the $SARX(P,r)_L$ models employed two distinct methodologies: the explicit formula and the NIE method. Both methods set the initial upper control limits b and an exponential smoothing parameter $(\lambda = 0.10)$ and ARL_0 = 500. Results indicate that both methods have similar ARL solutions, with an absolute percentage difference *Diff*(%) of less than 0.001. However, the explicit formula approach performs better than the NIE method in terms of CPU time.

The comparative analysis indicated a high degree of similarity between the ARL values obtained through the explicit formula and the NIE methods. Notably, the *Diff*(%) yielded a value of zero, leading to the conclusion that there was no discernible distinction between the ARL values derived from both methods. Particularly, the explicit formula method distinguishes itself with efficient computation, demonstrating a notably abbreviated processing time.

Table 1 ARL results for the explicit formulas and the NIE method on EWMA control chart for $\text{SARX}(P,r)_L$ model with $\alpha_0 = 1$, $\lambda = 0.10$, and $\text{ARL}_0 = 370$

Models	b	Parameters	ARL	α_1						
				1.01	1.03	1.05	1.10	1.20	1.30	1.40
SARX $(1,1)_{12}$	0.00363	$\phi_1 = 0.1$	Explicit	**334.560**	**274.864**	**227.465**	**145.930**	**67.000**	**34.707**	**19.848**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**334.560**	**274.864**	**227.465**	**145.930**	**67.000**	**34.707**	**19.848**
			(Sec.)	(2.247)	(2.309)	(2.262)	(2.262)	(2.278)	(2.247)	(2.262)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.00242	$\phi_1 = 0.1$	Explicit	**333.273**	**271.597**	**223.023**	**140.524**	**62.559**	**31.616**	**17.735**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**333.273**	**271.597**	**223.023**	**140.524**	**62.559**	**31.616**	**17.735**
			(Sec.)	(2.371)	(2.387)	(2.403)	(2.340)	(2.371)	(2.340)	(2.371)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(1,1)_{12}$	0.00328	$\phi_1 = 0.2$	Explicit	**334.308**	**274.099**	**226.391**	**144.586**	**65.871**	**33.912**	**19.298**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**334.308**	**274.099**	**226.391**	**144.586**	**65.871**	**33.912**	**19.298**
			(Sec.)	(2.356)	(2.418)	(2.356)	(2.402)	(2.403)	(2.356)	(2.402)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.00219	$\phi_1 = 0.2$	Explicit	**333.354**	**271.115**	**222.197**	**139.376**	**61.573**	**30.927**	**17.267**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**333.354**	**271.115**	**222.197**	**139.376**	**61.573**	**30.927**	**17.267**
			(Sec.)	(2.309)	(2.434)	(2.356)	(2.355)	(2.340)	(2.387)	(2.278)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,1)_{12}$	0.00328	$\phi_1 = 0.1$	Explicit	**333.308**	**274.099**	**226.391**	**144.586**	**65.871**	**33.912**	**19.298**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**333.308**	**274.099**	**226.391**	**144.586**	**65.871**	**33.912**	**19.298**
			(Sec.)	(2.340)	(2.309)	(2.325)	(2.293)	(2.340)	(2.371)	(2.402)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.00219	$\phi_1 = 0.1$	Explicit	**333.354**	**271.115**	**222.197**	**139.376**	**61.573**	**30.927**	**17.267**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**333.354**	**271.115**	**222.197**	**139.376**	**61.573**	**30.927**	**17.267**
			(Sec.)	(2.356)	(2.371)	(2.403)	(2.387)	(2.325)	(2.355)	(2.418)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,1)_{12}$	0.002962	$\phi_1 = 0.1$	Explicit	**333.798**	**273.128**	**225.151**	**143.146**	**64.714**	**33.110**	**18.752**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**333.798**	**273.128**	**225.151**	**143.146**	**64.714**	**33.110**	**18.752**
			(Sec.)	(2.387)	(2.355)	(2.434)	(2.372)	(2.355)	(2.371)	(2.356)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.00198	$\phi_1 = 0.1$	Explicit	**333.077**	**270.344**	**221.138**	**138.091**	**60.541**	**30.224**	**16.796**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**333.077**	**270.344**	**221.138**	**138.091**	**60.541**	**30.224**	**16.796**
			(Sec.)	(2.324)	(2.293)	(2.340)	(2.325)	(2.293)	(2.340)	(2.309)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Note: The results are expressed as an absolute percentage difference (Diff%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (seconds).

Table 1 (Continued)

Models	b	Parameters	ARL	α_1						
				1.01	1.03	1.05	1.10	1.20	1.30	1.40
SARX $(2,1)_{12}$	0.002962	$\phi_1 = 0.2$	Explicit	**333.798**	**273.128**	**225.151**	**143.146**	**64.714**	**33.110**	**18.752**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**333.798**	**273.128**	**225.151**	**143.146**	**64.714**	**33.110**	**18.752**
			(Sec.)	(2.247)	(2.278)	(2.262)	(2.293)	(2.277)	(2.231)	(2.356)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001976	$\phi_1 = 0.2$	Explicit	**332.382**	**269.781**	**220.679**	**137.806**	**60.418**	**30.164**	**16.763**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**332.382**	**269.781**	**220.679**	**137.806**	**60.418**	**30.164**	**16.763**
			(Sec.)	(2.247)	(2.231)	(2.309)	(2.247)	(2.246)	(2.324)	(2.246)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,1)_{12}$	0.002676	$\phi_1 = 0.2$	Explicit	**333.389**	**272.241**	**223.986**	**141.766**	**63.599**	**32.340**	**18.228**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**333.389**	**272.241**	**223.986**	**141.766**	**63.599**	**32.340**	**18.228**
			(Sec.)	(2.293)	(2.246)	(2.293)	(2.309)	(2.262)	(2.231)	(2.278)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001787	$\phi_1 = 0.2$	Explicit	**332.164**	**269.062**	**219.666**	**136.562**	**59.418**	**29.484**	**16.310**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	332.164	269.062	219.666	136.562	59.418	29.484	16.310
			(Sec.)	(2.247)	(2.340)	(2.262)	(2.278)	(2.293)	(2.278)	(2.262)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(1,2)_{12}$	0.001976	$\phi_1 = 0.1$	Explicit	**332.382**	**269.781**	**220.679**	**137.806**	**60.418**	**30.164**	**16.763**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	332.382	269.781	220.679	137.806	60.418	30.164	16.763
			(Sec.)	(2.262)	(2.247)	(2.277)	(2.309)	(2.371)	(2.278)	(2.262)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001321	$\phi_1 = 0.1$	Explicit	331.160	266.636	216.434	132.765	56.466	27.518	15.016
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	331.160	266.636	216.434	132.765	56.466	27.518	15.016
			(Sec.)	(2.262)	(2.340)	(2.293)	(2.294)	(2.262)	(2.309)	(2.293)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(1,2)_{12}$	0.001790	$\phi_1 = 0.2$	Explicit	**332.740**	**269.527**	**220.046**	**136.796**	**59.518**	**29.533**	**16.336**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	332.740	269.527	220.046	136.796	59.518	29.533	16.336
			(Sec.)	(2.246)	(2.278)	(2.371)	(2.247)	(2.294)	(2.277)	(2.309)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001194	$\phi_1 = 0.2$	Explicit	**330.654**	**265.696**	**215.258**	**131.458**	**55.489**	**26.882**	**14.604**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	**330.654**	**265.696**	**215.258**	**131.458**	**55.489**	**26.882**	**14.604**
			(Sec.)	(2.293)	(2.355)	(2.309)	(2.293)	(2.277)	(2.356)	(2.372)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Note: The results are expressed as an absolute percentage difference (Diff%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (seconds).

Table 1 (Continued)

Models	b	Parameters	ARL	α_1						
				1.01	1.03	1.05	1.10	1.20	1.30	1.40
SARX $(2,2)_{12}$	0.001787	$\phi_1 = 0.1$	Explicit	**332.164**	**269.062**	**219.666**	**136.562**	**59.418**	**29.484**	**16.310**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**332.164**	**269.062**	**219.666**	**136.562**	**59.418**	**29.484**	**16.310**
		$\beta_2 = 0.6$	(Sec.)	(2.308)	(2.325)	(2.355)	(2.294)	(2.340)	(2.308)	(2.387)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001195	$\phi_1 = 0.1$	Explicit	**330.941**	**265.926**	**215.443**	**131.571**	**55.536**	**26.904**	**14.616**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**330.941**	**265.926**	**215.443**	**131.571**	**55.536**	**26.904**	**14.616**
		$\beta_2 = 0.6$	(Sec.)	(2.308)	(2.387)	(2.309)	(2.324)	(2.371)	(2.340)	(2.293)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,2)_{12}$	0.001615	$\phi_1 = 0.1$	Explicit	**331.688**	**268.137**	**218.490**	**135.226**	**58.391**	**28.801**	**15.859**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**331.688**	**268.137**	**218.490**	**135.226**	**58.391**	**28.801**	**15.859**
		$\beta_2 = 0.6$	(Sec.)	(2.309)	(2.293)	(2.340)	(2.355)	(2.293)	(2.262)	(2.309)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001080	$\phi_1 = 0.1$	Explicit	**330.38**	**264.944**	**214.237**	**130.255**	**54.568**	**26.279**	**14.213**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**330.38**	**264.944**	**214.237**	**130.255**	**54.568**	**26.279**	**14.213**
		$\beta_2 = 0.6$	(Sec.)	(2.355)	(2.325)	(2.277)	(2.309)	(2.324)	(2.309)	(2.293)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,2)_{12}$	0.001614	$\phi_1 = 0.2$	Explicit	**331.476**	**267.965**	**218.351**	**135.140**	**58.355**	**28.783**	**15.850**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**331.476**	**267.965**	**218.351**	**135.140**	**58.355**	**28.783**	**15.850**
		$\beta_2 = 0.6$	(Sec.)	(2.293)	(2.262)	(2.293)	(2.309)	(2.324)	(2.262)	(2.372)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001079	$\phi_1 = 0.2$	Explicit	**330.063**	**264.691**	**214.033**	**130.132**	**54.517**	**26.255**	**14.201**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**330.063**	**264.691**	**214.033**	**130.132**	**54.517**	**26.255**	**14.201**
		$\beta_2 = 0.6$	(Sec.)	(2.294)	(2.308)	(2.293)	(2.340)	(2.293)	(2.340)	(2.309)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,2)_{12}$	0.001461	$\phi_1 = 0.2$	Explicit	**331.525**	**267.466**	**271.525**	**134.030**	**57.437**	**28.160**	**15.436**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**331.525**	**267.466**	**271.525**	**134.030**	**57.437**	**28.160**	**15.436**
		$\beta_2 = 0.6$	(Sec.)	(2.340)	(2.371)	(2.340)	(2.324)	(2.309)	(2.262)	(2.340)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.000977	$\phi_1 = 0.2$	Explicit	**330.129**	**264.213**	**213.237**	**129.073**	**53.667**	**25.692**	**13.835**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**330.129**	**264.213**	**213.237**	**129.073**	**53.667**	**25.692**	**13.835**
		$\beta_2 = 0.6$	(Sec.)	(2.356)	(2.371)	(2.293)	(2.340)	(2.262)	(2.309)	(2.324)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Note: The results are expressed as an absolute percentage difference (Diff%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (seconds).

Table 2 ARL results for the explicit formulas and the NIE method on EWMA control chart for SARX(P,r)$_L$ model with $\alpha_0 = 1$, $\lambda = 0.10$, and ARL$_0$ = 500

Models	b	Parameters	ARL	α_1 1.01	1.03	1.05	1.10	1.20	1.30	1.40
SARX $(1,1)_{12}$	0.004861	$\phi_1 = 0.1$	Explicit	**451.959**	**371.155**	**307.014**	**196.727**	**90.048**	**46.445**	**26.394**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**451.959**	**371.155**	**307.014**	**196.727**	**90.048**	**46.445**	**26.394**
			(Sec.)	(2.906)	(3.360)	(2.515)	(2.703)	(3.984)	(2.656)	(4.156)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.003232	$\phi_1 = 0.1$	Explicit	**449.804**	**366.346**	**300.648**	**189.14**	**83.892**	**42.181**	**23.491**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**449.804**	**366.346**	**300.648**	**189.14**	**83.892**	**42.181**	**23.491**
			(Sec.)	(3.078)	(4.766)	(2.812)	(3.000)	(3.063)	(2.578)	(3.282)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(1,1)_{12}$	0.004389	$\phi_1 = 0.2$	Explicit	**451.511**	**370.019**	**305.468**	**194.835**	**88.478**	**45.343**	**25.637**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**451.511**	**370.019**	**305.468**	**194.835**	**88.478**	**45.343**	**25.637**
			(Sec.)	(2.469)	(3.704)	(2.719)	(3.094)	(2.938)	(2.860)	(3.016)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.002919	$\phi_1 = 0.2$	Explicit	**449.143**	**365.057**	**299.001**	**187.248**	**82.407**	**41.174**	**22.817**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
			NIE	**449.143**	**365.057**	**299.001**	**187.248**	**82.407**	**41.174**	**22.817**
			(Sec.)	(2.656)	(2.734)	(3.156)	(2.703)	(2.673)	(3.531)	(4.095)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,1)_{12}$	0.004390	$\phi_1 = 0.1$	Explicit	**451.618**	**370.107**	**305.54**	**194.881**	**88.498**	**45.354**	**25.642**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**451.618**	**370.107**	**305.54**	**194.881**	**88.498**	**45.354**	**25.642**
			(Sec.)	(2.657)	(2.719)	(2.641)	(2.969)	(4.703)	(3.312)	(3.422)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.002922	$\phi_1 = 0.1$	Explicit	**449.625**	**365.448**	**299.321**	**187.447**	**82.493**	**41.217**	**22.840**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**449.625**	**365.448**	**299.321**	**187.447**	**82.493**	**41.217**	**22.840**
			(Sec.)	(3.031)	(2.938)	(3.250)	(3.359)	(2.860)	(3.313)	(2.938)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,1)_{12}$	0.003963	$\phi_1 = 0.1$	Explicit	**450.982**	**368.823**	**303.877**	**192.931**	**86.926**	**44.264**	**24.900**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**450.982**	**368.823**	**303.877**	**192.931**	**86.926**	**44.264**	**24.900**
			(Sec.)	(2.906)	(3.531)	(3.891)	(3.031)	(3.125)	(3.641)	(3.719)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.002640	$\phi_1 = 0.1$	Explicit	**449.072**	**364.25**	**297.754**	**185.619**	**81.055**	**40.245**	**22.193**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**449.072**	**364.25**	**297.754**	**185.619**	**81.055**	**40.245**	**22.193**
			(Sec.)	(2.641)	(3.000)	(3.843)	(3.219)	(3.281)	(4.484)	(3.078)
			Diff(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Note: The results are expressed as an absolute percentage difference (Diff%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (seconds).

Table 2 (Continued)

Models	b	Parameters	ARL	α_1 1.01	1.03	1.05	1.10	1.20	1.30	1.40
SARX $(2,1)_{12}$	0.003962	$\phi_1 = 0.2$	Explicit	**450.864**	**368.726**	**303.798**	**192.881**	**86.903**	**44.253**	**24.894**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**450.864**	**368.726**	**303.798**	**192.881**	**86.903**	**44.253**	**24.894**
			(Sec.)	(3.875)	(4.313)	(2.906)	(2.969)	(3.032)	(3.140)	(4.734)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.002639	$\phi_1 = 0.2$	Explicit	**448.894**	**364.106**	**297.637**	**185.546**	**81.024**	**40.230**	**22.184**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**448.894**	**364.106**	**297.637**	**185.546**	**81.024**	**40.230**	**22.184**
			(Sec.)	(2.890)	(2.922)	(4.500)	(2.703)	(2.500)	(2.562)	(4.406)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,1)_{12}$	0.003581	$\phi_1 = 0.2$	Explicit	**450.711**	**367.841**	**302.47**	**191.158**	**85.452**	**43.238**	**24.199**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**450.711**	**367.841**	**302.47**	**191.158**	**85.452**	**43.238**	**24.199**
			(Sec.)	(4.672)	(3.187)	(3.406)	(4.500)	(4.844)	(3.375)	(3.000)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.002385	$\phi_1 = 0.2$	Explicit	**448.417**	**362.975**	**296.131**	**183.77**	**79.628**	**39.291**	**21.561**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**448.417**	**362.975**	**296.131**	**183.77**	**79.628**	**39.291**	**21.561**
			(Sec.)	(3.000)	(4.406)	(4.234)	(2.719)	(2.734)	(3.062)	(3.781)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(1,2)_{12}$	0.002640	$\phi_1 = 0.1$	Explicit	**449.072**	**364.25**	**297.754**	**185.619**	**81.056**	**40.245**	**22.193**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	**449.072**	**364.25**	**297.754**	**185.619**	**81.056**	**40.245**	**22.193**
			(Sec.)	(2.844)	(4.156)	(3.656)	(3.062)	(2.953)	(4.423)	(3.313)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001762	$\phi_1 = 0.1$	Explicit	**447.088**	**359.687**	**291.733**	**178.597**	**75.614**	**36.623**	**19.809**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	**447.088**	**359.687**	**291.733**	**178.597**	**75.614**	**36.623**	**19.809**
			(Sec.)	(2.563)	(3.156)	(3.125)	(3.500)	(2.609)	(2.563)	(2.781)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(1,2)_{12}$	0.002385	$\phi_1 = 0.2$	Explicit	**448.417**	**362.975**	**296.131**	**183.77**	**79.628**	**39.291**	**21.561**
		$\beta_1 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	**448.417**	**362.975**	**296.131**	**183.77**	**79.628**	**39.291**	**21.561**
			(Sec.)	(2.734)	(4.219)	(3.985)	(3.922)	(3.968)	(4.250)	(3.266)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001593	$\phi_1 = 0.2$	Explicit	**446.601**	**358.563**	**290.255**	**176.893**	**74.319**	**35.775**	**19.259**
		$\beta_1 = 0.5$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_2 = 0.6$	NIE	**446.601**	**358.563**	**290.255**	**176.893**	**74.319**	**35.775**	**19.259**
			(Sec.)	(2.703)	(2.843)	(2.938)	(2.516)	(3.468)	(2.453)	(2.812)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Note: The results are expressed as an absolute percentage difference (Diff%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (seconds).

Table 2 (Continued)

Models	b	Parameters	ARL	α_1						
				1.01	1.03	1.05	1.10	1.20	1.30	1.40
SARX $(2,2)_{12}$	0.002388	$\phi_1 = 0.1$	Explicit	**449.008**	**363.452**	**296.519**	**184.009**	**79.730**	**39.340**	**21.587**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**449.008**	**363.452**	**296.519**	**184.009**	**79.730**	**39.340**	**21.587**
		$\beta_2 = 0.6$	(Sec.)	(0.406)	(0.297)	(0.344)	(0.297)	(0.344)	(0.281)	(0.203)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001595	$\phi_1 = 0.1$	Explicit	**447.190**	**359.035**	**290.636**	**177.123**	**74.414**	**35.820**	**19.282**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**447.190**	**359.035**	**290.636**	**177.123**	**74.414**	**35.820**	**19.282**
		$\beta_2 = 0.6$	(Sec.)	(0.234)	(0.453)	(0.328)	(0.188)	(0.500)	(0.325)	(0.312)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,2)_{12}$	0.002156	$\phi_1 = 0.1$	Explicit	**448.008**	**361.902**	**294.678**	**182.041**	**78.270**	**38.381**	**20.960**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**448.008**	**361.902**	**294.678**	**182.041**	**78.270**	**38.381**	**20.960**
		$\beta_2 = 0.6$	(Sec.)	(0.375)	(0.328)	(0.500)	(0.375)	(0.375)	(0.468)	(0.296)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001440	$\phi_1 = 0.1$	Explicit	**446.011**	**357.361**	**288.72**	**175.166**	**73.031**	**34.941**	**18.722**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**446.011**	**357.361**	**288.72**	**175.166**	**73.031**	**34.941**	**18.722**
		$\beta_2 = 0.6$	(Sec.)	(3.046)	(3.390)	(3.609)	(3.469)	(3.187)	(3.251)	(3.359)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,2)_{12}$	0.002160	$\phi_1 = 0.2$	Explicit	**448.879**	**362.604**	**295.248**	**182.39**	**78.418**	**38.452**	**20.998**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**448.879**	**362.604**	**295.248**	**182.39**	**78.418**	**38.452**	**20.998**
		$\beta_2 = 0.6$	(Sec.)	(3.938)	(3.219)	(4.688)	(3.672)	(3.360)	(3.468)	(3.625)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001440	$\phi_1 = 0.2$	Explicit	**446.011**	**357.361**	**288.72**	**175.166**	**73.031**	**34.941**	**18.722**
		$\phi_2 = 0.1$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**446.011**	**357.361**	**288.72**	**175.166**	**73.031**	**34.941**	**18.722**
		$\beta_2 = 0.6$	(Sec.)	(2.797)	(3.656)	(3.718)	(3.125)	(3.031)	(3.094)	(3.016)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
SARX $(2,2)_{12}$	0.001950	$\phi_1 = 0.2$	Explicit	**447.794**	**360.99**	**293.361**	**180.407**	**76.971**	**37.510**	**20.386**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.1$	NIE	**447.794**	**360.99**	**293.361**	**180.407**	**76.971**	**37.510**	**20.386**
		$\beta_2 = 0.6$	(Sec.)	(3.203)	(3.250)	(3.719)	(3.266)	(2.875)	(3.078)	(3.375)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	0.001303	$\phi_1 = 0.2$	Explicit	**445.860**	**356.512**	**287.474**	**173.626**	**71.836**	**34.160**	**18.217**
		$\phi_2 = 0.2$	(Sec.)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)	(0.010)
		$\beta_1 = 0.5$	NIE	**445.860**	**356.512**	**287.474**	**173.626**	**71.836**	**34.160**	**18.217**
		$\beta_2 = 0.6$	(Sec.)	(2.906)	(2.812)	(3.453)	(3.688)	(3.047)	(3.234)	(3.531)
			Dif(%)	0.000	0.000	0.000	0.000	0.000	0.000	0.000

Note: The results are expressed as an absolute percentage difference (Diff%) with the computational times in parentheses for the explicit formulas (seconds) and the NIE method (seconds).

6. Discussion and Conclusions

The study developed and calculated the ARL value using explicit formulas and the NIE method to detect process mean shifts in a $SARX(P,r)_L$ model with exponential white noise. Moreover, the demonstration of the existence and uniqueness of the ARL has been established through the validation of explicit formulas. A comparative analysis of the ARL using explicit formulas and the NIE method for monitoring mean shifts was made possible by the numerical investigation that determined the in-control ARL under various parameter configurations and levels of process mean shift. In summary, the absolute percentage difference between the ARLs derived from explicit formulas and those obtained through the NIE method exhibited similarity. However, it is noteworthy that the explicit formulas demonstrated a notable advantage in computational efficiency, requiring significantly less processing time compared to the NIE method.

In conclusion, this study derived the ARL utilizing explicit formulas for a $SARX(P,r)_L$ model on an EWMA control chart. The primary emphasis was focused on the exogenous variable within the $SARX(P,r)_L$ model. Exogenous variable incorporation was emphasized, underscoring its propensity to improve forecasting models' accuracy in comparison to models without exogenous variables. In future research, it would be interesting to see if more than one criterion can be used to assess control chart performance.

Acknowledgements

The authors gratefully acknowledge the editor and referees for their valuable comments and suggestions which greatly improve this paper. The research was funding by King Mongkut's University of Technology North Bangkok Contract no .KMUTNB-67-BASIC-08.

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