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Average Run Length Approximation on a Double Exponentially Weighted Moving Average Control Chart through the Numerical Integral Equation Approach

Pornphimol Doktoei, Yupaporn Areepong, and Saowanit Sukparungsee*
Department of Applied Statistics, Faculty of Applied Science, King Mongkut's University of Technology North Bangkok, Bangkok, Thailand.
*Corresponding author; e-mail: saowanit.s@sci.kmutnb.ac.th

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Abstract

In this paper, we suggest utilizing the midpoint rule, trapezoidal rule, Simpson's rule, and Gaussian rule in conjunction with the Numerical Integral Equation (NIE) method to estimate the Average Run Length (ARL). These techniques are applied to the Double Exponentially Weighted Moving Average (DEWMA) control chart in situations where the observations follow continuous distributions, like the Weibull and exponential distributions. Furthermore, we contrast the Exponentially Weighted Moving Average (EWMA) control chart's performance with that of the DEWMA control chart. Out-of-control Average Run Length (ARL_1) and CPU Times are the performance metrics. All of the methods perform similarly, according to the results. It is clear from the results that the DEWMA control chart performs better than the EWMA control chart. Additionally, a wide range of real-world datasets can be used to illustrate the efficacy of the suggested method by applying the NIE method to approximate the ARL.

Keywords: Average run length, numerical integral equation, DEWMA chart, EWMA chart, comparison.

1. Introduction

One way to measure and manage quality is through manufacturing process monitoring, or statistical process control, or SPC. Within his 1976 book Guide to Quality Control, Dr. Kaoru Ishikawa assembled a number of instruments for process enhancement (Ishikawa 1976). The Histogram, Pareto diagram, control chart, cause-and-effect diagram, check sheet, scatter diagram, and stratification are the seven well-known quality control (7-QC) tools. One of the most important SPC methods and a popular tool for production process monitoring is the control chart. The primary objective of a control chart is to notify users when a process deviates from control, allowing for the removal of assignable causes through corrective action. The control charting technique was really invented by Shewhart in 1931 (Shewhart 1931), and a plethora of other control chart types have been created since then. Though very user-friendly and useful for identifying large shifts in these parameters, Shewhart-type charts unfortunately perform poorly for small-to-moderate shifts in the process mean or variance. Due to their ability to analyze both historical and current data, memory-type control charts are recommended for the effective detection of this range of shifts. Among the most widely used memory-type control charts are the cumulative sum (CUSUM) and exponentially weighted moving average (EWMA) control charts, which were proposed by Page (1954) and Roberts (1959), respectively. Although they perform almost equally, practitioners frequently believe that the EWMA control chart is simpler to implement. To identify minute mean shifts in general processes, use the EWMA control chart. The development of various methods to enhance the EWMA control chart's ability to identify small-to-moderate shifts has involved extensive SPC research. Double exponentially weighted moving average (DEWMA) control chart was developed by Shamma et al. (1991) and Shamma and Shamma (1992) to increase the effectiveness of the EWMA control chart in identifying minor to moderate process shifts.

The average run length (ARL) of a control chart is commonly used to assess its efficiency. ARL_0 stands for the ARL of an in-control process, and ARL_1 stands for the ARL of an out-of-control process. The ARL_0, which indicates an out-of-control situation, is defined as the expectation of observations made before the first point, indicating that the process has deviated from its stable state. A small ARL_w is required for an effective control chart, and it is signaled when the process mean shifts. Numerous methods can be used to approximate the ARL, such as the Markov chain approach (MCA), the numerical integral equation (NIE), explicit formulas, the martingale approach, and Monte Carlo simulation (MC). Champ and Rigdon (1991) used Markov chain to estimate the ARL for quality control charts and NIE approaches to study CUSUM and EWMA control charts. Afterwards, Mastrangelo and Montgomery (1995) used the Monte Carlo simulation method to evaluate the effectiveness of EWMA control charts for processes that were serially correlated. In order to calculate the ARL for an EWMA control chart with a Laplace distribution and a CUSUM control chart with a hyperexponential distribution, Mititelu et al. (2010) used explicit formulas based on Fredholm-type integral equations.

With observations being continuous distributions, the objective of this study is to suggest a method for estimating the average run length (ARL) for the DEWMA control chart using the numerical integral equation (NIE) method, which incorporates the midpoint, trapezoidal, Simpson, and Gaussian rules. Additionally, the study compares the efficiency in terms of ARL_1 and the relative mean index (RMI) of the DEWMA control chart and the EWMA control chart. Lastly, a variety of real-world data can be utilized with the approximation ARL on the DEWMA control chart.

2. Control Charts and Properties

2.1. Control charts

The performance to identify a proven change in the EWMA and DEWMA control charts is taken into account in this study. Statistical process control (SPC) frequently uses these control charts to track and enhance processes.

2.1.1 The EWMA control chart

Roberts (1959) proposed the use of an exponentially weighted moving average (EWMA) control chart. Usually, the EWMA control chart tracks and identifies slight variations in the process mean. The EWMA statistic (Z_t) can be written as follows

$$Z_t = (1-\lambda_1)Z_{t-1} + \lambda_1 X_t \; ; \; t = 1,2,3,... \tag{1}$$

where λ_1 represents the EWMA control chart's smoothing constant $(0 < \lambda_1 \le 1)$, and X_t denotes an observation from a process. The asymptotic control limits of the EWMA control chart are given by

$$UCL = \mu_0 + L_1\sigma\sqrt{\frac{\lambda_1}{2-\lambda_1}}, \text{ and } LCL = \mu_0 - L_1\sigma\sqrt{\frac{\lambda_1}{2-\lambda_1}},$$

where μ_0 and σ are the mean and the standard deviation when an in-control process, respectively, and L_1 is the EWMA control chart's control chart coefficient. The stopping time of the EWMA control chart is given by

$$\tau_k = \inf\{t > 0 : Z_t > k\}, k > u,$$

where τ_k is the stopping time, and k is the upper control limit (UCL).

2.1.2 The DEWMA control chart

Shamma et al. (1991) and Shamma and Shamma (1992) proposed the Double Exponentially Weighted Moving Average (DEWMA) control chart. The following is one way to write the DEWMA statistic:

$$Z_t = (1-\lambda_1)Z_{t-1} + \lambda_1 X_t \text{ ; } t = 1,2,3,\ldots \text{ and } E_t = (1-\lambda_2)E_{t-1} + \lambda_2 Z_t \text{ ; } t = 1,2,3,\ldots \quad (2)$$

where λ_1 and λ_2 represent the smoothing constant of the EWMA control chart and the DEWMA control chart, respectively, and X_t denotes an observation from a process. The smoothing constant has a range of $0 < \lambda_1 \le 1$ and $0 < \lambda_1 < \lambda_2 \le 1$. The DEWMA control chart's asymptotic control limits are provided by

$$UCL = \mu_0 + L_2\sigma\sqrt{\frac{\lambda_2\left(2-2\lambda_2+\lambda_2^2\right)}{\left(2-\lambda_2\right)^3}},$$

$$LCL = \mu_0 - L_2\sigma\sqrt{\frac{\lambda_2\left(2-2\lambda_2+\lambda_2^2\right)}{\left(2-\lambda_2\right)^3}},$$

where μ_0 and σ are the mean and the standard deviation for an in-control process, respectively, and L_2 is the DEWMA control chart's control chart coefficient. The DEWMA control chart's stopping time is provided by

$$\tau_b = \inf\{t > 0 : E_t > b\}, b > u,$$

where τ_b is the stopping time, and b is the upper control limit (UCL).

2.2. Continuous distributions

This study takes into account the DEWMA control chart in scenarios where the observations are continuous distributions, like the Weibull and exponential distributions.

2.2.1 Exponential distribution

An exponential distribution is used to characterize the random variable X, and the probability density function is defined as follows

$$f(x;\beta) = \frac{1}{\beta}e^{-\frac{x}{\beta}}, x > 0, \beta > 0,$$

where β represents the scale parameter of the exponential distribution.

2.2.2 Weibull distribution

A Weibull distribution is used to characterize the random variable X, and the probability density function is defined as follows

$$f\left(x;\alpha,\beta\right)=\frac{\alpha}{\beta}\left(\frac{x}{\beta}\right)^{\alpha-1}e^{-\left(\frac{x}{\beta}\right)^{\alpha}},\ x>0,\ \alpha>0,\ \beta>0,$$

where α and β represent the shape and scale parameters of the Weibull distribution, respectively.

2.3. The numerical integral equation (NIE) method

This section considers the NIE methods for estimating the average run length (ARL) for the DEWMA control chart. Let $L(u)$ denotes the average run length (ARL) for the DEWMA control chart. To define function $L(u)$ is as follows

$$ARL=L\left(u\right)=E_{\infty}\left(\tau_b\right), \tag{3}$$

where τ_b is the stopping time, and $E_{\infty}(.)$ is the expectation. Then, following the approach described in Champ and Rigdon (1991), the initial values of the EWMA and DEWMA statistics are set to v and u, respectively. The formula for the $L(u)$ defined in (3) can be reformulated as follows

$$\begin{aligned}L\left(u\right)&=1-P\left(\frac{H_L-(1-\lambda_2)u+(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}<X_1<\frac{H_U-(1-\lambda_2)u+(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}\right)\\&\quad+\int_{\frac{H_L-(1-\lambda_2)u+(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}}^{\frac{H_U-(1-\lambda_2)u+(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}}\left(1+L\left((1-\lambda_2)u+\lambda_2\left((1-\lambda_1)v+\lambda_1 y\right)\right)\right)f\left(y\right)dy\\&=1+\int_{\frac{H_L-(1-\lambda_2)u+(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}}^{\frac{H_U-(1-\lambda_2)u+(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}}L\left((1-\lambda_2)u+\lambda_2\left((1-\lambda_1)v+\lambda_1 y\right)\right)f\left(y\right)dy.\end{aligned} \tag{4}$$

From (4) illustrates how the integration will change when the variables are changed.

$$L\left(u\right)=1+\frac{1}{\lambda_1\lambda_2}\int_{H_L}^{H_U}L\left(y\right)f\left(\frac{y-(1-\lambda_2)u-(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}\right)dy.$$

In this research, one side control chart is studied where the lower control limit (LCL) as $a=0$ and the upper control limit (UCL) denoted as b. The DEWMA statistic falls within the range $0\le E_t\le b$ for an in-control process and $E_t>b$ for an out-of-control process. The following can be used to express the formula for the $L(u)$:

$$L(u)=1+\frac{1}{\lambda_1\lambda_2}\int_0^b L(y)f\left(\frac{y-(1-\lambda_2)u-(1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}\right)dy. \tag{5}$$

The quadrature rule can be used to approximate the integral by a finite sum from (5). Weight $\{w_j,\ j=1,2,...,m\}$ and a set of points $\{a_j,\ j=1,2,...,m\}$ are given. The quadrature rule can be used to estimate an integral on the interval $[0,b]$ as follows:

$$\int_0^b W\left(y\right)f\left(y\right)dy\approx\sum_{j=1}^{m}w_j f\left(a_j\right),\ j=1,2,...,m \tag{6}$$

where $f(y)$ is a function to be integrated, and $W(y)$ is called a weight function. An approximation of the NIE method for function $L(u)$ is as follows

$$\tilde{L}(u) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j \tilde{L}(a_j) f\left(\frac{a_j - (1-\lambda_2)u - (1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}\right); i = 1, 2, ..., m.$$

The midpoint rule, trapezoidal rule, Simpson's rule, and Gaussian rule are the methods we suggest using the NIE method to estimate the average run length (ARL) in this study.

2.3.1 Midpoint rule

Given $f(A_j) = f\left(\frac{a_j - (1-\lambda_2)u - (1-\lambda_1)\lambda_2 v}{\lambda_1\lambda_2}\right)$, the interval $[0, b]$ is divided into m subintervals, each with a width equal to $h = b/m.$ The approximation of the ARL using the midpoint rule can be found as follows

$$\tilde{L}_M(u) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j L(a_j) f(A_j),$$

where $a_j = w_j\left(j - \frac{1}{2}\right)$ and $w_j = \frac{b}{m}; j = 1, 2, ..., m.$

2.3.2 Trapezoidal rule

The interval $[0, b]$ is divided into m subintervals, each with a width equal to $h = b/m.$ The approximation of the ARL using the Trapezoidal rule can be found as follows

$$\tilde{L}_T(u) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j L(a_j) f(A_j),$$

where $a_j = w_j j$ and $w_j = \frac{b}{m}; j = 1, 2, ..., m-1.$ In other cases, $w_j = \frac{b}{2m}.$

2.3.3 Simpson's rule

The interval $[0, b]$ is divided into $2m$ subintervals, each with a width equal to $h = b/m.$ The following is the approximate value of the ARL derived from Simpson's rule:

$$\tilde{L}_S(u) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j L(a_j) f(A_j)$$

where $a_j = w_j j,$ $w_j = \frac{4}{3}\left(\frac{b}{2m}\right); j = 1, 3, ..., 2m-1$ and $w_j = \frac{2}{3}\left(\frac{b}{2m}\right); j = 2, 4, ..., 2m-2.$ In other cases, $w_j = \frac{1}{3}\left(\frac{b}{2m}\right).$

2.3.4 Gaussian rule

From (6), where $W(y) = 1, -1 < y < 1.$ The following is the location of the ARL's Gaussian rule approximation:

$$\tilde{L}_G(u) \approx 1 + \frac{1}{\lambda_1\lambda_2}\sum_{j=1}^{m} w_j L(a_j) f(A_j).$$

3. Numerical Result

For the DEWMA control chart, this study presents the NIE method, which includes the Midpoint rule, Trapezoidal rule, Simpson's rule, and Gaussian rule when observations are continuous distributions, such as exponential and Weibull distributions that take processing time (CPU Times) into consideration when an in-control process is in place. The smoothing constant for the EWMA control chart is provided by $\lambda_1 = 0.1.$ For the DEWMA control chart, the smoothing constants are given by $\lambda_2 = 0.3, 0.5, 0.7,$ and 0.9, respectively. The ARL for an out-of-control process is presented with shift sizes $\delta = 0.01, 0.03, 0.05, 0.07, 0.1, 0.3, 0.5, 0.7, 1,$ and 2, respectively. Additionally, the performance of the DEWMA control chart and the EWMA control chart is compared based on the average run length values when an out-of-control process (ARL_1).

Additionally, the Relative Mean Index (RMI) is used to compare the performance efficiency of the DEWMA control chart and the EWMA control chart (Tang et al., 2018). Because a smaller RMI suggests that the ARL is shorter in relation to process variability, it suggests that the control chart performs more quickly and robustly when detecting shifts. RMI is described as

$$RMI(c) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{ARL_i(c) - ARL_i(\mathrm{s})}{ARL_i(\mathrm{s})}\right],$$

where $ARL_i(c)$ is denoted the ARL of the control chart for the shift size of row i and $ARL_i(\mathrm{s})$ is the control chart with the smallest ARL for the shift size of row i. The efficacy of a control chart under various modifications can also be assessed using performance metrics $(\delta_{\min} \le \delta \le \delta_{\max})$.

Tables 1 and 2, approximation of the ARL values on the DEWMA control chart using the NIE method given $\lambda_1 = 0.1, \lambda_2 = 0.7$ and $ARL_0 = 370$ when observations are continuous distributions, such as exponential and Weibull distributions, respectively was presented. The results suggest that the At every stage of the shift sizes, the NIE approach—which makes use of the midpoint and Trapezoidal rules—takes the shortest computational times. In order to compare performance with other control charts, the midpoint rule is chosen in the tables that follow.

Tables 3 and 4 present a comparison of the efficiency in terms of ARL_1 and RMI between the DEWMA and EWMA control charts, utilizing the midpoint rule in the case of exponentially distributed observations. The outcomes demonstrated that, for all shift sizes, the DEWMA control chart performed better than the EWMA control chart. Furthermore, when taking the RMI value into account. The results showed that the performance of the DEWMA control chart given $\lambda_2 = 0.5$ was the most effective in detecting changes.

Tables 5 and 6 show how the midpoint rule is used to compare the efficiency of the DEWMA control chart and the EWMA control chart when the observations follow a Weibull distribution. The outcomes demonstrated that, for all shift sizes, the DEWMA control chart performed better than the EWMA control chart. Furthermore, the results indicated that the DEWMA control chart provided $\lambda_2 = 0.5$ and $\lambda_2 = 0.7$ performed best in detecting changes when the RMI value was taken into account.

Table 1 ARLs of DEWMA control chart for exponential(4)

δ	Method			
	Midpoint	Trapezoidal	Simpson	Gaussian
0.00	370 **(22.344)**	370 (2.953)	370 (97.453)	370 (28.062)
0.01	188.33 **(21.891)**	188.33 (23.769)	188.33 (92.365)	188.33 (29.782)
0.03	95.405 (22.532)	95.405 **(22.025)**	95.405 (95.651)	95.405 (28.875)
0.05	64.113 **(21.766)**	64.113 (24.658)	64.113 (93.368)	64.113 (27.968)
0.07	48.406 **(22.104)**	48.406 (25.098)	48.406 (89.259)	48.406 (27.844)
0.10	35.525 **(20.422)**	35.525 (23.854)	35.525 (90.047)	35.525 (26.344)
0.30	13.299 (22.916)	13.299 **(22.815)**	13.299 (86.438)	13.299 (27.672)
0.50	8.493 **(21.687)**	8.493 (24.851)	8.493 (87.156)	8.493 (27.906)
0.70	6.393 **(21.641)**	6.393 (24.135)	6.393 (82.812)	6.393 (27.89)
1.00	4.801 **(19.25)**	4.801 (23.258)	4.801 (93.172)	4.801 (27.594)
2.00	2.922 (22.562)	2.922 **(22.367)**	2.922 (85.453)	2.922 (27.469)

Note: The CPU times (in seconds) are enclosed in parenthesis.

Table 2 ARLs of DEWMA control chart for Weibull(2, 4)

δ	Method			
	Midpoint	Trapezoidal	Simpson	Gaussian
0.00	370 **(1.525)**	370 (1.562)	370 (6.422)	370 (7.047)
0.01	153.797 **(1.509)**	153.797 (1.589)	153.797 (6.562)	153.797 (7.719)
0.03	71.318 **(1.657)**	71.318 (1.676)	71.318 (6.484)	71.318 (7.422)
0.05	46.647 **(1.562)**	46.647 (1.615)	46.647 (6.734)	46.647 (7.562)
0.07	34.78 (1.734)	34.78 **(1.654)**	34.78 (6.953)	34.78 (7.859)
0.10	25.291 **(1.594)**	25.291 (1.654)	25.291 (6.797)	25.291 (7.031)
0.30	9.408 **(1.547)**	9.408 (1.556)	9.408 (6.578)	9.408 (7.156)
0.50	6.049 (1.672)	6.049 **(1.589)**	6.049 (7.016)	6.049 (7.188)
0.70	4.587 **(1.593)**	4.587 (1.612)	4.587 (7.203)	4.587 (7.203)
1.00	3.480 **(1.547)**	3.480 (1.569)	3.480 (6.516)	3.480 (7.297)
2.00	2.173 **(1.588)**	2.173 (1.625)	2.173 (6.688)	2.173 (6.999)

Note: The CPU times (in seconds) are enclosed in parenthesis.

Table 3 Efficiency comparison in terms of ARL_1 and RMI between DEWMA control chart with EWMA control chart for exponential(2)

δ	DEWMA				EWMA
	$\lambda_2 = 0.3$	$\lambda_2 = 0.5$	$\lambda_2 = 0.7$	$\lambda_2 = 0.9$	
	b = 0.2383119	b = 0.30679	b = 0.415828342	b = 0.69476	k = 3.3181
0.00	370 (22.015)	370 (22)	370 (22.051)	370 (21.985)	370 (19.578)
0.01	118.848 (20.282)	**103.454** (22.376)	126.538 (22.547)	198.937 (21.875)	352.312 (18.235)
0.03	50.295 (21.875)	**42.829** (21.875)	55.142 (22.203)	103.589 (22.266)	320.166 (17.86)
0.05	31.851 (22.125)	**27.238** (21.906)	35.525 (22.547)	70.166 (20.984)	291.964 (18)
0.07	23.29 (21.875)	**20.093** (22.906)	26.348 (22.25)	53.13 (21.484)	267.136 (17.453)
0.10	16.595 (22.937)	**14.536** (22.515)	19.128 (21.797)	39.029 (21.969)	235.201 (18.453)
0.30	5.782 (21.859)	**5.546** (22.078)	7.271 (21.203)	14.436 (20.625)	117.655 (18.234)
0.50	**3.637** (22.594)	3.702 (21.907)	4.801 (21.609)	9.092 (21.125)	72.08 (35.422)
0.70	**2.755** (21.875)	2.912 (22.094)	3.73 (23.031)	6.763 (22.281)	50.366 (18.141)
1.00	**2.128** (21.703)	2.321 (22.063)	2.922 (22.032)	5.008 (22.078)	34.235 (18.641)
2.00	**1.472** (22.047)	1.642 (22.062)	1.971 (22.593)	2.967 (22.313)	16.85 (31.515)
RMI	0.084	**0.028**	0.314	1.417	12.792

Note: The CPU times (in seconds) are enclosed in parenthesis.

Table 4 Efficiency comparison in terms of ARL_1 and RMI between DEWMA control chart with EWMA control chart for exponential (4)

δ	DEWMA				EWMA
	$\lambda_2 = 0.3$	$\lambda_2 = 0.5$	$\lambda_2 = 0.7$	$\lambda_2 = 0.9$	
	b = 0.47662379	b = 0.61358	b = 0.831656683	b = 1.38951999	k = 6.6361
0.00	370 (20.541)	370 (22.234)	370 (22.344)	370 (22.305)	370 (16.906)
0.01	179.977 (21.234)	**161.426** (23.016)	188.33 (21.891)	258.692 (22.656)	360.989 (16.078)
0.03	88.683 (21.641)	**76.236** (22.656)	95.405 (22.532)	161.654 (21.859)	343.82 (16.109)
0.05	58.787 (21.328)	**50.104** (22.578)	64.113 (21.766)	117.654 (22.328)	327.761 (15.765)
0.07	43.941 (21.938)	**37.426** (22.156)	48.406 (22.104)	92.543 (22.078)	312.725 (16.532)
0.10	31.851 (21.672)	**27.238** (22.688)	35.525 (20.422)	70.166 (22.781)	291.923 (16.609)
0.30	11.231 (21.14)	**10.093** (21.937)	13.299 (22.516)	27.166 (22.906)	193.075 (16.625)
0.50	6.871 (21.782)	**6.463** (22.078)	8.493 (21.687)	17.054 (23.051)	136.779 (15.766)

Table 4 (Continued)

δ	DEWMA				EWMA
	$\lambda_2 = 0.3$	$\lambda_2 = 0.5$	$\lambda_2 = 0.7$	$\lambda_2 = 0.9$	
	b = 0.47662379	b = 0.61358	b = 0.831656683	b = 1.38951999	k = 6.6361
0.70	5.008 (22.312)	**4.888** (22.047)	6.393 (21.641)	12.543 (21.563)	102.458 (16.985)
1.00	**3.637** (21.828)	3.702 (22.078)	4.801 (19.25)	9.092 (22.656)	72.074 (18.031)
2.00	**2.128** (22.103)	2.321 (22.375)	2.922 (22.062)	5.008 (21.937)	34.233 (17.938)
RMI	0.100	**0.011**	0.293	1.387	11.952

Note: The CPU times (in seconds) are enclosed in parenthesis.

Table 5 Efficiency comparison in terms of ARL_1 and RMI between DEWMA control chart with EWMA control chart for Weibull(1.5, 3)

δ	DEWMA				EWMA
	$\lambda_2 = 0.3$	$\lambda_2 = 0.5$	$\lambda_2 = 0.7$	$\lambda_2 = 0.9$	
	b = 0.86335061	b = 0.742514373	b = 0.738183945	b = 0.89205076	k = 321034
0.00	370 (1.797)	370 (1.718)	370 (1.75)	370 (1.813)	370 (1.391)
0.01	213.906 (1.656)	**139.716** (1.594)	140.528 (1.688)	200.947 (1.719)	351.995 (1.281)
0.03	114.929 (1.594)	**62.504** (1.609)	63.157 (1.671)	105.12 (1.609)	319.441 (1.344)
0.05	77.922 (1.594)	**40.404** (1.704)	40.969 (1.656)	71.253 (1.656)	290.925 (1.281)
0.07	58.588 (1.625)	**29.931** (1.766)	30.446 (1.563)	53.936 (1.563)	265.862 (1.218)
0.10	42.362 (1.609)	**21.631** (1.453)	22.101 (1.407)	39.574 (1.64)	233.688 (1.265)
0.30	13.884 (1.656)	**7.907** (1.609)	8.269 (1.531)	14.469 (1.688)	116.245 (1.297)
0.50	7.915 (1.453)	**5.051** (1.625)	5.367 (1.781)	9.008 (1.641)	71.447 (1.344)
0.70	5.455 (1.656)	**3.825** (1.656)	4.107 (1.813)	6.631 (1.688)	50.348 (1.312)
1.00	3.726 (1.625)	**2.911** (1.859)	3.156 (1.703)	4.844 (1.687)	34.766 (1.375)
2.00	1.992 (1.609)	**1.869** (1.609)	2.038 (1.875)	2.78 (1.563)	17.828 (1.172)
RMI	0.631	**0.000**	0.043	0.701	8.801

Note: The CPU times (in seconds) are enclosed in parenthesis.

Table 6 Efficiency comparison in terms of ARL_1 and RMI between DEWMA control chart with EWMA control chart for Weibull (2, 4)

δ	DEWMA				EWMA
	$\lambda_2 = 0.3$	$\lambda_2 = 0.5$	$\lambda_2 = 0.7$	$\lambda_2 = 0.9$	
	b = 1.805842	b = 1.269106034	b = 1.077138695	b = 1.10306283	k = 4.6176656
0.00	370 (1.687)	370 (1.546)	370 (1.625)	370 (1.75)	370 (1.5)
0.01	289.444 (1.594)	176.227 (1.719)	**153.797** (1.609)	203.057 (1.781)	352.318 (1.297)
0.03	200.034 (1.765)	86.175 (1.468)	**71.318** (1.657)	106.815 (1.531)	320.281 (1.266)
0.05	151.598 (1.734)	57.094 (1.672)	**46.647** (1.562)	72.517 (1.719)	292.143 (1.39)
0.07	121.264 (1.547)	42.726 (1.547)	**34.78** (1.734)	54.922 (1.75)	267.354 (1.297)
0.10	92.42 (1.75)	31.056 (1.703)	**25.291** (1.594)	40.299 (1.5)	235.451 (1.156)
0.30	32.338 (1.625)	11.187 (1.516)	**9.408** (1.547)	14.672 (1.406)	118.235 (1.094)
0.50	17.921 (1.516)	6.962 (1.625)	**6.049** (1.672)	9.088 (1.797)	73.263 (1.297)
0.70	11.795 (1.579)	5.138 (1.641)	**4.587** (1.593)	6.657 (1.578)	52.063 (1.157)
1.00	7.453 (1.859)	3.777 (1.563)	**3.480** (1.547)	4.828 (1.718)	36.397 (1.359)
2.00	3.154 (1.593)	2.227 (1.75)	**2.173** (1.688)	2.72 (1.562)	19.244 (1.172)
RMI	1.764	0.160	**0.000**	0.470	7.539

Note: The CPU times (in seconds) are enclosed in parenthesis.

4. Real Application

This section will apply the proposed control chart to real- world data. A performance comparison of two control charts, the DEWMA control chart and the EWMA control chart is presented using the midpoint rule.

We tested the real data distribution to the exponential and Weibull distributions. The results are shown as follows.

Distribution	Parameters	p-value
exponential	$\beta = 0.9288501,$	3.776e-10
Weibull	$\alpha = 3.976466,\ \beta = 1.187820$	0.1227

From the test, the dataset comprises observations are Weibull distribution with parameters $\alpha = 3.976466$ and $\beta = 1.187820,$ representing the average time a passenger waits for a subway beyond schedule (New York State Open Data 2015). These observations were obtained from 50 Type B subway line W passengers in New York State. The efficiency of the DEWMA control chart is shown in Table 7 and Figure 1.

Table 7, comparison of the efficiency in terms of ARL_1 and RMI for the DEWMA control chart with the EWMA control chart by using the midpoint rule given $\lambda_1 = 0.1, \lambda_2 = 0.5,$ and $ARL_0 = 370$

when dataset of real observations are Weibull distribution. The performance of the DEWMA control chart outperformed the EWMA control chart across all shift sizes, according to the results.

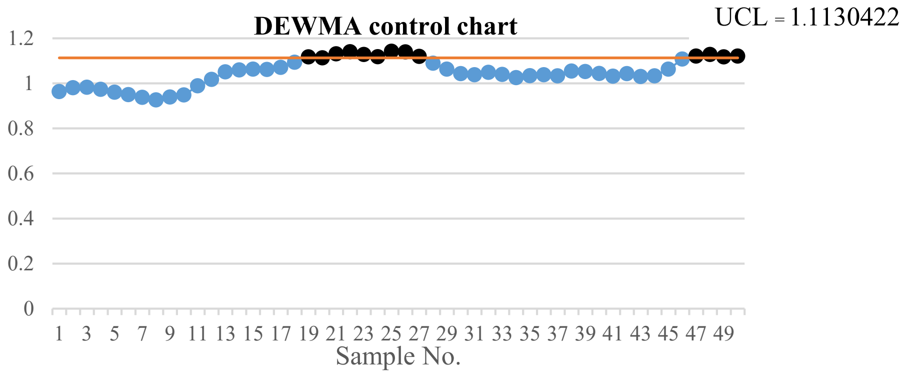
Figure 1(a), the DEWMA control chart detected the shift at the 19th to 27th and 47th to 50th observations. In Figure 1(b), the EWMA control chart shows that no observations are out of the control limit.

Table 7 Efficiency comparison in terms of ARL_1 and RMI between DEWMA control chart with EWMA control chart for real data under Weibull (3.976466, 1.187820)

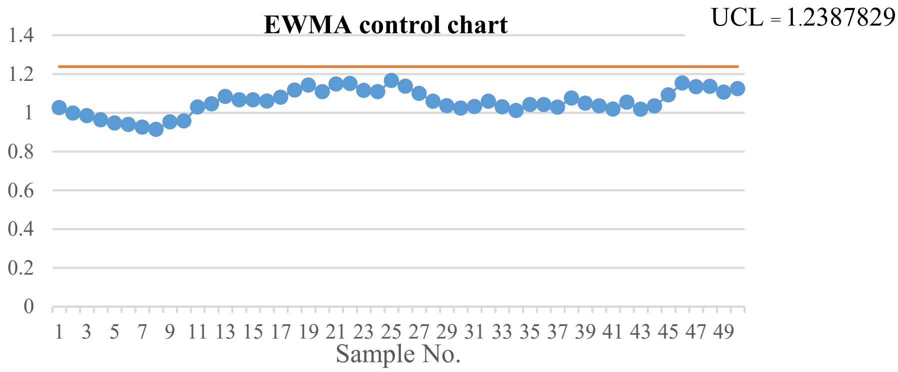
δ	DEWMA	EWMA
	b = 1.1130422	k = 1.2387829
0.00	370 (1.457)	370 (8.328)
0.01	**212.564** (1.375)	275.356 (8.515)
0.03	**112.372** (1.478)	168.04 (8.391)
0.05	**74.993** (1.329)	114.328 (8.735)
0.07	**55.573** (1.375)	84.662 (8.14)
0.10	**39.42** (1.531)	60.489 (8.265)
0.30	**12.094** (1.516)	23.78 (8.063)
0.50	**6.755** (1.5)	16.386 (7.438)
0.70	**4.586** (1.547)	12.87 (6.907)
1.00	**3.133** (1.328)	9.928 (6.968)
2.00	**1.884** (1.454)	5.895 (6.542)
RMI	**0.000**	1.087

5. Discussion and Conclusions

The goal of this study is to examine how the numerical integral equation (NIE) method, specifically the midpoint, trapezoidal, Simpson, and Gaussian rules for the DEWMA control chart, can be used to estimate the average run length (ARL). As can be seen from the results, the midpoint and trapezoidal rules yielded the ARL values of the DEWMA control chart that showed the quickest computation times. The ARL_1 and RMI are used as the efficiency criteria in this research. The performance of the DEWMA control chart is superior to the EWMA control chart in all shift sizes, as evidenced by the analysis results from the simulation data matching those from the real data. We recommend $\lambda_2 = 0.5$ because the RMI value is minimal. Future studies, however, might examine different metrics, like the Performance Comparison Index (PCI), Average Extra Quadratic Loss (AEQL), Standard Deviation of the Run Length (SDRL), and others, for evaluating the performance of control charts. This more thorough analysis might offer a more thorough comprehension of the efficacy of control charts across a range of metrics.



(a) DEWMA control chart



(b) EWMA control chart

Figure 1 Control charts of dataset of real observations are Weibull distribution

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References

Champ CW, Rigdon SE. A comparison of the markov chain and the integral equation approaches for evaluating the run length distribution of quality control charts. Commun Stat Simul Comput. 1991; 20(1): 191-204.

Ishikawa K. Guide to quality control. Japan: Asian Productivity Organization; 1976.

Mititelu G, Areepong Y, Sukparungsee S, Novikov AA. Explicit analytical solutions for the average run length of CUSUM and EWMA charts. East West J Math. 2010; Special edition: 253-265.

Mastrangelo CM, Montgomery DC. SPC with correlated observations for the chemical and process industries. Qual Reliab Eng Int. 1995; 11: 79-89.

New York State Open Data. MTA subway customer journey-focused metrics [serial on the Internet]. 2015 [cited 16 Nov. 2023]. Available from: https://data.world/data-ny-gov/r7qk-6tcy.

Page ES. Continuous inspection schemes. Biometrika. 1954; 41(1-2): 100-115.

Roberts SW. Control chart tests based on geometric moving average. Technometrics. 1959; 1(3): 239-250.

Shamma SE, Amin RW, Shamma AK. A double exponentially weighted moving average control procedure with variable sampling intervals. Commun Stat Simul Comput. 1991; 20(2-3): 511-528.

Shamma SE, Shamma AK. Development and evaluation of control charts using double exponentially weighted moving averages. Qual Technol Quant Manag. 1992; 9(6): 18-25.

Shewhart WA. Economic control of quality of manufactured product. New York: D. Van Nostrand Company, Inc; 1931.

Tang A, Castagliola P, Sun J, Hu X. Optimal design of the adaptive EWMA chart for the mean based on median run length and expected median run length. Qual Technol Quant Manag. 2018; 16(4): 439-458.