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A New Class of Distributions for Modelling Continuous Positively Skewed Data Sets

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Abstract

In this paper, we proposed a new class of distributions by introducing a new constant in the existing model. We discuss general properties of the family such as density function, quantile function and hazard rate function. We then discuss a member of the family considering the exponential distribution as baseline distribution. Various properties of the model such as quantile function, moments, moment generating function, order statistics, stress-strength parameter, and mean residual life function are discussed. We also discussed the mean, variance, skewness and kurtosis of the proposed model numerically. The expression for Rényi and Shannon entropies are also derived. The different methods of estimation such as maximum likelihood estimation, maximum product spacing and least squares estimates are used for the estimation of the unknown parameters of the proposed distribution.. The simulation study is performed to study the behaviour of the estimates based on their mean squared errors. Lastly, we apply our proposed model to two real data sets.

Keywords: Generalized probability distribution, moments, quantile function, stress-strength parameter, mean residual life function, Rényi entropy, Shannon entropy, maximum likelihood estimation, least squares estimation, maximum product spacing estimation.

1. Introduction

In statistics, a number of probability models have been proposed for modeling lifetime data sets. It can be seen from the literature that adding an extra parameter that controls skewness of the distribution makes it flexible to a great extent. We know that the exponential distribution is a simplest and useful probability model but it cannot be used in many real life situations, specially when the hazard rate is time dependent. Keeping such type of problems in mind, many authors introduced shape parameters in it in different ways to make the exponential distribution flexible to fit non-constants hazard rate data. Readers may follow Gupta (2001) and Rinne (2008) for detailed theory and applications of the extensions of the exponential distribution. Azzalini (1985) introduced a parameter λ to the standard normal distribution and derived the skew normal distribution defined by the following probability density function (pdf),

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x);\; x \in R,$$

where $\phi(x)$ and $\Phi(x)$ denote the pdf and cdf of the normal distribution, respectively. λ plays a significant role for controlling the skewness. It can be easily seen that $f(x;\lambda)$ is more flexible than the normal distribution, because it takes the different skewed shapes for its pdf depending upon the values of λ.

We can also mention here that both Weibull and Gamma distributions have time dependent (monotone) hazard rate function (hrf) and overcome the drawbacks of exponential distribution. But these distributions cannot be used in those situations where the hrf is non-monotone. For example, in case of human life, during first year of life the probability of infant death is high, and as the time passes, this probability decreases, remains constants and after a certain time again the probability increases due to the aging. Therefore, in this case, the use of the Weibull or Gamma distributions is inappropriate. To overcome such problems, Mudholkar (1995) and (1996) proposed the Exponentiated Weibull Distribution (EWD), having bathtub-shaped hrf. Many other extended distributions such as additive Weibull distribution by Xie (1995), generalized modified Weibull by Carrasco (2008), and exponential-Weibull distribution by Cordeiro (2014) and Lehmann type family by Gupta (1998) are available in literature. For generalized classes of distributions, we also follow Marshall (1997), Lee (2013), Alzaatreh (2014), Jones (2015) and references cited theirin.

In this article, we propose an extension of the family introduced by Kumar (2017), in which they call it as the minimum guarantee transformation (MGT). If $G(x)$ and $g(x)$ are the baseline cdf and pdf, respectively, then the MGT is defined by the cummulative distribution function (cdf),

$$F(x) = e^{1-\frac{1}{G(x)}};\ x \in R, \tag{1}$$

and the pdf is

$$f(x) = \frac{e^{1-\frac{1}{G(x)}}}{(1-G(x))^2} g(x). \tag{2}$$

Further they considered the exponential distribution as baseline distribution and called the proposed model as minimum guarantee exponential (MG_{exp}) distribution. They discussed the properties, problem of sample generation, parameter estimation aand application for this distribution.

This paper aims to propose a more general form of the MG_{exp} distribution by replacing the exponential with a new constant. Also the aim is to discuss the various statistical properties and applications of the proposed model. This distribution may provide a better fitting than the existing distributions for real life data sets.

Further, sections of this paper are organized as follows: In Section 2, The new transformation is proposed and also some of it's statistical properties are discussed. In Section 3, we propose a new model by considering the exponential distribution as the baseline distribution and some statistical properties of the proposed model such as moments and moment generating function, order statistics, mean residual life time, stress-strength parameter and entropies are discussed. In Section 4, we study the problem of estimation by using the maximum likelihood estimates (MLEs), maximum product spacing estimates (MPSEs) and least squares estimates (LSEs) of the parameters. Simulation study is also performed in this Section to show the behavior of the estimators of the parameters. In the next Section 5, real data sets are taken to show the applicability of the proposed model. Finally, the conclusion of this paper is given in Section 6.

2. Proposed Distribution and Interpretations

As it pointed out in the previous section that we replace exponent power to a constant, say $\alpha > 0$ and introduce an extension of the MGT distributions. The cdf and pdf of the proposed distribution are as follows

$$F(x) = \alpha^{1-\frac{1}{G(x)}}, \quad x \in R, \alpha > 0, \tag{3}$$

$$f(x) = \frac{\log(\alpha)\alpha^{1-\frac{1}{G(x)}}}{(G(x))^2} g(x), \quad x \in R, \alpha > 0. \tag{4}$$

We can also rewrite them in following forms

$$F(x) = \begin{cases} \alpha^{1-\frac{1}{G(x)}}, & \text{if } \alpha > 1, \quad x \in R, \\ G(x), & \text{if } \alpha = 1, \quad x \in R, \end{cases} \tag{5}$$

and

$$f(x) = \begin{cases} \dfrac{\log(\alpha)\alpha^{1-\frac{1}{G(x)}}}{(G(x))^2} g(x), & \text{if } \alpha = 1, \quad x \in R, \\ g(x), & \text{if } \alpha = 1, \quad x \in R. \end{cases} \tag{6}$$

Here, we show two interpretations of the proposed distribution. Alzaatreh (2013) introduced the following method for generating families of distributions. Let X be a random variable with pdf $g(x)$ and cdf $G(x)$. Let T be another continuous random variable with pdf $r(t)$ defined on the interval $[a, b]$. Then the cdf of a new family of distributions can be defined as

$$F(x) = \int_a^{W(G(x))} r(t)dt, \tag{7}$$

where, $W(G(x))$ is the function of cdf $G(x)$, satisfies the following properties,

$$\begin{cases} W(G(x)) \in [a, b], \\ W(G(x)) \quad \text{is differentiable and monotonically non-decreasing,} \\ W(G(x)) \to a \text{ as } x \to -\infty \text{ and } W(G(x)) \to b \text{ as } x \to \infty. \end{cases} \tag{8}$$

Equation (3) can be written as $F(x) = R\{W(G(x))\}$ and R(t) denotes the cdf of T. The pdf corresponding to the $F(x)$ is given by

$$f(x) = \left\{\frac{d}{dx} W(G(x))\right\} \quad r\{W(G(x))\}. \tag{9}$$

This pdf $f(x)$ is called "$T - X$" family of distributions. Let us consider $W(G(x)) = \alpha^{1-\frac{1}{G(x)}}$, then using "$T - X$" family of transformation, we get the cdf and pdf of new family of distribution as,

$$F(x) = R\left\{\alpha^{1-\frac{1}{G(x)}}\right\}, \tag{10}$$

$$f(x) = \left\{\frac{d}{dx}\alpha^{1-\frac{1}{G(x)}}\right\} \quad r(\alpha^{1-\frac{1}{G(x)}}). \tag{11}$$

Let $T \sim U(0, 1)$, then the cdf and pdf of T are $R(t) = t$, and $r(t) = 1$ respectively. A weighted class of distributions by Patil (1978) is given as

$$f^w(x) = \frac{w(x)f(x)}{w},$$

where $w(x)$ is a weight function and $w = \int w(x)f(x)dx$ is a normalizing constant such that $\int f^w(x)dx = 1$. Thus, we can see that the proposed distribution is a weighted distribution with $w(x) = \frac{\alpha^{1-\frac{1}{G(x)}}}{(G(x))^2}$. There are a number of articles that showed the applications of the weighted distributions in various fields such as reliabiity, medicine, ecology and branching processes. For a detailed theory and applications, see Rao (1965), Patil (1978), Patil (1986), Sharma (2018) and references cited therein.

Let hrf and survival function of the proposed distribution are as follows,

$$h(x) = \begin{cases} \dfrac{\log(\alpha)\alpha^{1-\frac{1}{G(x)}}}{(G(x))^2(1-\alpha^{1-\frac{1}{G(x)}})} g(x), & \text{if } \alpha > 1, \\ \frac{g(x)}{1-G(x)}, & \text{if } \alpha = 1. \end{cases} \tag{12}$$

$$S(x) = \begin{cases} 1 - \alpha^{1-\frac{1}{G(x)}}, & \text{if } \alpha > 1, \\ 1 - G(x), & \text{if } \alpha = 1. \end{cases} \tag{13}$$

Let x_p be the p^{th} quantile of the proposed distribution, then it's expression is given by

$$x_p = F^{-1}\left\{\frac{\log(\alpha)}{\log(\alpha) - \log(p)}\right\}, \tag{14}$$

where $F^{-1}(\cdot)$ is the quantile function of the baseline distribution. We can easily generate random numbers from the proposed distribution using the baseline distribution.

3. Exponential Based Member and It's Properties

In this section, we introduce a member of the family given in (3) based on the exponentila distribution. The cdf of proposed distribution is given by,

$$F(x) = \begin{cases} \alpha^{1-\frac{1}{1-e^{-\lambda x}}}, & \text{if } \alpha > 1, x > 0, \\ 1 - e^{-\lambda x}, & \text{if } \alpha = 1. \end{cases} \tag{15}$$

The pdf corresponding to above proposed cdf is given as,

$$f(x) = \begin{cases} \dfrac{\log(\alpha)\alpha^{1-\frac{1}{1-e^{-\lambda x}}}}{(1-e^{-\lambda x})^2}\lambda e^{-\lambda x}, & \text{if } \alpha > 1, x > 0, \\ \lambda e^{-\lambda x}, & \text{if } \alpha = 1. \end{cases} \tag{16}$$

Also, the expression for the hrf and survival functions for the proposed model are given by,

$$h(x;\lambda,\theta) = \begin{cases} \dfrac{\log(\alpha)\alpha^{1-\frac{1}{1-e^{-\lambda x}}}}{(1-e^{-\lambda x})^2(1-\alpha^{1-\frac{1}{1-e^{-\lambda x}}})}\lambda e^{-\lambda x}, & \text{if } \alpha > 1, \\ \lambda, & \text{if } \alpha = 1. \end{cases} \tag{17}$$

$$S(x) = \begin{cases} 1 - \alpha^{1-\frac{1}{1-e^{-\lambda x}}}, & \text{if } \alpha > 1, \\ e^{-\lambda x}, & \text{if } \alpha = 1. \end{cases} \tag{18}$$

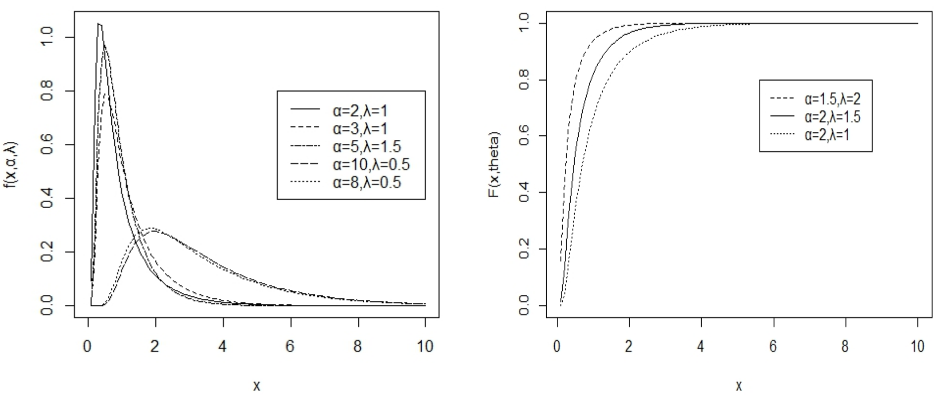

Figure 1 Plots of the pdf and cdf for different values of α, λ

Plots of the pdf and hrf are given in Figures 1 and 2, respectively. The proposed distribution is positively skewed and its hrf is upside-down bathtub shaped.

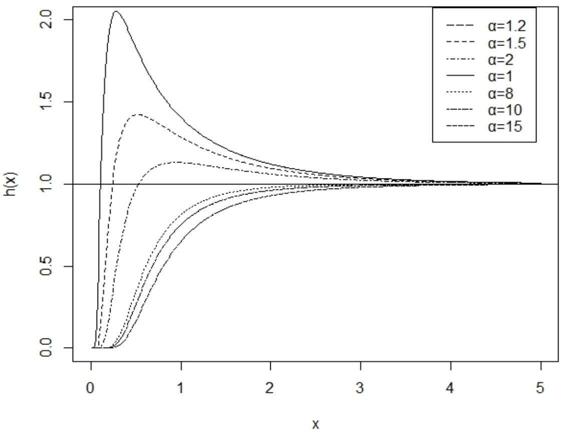

Figure 2 Plots of the hazard rate function for different values of α and $\lambda = 1$

3.1. Quantile function

The quantile function, say x_p, defined by $F(x_p) = p$ is the root of the equation.

$$\alpha^{1-\frac{1}{1-e^{-\lambda x_p}}} = p,$$
$$x_p = -\frac{1}{\lambda}\log\left\{\frac{\log(p)}{\log(p)-\log(\alpha)}\right\}.$$

3.2. Moments and moment generating function

In this section the rth moment about origin and moment generating function are obtained by using the following series expansion:

$$\alpha^z = \sum_{k=0}^{\infty}\frac{(\log\alpha)^k z^k}{k!}. \tag{19}$$

The r^{th} moment about origin of the proposed distribution can be obtained as follows:

$$\mu_r' = E[X^r],$$
$$= \int_0^{\infty} x^r \frac{\log(\alpha)\alpha^{1-\frac{1}{1-e^{-\lambda x}}}}{(1-e^{-\lambda x})^2}\lambda e^{-\lambda x}dx.$$

Using the series expansion given in (21) μ_r' is obtained as follows:

$$\mu_r' = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+2}{j}\frac{(-1)^k(\log(\alpha))^{k+1}}{k!}\frac{\Gamma r+1}{\lambda^r(j+k+1)^{r+1}}, \quad r = 1,2,3,... \tag{20}$$

Putting $r = 1$, we get the first non-central moments (μ_1'), which is mean of the distribution, and taking $r = 2$, the second non-central moment (μ_2') can be obtained, and hence the variance of the distribution can be obtained as $\mu_2 = \mu_2' - (\mu_1')^2$. The formula for skewness (s_k) based on moments and the coefficient of kurtosis β_2 are given as follows:

$$s_k = \frac{\sqrt{\beta_1}(\beta_2+3)}{2(5\beta_2-6\beta_1-9)},$$

where, $\beta_1 = \frac{\mu_3^2}{\mu_2^3}$ and $\beta_2 = \frac{\mu_4}{\mu_2^2}$.

Table 1 The values of mean, skewness and kurtosis of the proposed distribution for different combinations of the parameters

(α, λ)	Mean	Variance	Skewness	Kurtosis
(1.5,0.5)	1.4289	2.2235	1.3683	15.9121
(2.5,0.5)	2.2443	3.3278	1.0444	10.6178
(3.5,0.5)	2.6256	3.7674	0.9445	9.3643
(4.5,0.5)	2.8655	4.0194	0.8919	8.7694
(5.5,0.5)	3.0369	4.188	0.8584	8.4115
(10.5,0.5)	3.5016	4.6004	0.7811	7.6491
(1.5,1.5)	0.4763	0.2471	1.3683	15.9121
(1.5,2.5)	0.2858	0.0889	1.3684	15.9122
(1.5,3.5)	0.2041	0.0454	1.3683	15.9121
(1.5,4.5)	0.1588	0.0274	1.3709	15.9174
(1.5,5.5)	0.1299	0.0184	1.3683	15.9121
(1.5,10.5)	0.068	0.005	1.3683	15.9123

We provide the values of mean, variance, skewness and kurtosis of the proposed distribution in Table 1. From the table, it can be seen that skewness is positive for all the parameter combinations, therefore the proposed distribution is positively skewed. β_2 is greater than three in every case, hence, the proposed distribution exhibits the shapes with higher than the normal curve.

Moment generating function (MGF) of the proposed distribution is given by:

$$\begin{aligned} M_X(t) &= E[e^{tX}]; t \in R \\ &= \int_0^\infty e^{tx} \frac{\log(\alpha)\alpha^{1-\frac{1}{1-e^{-\lambda x}}}}{(1-e^{-\lambda x})^2} \lambda e^{-\lambda x} dx \\ &= \sum_{i=0}^\infty \sum_{j=0}^\infty \frac{(-1)^i (\log(\alpha))^{(i+1)}}{i!} \binom{i+2}{j} \frac{\lambda}{\lambda(i+j+1)-t}. \end{aligned} \tag{21}$$

3.3. Order statistics

Let us consider a random sample $X_1, X_2, ..., X_n$ of size n has been drawn from the proposed distribution and $X_{(1)} \leq X_{(2)} \leq, ..., \leq X_{(n)}$ are the order statistics, then the pdf of i^{th} order statistic is defined as:

$$f_i(x) = \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1} f(x)[1-F(x)]^{n-i}.$$

Putting the values from (15) and (16) and using the series expansion the pdf of i^{th} order statistics can be obtained as follows:

$$f_i(x) = \frac{\lambda \log(\alpha)}{\beta(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} (-1)^{j+k} \binom{n-i}{j} \frac{(\log(\alpha))^k (i+j)^k}{k!} \frac{e^{-(1+k)\lambda x}}{(1-e^{-\lambda x})^{k+2}}. \tag{22}$$

The expression for the s^{th} moment of $X_{(i)}$ can be given as

$$\begin{aligned} E(X_{(i)}^s) = \frac{\lambda \log(\alpha)}{\beta(i, n-i+1)} \sum_{j=0}^{n-i} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} (-1)^{j+k} \binom{n-i}{j} \binom{k+2}{l} \\ \frac{(\log(\alpha))^k (i+j)^k}{k!} \frac{\Gamma(s+1)}{(\lambda(k+l+1))^{s+1}}. \end{aligned} \tag{23}$$

3.4. Stress-strength parameter

Let X_1 be a random variable which follows the proposed distribution with parameters (α_1, λ_1) and X_2 be another random variable having the same distribution with parameters (α_2, λ_2), then the stress-strength parameter R is calculated as:

$$R = \int_{-\infty}^{\infty} f_1(x) F_2(x) dx \tag{24}$$

where f_1 is the pdf of X_1 and $F_2(x)$ be the cdf of X_2. After putting the values from (16) and (17) the above equation reduces to

$$R = \int_0^{\infty} \frac{\log(\alpha_1)\alpha_1^{1-\frac{1}{1-e^{-\lambda_1 x}}}}{(1-e^{-\lambda_1 x})^2} \lambda_1 e^{-\lambda_1 x} \alpha_2^{1-\frac{1}{1-e^{-\lambda_2 x}}} dx.$$

Using the series expansion (21) and solving the above equation we get the expression for stress-strength parameter as follows:

$$R = \lambda_1 \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} (-1)^{i+j} \frac{(\log(\alpha_1))^{i+1}}{i!} \frac{(\log(\alpha_2))^{j}}{j!} \binom{i+2}{k} \times \binom{j}{l} \frac{1}{\lambda_1(i+k)+\lambda_2(j+l)}. \tag{25}$$

3.5. Mean residual life function

Let us consider a unit of age t, then the remaining life of the unit after time t will be a random quantity. Mean residual life is defined as the expected value of the random residual given that the unit has survived up to age t. Mean residual life is calculated for each t, therefore, we talk of mean residual life function. Let X be a random variable which has survival function (19), then the mean residual life function, say $m(t)$ is given by:

$$m(t) = \frac{1}{S(t)} \left[E(t) - \int_0^t x f(x) dx \right] - t. \tag{26}$$

The value of $\int_0^t x f(x) dx$ is obtained as,

$$\int_0^t x f(x) dx = \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \lambda (-1)^k \frac{(\log(\alpha))^{k+1}}{k!} \times \binom{k+2}{l} \frac{\gamma(\lambda t(k+l+1), 2)}{\left\{\lambda(k+l+1)\right\}^2} \tag{27}$$

where $\gamma(m,n) = \int_0^m x^{n-1} e^{-x} dx$ is known as lower incomplete gamma function. Now, the mean residual function is given by

$$m(t) = \frac{\sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \binom{k+2}{j} \frac{(-1)^k (\log(\alpha))^{k+1}}{k!} \frac{1}{\lambda(j+k+1)^2} - \int_0^t x f(x) dx}{1-\alpha^{1-\frac{1}{1-e^{-\lambda t}}}} - t.$$

3.6. Entropy

Entropy is well known to the measure of uncertainty of a probability distribution. Many relationships between entropy and it's associated probability distribution are available in the literature. There are various entropies proposed by different authors based on the properties of entropy. Among these proposed entropies Rényi and Shannon entropies are the two famous entropies. Expression for the Rényi entropy Renyi (1961) is given as

$$RE_x(\nu) = \frac{1}{1-\nu} \log\left\{ \int_{-\infty}^{\infty} f^{\nu}(x) dx \right\} \qquad \nu > 0, \nu \neq 1.$$

Putting the value of pdf given in (16) and using the series expansion, We can get the R$\acute{e}$nyi entropy as under

$$\begin{aligned}RE_x(\nu) &= \frac{1}{1-\nu}\log\int_{-\infty}^{\infty}\left\{\frac{\log(\alpha)\alpha^{1-\frac{1}{1-e^{-\lambda x}}}}{(1-e^{-\lambda x})^2}\lambda e^{-\lambda x}\right\}^{\nu}dx\\ &= \frac{1}{1-\nu}\log\left\{(log(\alpha))^{\nu}\lambda^{\nu}\int_{-\infty}^{\infty}\sum_{i=0}^{\infty}(-1)^i e^{-\lambda x}\frac{(\log(\alpha))^i}{i!}\frac{\nu^i e^{-i\lambda x}}{(1-e^{-\lambda x})^{2\nu+i}}\right\}\\ &= \frac{\nu}{1-\nu}\log(\lambda log(\alpha)) + \frac{1}{1-\nu}\log\left\{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\binom{2\nu+i}{j}\frac{(-1)^i(\nu log(\alpha))^i}{i!}\frac{1}{\lambda(i+j+1)}\right\}. \end{aligned} \quad (28)$$

The other famous measure of entropy is Shannon entropy Shannon (1951) defined as

$$SE_x = E[-\log f(x)]. \quad (29)$$

Using (17) expression for the Shannon entropy is given by

$$\begin{aligned}SE_x = -\lambda\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^k\frac{(\log\alpha)^{k+1}}{k!}&\left[\binom{k+2}{j}\left\{\frac{1}{\lambda(j+k+1)^2}-\frac{1}{\lambda(j+k+2)}\right\}\right.\\ &\left.+\binom{k+1}{j}\times\left(\frac{1}{\lambda(j+k+1)}\right)\right].\end{aligned} \quad (30)$$

4. Estimation of the Parameters by Different Methods

4.1. Maximum likelihood estimation

Let $X_1, X_2, ..., X_n$ be a random sample of size n from the proposed distribution defined by the (17), then the likelihood function is given by

$$l = \prod_{i=1}^{n}\frac{\log(\alpha)\alpha^{1-\frac{1}{1-e^{-\lambda x_i}}}}{(1-e^{-\lambda x_i})^2}\lambda e^{-\lambda x_i},$$

and the corresponding loglikelihood function can be written as

$$\log l = n\log\lambda + n\log(\log(\alpha)) - \lambda\sum_{i=1}^{n}x_i + \log(\alpha)\sum_{i=1}^{n}(1-\frac{1}{1-e^{-\lambda x}}) - 2\sum_{i=1}^{n}log(1-e^{-\lambda x_i}). \quad (31)$$

Now, to obtain the maximum likelihood estimates of the parameters, we maximize the loglikelihood with respect to α and λ. Differentiating (31) with respect to α and λ, we get the log-likelihood equations as

$$\frac{d\log l}{d\alpha} = \frac{n}{\alpha\log\alpha} + \sum_{i=1}^{n}\frac{1}{\alpha}\left\{1-\frac{1}{1-e^{-\lambda x_i}}\right\} = 0, \quad (32)$$

$$\frac{d\log l}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{\lambda\log(\hat{\alpha})e^{-\lambda x_i}}{(1-e^{-\lambda x_i})^2} - 2\sum_{i=1}^{n}\frac{\lambda e^{-\lambda x_i}}{1-e^{-\lambda x_i}} = 0. \quad (33)$$

From (32), we get the MLE of α for the given λ as

$$\hat{\alpha} = \exp\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{e^{-\lambda x_i}}{1-e^{-\lambda x_i}}\right)}\right). \quad (34)$$

Now, substituting (34) in (33), we can obtain the MLE of λ by solving the following non-linear equation

$$\frac{d\log l}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n}x_i + \sum_{i=1}^{n}\frac{\lambda e^{-\lambda x_i}}{(1-e^{-\lambda x_i})^2}\left(\frac{n}{\sum_{i=1}^{n}\left(\frac{e^{-\lambda x_i}}{1-e^{-\lambda x_i}}\right)}\right) - 2\sum_{i=1}^{n}\frac{\lambda e^{-\lambda x_i}}{1-e^{-\lambda x_i}} = 0. \quad (35)$$

Since, (35) is not in closed form, therefore, we use numerical techniques such as Newton Raphson method to solve this equation. The elements of Fisher's Information matrix are

$$\frac{d^2 \log l}{d\alpha^2} = -\frac{n(1+\log(\alpha))}{(\alpha \log(\alpha))^2} - \frac{1}{\alpha^2}\sum_{i=1}^{n}\left\{1 - \frac{1}{1-e^{-\lambda x}}\right\} \tag{36}$$

$$\frac{d^2 \log l}{d\lambda^2} = \frac{-n}{\lambda^2} - \sum_{i=1}^{n}\frac{\lambda^2 e^{-\lambda x}}{(1-e^{-\lambda x})^3}\left\{\log(\alpha)(1+e^{-\lambda x}) - 2(1-e^{-\lambda x})\right\} \tag{37}$$

$$\frac{d^2 \log l}{d\alpha d\lambda} = \sum_{i=1}^{n}\frac{\lambda}{\alpha}\frac{e^{-\lambda x}}{(1-e^{-\lambda x})^2}. \tag{38}$$

Fisher information matrix is given by

$$I(\hat{\alpha},\hat{\lambda}) = \begin{bmatrix} -\frac{\partial^2 \log(L)}{\partial\alpha^2} & -\frac{\partial^2 \log(L)}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2 \log(L)}{\partial\alpha\partial\lambda} & -\frac{\partial^2 \log(L)}{\partial\lambda^2} \end{bmatrix}_{(\hat{\alpha},\hat{\lambda})}.$$

4.2. Maximum product spacing (MPS) method

This method of estimation is proposed by Cheng (1983). Let $x_1, x_2, ..., x_3$ be a random sample from a distribution having pdf $f(x,\alpha,\lambda)$ and we want to estimate the parameter θ. Also let $x_{(1)}, x_{(2)}, ..., x_{(i)}, ..., x_{(n)}$ denote the order statistics,then the spacings D_i^s can be defined as $D_1 = F(x_{(1)};\alpha,\lambda)$, $D_{n+1} = 1 - F(x_{(n)};\alpha,\lambda)$ and $D_i = F(x_{(i)};\alpha,\lambda) - F(x_{(i-1)};\alpha,\lambda)$, $i = 2,3,...,n$, where $\sum_{i=1}^{n} D_i = 1$. This method cosists in finding the value of α, and λ that maximizes the geometric mean (say) G of the spacings, i.e.,

$$G = \left\{\prod_{i=1}^{n+1} D_i\right\}^{\frac{1}{n+1}} \tag{39}$$

or equivalently, we can say that

$$\begin{aligned}\log(G) &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log D_i \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1}\log\Big[F(x_{(i)};\alpha,\lambda) - F(x_{(i-1)};\alpha,\lambda)\Big].\end{aligned} \tag{40}$$

The basic concept behind this method of estimation is that differences between the cdfs at neighbouring points should be identically distributed. The MPSEs can be obtained by solving the following non-linear equations

$$\frac{\partial}{\partial\alpha}\log(G) = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_\alpha(x_{(i)};\alpha,\lambda) - F'_\alpha(x_{(i-1)};\alpha,\lambda)}{F(x_{(i)};\alpha,\lambda) - F(x_{(i-1)};\alpha,\lambda)}\right] \tag{41}$$

$$\frac{\partial}{\partial\lambda}\log(G) = \frac{1}{n+1}\sum_{i=1}^{n+1}\left[\frac{F'_\lambda(x_{(i)};\alpha,\lambda) - F'_\lambda(x_{(i-1)};\alpha,\lambda)}{F(x_{(i)};\alpha,\lambda) - F(x_{(i-1)};\alpha,\lambda)}\right] \tag{42}$$

where, $F'_\alpha(x;\alpha,\lambda) = \frac{e^{-\lambda x}}{e^{-\lambda x}-1}\alpha^{-\frac{1}{1-e^{-\lambda x}}}$ and $F'_\lambda(x;\alpha,\lambda) = \frac{\log(\alpha)xe^{-\lambda x}}{(1-e^{-\lambda x})^2}\alpha^{1-\frac{1}{1-e^{-\lambda x}}}$.

4.3. Least square estimates (LSE)

Let $x_{(1)}, x_{(2)}, ..., x_{(n)}$ be the ordered samples of size n drawn from the proposed pdf given in (17). Then, we can define the expectation of empirical cdf as

$$E[F(x_{(i)})] = \frac{i}{n+1}, \quad i = 1,2,3,...n. \tag{43}$$

We can obtain the LSE_s of α and λ by minimizing

$$S(\alpha,\lambda)=\sum_{i=1}^{n}\left(F(x_{(i)};\alpha,\lambda)-\frac{i}{n+1}\right)^{2}. \tag{44}$$

Therefore, we can find the estimates $\hat{\alpha}$ and $\hat{\lambda}$ by solving the following equations:

$$\frac{\partial}{\partial\alpha}S(\alpha,\lambda)=\sum_{i=1}^{n}F_{\alpha}^{'}(x_{(i)};\alpha,\lambda)\left(F(x_{(i)};\alpha,\lambda)-\frac{i}{n+1}\right)=0 \tag{45}$$

$$\frac{\partial}{\partial\lambda}S(\alpha,\lambda)=\sum_{i=1}^{n}F_{\lambda}^{'}(x_{(i)};\alpha,\lambda)\left(F(x_{(i)};\alpha,\lambda)-\frac{i}{n+1}\right)=0 \tag{46}$$

where, $F_{\alpha}^{'}(x_{(i)};\alpha,\lambda)$ and $F_{\lambda}^{'}(x_{(i)};\alpha,\lambda)$ are given above. These equations can be solved by using the technique of numerical methods such as Newton Raphson method.

4.4. Simulation study

Simulation study has been performed in this section. Behavior of the MLE, MPS and LSE with varying sample size is given in Table 2. Here we have taken $\alpha=1.5$ and $\lambda=1$. We have generated the sample of sizes $n = 20,30,40,50,60,70,80,90$ and 100 from the proposed distribution and obtained the estimates based on these samples. We have repeated this procedure 5000 times. The average values of the parameters and MSE are given in Table 2.

Table 2 Average values of the estimates and MSE (below the estimate of each cell) with varying sample size for $\alpha=1.5$ and $\lambda=1$

		α			λ	
n	MLE	MPS	LSE	MLE	MPS	LSE
20	2.3978	1.5935	2.1962	1.4146	0.8611	1.2756
	3.7038	3.1015	3.9693	0.7589	0.4479	0.7621
30	1.9045	1.4837	1.7984	1.2366	0.8518	1.0089
	1.8892	0.5571	3.4774	0.3941	0.2719	0.5399
40	1.7678	1.4613	1.7168	1.1746	0.8706	1.0079
	0.5846	0.1927	2.4357	0.2586	0.2065	0.3822
50	1.6964	1.4563	1.6259	1.1402	0.8865	0.9996
	0.2838	0.1147	0.5566	0.2002	0.1583	0.3103
60	1.6526	1.4542	1.6143	1.1174	0.8909	0.9803
	0.1992	0.088	0.544	0.1557	0.1381	0.2819
70	1.6247	1.4473	1.6078	1.0953	0.8925	1.0304
	0.1313	0.0615	0.2552	0.1243	0.1082	0.2228
80	1.6118	1.4524	1.5726	1.0899	0.9019	1.0122
	0.1081	0.0566	0.1867	0.1101	0.0974	0.1978
90	1.5979	1.4553	1.5897	1.0774	0.9119	1.0297
	0.0893	0.0462	0.1847	0.0969	0.0836	0.1679
100	1.5809	1.4522	1.5628	1.0654	0.9113	1.0123
	0.07	0.0423	0.1291	0.0827	0.0788	0.1456

This table shows that as the sample size increases, the MSE based on MLE, MPS, and LSE decrease for the given values of the parameters, so the estimators are consistent. The formula for the average values of the estimates and MSE are given by

$$Average(\hat{\alpha})=\frac{1}{5000}\sum_{i=1}^{5000}\hat{\alpha_i},\quad MSE(\hat{\alpha})=\frac{1}{5000}\sum_{i=1}^{5000}(\alpha-\hat{\alpha_i})^2.$$

From Table 2, we observe that the MSE of the parameters based on MPS are less than that obtained using MLE for all the sample sizes. It can also be seen that as the sample size increases the the difference between MSE based on both the methods become lower. We can also see that the MSE of least square estimators are greater than those of MLE and MPS estimators for all the sample sizes, and the differences between MSE decrease as sample size increases.

5. Real Data Application

We have considered two real data sets to show the applicability and importance of the proposed model. The first data set reported by Efron (1988) represents the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT). Second data corresponds to 46 observations reported on active repair times (hours) for an airborne communication transceiver discussed by Alzaatreh (2014). We have taken a several models for the comparison purpose. These models are Gamma, Lindley, Weibull, exponentiated exponential distribution (EED), generalized DUS distribution (GDUSED) , and generalized Lindley distribution (GLD). Estimates of the parameters for both the data sets are given Table 3 and Table 4, respectively. To show fitting of the distributions for the considered data sets, Kolmogrov Smirnov (K-S) test is used. The AIC (Akaike information criterion) and BIC (Bayesian information criterion) are considered for model discriminations. The formula for AIC, BIC and KS-statistics are given by

$$AIC = 2k - 2\log(\hat{L}), \quad BIC = k\log(n) - 2\log(\hat{L})$$
$$D = \sup_{x} \mid F_n(x) - F(x) \mid,$$

respectively, where n is sample size, k is the number of parameters and $\hat{L}$ is the maximized value of L, $F_n(.)$ is the empirical cdf of the data under consideration. The smaller values of AIC, BIC, and the K-S test statistic indicate the better fit of the distributions. From the Tables 3 and 4, we can observe that p-value of the proposed model for both the data sets are 0.741 and 0.764 respectively, which support the hypothesis that our proposed model fits both the data sets very well. The values of AIC and BIC are lowest for the proposed model in both the cases as compared to the other considered models, which means our proposed model is the best model among all the considered models for both the given data sets.

Table 3 The values of MLEs, AIC, BIC and K-S statistics with p-values for all the considered models

Head and Neck cancer data						
Model	MLEs		AIC	BIC	K-S	p-value
	α	λ				
Lindley	0.0089		581.1628	582.947	0.22	0.025
Proposed model	1.1424	0.0016	560.4722	564.0406	0.099	0.741
Gamma	218.3115	1.0237	568.0038	571.5722	0.147	0.268
Weibull	216.1174	0.9409	567.6833	571.2517	0.131	0.405
EED	1.0712	0.0047	567.9101	571.4784	0.125	0.463
GDUSED	0.8312	0.0051	571.2624	574.8308	0.158	0.19
GLD	0.5006	0.0058	570.071	573.6393	0.165	0.163

Table 4 The values of MLE_s, AIC, BIC and K-S statistics with p-values for all the considered models

Airborne data						
Model	MLEs		AIC	BIC	K-S	p-value
	α	λ				
Lindley	0.4664		221.9694	223.7981	0.234	0.013
Proposed model	1.115	0.0911	203.0663	206.7236	0.099	0.764
Gamma	3.8685	0.9323	213.8619	217.5192	0.145	0.285
Weibull	0.8986	3.3913	212.9394	216.5967	0.12	0.517
EED	0.9583	0.2694	213.9658	217.6231	0.152	0.239
GDUSED	0.7411	0.2907	217.5301	221.1874	0.158	0.201
GLD	0.6643	0.3677	219.6976	223.3548	0.166	0.157

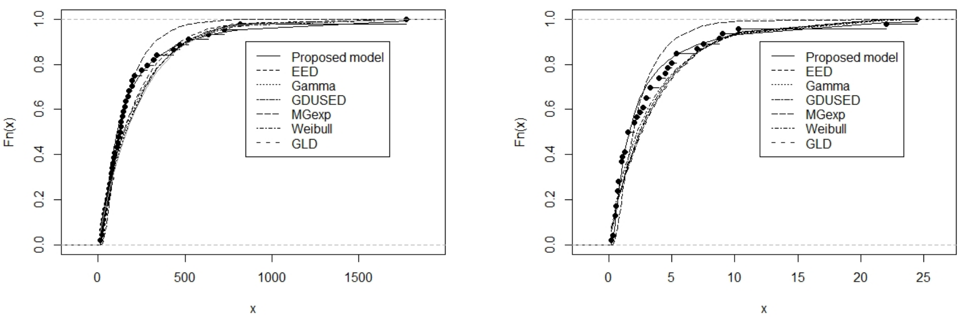

Figure 3 Plots of the Empirical cdf and fitted cdf for all the considered dataset

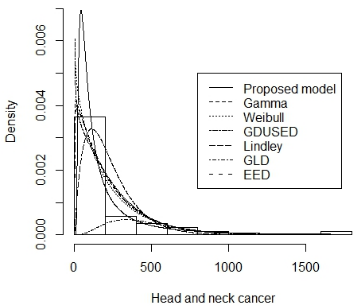

Figure 4 Fitted pdf plots of the considered distribution for the head and neck cancer data set

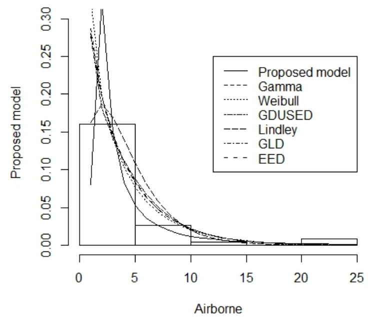

Figure 5 Fitted pdf plots of the considered distribution for the Airborne data

6. Discussion and Conclusions

In the present paper, we proposed a family of distributions. We discussed the some statistical properties of the distribution. Then we have applied this transformation the exponential distribution and derived the various characteristics such as moments, moment generating function, order statistics, stress-strength parameter, mean residual life function and entropy of the new proposed distribution. MLE, MPS, and LSE methods are used for the estimation of the parameters and information matrix is also given. Simulation study shows that the estimators are consistent and MPS performs better than MLE and LSE in terms of MSE for varying sample sizes. We have taken two real data sets for the application purpose and the empirical cdfs are plotted in Figure 3. Results based on the two considered data sets shows that our proposed model is better than the other considered models as the values of AIC and BIC are lowest for the proposed model. Thus the above discussion concludes that our proposed model is a flexible model and is applicable to deal with such type of data sets available in different fields such as in medical field, engineering and social sciences, etc.

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