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Setbacks of Joint Location and Dispersion Control Charts

Samson Offorma Ugwu* **and Akinenyene Udo Udum**
Department of Statistics, University of Nigeria Nsukka, Enugu-State, Nigeria.
*Corresponding author; e-mail: offorma.ugwu@unn.edu.ng

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Abstract

It is a common practice to monitor or control a process with control charts. The Shewhart $\bar{X}$- and R- or S-charts are the most common in use for monitoring process location and dispersion respectively. The literature reveals that until lately, the tradition has been to apply the charts for location and dispersion independently, but now, some works considered them jointly. The use of three-sigma limits, estimated parameter, and multiple charting have been shown to affect the joint chart schemes by deteriorating the performance. In the literature, works exist on joint charts for $\bar{X}$- and R-charts when the process parameters are known and the process is in a state of control, on $\bar{X}$- and R-charts when the process parameters are unknown and the process is in a state of control and on $\bar{X}$- and S-charts (here, S-chart has one-sided control limit) for both when the process parameters are known and unknown and the process is in a state of control. For the works so mentioned, the in-control average run length was used as the sole index for measuring the charts performance. Similar works on such joint charts for both the in-control and out-of-control states with estimated parameters where the performance is evaluated in terms of the average run and the median run lengths lack in the literature and this work will fill the gap. Therefore, in this work, a joint $\bar{X}$ and S^2-charts will be extensively considered when the S^2-chart is with both one-sided and two-sided control limits using the information from the unconditional run length (RL) cumulative distribution function (cdf) and its percentiles (mainly the median). New control limit constants will be provided to guarantee the desired in-control performance for the joint chart.

Keywords: Joint chart, multiple charting, parameter estimation, average run length (ARL), median run length (MRL).

1. Introduction

Control charts are indispensable tools for monitoring process quality in firms and industries as they detect special causes of variations in the process, Jardim et al. (2018). The process parameters like the mean and standard deviation are used to determine the control limits of the charts, however, in practical situations, these process parameters are unknown and have to be estimated from the in-control Phase I samples(m), Loureiro et al. (2017). It is well known in the literature that when estimated limits are used instead of the known control limits values, the behavioral properties of the control charts both the in- and out-of-control situations are substantially negatively affected. This effect is especially large when a small to moderate number of Phase I samples, m is used for the parameter estimation, Goedhart et al. (2017). Goedhart et al. (2016), Quesenberry (1993), Chen (1997), Chakraborti (2000) and Chakraborti (2006) are some of the several works in the literature that have studied the effect of the estimated parameter(s) on the performance of control charts especially as it concerns its probability of alarm and the run length distribution.

It is a common practice to measure the performance of a control chart in terms of the run-length (RL) distribution, that is, the distribution of the number of observations (or samples) until an alarm. This is often done either by its mean, that is, the average of the run-length distribution (ARL), or in terms of the entire (RL) distribution and other associated measures, Moskowitz et al. (1994). But it is well known in the literature that the RL distribution is right-skewed and in a right-skewed distribution, the mean is in error larger than the measure of the center of the distribution, (see in Gupta, 2013, p.510), therefore, the median is a better measure of the central tendency in such distributions. In line with this remark,Chakraborti (2006) noted that one might prefer the median run length (MRL) or some other quantiles instead of the ARL as a measure of typical chart performance. Chakraborti (2007) also noted that since the RL random variable takes on only positive integer values, the distributional shape is significantly right-skewed and advocated the use of other more representative measures of location other than the ARL for the assessment of charts performance and suggested the percentile of the RL distribution. Han (2019) stated that the MRL is the 50th percentage point of the RL distribution which denotes the median number of samples drawn by the control chart until it issues an alarm and which as well means that the fifty percents of the RLs lie below the observed point. The percentiles provide wider information about the (RL) distribution and hence about the performance of a control chart not provided by the mean or the expected value and the choice of assessing charts performance using the percentiles is becoming a common one now. In this regard, see Shmueli and Cohen (2003), Khoo (2004), Radson and Boyd (2005), Chakraborti (2006) and Chakraborti (2007) on the RL distribution and its percentiles of the Shewhart $\bar{X}$-chart, Shu et al. (2012) for similar work on CUSUM control chart under changes in variances, Boone and Chakraborti (2012) on the Hotelling's chi-square and two simple Shewhart multivariate non-parametric control charts based on sign and singed-rank tests for known and estimated parameters and Diko et al. (2019) on the adjusted Shewhart, CUSUM and EWMA control charts for sustained shifts in the process mean.

The vast majority of control charts existing in the literature are designed to monitor a single process parameter, such as the mean or the variance at a time, McCracken and Chakraborti (2013). They can either be location charts to monitor changes in the process mean or dispersion charts to monitor the process variability. However, changes can occur simultaneously in both the location and variability parameters of the quality of interest, therefore, controlling for both the process location and variability at the same time is ideal, Zaman and Lee (2016). Gan et al. (2004), specifically stated that it is more reasonable to combine the mean and variance information on joint monitoring charts scheme when special causes in the process can cause both the mean and variance to shift simultaneously. Here, the process is considered to be in control (IC) whenever the charting statistics of both charts display randomness and equally plot within their respective limits but out-of-control (OOC) whenever either or both the charting statistics of the charts display non-randomness or at least one of the charts signals OOC, Diko et al. (2015). Of late, there are growing cases in the literature where the process mean and the variance are simultaneously monitored via the use of joint charts of location and dispersion. For instance, under unconditional perspective, Diko (2014) considered the design and performance of the $\bar{X}$- and R- charts as they are applied jointly when the process is IC and the process parameters are known (Case KK). Diko et al. (2015) under the same perspective also looked at the design and the performance of the $\bar{X}$- and R-charts as they are applied jointly when the process is IC still but the parameters are unknown and estimated from m Phase I samples (Cases KU and UU) using $\frac{\bar{R}}{d_2}$ as the Phase I estimator of the process standard deviation. Here, Case KU stands for the situation where the process mean is known but the variance is unknown while Case UU stands for the situation where both process mean and the variance are unknown. Therefore, the work of Diko et al. (2015) is basically on studying the effects of parameter estimation on the $\bar{X}$- and R- charts applied jointly and the provision of new control charting constants expected to take into cognizant the effects of the use of the three-sigma control limits, the multiplicity effect of the joint charting scheme and the effects of parameter estimation to deliver the desired in-control performance.

Loureiro et al. (2017) under the conditional perspective studied the effects of parameter estimation on the performance of $\bar{X}$- and S-charts applied jointly when the process is IC using probability limits and the square root of the pooled variance (S_p) used as the Phase I estimator of the process standard deviation. The choice of the Phase I estimator is based on the recommendation of Mahmoud et al. (2010). This work will be different from the ones already in the literature in the sense that, even though the unconditional perspective as in Diko (2014) and Diko et al. (2015) will be used to study the effect of parameter estimation on $\bar{X}$- and S^2-charts applied jointly $\left(\bar{X}, S^2\right)$- when the process mean is known and the variance is unknown (Case KU), the study will cover both the IC and OOC conditions in the process using S_p and $\frac{\bar{S}}{C_4}$ as the Phase I estimators but like in Loureiro et al. (2017), the probability limits will be used in this work. The study will also, unlike in others in the literature, consider $\left(\bar{X}, S^2\right)$-chart when S^2-chart has a one-sided upper limit and when it has two-sided limits. To avoid using the ARL as the sole index in evaluating the performance of the chart, the distributions of the RL percentiles of $\left(\bar{X}, S^2\right)$-chart under the two condition limits of S^2-chart will be derived and evaluated to furnish the work with the MRL. New control charting constants will be provided to guarantee the traditionally desired in-control performance of 370 and 500 ARLs for the chart in the said case also. All these objectives raised here which this work pursues to achieve are currently not available in the literature and as such will be the contribution of the work.

The rest of the paper hereafter is ordered as follows: A review of the conditional probability of alarm of $\bar{X}$-chart for Case KU is presented in Section 2. Section 3 is the review of the conditional probabilities of alarm of the one- and two-sided S^2-control charts. We will present the unconditional probability and average run length of the combined chart, $(\bar{X}, S^2)$ in Section 4. In Section 5, we discuss the new control limits of the $(\bar{X}, S^2)$-chart corrected for parameter estimation and multiplicity effects. Results, discussions, and a numerical example are presented in Section 6 while Section 7 is the conclusion of the work.

2. Rview of the Conditional Probability of Alarm of $\bar{X}$-chart for Case KU

To derive the conditional probability of alarm in this context, a situation is defined as, let the mean standard $\mu = \mu_0$ be given but the standard deviation σ_0 is unknown, that is, the Case KU, the process standard deviation is typically estimated from m Phase I samples each of size n when the process is IC, Jardim et al. (2018). Then, with the estimates, the α estimated probability limits of the chart become; $\mu_0 \pm Z_{\alpha/2}\frac{\hat{\sigma_0}}{\sqrt{n}}$, where $\hat{\sigma_0}$ is the estimate of the process standard deviation and can be obtained by using the square root of the pooled variance, S_p or the average of the sample standard deviations, $\frac{\bar{S}}{c_4(n)}$. When these estimated limits are plugged into the design of the chart, the RL random variable (N) is no longer geometric, Quesenberry (1993), however, when conditioned on $\hat{\sigma_0}$, N becomes geometric with the probability of success (alarm) of the chart given by $1 - \beta\left(\delta, n, \hat{\sigma_0}\right)$, where

$$\beta\left(\delta, n, \hat{\sigma_0}\right) = \Phi\left(-\delta\sqrt{n} + Z_{\alpha/2}\frac{\hat{\sigma_0}}{\sqrt{n}}\right) - \Phi\left(-\delta\sqrt{n} - Z_{\alpha/2}\frac{\hat{\sigma_0}}{\sqrt{n}}\right).$$

By using the fact that $Y = m\left(n-1\right)S_p^2/\sigma_0^2 \sim \chi^2_{m(n-1)}$ and by letting $b_0 = m(n-1)$, the conditional probability of alarm and the ARL of the $\bar{X}$-chart can be rewritten as in Equations (1) and (2),

$$1 - \left\{\Phi\left(-\delta\sqrt{n} + \frac{a_0 Z_{\frac{\alpha}{2}}}{\sqrt{b_0}}\sqrt{Y}\right) - \Phi\left(-\delta\sqrt{n} - \frac{a_0 Z_{\frac{\alpha}{2}}}{\sqrt{b_0}}\sqrt{Y}\right)\right\} \tag{1}$$

and

$$\left[1 - \left\{\Phi\left(-\delta\sqrt{n} + \frac{a_0 Z_{\frac{\alpha}{2}}}{\sqrt{b_0}}\sqrt{Y}\right) - \Phi\left(-\delta\sqrt{n} - \frac{a_0 Z_{\frac{\alpha}{2}}}{\sqrt{b_0}}\sqrt{Y}\right)\right\}\right]^{-1} \tag{2}$$

but bearing in mind that when $\hat{\sigma_0}$ is estimated with S_p, a_0=1 and $b_0 = m(n-1)$ but when estimated with $\frac{\bar{S}}{c_4(n)}$, $a_0 = \sqrt{V(T)+1}$ and $b_0 = \frac{1}{2}\left(1+\frac{1}{V(T)}\right)$, where $V(T) = \frac{1-c_4^2(n)}{mc_4^2(n)}$. The values of $c_4(n)$ is tabulated in many quality control textbooks, (see in Montgomery, 2013, p.720), the values of a_0 and b_0 will be obtained according to the well-known Patnaik (1950) approximation of the $\frac{\bar{S}}{c_4(n)}$ estimator and Y is a chi-square variable from a chi-square distribution with b_0 degrees of freedom.

3. Review of the Conditional Probabilities of Alarm of the One- and Two-sided S^2-control Charts

3.1. The one-sided upper S^2-chart

Let $m > 1$ Phase I independent random samples each of size n with which to estimate the unknown Phase I variance $\left(\sigma_0^2\right)$ be assumed available. The observations are, as well, assumed to follow a normal distribution with mean u_0 and variance σ_0^2, both unknown. Let the Phase II subgroup samples each of size n be equally assumed to follow a normal distribution with unknown mean (μ_1) and variance $\left(\sigma^2\right)$. Define the standard deviation ratio and the error factor of estimate as $\gamma = \sigma/\sigma_0$ and $W = \hat{\sigma}_0/\sigma_0$ respectively. The process is IC when $\sigma = \sigma_0$ and $\gamma = 1$ and OOC when $\sigma > \sigma_0$ and $\gamma > 1$. Let delta (δ) be the size of the shift in $\bar{X}$-chart to cause an OOC in the process mean and equally be the size of the shift in γ that makes it greater than one to cause an OOC in S^2-chart. According to Jardim et al. (2020), with these assumptions and definitions, the α estimated upper probability limit of the one-sided upper S^2-control chart is given by $\widehat{UCL}_{One-sided,S^2-chart} = \frac{\chi^2_{n-1,\alpha}}{n-1}\hat{\sigma}_0^2$ and the conditional probability of alarm (CPL) of the chart is given by

$$P(S^2 > \widehat{UCL}_{One-sided,S^2-chart}) = P\left\{S^2 > \frac{\hat{\sigma}_0^2\chi^2_{n-1},\alpha}{n-1}\right\},$$

$$CPL_{One-sided,S^2-chart} = P\left\{\frac{(n-1)S^2}{\sigma^2} > \frac{W^2}{\gamma^2}\chi^2_{n-1},\alpha\right\},$$

$$CPL_{One-sided,S^2-chart} = 1 - F_{\chi^2_{n-1}}\left(\frac{W^2}{\gamma^2}\chi^2_{n-1},\alpha\right).$$

When $\hat{\sigma_0}$ is estimated with S_p and $\frac{\bar{S}}{c_4(m)}$, W follows a scaled chi-square distribution, $\frac{a_0\sqrt{\chi^2}}{\sqrt{b_0}}$ with b_0 degrees of freedom where a_0 and b_0 are as defined already in Section 2 for the two estimators. Therefore, the conditional probability of alarm of the one-sided S^2-chart is given by

$$CPL_{1S^2-chart}(Y,\gamma) = 1 - F_{\chi^2_{n-1}}\left\{\frac{a_0^2 Y}{\gamma^2 b_0}\chi^2_{n-1},\alpha\right\}. \tag{3}$$

3.2. Two-sided S^2-chart

By using the same assumptions, definitions and procedures considered above for the one-sided S^2-chart, the α estimated probability limits of the two-sided S^2-control chart are given by $\widehat{UCL}_{two} = \hat{\sigma}_0^2\chi^2_{n-1},\frac{\alpha}{2}/(n-1)$ and $\widehat{LCL}_{two} = \hat{\sigma}_0^2\chi^2_{n-1},(1-\frac{\alpha}{2})/(n-1)$ and for $\sigma > \sigma_0$ or $\sigma < \sigma_0$, the conditional probability of alarm is given by

$$P(S_i^2 > \widehat{UCL}) or P(S_i^2 < \widehat{LCL}) = 1 - \left\{F_{\chi^2_{n-1}}\left(\frac{W^2}{\gamma^2}\chi^2_{n-1},1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{W^2}{\gamma^2}\chi^2_{n-1},\frac{\alpha}{2}\right)\right\},$$

$$CPL_{Two-sided,S^2-chart}(Y,\gamma) = 1 - \left\{F_{\chi^2_{n-1}}\left(\frac{a_0^2 Y}{\gamma^2 b_0}\chi^2_{n-1},1-\frac{\alpha}{2}\right) - F_{\chi^2_{n-1}}\left(\frac{a_0^2 Y}{\gamma^2 b_0}\chi^2_{n-1},\frac{\alpha}{2}\right)\right\}. \tag{4}$$

From henceforth, the subscripts 1, S^2-chart and 2, S^2-chart will be used for the one-sided upper and two-sided limits of S^2-chart.

4. Unconditional Probability and Average Run Length of the Combined Chart $(\bar{X}, S^2)$

Diko (2014), Diko et al. (2015) and Loureiro et al. (2017) already reported that the charting statistics of $\bar{X}$-and R-charts, $\bar{X}$-and S-charts are independent under normality, therefore, the charting statistics of $\bar{X}$-and S^2-charts are equally independent under the same normality condition. Utilizing these independence properties of the two charts, the nominal joint false-alarm rate of $(\bar{X}, S^2)$-chart is a function of the nominal false-alarms of the $\bar{X}$-and S^2-charts and is written as

$$\alpha_{(\bar{X},S^2)} = 1 - [(1-\alpha_{\bar{X}})(1-\alpha_{S^2})]$$

where $\alpha_{\bar{X}}$ and α_{S^2} are the respective nominal false-alarms for the $\bar{X}$-and S^2-charts being combined. Therefore, if $\alpha_{\bar{X}} = \alpha_{S^2} = 0.0027$ which gives the usually desired 370 IC ARL performance of a control chart, the nominal false-alarm of the $(\bar{X}, S^2)$-chart becomes, $\alpha_{(\bar{X},S^2)}$= 1- [(1-0.0027)(1-0.0027)] = 0.0054 which yields an IC ARL performance of 185 approximated to the nearest whole number. Since the control limits of the $\bar{X}$- and S^2- charts are functions of the estimators of the process parameters, their alarm probabilities are equally functions of those estimators. And since the estimators are random variables, the probabilities of alarm of the individual charts as well as that of the joint, $(\bar{X}, S^2)$-chart are random variables, therefore, taking the expectation of them over the distribution of the estimators gives unconditional probabilities of the charts. Recall that the conditional probabilities of alarms for the $\bar{X}$- and S^2- charts are given in Equations (1), (3), and (4). Therefore, leveraging on the independence of $\bar{X}$- and S^2- charts, the conditional probability of alarm, CPL for the $(\bar{X}, S^2)$-chart with the one-sided and two-sided S^2-charts become as in Equations (5) and (6).

$$CPL_{(\bar{X},1,S^2-chart)}(Y,\delta,\gamma,m,n) = 1 - [(1-CPL_{\bar{X}}(Y,\delta,m,n)][(1-CPL_{1,S^2-chart}(Y,\gamma,m,n)] \quad (5)$$

and

$$CPL_{(\bar{X},2,S^2-chart)}(Y,\delta,\gamma,m,n) = 1 - [(1-CPL_{\bar{X}}(Y,\delta,m,n)][(1-CPL_{2,S^2-chart}(Y,\gamma,m,n)], \quad (6)$$

where Y, γ , m, and n are as defined before. As already hinted, the unconditional probability of alarm, UPL, and the unconditional average run length, UARL for the joint chart is obtained by taking the expectation of the conditional probability of alarms in Equations (5) and (6) over the distribution of Y. Therefore, the UPLs and UARLs for the joint chart for Case KU in both one-sided upper and two-sided S^2-chart are given as

$$\begin{aligned}
UPL_{(\bar{X},1,S^2-chart)}(\delta,\gamma,m,n) &= \int_0^\infty \{1-[(1-CPL_{\bar{X}}(y,\delta,m,n)] \\
&\quad [(1-CPL_{1,S^2-chart}(y,\gamma,m,n)]\} f_{\chi^2_{b_0}}(y)dy \\
UARL_{(\bar{X},1,S^2-chart)}(\delta,\gamma,m,n) &= \int_0^\infty \{1-[(1-CPL_{\bar{X}}(y,\delta,m,n)] \\
&\quad [(1-CPL_{1,S^2-chart}(y,\gamma,m,n)]\}^{-1} f_{\chi^2_{b_0}}(y)dy \qquad (7) \\
UPL_{(\bar{X},2,S^2-chart)}(\delta,\gamma,m,n) &= \int_0^\infty \{1-[(1-CPL_{\bar{X}}(y,\delta,m,n)] \\
&\quad [(1-CPL_{2,S^2-chart}(y,\gamma,m,n)]\} f_{\chi^2_{b_0}}(y)dy
\end{aligned}$$

and

$$\begin{aligned}
UARL_{(\bar{X},2,S^2-chart)}(\delta,\gamma,m,n) &= \int_0^\infty \{1-[(1-CPL_{\bar{X}}(y,\delta,m,n)] \\
&\quad [(1-CPL_{2,S^2-chart}(y,\gamma,m,n)]\}^{-1} f_{\chi^2_{b_0}}(y)dy, \qquad (8)
\end{aligned}$$

respectively.

Equations (7) and (8) will be evaluated at different values of $\delta, \gamma, m,\ n$ under each of the two Phase I estimators of the process standard deviation; S_p and $\frac{\bar{S}}{c_4(m)}$ with the view to study the unconditional performance of the $\left(\bar{X}, S^2\right)$-chart at both the IC and OOC conditions, look at the effect of the sizes of m and n used in estimating the parameter and to study the effect of the different Phase I estimators on the chart. It is important to note here that when $\delta = 0$ and $\gamma = 1$, $\left(\bar{X}, S^2\right)$-chart is operating under an IC state and the UARL will be written as $UARL_{\bar{x},1,S^2-chart}\left(0,1,m,n\right)$ or $UARL_{\bar{x},2,S^2-chart}\left(0,1,m,n\right)$ from henceforth. The results of the evaluation are presented in Table 1.

4.1. Exact (unconditional) cumulative distribution functions of the run lengths of the $\left(\bar{X}, S^2\right)$-chart

Recall that given $W = \hat{\sigma_0}/\sigma_0$, and that $W \sim \frac{a_0\sqrt{\chi^2}}{\sqrt{b_0}}$ where $\chi^2 = Y$, a Chi-sqaure random variable, the random variable, N has a geometric distribution with the probability of success (alarm) given by

$$CPL_{\left(\bar{X},1,S^2-chart\right)}(Y,\delta,\gamma,m,n) = 1 - [(1 - CPL_{\bar{X}}(Y,\delta,m,n)][(1 - CPL_{1,S^2-chart}(Y,\gamma,m,n)]$$

or

$$CPL_{\left(\bar{X},2,S^2-chart\right)}(Y,\delta,\gamma,m,n) = 1 - [(1 - CPL_{\bar{X}}(Y,\delta,m,n)][(1 - CPL_{2,S^2-chart}(Y,\gamma,m,n)]$$

depending on whether S^2 is a one-sided or a two-sided chart. Now, by using the conditioning-unconditioning approach in Chakraborti (2000), the exact cumulative RL distribution for $\left(\bar{X}, S^2\right)$-chart for when S^2-chart is one-sided upper and two-sided limits are given as

$$P\left(N \le a\right) = 1 - \int_0^\infty [(1 - CPL_{\bar{X}}(y,\delta,m,n)][(1 - CPL_{1,S^2-chart}(y,\gamma,m,n)]^a f_{\chi^2_{b_0}}(y)dy \tag{9}$$

and

$$P\left(N \le a\right) = 1 - \int_0^\infty [(1 - CPL_{\bar{X}}(y,\delta,m,n)][(1 - CPL_{2,S^2-chart}(y,\gamma,m,n)]^a f_{\chi^2_{b_0}}(y)dy. \tag{10}$$

4.2. The RL percentiles of the $\left(\bar{X}, S^2\right)$-chart

One can study the statistical properties of the $\left(\bar{X}, S^2\right)$-chart including the various performance characteristics by studying the behavior of the RL cdfs in Equations (9) and (10) through the shape of the curves for the different values of m under an IC and OOC states. Another instance is to calculate the $100pth$ RL percentiles using the idea that it is the smallest positive integer a so that the cdf at "a" is at least equal to p Chakraborti (2007). The performance of the $\left(\bar{X}, S^2\right)$-chart is studied here through this means. Based on these unconditional RL percentiles, Equations (9) and (10) can be redefined as

$$1 - \int_0^\infty [(1 - CPL_{\bar{X}}(y,\delta,m,n)][(1 - CPL_{1,S^2-chart}(y,\gamma,m,n)]^a f_{\chi^2_{b_0}}(y)dy \ge p \tag{11}$$

and

$$1 - \int_0^\infty [(1 - CPL_{\bar{X}}(y,\delta,m,n)][(1 - CPL_{2,S^2-chart}(y,\gamma,m,n)]^a f_{\chi^2_{b_0}}(y)dy \ge p \tag{12}$$

for the $\left(\bar{X}, S^2\right)$-chart when S^2-chart is with one-sided-upper and when it is with two-sided limits respectively.

Equations (11) and (12) will be evaluated and presented in Tables 1 and 2 at different values of δ, γ, m, n and at $p = 50th$ (MRL) under each of the two Phase I estimators of the process standard deviation; S_p and $\frac{\bar{S}}{c_4(m)}$ with the view to bringing in the percentiles in the study of the performance of the $(\bar{X}, S^2)$-chart at both the IC and OOC conditions, to look at the effect of the sizes of m (number of Phase-I-samples) and n used in estimating the parameter and to study the effect of the different Phase-I-estimator on the chart. The presentation is of the form, ARL (MRL) in Tables 1 and 2. Note that when the parameter is not estimated, the RL percentiles of the individual $\bar{X}$ and S^2-charts can be defined to be the smallest positive integer $'a'$ such that the cdf of the geometric distribution is at least equal to p. That is,

$$1-(1-\beta)^a \geq p$$

and

$$a = \frac{In\,(1-p)}{In\,(1-\beta)} \tag{13}$$

where β is the nominal false alarm rate of any individual chart. Therefore, $\beta = \alpha_{\bar{X}} = \alpha_{S^2}$= 0.0027 because this performance is desired in this work. As the usual nominal false alarm rate is 0.0027, by setting p= 0.5, the result of the evaluation is 257 which is the MRL for the IC state of each chart. However, recall that when the two charts, $\bar{X}$ and S^2- are combined into $(\bar{X}, S^2)$-chart, the nominal false alarm rate is determined by $\alpha_{(\bar{X}-S^2)} = 1 - [(1-\alpha_{\bar{X}})(1-\alpha_{S^2})]$. Therefore, for the usually desired 0.0027 false alarm rate of a control chart, $(\bar{X}, S^2)$-chart has a nominal false alarm of 0.0054. Substituting the 0.0054 in place of β in equation (13) evaluates it to be 128 to the nearest whole number. Therefore, if the parameter is not estimated, either $\bar{X}$ or S^2-charts is expected to have an IC MRL of 257 while $(\bar{X}, S^2)$-chart has an IC MRL of 128. This IC MRL of 128 is quite smaller than the expected value of 257 for the usual control charts and as such, the control limits of component charts should be adjusted to make the joint chart deliver the desired performance.

5. New Control Limits Corrected for Parameter Estimation and Multiplicity Effects

It is obvious from Tables 1 and 2 that while the number of Phase-I-samples are small to moderate sizes; m = 20, 50, 100, and 500, the IC ARLs are all in error above the IC ARL of 185 expected of the $(\bar{X}, S^2)$-chart when the IC nominal false alarm of 0.0027 is used for the component charts, however, the value continues to approach the target as m grows larger. Therefore, like already has been pointed out, there is a need to select the control limits of the components charts of the $(\bar{X}, S^2)$-chart in such a way that at a given Phase I sample, the chart will not only deliver 185 expected of it but delivers the traditionally desired IC performance of 370 and 257 ARL and MRL respectively after taking care of the effects of parameter estimation and multiple charting. To do this, the conditioning technique of the Chakraborti (2000) and the use of probability limits will be adopted to address the issue of the parameter estimation and three-sigma limits respectively and there will be a correction for the effect of multiple charting. This will be done by replacing α with $\alpha_{(m)}$ in Equations (5) and (6) and solving for it in the following system of equations. Therefore, for some given values of $ICARL$, m, and n, we solve

$$\int_0^\infty \left\{1 - [(1 - CPL_{\bar{X}}(y,\delta,m,n)][(1 - CPL_{1,S^2-chart}(y,\gamma,m,n)]\right\}^{-1} f_{\chi^2_{b_0}}(y)dy \geq ICARL \tag{14}$$

and

$$\int_0^\infty \left\{1 - [(1 - CPL_{\bar{X}}(y,\delta,m,n)][(1 - CPL_{2,S^2-chart}(y,\gamma,m,n)]\right\}^{-1} f_{\chi^2_{b_0}}(y)dy \geq ICARL \tag{15}$$

for $\alpha_{(m)}$ using R statistical software. Note that the expressions $CPL_{\bar{X}}(Y,\delta,m,n)$, $CPL_{1,S^2-chart}(Y,\gamma,m,n)$ and $CPL_{2,S^2-chart}(Y,\gamma,m,n)$ as used in (5) and (6) are as already defined in Equations (1), (3) and (4). Once $\alpha_{(m)}$ is found, the corrected probability limits for the $\bar{X}$- and S^2-charts are found from the corresponding percentiles of the standard normal distribution and the chi-square distribution with b_0 degrees of freedom.

Table 1 ARLs and MRLs of $\left(\bar{X}, S^2\right)$-chart in case KU when S^2-chart at various values of m, δ, γ and at $\alpha = 0.0027$ for component charts and the standard deviation is estimated by S_p

When S^2-chart has one-sided upper control limit in $\left(\bar{X}, S^2\right)$-chart								
	m							
Shifts sizes	V	X	XX	L	C	D	M	$\bar{X}$
0.00 in δ and γ	**1667.27(94)**	**500.25(106)**	294.24(**115**)	222.64(**122**)	202.98(**125**)	188.79(**128**)	187.10(**128**)	185.60(**128**)
0.05 in δ	1513.44(92)	471.78(103)	284.76(112)	214.99(118)	196.47(121)	183.05(124)	181.45(124)	180.03(124)
0.08 in δ	1320.60(88)	433.02(99)	267.09(107)	203.90(113)	186.96(116)	174.63(118)	173.16(119)	171.85(119)
0.10 in δ	1179.07(85)	402.16(95)	252.42(102)	194.49(108)	178.83(111)	167.40(113)	166.03(114)	164.81(114)
0.05 in γ	1348.50(65)	354.55(73)	202.96(80)	150.82(83)	137.46(85)	127.92(86)	126.78(87)	125.78(87)
0.08 in γ	1094.01(52)	271.30(58)	155.71(61)	116.80(65)	106.87(66)	99.78(67)	98.93(68)	98.19(68)
0.10 in γ	911.94(44)	222.72(49)	129.56(52)	98.24(55)	90.23(56)	84.50(57)	83.82(57)	83.21(57)
0.05 in δ and γ	1237.87(64)	339.39(72)	197.03(77)	147.28(81)	134.45(83)	125.26(85)	124.17(86)	123.21(85)
0.08 in δ and γ	914.09(50)	249.26(55)	147.03(57)	111.47(62)	102.29(64)	95.70(65)	94.92(65)	94.22(65)
0.10 in δ and γ	720.78(42)	200.07(47)	120.34(50)	92.43(52)	85.18(53)	79.96(54)	79.34(54)	78.79(54)
When S^2-chart has two-sided control limit in $\left(\bar{X}, S^2\right)$-chart								
0.00 in δ and γ	169.08(92)	179.76(103)	184.80(**108**)	186.44(**119**)	186.30(**123**)	185.68(**127**)	185.56(**128**)	185.44(**128**)
0.05 in δ	165.62(91)	175.57(102)	180.02(105)	181.21(115)	1880.89(120)	180.15(123)	180.02(124)	179.89(124)
0.08 in δ	160.45(89)	169.33(100)	172.95(101)	173.48(110)	172.94(114)	172.02(118)	171.87(118)	171.72(119)
0.10 in δ	155.92(86)	163.88(97)	166.80(97)	166.80(106)	166.07(110)	165.03(113)	164.86(114)	164.70(114)
0.05 in γ	161.94(66)	165.27(75)	163.19(87)	158.01(97)	154.92(100)	151.68(103)	151.23(104)	150.79(104)
0.08 in γ	153.41(53)	151.61(58)	145.17(76)	136.53(82)	132.23(85)	128.14(87)	127.58(88)	127.08(88)
0.10 in γ	146.25(46)	141.07(51)	132.10(67)	121.83(72)	117.13(75)	112.86(77)	112.30(77)	111.78(77)
0.05 in δ and γ	159.07(65)	161.92(73)	159.56(87)	154.26(95)	151.17(98)	147.97(101)	147.52(101)	147.10(102)
0.08 in δ and γ	147.32(52)	144.76(56)	138.03(72)	129.49(78)	125.37(80)	121.49(83)	120.97(83)	120.49(83
0.10 in δ and γ	138.03(43)	132.08(48)	123.01(63)	113.17(68)	108.81(70)	104.90(71)	104.39(71)	103.92(72)

Table 2 ARLs and MRLs of $(\bar{X}, S^2)$-chart in case KU when S^2-chart at various values of m, δ, γ and at $\alpha = 0.0027$ for component charts and the standard deviation is estimated by $\frac{\bar{S}}{C_4(n)}$

When S^2-chart has one-sided upper control limit in $(\bar{X}, S^2)$-chart						
	m					
Shifts sizes	XX	L	C	D	M	$\bar{X}$
0.00 in δ and γ	279.19(**119**)	217.40(**124**)	200.63(**126**)	185.35(**128**)	186.89(**128**)	185.58(**128**)
0.05 in δ	268.11(116)	210.10(120)	194.26(122)	182.64(124)	181.25(124)	180.01(124)
0.08 in δ	252.30(111)	199.50(115)	184.96(117)	174.26(118)	172.98(119)	171.83(119)
0.10 in δ	239.09(106)	190.48(110)	177.00(112)	167.06(114)	165.86(114)	164.80(114)
0.05 in γ	189.46(81)	147.09(84)	135.81(85)	127.61(87)	126.64(87)	125.77(87)
0.08 in γ	145.44(64)	113.99(66)	105.63(67)	99.55(68)	98.83(68)	98.18(68)
0.10 in γ	121.26(54)	95.97(56)	89.23(58)	84.31(59)	83.73(59)	83.20(59)
0.05 in δ and γ	184.29(80)	143.72(82)	132.87(84)	124.97(85)	124.03(85)	123.19(85)
0.08 in δ and γ	137.84(61)	108.92(64)	101.15(65)	95.49(67)	94.82(67)	94.21(67)
0.10 in δ and γ	113.14(51)	90.41(53)	84.28(54)	79.79(56)	79.26(56)	78.78(56)
When S^2-chart has two-sided control limit in $(\bar{X}, S^2)$-chart						
0.00 in δ and γ	188.39(**113**)	187.80(**121**)	186.92(**124**)	185.79(**127**)	185.62(**128**)	185.45(**128**)
0.05 in δ	183.43(110)	182.47(117)	181.47(121)	180.25(124)	180.06(124)	179.89124)
0.08 in δ	176.09(105)	174.61(112)	173.44(115)	172.11(118)	171.91(118)	171.73(118)
0.10 in δ	169.70(101)	167.82(108)	166.52(111)	165.09(113)	164.89(114)	164.70(114)
0.05 in γ	165.25(93)	158.40(99)	154.98(102)	151.66(103)	151.21(103)	150.29(103)
0.08 in γ	146.18(79)	136.39(84)	132.03(86)	128.07(87)	127.55(88)	127.07(88)
0.10 in γ	132.47(70)	121.44(75)	116.82(77)	112.26(80)	112.25(80)	111.78(80)
0.05 in δ and γ	161.50(91)	154.61(96)	151.21(99)	147.95(100)	147.51(100)	147.10(100)
0.08 in δ and γ	138.88(75)	129.33(87)	125.17(83)	121.43(85)	120.94(87)	120.49(87)
0.10 in δ and γ	123.24(66)	112.79(69)	108.52(82)	104.83(81)	104.35(81)	103.92(81)

6. Results and Discussions

First, note that the U-S^2-chart and L-S^2-chart as used in Tables 3-6 in order to manage the sizes of the tables and have a more compact presentation of them stand for the upper control limits and the lower control limits of the S^2-chart respectively.

From Table 1, it can be seen that in each size of δ, γ , and m, the ARL is larger than the MRL. This is a serious pointer to the fact the RL distribution is right-skewed and a reminder that other better measures of chart performance be considered. Table 1 also shows that once the parameter is estimated with small to moderate sizes of Phase-I-samples, the $(\bar{X}, S^2)$-chart could not deliver the IC ARL and MRL of 185 and 128 expected of it, this is seen for an example as the IC ARL for m =5, 10, which are 1667.27 and 500.25 are far larger in error than the expected value and the IC MRL for the corresponding values of m are 94 and 106, which are also in error, far lower than the expected value but as the number of the Phase-I-sample grows to at least 500, the chart is seen to deliver an expected IC MRL of 128. However, even at this stage of increased number of the Phase-I-samples, the IC ARL is still above the 185 expected of it when the parameter is not estimated. This means that in $(\bar{X}, S^2)$-chart, the in-control MRL when the parameter is estimated converges to its expected value faster than the ARL to its expected value at an increasing number of Phase-1-sample. Therefore, to make the $(\bar{X}, S^2)$-chart operating under parameter estimation to deliver the expected values, the control limits of the components charts that make up the $(\bar{X}, S^2)$-chart should be adjusted functional on the Phase-I-samples. This was discussed in Section 5 and the results of the new limits for the component charts necessary to deliver the desired IC performance for the $(\bar{X}, S^2)$-chart presented in Tables 3-6. As expected of any chart, $(\bar{X}, S^2)$-chart continues raising more alarms as the sizes of OOC grow. Similar observations are noted for the $(\bar{X}, S^2_{2,S^2-chart})$-chart in section two of Table 1, however, $(\bar{X}, S^2_{1,S^2-chart})$-chart outperformed the later. For instance, considering the IC first row in

Table 3 Corrected Limits of the $\bar{X}$and S^2-one sided-upper component charts in $\left(\bar{X}, S^2\right)$-chart for case KU and standard deviation estimated by S_p to deliver the desired in-control average run length of 370 and 500

		$ICARL_0$=CCCLXX			$ICARL_0$ =D		
n	m	p	$\bar{X}$-chart	U-S^2-chart	p	$\bar{X}$-chart	U-S^2-chart
V	V	0.0063333	2.730	14.323	0.0053083	2.788	14.724
	X	0.0033648	2.932	15.756	0.0027100	2.999	16.243
	XX	0.0022348	3.057	16.675	0.0017304	3.133	17.247
	XXX	0.0019134	3.103	17.023	0.0014612	3.182	17.624
	L	0.0016846	3.141	17.307	0.0012656	3.224	17.944
	LXXV	0.0015575	3.164	17.482	0.0011722	3.246	18.114
	C	0.0015067	3.173	17.556	0.0011271	3.257	18.201
	D	0.0013796	3.199	17.752	0.0010261	3.283	18.410
X	V	0.0033987	2.929	15.733	0.0027100	2.999	16.243
	X	0.0023154	3.046	16.596	0.0017874	3.123	17.175
	XX	0.0018134	3.119	17.143	0.0013760	3.200	17.758
	XXX	0.0016548	3.146	17.347	0.0012463	3.228	17.978
	L	0.0015227	3.170	17.532	0.0011461	3.252	18.164
	LXXV	0.0014699	3.181	17.611	0.0010960	3.265	18.264
	C	0.0014435	3.186	17.651	0.0010735	3.271	18.310
	D	0.0013642	3.202	17.777	0.0010146	3.286	18.435

Table 4 Corrected Limits of the $\bar{X}$and S^2-one sided-upper component charts in $\left(\bar{X}, S^2\right)$-chart for case KU and standard deviation estimated by $\frac{\bar{S}}{C_4(n)}$ to deliver the desired in-control average run length of 370 and 500

		$ICARL_0$=CCCLXX			$ICARL_0$ =D		
n	m	p	$\bar{X}$-chart	U-S^2-chart	p	$\bar{X}$-chart	U-S^2-chart
V	V	0.0056514	2.767	14.582	0.0046865	2.828	15.007
	X	0.0031099	2.957	15.933	0.0024645	3.028	16.456
	XX	0.0021276	**3.072**	**16.785**	0.0016367	3.149	17.371
	XXX	0.0018448	3.114	17.104	0.0014024	3.194	17.716
	L	0.0016386	3.149	17.369	0.0012322	3.231	18.003
	LXXV	0.0015385	3.167	17.510	0.0011522	3.250	18.153
	C	0.0014915	3.176	17.579	0.0011113	3.261	18.233
	D	0.0013804	3.199	17.751	0.0010230	3.284	18.416
X	V	0.0033944	2.930	24.642	0.0026897	3.001	25.267
	X	0.0023013	3.043	25.683	0.0017728	3.126	26.376
	XX	0.0018048	3.121	26.328	0.0013684	3.201	27.058
	XXX	0.0016486	3.147	26.567	0.0012402	3.229	27.315
	L	0.0015285	3.169	26.766	0.0011401	3.253	27.535
	LXXV	0.0015285	3.169	26.766	0.0011431	3.254	27.529
	C	0.0014394	3.187	26.925	0.0010701	3.271	27.701
	D	0.0013694	3.201	27.055	0.0010150	3.286	27.838

Table 5 Corrected Limits of the $\bar{X}$and S^2-Two-sided component charts in $\left(\bar{X}, S^2\right)$-chart for case KU and the standard deviation is estimated by S_p to deliver the desired in-control average run length of 370 and 500

		$ICARL_0$=CCCLXX				$ICARL_0$ =D			
n	m	p	$\bar{X}$-chart	U-S^2-chart	L-S^2-chart	p	$\bar{X}$-chart	U-S^2-chart	L-S^2-chart
V	V	0.0012399	3.230	19.524	0.071	0.0009198	3.314	20.181	0.061
	X	0.0013219	3.211	19.383	0.074	0.0009823	3.296	20.037	0.063
	XX	0.0013571	3.204	19.325	0.075	0.0010096	3.288	19.976	0.064
	XXX	0.0013649	3.202	19.312	0.075	0.0010135	3.287	19.968	0.064
	L	0.0013649	3.202	19.312	0.075	0.0010135	3.287	19.968	0.064
	LXXV	0.0013649	3.202	19.312	0.075	0.0010096	3.288	19.976	0.064
	C	0.0013649	3.202	19.312	0.075	0.0010096	3.288	19.976	0.064
	D	0.0013671	3.201	19.308	0.075	0.0010018	3.298	19.993	0.064
X	V	0.0011072	3.262	29.405	0.996	0.0008183	3.347	30.198	0.926
	X	0.0012165	3.235	29.163	1.019	0.0008764	3.327	30.003	0.941
	XX	0.0012829	3.220	29.026	1.033	0.0009471	3.306	29.805	0.959
	XXX	0.0013063	3.215	28.980	1.037	0.0009667	3.300	29.752	0.964
	L	0.0013258	3.210	28.942	1.041	0.0009701	3.299	29.743	0.965
	LXXV	0.0013336	3.209	28.927	1.042	0.0009842	3.295	29.707	0.968
	C	0.0013375	3.208	28.919	1.043	0.0009878	3.294	29.697	0.969
	D	0.0013492	3.205	28.897	1.045	0.0009881	3.294	29.696	0.969

Table 6 Corrected Limits of the $\bar{X}$ and S^2-Two-sided component charts in $\left(\bar{X}, S^2\right)$-chart for case KU and the standard deviation is estimated by $\frac{\bar{S}}{C_4(n)}$ to deliver the desired in-control average run length of 370 and 500

		$ICARL_0$=CCCLXX				$ICARL_0$ =D			
n	m	p	$\bar{X}$-chart	U-S^2-chart	L-S^2-chart	p	$\bar{X}$-chart	U-S^2-chart	L-S^2-chart
V	V	0.0013220	3.211	19.382	0.074	0.0009829	3.295	20.035	0.063
	X	0.0013725	3.201	19.230	0.075	0.0010193	3.285	19.955	0.065
	XX	0.0013865	3.197	19.277	0.075	0.0010291	3.282	19.934	0.065
	XXX	0.0013837	3.198	19.282	0.075	0.0010277	3.283	19.937	0.065
	L	0.0013767	3.199	19.283	0.075	0.0010235	3.284	19.946	0.065
	LXXV	0.0013712	3.201	19.302	0.075	0.0010165	3.286	19.961	0.064
	C	0.0013669	3.202	19.309	0.075	0.0010137	3.287	19.967	0.064
	D	0.0013556	3.204	19.327	0.075	0.0010025	3.289	19.992	0.064
X	V	0.00114825	3.251	29.311	1.005	0.0008483	3.337	30.086	0.933
	X	0.0012421	3.230	29.110	1.002	0.0009198	3.314	29.880	0.952
	XX	0.0012982	3.216	28.996	1.036	0.0009619	3.301	29.765	0.963
	XXX	0.0013178	3.212	28.957	1.040	0.0009759	3.297	29.728	0.966
	L	0.0013318	3.209	28.930	1.042	0.0009857	3.295	29.703	0.968
	LXXV	0.0013388	3.207	28.916	1.043	0.0009913	3.293	29.688	0.969
	C	0.0013430	3.207	28.904	1.044	0.0009941	3.292	29.681	0.970
	D	0.0013501	3.205	28.895	1.046	0.0009999	3.291	29.666	0.972

each sections of the Table 1, it is discovered that the IC MRL is larger in $(\bar{X}, S^2_{1,S^2-chart})$-chart than in $(\bar{X}, S^2_{2,S^2-chart})$-chart which implies a better performance in the former than in the later. Also, the $(\bar{X}, S^2_{1,S^2-chart})$-chart raised more OOC alarms (smaller MRL) than the $(\bar{X}, S^2_{2,S^2-chart})$ which still makes the former a chart with a better performance than the later. This is seen when one considers the shift-rows with 0.10 in δ, 0.10 in γ, and 0.10 in both δ and γ which represents a substantial shift in the IC states of the chart in both sections of Table 1, all these rows show $(\bar{X}, S^2_{1,S^2-chart})$-chart with smaller OOC MRL than $(\bar{X}, S^2_{2,S^2-chart})$-chart and it is expected that a better chart in terms of performance should raise more alarms (smaller OOC MRL) when a shift from IC state is introduced.

To look at the performance of the $(\bar{X}, S^2)$-chart under the S_p and $\frac{\bar{S}}{C_4(n)}$ Phase-1-estimators of the process standard deviation, the IC rows of Tables 1 and 2 for the two estimators respectively are selected and examined. From Table 1, the IC MRL of the $(\bar{X}, S^2_{1,S^2-chart})$-chart for m = 20, 50, 100, 500, 1000 and 10,000 are 115, 122, 125, 128, 128, and 128 and the corresponding values in Table 2 are 119, 124, 126, 128, 128, and 128 which are almost the same. Also, in Table 1, the IC MRL of the $(\bar{X}, S^2_{2,S^2-chart})$-chart for the same values of m are 108, 119, 123, 127, 128, and 128 while the corresponding values in Table 2 are 113, 121, 124, 127, 128, and 128 which are also almost the same too. Therefore, the $(\bar{X}, S^2)$-chart performs almost the same under the two Phase-I-estimators considered and the use of any of them should be a matter of choice and perhaps on the basis of the easier one to apply.

The drastic fall-short of the provision of the 370 and 257 traditionally desired IC ARL and MRL performance of a control chart by the joint charts has been attributed to the effects of the use of the three-sigma control limits, the multiplicity effects of the joint charting scheme and the effects of parameter estimation. Therefore, for $(\bar{X}, S^2)$-chart to be of this desired performance, the control limits of the component charts involved have been adjusted to guarantee such performance in the presence of these setbacks of the joint charts. In this work, the probability limits are used instead of the three-sigma limits, therefore, the provision of the new charting constants for the $\bar{X}$- and S^2-charts to enable $(\bar{X}, S^2)$-chart deliver the desired 370 or 257 in-control ARL or MRL as presented in Tables 3, 4, 5 and 6 will be the ones based on taking into cognizant of the multiplicity effects of the joint charting scheme and that of the parameter estimation.

It suffices to mention here that the design of the $\left(\bar{X}, S^2\right) - chart$ can be applied in Case UU by making use of the appropriate control limits of the $\bar{X}$-chart for when both process parameters are unknown while combining the charts, $\left(\bar{X}, S^2\right)$. This is because Case UU involves variance estimation since it is unknown. However, the chart can't be applied in Case UK since, here, the process variance is not to be estimated here but known already and it was defined that $W = \hat{\sigma}_0/\sigma_0$ and that W follows a scaled chi-square distribution, that is, $W \sim \frac{a_0\sqrt{\chi^2}}{\sqrt{b_0}}$ with b_0 degrees of freedom. This is because the expression of W involves estimating the process variance.

6.1. Numerical application of the $\left(\bar{X}, S^2\right)$-control chart with the corrected limits

This subsection demonstrates the application of the chart based on the real-life data set concerning the semiconductor measurements as described and reported in (Montgomery, 2009, p.240). Here, the quality characteristic of interest to be controlled is the flow width of the resist. It was reported that the flow width measurement follows a normal distribution with an IC mean (μ_0) of 1.5 microns and an IC standard deviation (σ_0) of 0.15 microns. However, oftentimes, these parameters are unknown and should be estimated.

To demonstrate the application of the chart, let us assume that the in-control standard deviation (σ_0) is unknown and must be estimated. We chose the first 20 subgroups, each of size 5 of the flow width measurements for the estimation of the unknown σ_0 and the remaining five subgroups each of the same size will be used in Phase-II-analysis. With the data, the application of the chart follows the following steps:

1. Calculate the sample variance and the sample standard deviation for each sample and denote them by s_i^2 and s_i respectively, for i= 1,2,...,m like

Subgroups	Observations	Sample Variance	Sample Standard deviation
I	$x_{1,1}\ x_{1,2}\ldots\ldots x_{1,n}$	$s_1^2=\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{1,j}-\bar{X}_1\right)^2$	$s_1=\sqrt{\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{1,j}-\bar{X}_1\right)^2}$
II	$x_{2,1}\ x_{2,2}\ldots\ldots x_{2,n}$	$s_2^2=\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{2,j}-\bar{X}_2\right)^2$	$s_2=\sqrt{\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{2,j}-\bar{X}_2\right)^2}$
III	$x_{3,1}\ x_{3,2}\ldots\ldots x_{3,n}$	$s_3^2=\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{3,j}-\bar{X}_3\right)^2$	$s_3=\sqrt{\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{3,j}-\bar{X}_3\right)^2}$
$\vdots$	$\vdots\ \vdots\ \ddots\ \vdots$	$\vdots$	$\vdots$
$\vdots$	$\vdots\ \vdots\ \ddots\ \vdots$	$\vdots$	$\vdots$
m	$x_{m,1}x_{m,2}\ldots\ldots x_{m,n}$	$s_m^2=\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{m,j}-\bar{X}_m\right)^2$	$s_m=\sqrt{\frac{1}{n-1}\Sigma_{j=1}^{n}\left(X_{m,j}-\bar{X}_m\right)^2}$

2. With the data, the variance and standard deviation formulas, obtain S_p and $\frac{\bar{S}}{c_4(n)}$ estimators of σ_0 as $S_p=\sqrt{\Sigma_{i=1}^{m}\left(s_i^2\right)/m}$ and $\frac{\bar{S}}{c_4(n)}=\Sigma_{i=1}^{m}\left(s_i\right)/mc_4(n)$ respectively.
3. For this illustrative example, we consider the $\left(\bar{X},S^2\right)$-chart when S^2-chart has one-sided upper limit, $\frac{\bar{S}}{c_4(n)}$ used as the in-control Phase-I-estimator of the process standard deviation for m=20, $n=5$. Recall that the control limits of the $\bar{X}$ and S_1^2-charts when the process standard deviation is not given are $\mu_0\pm Z_{\alpha/2}\frac{\hat{\sigma_0}}{\sqrt{n}}$ and $\frac{\chi^2_{n-1,\alpha}}{n-1}\hat{\sigma}_0^2$ respectively and it has been provided in Table 4 that to use the charts jointly and for the joint chart to deliver an IC ARL of 370, $Z_{\alpha/2}$ = 3.072 and $\chi^2_{n-1,\alpha}$=16.785. Therefore, the control limits of the $\bar{X}$-component chart are obtained as $1.5\pm\frac{3.072\times0.1511}{\sqrt{5}}$ = 1.7076 or 1.2924, for the S^2-component chart, the control limit is $\frac{16.785\times(0.1511)^2}{5-1}$ = 0.0958.
4. Using the control limits estimated in 3, at any point that either the sample mean of any of the remaining samples for the Phase-II-analysis goes above 1.7066 or 1.2934 or that the sample variance of any of the remaining samples for the Phase-II-analysis goes above 0.0958, the $\left(\bar{X},S^2\right)$-chart declares the process out-of-control. In the case of the example from (see in Montgomery, 2009, p.240), none of the sample observation was out-of-control.

7. Conclusion

$\left(\bar{X},S^2\right)$-chart even without parameter estimation can't deliver the desired IC ARL of 370 when the nominal false alarm rate of 0.0027 is used in the component charts but 185 due to the multiple charting issue. Under parameter estimation with small to moderate values of Phase-I-sample, $\left(\bar{X},S^2\right)$-chart perform poorly, returning in errors IC ARL values larger than the expected 185 in the case where S^2-chart is one-sided and values smaller when S^2-chart is two-sided. $(\bar{X},S^2_{1,S^2-chart})$-chart is found to have longer IC MRL values than the $(\bar{X},S^2_{2,S^2-chart})$-chart, therefore, outperformed the latter and equally raised more OOC alarms in the presence of shifts from the IC states of the component charts. $\left(\bar{X},S^2\right)$-chart performed almost equally well when S_p and $\frac{\bar{S}}{C_4(n)}$ are used as the Phase I estimators and the authors remarked that the use of any should be based on choice and easiness of application. New control limits for the component charts are provided to guarantee the provision of the desired IC performance of 370 and 500 at various values of Phase I samples.

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