Thailand Statistician
January 2025; 23(1): 72-96
http://statassoc.or.th
Contributed paper

Modified Topp-Leone Distribution: Properties, Classical and Bayesian Estimation with Application to COVID-19 and Reliability Data

Bhupendra Singh[a], Shikhar Tyagi[b], Ravindra Pratap Singh[b], and Abhishek Tyagi[a]*

[a] Department of Statistics, Chaudhary Charan Singh University, Meerut, India
[b] Department of Statistics, Central University of Rajasthan, Rajasthan, India
*Corresponding author; e-mail: abhishektyagi033@gmail.com

Received: 23 September 2021
Revised: 16 July 2022
Accepted: 4 September 2022

Abstract

In this article, we have proposed a new continuous model called Modified Topp-Leone distribution. One of the main features of this model is that it has only one parameter but contains varieties of shapes for density and hazard rate functions. We have discussed its various impressive properties like heavy-tailed behavior, mode, moments, quantile, median, skewness, kurtosis, moment generating function, mean deviation, various inequality measures, mean residual life, expected inactivity time function, weighted moments, entropies, and various other important reliability characteristics including stress-strength reliability, hazard rate, survival function, stochastic ordering, and order statistics. The parameter estimation of the proposed model is discussed in the classical and Bayesian paradigm. In classical point estimation, we have used the method of maximum likelihood, ordinary and weighted least squares, Cramer-Von-Mises, and the method of maximum product of spacings. The asymptotic distribution of the maximum likelihood estimator is also provided and it is used to develop the asymptotic confidence interval. In Bayesian estimation, we have used informative and non-informative priors under symmetric and asymmetric loss functions to obtain the Bayes estimator of the unknown parameter. The highest posterior density interval of the parameter is also obtained. An extensive simulation study is presented to the assessment of the different estimation procedures. In the end, three real datasets are examined to show the utility of the proposed model in the real world.

Keywords: Least squares estimation, lifetime data, maximum likelihood estimation, maximum product of spacings, Topp-Leone distribution

1. Introduction

Lifetime data modelling is important in a variety of applied fields, including biomedical, reliability engineering, economics, and actuarial science. In view of this, an enormous amount of attention in statistical literature is given to lifetime distributions and their applicability to real-world occurrences. The continuous lifetime distributions have evolved rapidly during the previous few decades. Gupta and Kundu (1999) introduced generalized exponential distribution to provide more flexibility over baseline exponential distribution. This model has decreasing and unimodal shapes for the density function and its hazard rate can take increasing and decreasing shapes. Nadarajah and Kotz (2006) proposed beta exponential distribution by using the beta-G class. Its density can have decreasing and unimodal shapes whereas the hazard rate can exhibit decreasing and increasing shapes. Nadarajah et al. (2011) developed Nadarajah-Haghighi distribution to model increasing, decreasing, and constant hazard rate function (HRF). Bidram et al. (2013) presented new generalized exponential geometric distribution. They showed that this new generalization of the generalized exponential geometric model can be used to fit increasing, decreasing or bathtub (BT) shaped failure rate. Chaubey and Zhang (2015) pioneered exponentiated Chen distribution with increasing and BT shapes hazard function. Ekhosuehi and Opone (2018) coined three-parameter generalized Lindley distribution and they have illustrated that the proposed model is useful to model increasing and decreasing failure rates. Choudhary et al. (2021) developed new extended modified Weibull distribution with increasing, decreasing and BT shaped hazard function. El-Morshedy et al. (2021) derived type I half-logistic odd Weibull-G family and discussed its usefulness in the engineering field. Recently, Tyagi et al. (2022) proposed power xgamma distribution with applications to cancer data.

Inverted (inverse) distributions are another type of distribution that has been proposed in the literature by employing inverse transformations to well-known random variables (RVs). Such distributions have distinct features of density and hazard rate shape, and they may also be used to the (lifetime) phenomena, which the non-inverted distribution cannot study correctly. For further information on the concept, theory and discussion about inverted distributions, one can refer to Keller and Kamath (1982), Sheikh et al. (1987), and Lehmann and Shaffer (2012). There are well-known univariate continuous inverted models in the existing literature, for example, inverted Rayleigh distribution proposed by Voda (1972); inverted Weibull distribution introduced by Keller and Kamath (1982); inverted gamma model pioneered by Lin et al. (1989); inverted Lindley distribution developed by Sharma et al. (2015); inverted power Lindley by Barco et al. (2017), and inverted Nadarajah-Haghighi distribution invented by Tahir et al. (2018), and the references cited therein.

Topp and Leone (1955) introduced the Topp-Leone (TL) distribution and described how it might be used to represent the lifetime data. After a long silence, Nadarajah and Kotz (2003) show that even with a single parameter it can produce a U-shaped hazard rate. The data from human populations may be effectively modelled using lifetime distributions with a U-shaped hazard rate. This is attributed to high newborn mortality rates (due to diseases and birth abnormalities), a nearly constant death rate until the thirties, and a high death rate thereafter. The same patterns may be seen in certain manufactured goods as well. The TL distribution, unlike the lognormal and gamma distributions, has a closed-form distribution function and a HRF. The TL distribution's enticing properties have lately piqued academics' interest, prompting them to examine it for modelling and predicting lifetime data. Some of the recent studies concerning the TL model and its modifications can be found in the works of Al-Zahrani and Alshomrani (2012), Genc (2013), Bayoud (2015), Reyad and Othman (2017), and Sharma (2018). Recently, Chesneau et al. (2021) proposed an extended TL family of distributions as an alternative to beta and Kumaraswamy type distributions and Ikechukwu et al. (2021) developed Type II TL generalized power Ishita distribution.

Although the TL distribution with one parameter produces a U-shaped hazard rate, it is not much flexible due to the domain restriction of RV X to (0, 1). Therefore, in many fields including lifetime data analysis, it is not widely applicable. Such limitations motivate us to develop a new form of TL distribution. Therefore, in this article, by using an inverse transformation of type $Y = X/(1-X)$, we have proposed a new type of TL distribution, the so-called Modified Topp-Leone (MTL) distribution. The main objectives of proposing the MTL distribution using the transformation $Y = X/(1-X)$ are as follows:

- With this transformation, our primary goal is to broaden the domain of the RV while keeping the same number of parameters, so that a greater variety of data can be studied.

- For mathematical and computational ease, our aim is to invent a distribution whose various distribution characteristics, including probability density function (PDF), cumulative density function (CDF), stress-strength reliability (SSR), etc., have nice closed-form expressions. Such qualities are rarely observed in well-known existing models.

- To design a model whose PDF and HRF may take on multiple forms for different values of the parameter, allowing us to fit a broad variety of real data from different fields.
- Another important objective of this article is to construct a model that can consistently produce better fits than other developed continuous distributions using the same baseline model and other popular continuous distributions available in the existing literature.

The rest of the structure of this article is as follows. In Section 2, the MTL distribution is presented. Section 3 contains various important statistical properties of the proposed model. In Section 4, various classical point estimators as well as asymptotic confidence intervals (ACIs) have been discussed. Section 5 deals with the Bayesian estimation under informative and non-informative priors with different loss functions. The highest posterior density (HPD) intervals are also provided in Section 5. A comprehensive simulation study is presented to access the behavior of different estimation techniques in Section 6. In Section 7, three real data examples are used to show the practicability of the proposed model to study real-world phenomena. In the end, some concluding remarks are given in Section 8.

2. Modified Topp-Leone Distribution

The PDF and CDF of continuous TL distribution are as follows:

$$f(x,\alpha) = 2\alpha(1-x)(2x-x^2)^{\alpha-1}; 0<x<1, \alpha>0, \tag{1}$$

$$F(x) = (2x-x^2)^{\alpha}; 0<x<1, \alpha>0. \tag{2}$$

An RV Y is said to follow MTL distribution if we use the transformation $Y = X/(1-X)$, where X has TL distribution with PDF (1). The PDF of RV Y can be written as

$$f(y,\alpha) = 2\alpha(1+y)^{-2\alpha-1}(2y+y^2)^{\alpha-1}; 0<y<\infty, \tag{3}$$

where $\alpha > 0$ is a shape parameter responsible for the different shapes of density in (3). The limiting behavior of the PDF can be obtained as:

$$\lim_{y\to\infty} f(y) = 0 \quad \text{and} \quad \lim_{y\to 0} f(y) = \begin{cases} 2; & \alpha = 1 \\ 0; & \alpha > 1 \\ \infty; & \alpha < 1 \end{cases}.$$

The CDF corresponding to the PDF can be written as

$$F(y) = \left(\frac{2y+y^2}{(1+y)^2}\right)^{\alpha}; y>0, \alpha>0. \tag{4}$$

Figure 1 presents the different shapes of PDF and CDF of MTL distribution. From this figure, we can easily observe that the density of the MTL model can be decreasing and unimodal (rightly skewed). Also, we have noticed that as we increase the value of α, the shape of the PDF rapidly changes from reversed-J to upside-down bathtub-shaped. For large values of α, the shape of the CDF gradually increases towards 1 compared to smaller values of α.

3. Statistical Properties

3.1. Heavy-tailed behavior

The tails of heavy-tailed distributions are not exponentially bounded, implying that they have longer tails than the exponential distribution. Mathematically, a distribution with CDF $F(y)$ is said to be heavy-tailed, if the following equation holds,

$$\lim_{y\to\infty} \exp(\kappa y)[1-F(y)] = \infty;\ \kappa > 0.$$

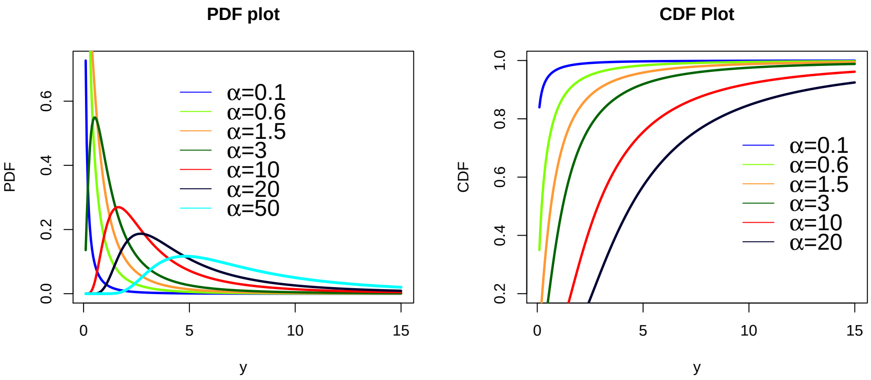

Figure 1 The various shapes of PDF and CDF of MTL distribution for different values of α

Using the above equation for the CDF of MTL distribution, we get

$$\lim_{y\to\infty} \exp(\kappa y)[1 - F(y)] = \lim_{y\to\infty} \exp(\kappa y) \left[\frac{(1+y)^{2\alpha} - (2y+y^2)^{\alpha}}{(1+y)^{2\alpha}}\right] = \infty.$$

Hence, we can say that the MTL distribution is heavy-tailed. Also from Figure 1 (left panel), we can easily observe that the proposed model has a heavy-tailed (right-tailed) curve.

3.2. Mode

The mode of a probability distribution is the point at which the PDF attains its maximum value. The mode of MTL distribution can be obtained by solving the following equation

$$\frac{\partial \log f}{\partial y} = \frac{\alpha - 1}{y} - \frac{2\alpha + 1}{1 + y} + \frac{\alpha - 1}{2 + y} = 0.$$

Now, by solving the above equation for y, we can get the mode of the MTL distribution as

$$Mode = -1 + \frac{1}{3}\sqrt{6\alpha + 3}; \alpha > 1. \tag{5}$$

Hence, MTL distribution is a unimodal distribution. For some specific values of the parameter, Table 1 lists descriptive statistics of the MTL distribution, including mode. This table concludes that the mode increases as the value of α increases.

Table 1 Descriptive statistics of MTL distribution for some choices of parameter α

α	Mean	Median	Mode	MD(μ)	MD(m)	Skewness	Kurtosis
0.5	0.5707	0.1547	-	0.6613	0.1075	0.4910	1.7855
1.5	1.3561	0.6439	0.1547	1.2537	0.3417	0.3558	1.6156
2.5	1.9452	1.0321	0.4142	1.6492	0.4890	0.3342	1.5956
3.5	2.4361	1.3592	0.6329	1.9671	0.6029	0.3257	1.5879
5.5	3.2521	1.9060	1.0000	2.4836	0.7832	0.3185	1.5813

3.3. Quantiles, median, skewness and kurtosis

One method to specify a probability distribution is to use the quantile function (QF). In both statistical applications and Monte Carlo techniques, it is a frequently used statistical tool. The QF of the MTL distribution, denoted by $Q(u)$, can be derived by inverting the CDF (4) as follows:

$$Q(u) = -1 + (1 - u^{1/\alpha})^{-1/2}; 0 < u < 1, \tag{6}$$

where U follows Uniform distribution with support (0, 1). Particularly, the first quartile Q_1, the second quartile Q_2 (or median), and the third quartile Q_3 can be obtained easily by putting 0.25, 0.50, and 0.75, respectively.

Kenney and Keeping (1962) and Moors (1988) provided the famous expressions for skewness and kurtosis based on the quantiles. One of the most notable characteristics of these measures is that they are less influenced by outliers and may be computed even for distributions without moments. The expression of skewness (Sk) by Kenney and Keeping (1962) is given by

$$\mathrm{Sk} = \frac{\mathrm{Q}(\frac{3}{4}) + \mathrm{Q}(\frac{1}{4}) - 2\mathrm{Q}(\frac{1}{2})}{\mathrm{Q}(\frac{3}{4}) - \mathrm{Q}(\frac{1}{4})}.$$

The Moors kurtosis proposed by Moors (1988) can be presented as

$$\mathrm{Ku} = \frac{\mathrm{Q}(\frac{7}{8}) - \mathrm{Q}(\frac{5}{8}) + \mathrm{Q}(\frac{3}{8}) - \mathrm{Q}(\frac{1}{8})}{\mathrm{Q}(\frac{3}{4}) - \mathrm{Q}(\frac{1}{4})}.$$

Using Equation (6) in the above expressions, one can easily obtain the Sk and Ku for the MTL distribution. For some specific values of the parameter, Table 1 shows the median, skewness, and kurtosis of the MTL distribution. We can infer from this table that

- Median increases as the value of α increases.
- Skewness and kurtosis decrease as we increase the value of α.
- From the calculated values of the measure of symmetry, we can say that the MTL distribution is positively skewed.

3.4. Moments and moment generating function

The moments of a probability distribution are important for measuring its different properties such as mean, variance, skewness, kurtosis, and so on. The r^{th} row moment, say μ'_r, of the MTL distribution can be obtained as

$$\begin{aligned}\mu'_r = E[Y^r] &= \int_0^\infty y^r f(y,\alpha)dy \\ &= 2\alpha \int_0^\infty \frac{y^{\alpha+r-1}(2+y)^{\alpha-1}}{(1+y)^{2\alpha+1}}dy \\ &= 2\alpha \int_0^\infty \frac{y^{\alpha+r-1}(1+\frac{1}{(1+y)})^{\alpha-1}}{(1+y)^{\alpha+2}}dy.\end{aligned}$$

If $|s| < 1$ and d is a real non-integer, then we have

$$(1+s)^d = \sum\nolimits_{j=0}^{\infty} \begin{pmatrix} d \\ j \end{pmatrix} s^j. \tag{7}$$

If d is a positive integer, the above equation reduces to the binomial expansion. Using the above power-series in (7), μ'_r can be written as

$$\mu'_r = 2\alpha \sum_{j=0}^{\infty} \begin{pmatrix} \alpha-1 \\ j \end{pmatrix} \int_0^\infty \frac{y^{\alpha+r-1}}{(1+y)^{\alpha+2+j}}dy.$$

Finally, the expression for the r^{th} row moment of MTL distribution is

$$\mu'_r = 2\alpha \sum_{j=0}^{\infty} \begin{pmatrix} \alpha-1 \\ j \end{pmatrix} B(\alpha+r, 2+j-r);\ r = 1, 2, ..., \tag{8}$$

where $B(a, b)$ is the Beta function with parameters a and b . From Equation (8), it is observable that the r^{th} row moment exists if $2 + j > r$. In particular, the mean of the MTL distribution can be obtained by putting $r = 1$ in Equation (8) as

$$\mu_1' = E(Y) = 2\alpha \sum_{j=0}^{\infty} \binom{\alpha - 1}{j} B(\alpha + 1, j + 1). \tag{9}$$

Unfortunately, MTL distribution does not have finite moments of order greater than or equal to two due to the condition $2 + j > r; r = 2, 3, ...$

Table 1 includes the mean of the MTL distribution for some choices of the parameter α. This table shows that as the value of α is increased, the mean increases as well.

The moment generating function (MGF) of the MTL distribution can be obtained as

$$\begin{aligned} M_Y(t) = E[e^{ty}] &= 2\alpha \int_0^{\infty} e^{ty} \frac{y^{\alpha-1}(2+y)^{\alpha-1}}{(1+y)^{2\alpha+1}} dy \\ &= 2\alpha \sum_{k=0}^{\infty} \frac{t^k}{k!} \int_0^{\infty} \frac{y^{k+\alpha-1}(2+y)^{\alpha-1}}{(1+y)^{2\alpha+1}} dy. \end{aligned}$$

Using power-series (7), $M_Y(t)$ becomes

$$M_Y(t) = 2\alpha \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^k}{k!} \binom{\alpha - 1}{j} B(\alpha + k, 2 + j - k). \tag{10}$$

Here, it is observable that the MGF (10) holds when $2 + j > k$.

3.5. Mean deviation

The average absolute deviations from the mean and the median are known as the mean deviation about the mean and the mean deviation about the median. These tools are critical for capturing the spread of a population from a central point (like mean, median or mode). If $\mu = E[X]$ and M represents the mean and median, respectively, then mean deviation about the mean (denoted by $MD(\mu)$) and mean deviation about the median (symbolized by $MD(m)$) can be defined as

$$MD(\mu) = E|y - \mu| = \int_0^{\infty} |y - \mu| f(y) dy$$

and

$$MD(m) = E|y - m| = \int_0^{\infty} |y - m| f(y) dy,$$

respectively. The above measures can be calculated by the following expressions:

$$\begin{aligned} E|y - \delta| &= \int_0^{\delta} (\delta - y) f(y) dy + \int_{\delta}^{\infty} (y - \delta) f(y) dy \\ &= 2 \int_0^{\delta} (\delta - y) f(y) dy = 2 \left\{ \delta F(\delta) - \int_0^{\delta} y f(y) dy \right\}, \end{aligned} \tag{11}$$

where $\delta = \mu$ or M. Now, in the case of MTL distribution, consider

$$I = \int_0^{\delta} y f(y) dy = 2\alpha \int_0^{\delta} \frac{y^{\alpha+1-1}(2+y)^{\alpha-1}}{(1+y)^{2\alpha+1}} dy. \tag{12}$$

Using power-series (7), expression (12) becomes

$$I=\int_0^\delta yf(y)dy=2\alpha\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ j\end{array}\right)\int_0^\delta\frac{y^{\alpha+1-1}}{(1+y)^{\alpha+2+j}}dy.$$

By transforming $y=\frac{z}{1-z}$ in the above integral, we get

$$I=2\alpha\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ j\end{array}\right)B(\delta/(1+\delta),\alpha+1,j+1),$$

where $B(\delta/(1+\delta),a,b)$ is the incomplete Beta function. Therefore, using the final value of I in Equation (11), we get the required measures for MTL distribution as

$$MD(\mu)=2\left[\mu F(\mu)-2\alpha\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ j\end{array}\right)B(\mu/(1+\mu),\alpha+1,j+1)\right],$$

and

$$MD(m)=2\left[mF(m)-2\alpha\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ j\end{array}\right)B(m/(1+m),\alpha+1,j+1)\right].$$

In Table 1, we have calculated $MD(\mu)$ and $MD(m)$ for some selected values of the unknown parameter α. As the value of α increases, so does the value of $MD(\mu)$ and $MD(m)$. It also confirms the theoretical finding that $MD(m)$ is less than $MD(\mu)$.

3.6. Inequality measures of MTL distribution

Inequality measures are frequently employed in the socio-economic science to investigate the distribution of income and wealth. The most widely used inequality measures in the literature are Lorenz (Lorenz, 1905), Bonferroni (Bonferroni, 1930), and Zenga (Zenga, 2007) curves. These three curves have the common property that they can be described using only the population mean and the means of specific subgroups. The significant use of inequality curves is that they may be used to establish various orderings that allow for inequality-based comparisons of distributions. Such comparisons within the same model enable an understanding of how inequality is affected by distribution parameters. In this segment, we develop Lorenz, Bonferroni, and Zenga curves for MTL distribution.

The Lorenz, Bonferroni, and Zenga curves are presented, respectively, by

$$L(F(y))=\frac{\int_0^y zf(z)dz}{\mu}=F(y).\frac{\bar{\mu}}{\mu},\quad B(F(y))=\frac{\bar{\mu}}{\mu},\quad\text{and}\quad A(y)=1-\frac{\bar{\mu}}{\mu^+},$$

where $\bar{\mu}=\frac{\int_0^y zf(z)dz}{F(y)}=\frac{E_{Y\leq y}(Y)}{F(y)}$ and $\mu^+=\frac{\int_y^\infty zf(z)dz}{1-F(y)}=\frac{E(Y)-E_{Y\leq y}(Y)}{1-F(y)}$ are the lower and upper means, respectively. In the case of MTL distribution, we have

$$\begin{aligned}\bar{\mu}&=\tfrac{1}{F(y)}\int_0^y zf(z)dz=\tfrac{2\alpha}{F(y)}\int_0^y\tfrac{z^{\alpha+1-1}(2+z)^{\alpha-1}}{(1+z)^{2\alpha+1}}dz\\&=\tfrac{2\alpha}{F(y)}\sum_{j=0}^{\infty}\left(\begin{array}{c}\alpha-1\\ j\end{array}\right)B(y/(1+y),\alpha+1,j+1),\end{aligned}$$

and

$$\mu^+=\tfrac{1}{1-F(y)}\int_y^\infty zf(z)dz=\tfrac{2\alpha}{S(y)}\int_y^\infty\tfrac{z^{\alpha+1-1}(2+z)^{\alpha-1}}{(1+z)^{2\alpha+1}}dz.$$

Using power-series (7), the above expression becomes

$$\mu^{+} = \tfrac{2\alpha}{S(y)} \sum_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) \int_{y}^{\infty} \tfrac{z^{\alpha+1-1}}{(1+z)^{\alpha+2+j}} dz.$$

By putting $z = \frac{u}{1-u}$, we get

$$\begin{aligned} \mu^{+} &= \tfrac{2\alpha}{S(y)} \sum_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) \int_{0}^{1/(1+y)} z^{j+1-1} (1+z)^{\alpha+1-1} dz \\ &= \tfrac{2\alpha}{S(y)} \sum_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) B(1/(y+1); j+1, \alpha+1). \end{aligned}$$

Thus, for MTL distribution Lorenz, Bonferroni, and Zenga curves are respectively given by

$$L_F(y;\alpha) = \tfrac{2\alpha}{\mu} \sum_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) B(y/(1+y), \alpha+1, j+1),$$

$$B_F(y;\alpha) = \tfrac{2\alpha}{\mu F(y)} \sum_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) B(y/(1+y), \alpha+1, j+1),$$

$$A_F(y;\alpha) = 1 - \frac{S(y) \sum\limits_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) B(y/(1+y), \alpha+1, j+1)}{F(y) \sum\limits_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) B(1/(1+y), j+1, \alpha+1)}.$$

3.7. Mean residual life function

The mean residual life (MRL) function is used extensively in a wide variety of areas, including reliability engineering, survival analysis, and biomedical research since it represents the ageing mechanism. It is well known that the MRL function characterizes the distribution function F uniquely since it contains all of the model's information. Let T be an RV with distribution function F and if $E[T] < \infty$, then the MRL function of T, symbolized by $m(t)$ can be defined by

$$m(t) = E(T - t | T > t) = \frac{1}{1 - F(t)} \int_{t}^{\infty} (u - t) f(u) du.$$

If Y has MTL distribution with parameter α , then the MRL function of Y is

$$\begin{aligned} m(y) =& E(Y - y | Y > y) \\ =& \frac{2\alpha}{S(y)} \sum_{j=0}^{\infty} \left(\begin{array}{c} \alpha - 1 \\ j \end{array} \right) \left[B\left(\tfrac{1}{y+1}; j+1, \alpha+1 \right) - tB\left(\tfrac{1}{y+1}; j+2, \alpha \right) \right]. \end{aligned}$$

3.8. Expected inactivity time function

The expected inactivity time function or mean past life function (MPL) can be explained as the expectation of the conditional RV $(t - T | T \leq t)$ i.e., expected time elapsed from the failure of a component given that its lifetime is less than or equal to time t. As with the MRL function, the MPL function has applications in a wide variety of areas, including reliability theory and survival analysis, actuarial research, and forensic science. Symbolically, MPL function is defined as

$$m^{*}(t) = E(t - T | T \leq t) = \frac{1}{F(t)} \int_{0}^{t} (t - u) f(u) du.$$

In the case of proposed distribution, the MPL function can be obtained as

$$\begin{aligned} m^*(y) &= E(y - Y|Y \le y) \\ &= \frac{2\alpha}{F(y)} \sum_{j=0}^{\infty} \binom{\alpha-1}{j} \left[B\left(\tfrac{y}{1-y}; \alpha+1, j+1,\right) - tB\left(\tfrac{y}{1-y}; \alpha, j+2\right) \right]. \end{aligned}$$

3.9. Probability weighted moment

The probability-weighted moments (PWMs) are the generalization of the ordinary moments and may be derived for any probability distribution whose conventional moment exist. Typically, the PWM technique is used in the estimation of unknown parameter of a probability distribution that has a closed-form for the inverse CDF. If the RV Y follows MTL distribution then the PWM is given by

$$\begin{aligned} \eta_{r,s} &= E\left[Y^r F^s(y,\alpha)\right] \\ &= \int_0^\infty y^r F^s(y,\alpha) f(y) dy \\ &= 2\alpha \int_0^\infty \frac{y^{r+\alpha s+\alpha-1} (2+y)^{\alpha s+\alpha-1}}{(1+y)^{2\alpha s+2\alpha+1}} dy. \end{aligned}$$

Using power-series (7), $\eta_{r,s}$ becomes

$$\eta_{r,s} = 2\alpha \sum_{j=0}^{\infty} \binom{\alpha s+\alpha-1}{j} \int_0^\infty \frac{y^{r+\alpha s+\alpha-1}}{(1+y)^{\alpha s+\alpha+j+2}} dy.$$

Hence,

$$\eta_{r,s} = 2\alpha \sum_{j=0}^{\infty} \binom{\alpha s+\alpha-1}{j} B(\alpha s+\alpha+r, 2+j-r).$$

If we put $s = 0$ in $\eta_{r,s}$ we get the expression for r^{th} row moment. Also, $\eta_{r,s}$ exist when $2+j>r$.

3.10. Entropy

In information theory, the entropy of an RV is the average level of information, surprise, or uncertainty contained in the possible outcomes of that variable. One of the important entropy is Rnyi entropy (RE) (Renyi, 1961). It is a crucial measure of complexity and uncertainty and is used in many fields including problems identification in statistics, statistical inference, physics, econometrics, and pattern recognition in computer science. For the MTL distribution, the RE can be defined as $(\rho > 0, \rho \neq 1)$

$$\begin{aligned} I_R(\rho) &= \frac{1}{1-\rho} \log \left[\int_0^\infty f^\rho(y) dy \right] \\ &= \frac{1}{1-\rho} \log \left[(2\alpha)^\rho \int_0^\infty \frac{y^{\rho\alpha-\rho} (2+y)^{\rho\alpha-\rho}}{(1+y)^{2\rho\alpha+\rho}} dy \right] \end{aligned}$$

Using power series (7) and proceeding with the simple steps of integration we get,

$$I_R(\rho) = \frac{1}{1-\rho} \log \left[(2\alpha)^\rho \sum_{j=0}^{\infty} \binom{\rho\alpha-\rho}{j} B(\rho\alpha-\rho+1, 3\rho+j-1) \right].$$

The above expression of RE for MTL distribution exists when $3\rho+j>1$ and $\rho(1-\alpha)<1$.

A famous entropy called Shannon entropy (ShE) can be obtained as a particular case of RE as $\rho \to 1$, where $ShE = -E[\log f(y;\alpha)]$. Another important entropy in the literature is ω-entropy and it is defined as

$$H_\omega(Y) = \frac{1}{\omega-1}\log\left(1 - \int_0^\infty f^\omega(y)dy\right),\ \omega > 0\, and\, \omega \neq 1.$$

Therefore, in the case of the proposed distribution, the ω-entropy is

$$H_\omega(Y) = \frac{1}{\omega-1}\log\left(1 - (2\alpha)^\omega \sum_{j=0}^{\infty}\left(\begin{array}{c}\omega\alpha-\omega\\ j\end{array}\right)B(\omega\alpha-\omega+1, 3\omega+j-1)\right).$$

3.11. Stress-strength reliability (SSR)

Suppose the RV Y_1 be the strength of a system subjected to stress Y_2, given that Y_1 and Y_2 are stochastically independent RVs, then the parameter $R = P[Y_2 < Y_1]$ is called SSR. The SSR parameter has a wide range of applications in reliability theory, particularly in engineering concepts such as various structures, ageing of concrete pressure vessels, static fatigue of ceramic components, fatigue failure of aircraft structures, and the deterioration of rocket motors, etc. Because of the broader scope of SSR, a significant amount of work has been done in recent years on the study of SSR parameter (Goel and Singh, 2020). In the context of the proposed model if $Y_1 \sim MTL(\alpha_1)$ and $Y_2 \sim MTL(\alpha_2)$ distributions, then SSR is

$$\begin{aligned} R = P[Y_2 < Y_1] &= \int_0^\infty P[Y_2 < Y_1 | Y_1 = y_1] f_{Y_1}(y_1)dy_1 \\ &= \int_0^\infty F_{Y_2}(y_1) f_{Y_1}(y_1)dy_1 \\ &= 2\alpha_1 \int_0^\infty \frac{y_1^{\alpha_1+\alpha_2-1}(2+y_1)^{\alpha_1+\alpha_2-1}}{(1+y_1)^{2(\alpha_1+\alpha_2)+1}}dy_1. \end{aligned}$$

Hence, the SSR parameter is given by

$$R = \frac{\alpha_1}{\alpha_1+\alpha_2}.$$

We can further explore the estimation of SSR parameter R using different techniques (Goel and Singh, 2020).

3.12. Stochastic dominance

Stochastic ordering is a vital means of assessing comparable behavior in reliability theory and other disciplines. Suppose Y and Z be two RVs with respective CDFs, survival functions (SFs), and PDFs, as $F_Y(y)$, $F_Z(z)$, $S_Y(y) = 1 - F_Y(y)$, $S_Z(z) = 1 - F_Z(z)$, $f_Y(y)$ and $f_Z(z)$. The RV Y is said to be smaller than Z in the

1. Stochastic order (indicated by $Y \leq_{st} Z$) if $S_Y(y) \leq S_Z(y)$ for all y;

2. Hazard rate order (indicated by $Y \leq_{hr} Z$) if $S_Y(y)/S_Z(y)$ is decreasing in $y \geq 0$;

3. Likelihood ratio order (indicated by $Y \leq_{lr} Z$) if $f_Y(y)/f_Z(y)$ is decreasing in $y \geq 0$;

4. Reversed hazard rate order (indicated by $Y \leq_{rhr} Z$) if $F_Y(y)/F_Z(y)$ is decreasing in $y \geq 0$.

From Shaked and Shanthikumar (2007), the stochastic orders in (a)-(d) are linked and have the following implications:

$$(Y \leq_{rhr} Z) \Leftarrow (Y \leq_{lr} Z) \Rightarrow (Y \leq Z) \Rightarrow (Y \leq Z).$$

The following theorem shows that the MTL distributions are ordered with respect to the strongest likelihood ratio ordering.

Theorem 1 *Suppose $Y \sim MTL(\alpha_1)$ and $Z \sim MTL(\alpha_2)$. If $\alpha_1 < \alpha_2$, then $Y \leq_{lr} Z$.*

Proof: Firstly, the density ratio

$$\frac{f_Y(y,\alpha_1)}{f_Z(y,\alpha_2)} = (2y+y^2)^{\alpha_1-\alpha_2}(1+y)^{2(\alpha_1-\alpha_2)}.$$

Taking log on both sides we get

$$\log\left(\frac{f_Y(y,\alpha_1)}{f_Z(y,\alpha_2)}\right) = (\alpha_1-\alpha_2)\log(2y+y^2) + 2(\alpha_1-\alpha_2)\log(1+y).$$

If $\alpha_1 < \alpha_2$, we have $\frac{d}{dx}\log\left(\frac{f_Y(y,\alpha_1)}{f_Z(y,\alpha_2)}\right) = \frac{2(\alpha_1-\alpha_2)}{(2x+x^2)(1+x)} < 0$, this implies that $\frac{f_Y(y,\alpha_1)}{f_Z(y,\alpha_2)}$ is decreasing in y and therefore $Y \leq_{lr} Z$.

3.13. Order statistics

In many areas of statistical theory and practice, order statistics (OSts) play a key role, thus their study becomes vital for a newly formed distribution. Let $Y_1, Y_2, ..., Y_n$ be a random sample from the MTL distribution, and $Y_{(1)} \leq Y_{(2)} \leq ... \leq Y_{(n)}$ be the corresponding OSts. The PDF of u^{th} OSts is widely known as the following:

$$g_{(u)}(y) = \frac{f(y,\alpha)}{B(u,n-u+1)}\sum_{j=0}^{n-u}(-1)^j\binom{n-u}{j}[F(y,\alpha)]^{u+j-1}. \tag{13}$$

Using Equations (3) and (4) in (13), we have

$$g_{(u)}(y) = \frac{2\alpha}{B(u,n-u+1)}\sum_{j=0}^{n-u}(-1)^j\binom{n-u}{j}\frac{(2y+y^2)^{\alpha(u+j)-1}}{(1+y)^{2\alpha(u+j)+1}}. \tag{14}$$

One interesting point that should be noted from Equation (14) is that the PDF of u^{th} OSts can be written as the finite mixture of the MTL distribution.

Particularly, the PDF of the smallest OSts $Y_{(1)} = \min(Y_1, Y_2, ..., Y_n)$ can be obtained by putting $u = 1$ in Equation (14) as

$$g_{(1)}(y) = 2n\alpha\sum_{j=0}^{n-1}(-1)^j\binom{n-1}{j}\frac{(2y+y^2)^{\alpha(1+j)-1}}{(1+y)^{2\alpha(1+j)+1}}.$$

Moreover, the PDF of the largest OSts $Y_{(n)} = \max(Y_1, Y_2, ..., Y_n)$ can be derived by substituting $u = n$ in Equation (14) as

$$g_{(n)}(y) = 2n\alpha\frac{(2y+y^2)^{n\alpha-1}}{(1+y)^{2n\alpha+1}},$$

that means if $Y_k; k = 1, 2, ..., n$ are independently and identically distributed (IID) RVs from MTL (α) distribution, then maximum (Y_k) also follows MTL distribution with parameter $n\alpha$. This property is very important to analyze the reliability characteristics of parallel systems.

3.14. Survival and hazard rate functions

The SF and HRF of the MTL distribution are respectively given by

$$S(y) = 1 - \left(\frac{2y + y^2}{(1+y)^2}\right)^{\alpha}; y > 0, \alpha > 0,$$

$$r(y) = \frac{f(y)}{S(y)} = \frac{2\alpha(1+y)^{-2\alpha-1}(2y+y^2)^{\alpha-1}}{1 - \left((2y+y^2)(1+y)^{-2}\right)^{\alpha}}; y > 0, \alpha > 0.$$

Figure 2 depicts various plots of SF and HRF of the proposed model. From the HRF plot, it is easily visible that the HRF of the MTL distribution can be decreasing, increasing followed by decreasing, upside-down bathtub shaped, and it approaches towards 0 as y gets larger. Numerically, the limiting behavior of the HRF can be stated as

$$\lim_{y\to\infty} r(y) = 0 \quad \text{and} \quad \lim_{y\to 0} r(y) = \begin{cases} 2; & \alpha = 1 \\ 0; & \alpha > 1 \\ \infty; & \alpha < 1 \end{cases}.$$

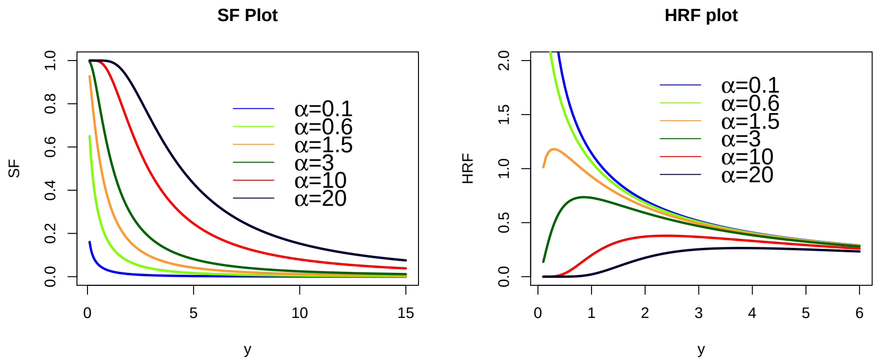

Figure 2 The various shapes of SF and HRF of MTL distribution for different values of α

4. Classical Estimation

In this section, we perform the point and interval estimation of the unknown parameter of MTL distribution through the different important classical methods.

4.1. Method of maximum likelihood estimation

The method of maximum likelihood is one of the most often used classical point estimation techniques. The maximum likelihood (ML) estimate is the point in the parameter space that maximizes the likelihood function. Its logic is both intuitive and adaptable, and as a result, it has become a dominating approach to statistical inference.

Let $y_1, y_2, ..., y_n$ be a random sample of size n from MTL distribution with parameter α, then the likelihood function can be written as

$$L(\underline{y}, \alpha) = 2^n \alpha^n \prod_{i=1}^{n} \frac{y_i^{\alpha-1}(2+y_i)^{\alpha-1}}{(1+y_i)^{2\alpha+1}}. \tag{15}$$

The corresponding log-likelihood (LL) function is

$$\log L = n\log 2 + n\log\alpha + (\alpha-1)\left(\sum\nolimits_{i=1}^{n}\log y_i + \sum\nolimits_{i=1}^{n}\log(2+y_i)\right) - (2\alpha+1)\sum\nolimits_{i=1}^{n}\log(1+y_i). \quad (16)$$

By setting the LL equation $\frac{\partial \log L}{\partial \theta}$ equals to zero, the ML estimator of α can be obtained as

$$\hat{\alpha} = n\Big[\sum\nolimits_{i=1}^{n}\log\Big\{(1+y_i)^2\Big/y_i(2+y_i)\Big\}\Big]^{-1}. \quad (17)$$

It is interesting to observe that the ML estimator of α in Equation (17) has closed-form expression.

Also, the ML estimator $\hat{\alpha}$ of α is consistent and asymptotically normal with $\sqrt{n}\,(\hat{\alpha}_{ML}-\alpha)\to N(0,\zeta^{-1}(\alpha))$, here $\zeta(\alpha)$ is

$$\zeta(\alpha) = E\left(-\tfrac{\partial^2}{\partial\alpha^2}\log f(y;\alpha)\right) = \frac{n}{\alpha^2}.$$

Therefore, the $100\times(1-\gamma)\%$ ACI for the parameter α is

$$\hat{\alpha} \mp \mathrm{Z}_{\gamma/2}\frac{\hat{\alpha}}{\sqrt{n}},$$

where $\mathrm{Z}_{\gamma/2}$ is the upper $\gamma/2$ quantile of the standard normal distribution and $\hat{\alpha}$ is an estimate of α given in Equation (17).

4.2. Method of ordinary and weighted least squares estimation

Here, we present the regression-based estimation methods for estimating the model parameter. These approaches are known as the ordinary least squares (OLS) and the weighted least squares (WLS) estimation, and they were first suggested by Swain et al. (1988). The OLS and WLS estimators depend on the combination of the non-parametric and parametric distribution functions.

Let $Y_1, Y_2, ..., Y_n$ be a random sample from the CDF $F(\cdot)$ in (4), and suppose $Y_{1:n} \le Y_{2:n} \le ... \le Y_{n:n}$ be the corresponding ordered values. Then, the OLS estimator of α, say $\hat{\alpha}_{OLS}$ can be achieved by minimizing

$$V(\alpha) = \sum_{i=1}^{n}\left[F\left(y_{i:n};\alpha\right) - \frac{i}{n+1}\right]^2,$$

with respect to α. Evenly, $\hat{\alpha}_{OLS}$ can be determined by solving

$$\sum_{i=1}^{n}\left[F\left(y_{i:n};\alpha\right) - \frac{i}{n+1}\right]\xi(y_{i:n};\alpha) = 0,$$

where

$$\xi(y_{i:n};\alpha) = \left(\frac{2y_{i:n}+y_{i:n}^2}{\left(1+y_{i:n}\right)^2}\right)^{\alpha}\log\left(\frac{2y_{i:n}+y_{i:n}^2}{\left(1+y_{i:n}\right)^2}\right). \quad (18)$$

The WLS estimator $\hat{\alpha}_{WLS}$ of α can be found by minimizing

$$W(\alpha) = \sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(y_{i:n};\alpha\right) - \frac{i}{n+1}\right]^2.$$

The estimator $\hat{\alpha}_{WLS}$ can also be obtained by simplifying the following equation

$$\sum_{i=1}^{n}\frac{(n+1)^2(n+2)}{i(n-i+1)}\left[F\left(y_{i:n};\alpha\right) - \frac{i}{n+1}\right]\xi(y_{i:n};\alpha) = 0.$$

4.3. Method of Cramer-Von-Mises estimation

The method of minimum distance estimates the unknown parameter by minimizing the distance between the theoretical and empirical cumulative functions. Here, we use the Cramer-von-Mises type minimum distance estimator due to the empirical evidence that the bias of the estimator is smaller than the other minimum distance estimators (Macdonald, 1971). The Cramer-von-Mises (CVM) estimator, say, $\hat{\alpha}_{CVM}$ of the parameter α is obtained by minimizing the following function with respect to the parameter α,

$$C(\alpha) = \frac{1}{12n} + \sum_{i=1}^{n} \left[F\left(y_{i:n}|\alpha\right) - \frac{2i-1}{2n} \right]^2 .$$

The estimator $\hat{\alpha}_{CVM}$ can also be derived by simplifying the following non-linear equation

$$\sum_{i=1}^{n} \left[F\left(y_{i:n};\alpha\right) - \frac{2i-1}{2n} \right] \xi(y_{i:n};\alpha) = 0,$$

where $\xi(y_{i:n};\alpha)$ is given in Equation (18).

4.4. Method of maximum product of spacings estimation

For estimating the unknown parameters of continuous univariate distributions, Cheng and Amin (1979) developed the maximum product spacing (MPS) method as an alternative to the method of maximum likelihood. Ranneby (1984) separately developed this approach as an approximation to the KullbackLeibler measure of information. Cheng and Amin (1983) demonstrated that this approach is equally efficient as the method of maximum likelihood and consistent under more general conditions, which motivate our choice.

Using the same notations mentioned in Section 4.2, define the uniform spacings of a random sample from MTL distribution as

$$D_i(\alpha) = F\left(y_{i:n};\alpha\right) - F\left(y_{i-1:n};\alpha\right), \qquad i = 1, 2, ..., n,$$

where $F\left(y_{0:n};\alpha\right) = 0$ and $F\left(y_{n+1:n}|\alpha\right) = 1$. Obviously, $\sum_{i=1}^{n+1} D_i(\alpha) = 1$. The MPS estimator $\hat{\alpha}_{MPS}$ of the parameter α is obtained by maximizing the geometric mean of the spacings with respect to α , or, equivalently, by maximizing the following function

$$H(\alpha) = \frac{1}{n+1} \sum_{i=1}^{n+1} \log(D_i(\alpha)).$$

The estimator $\hat{\alpha}_{MPS}$ can also be derived by simplifying the following non-linear equation

$$\frac{1}{n+1} \sum_{i=1}^{n+1} \frac{1}{D_i(\alpha)} \left[\xi(y_{i:n};\alpha) - \xi(y_{i-1:n};\alpha) \right] = 0,$$

where $\xi(y_{i:n};\alpha)$ is given in Equation (18).

5. Bayesian Estimation

Bayesian estimation becomes crucial if the experimenter has some prior knowledge regarding the unknown parameter of the model. This information is represented by prior distribution, and after incorporating this prior knowledge with experimental information in hand we draw the posterior inferences about the unknown parameter. Here, we perform the Bayesian estimation by assuming prior as gamma distribution with the following density

$$g(\alpha) = \frac{a^b}{\Gamma b} \alpha^{b-1} e^{-a\alpha}; \alpha > 0, (a, b) > 0, \tag{19}$$

here, a and b are known as hyper-parameters. In the case of $a = b = 0$, the prior distribution (19) switches from an informative prior (IP) to a non-informative prior (NIP).

From the economical aspect, the selection of an appropriate loss function is very crucial in the Bayesian paradigm. There are various loss functions available in the literature, (Robert, 2007). Here, we use the most common symmetric loss function, named a squared error loss function (SELF). It is equally penalized over as well as underestimation due to its symmetric nature. If $\hat{\alpha}$ is an estimate of α, then, the SELF has the form $L(\alpha, \hat{\alpha}) \propto (\hat{\alpha} - \alpha)^2$. Thus, the Bayes estimator of α under SELF is $\hat{\alpha}_S = E\left(\alpha|\underline{y}\right)$, where the expectation is to be taken with respect to the posterior distribution of α.

The symmetric nature of a loss function becomes an issue if the over (under) estimation results in more severe consequences than under (over) estimation (Calabria and Pulcini, 1996). Therefore, in this context, we also use an asymmetric loss function known as a generalized entropy loss function (GELF) (Calabria and Pulcini, 1996) which is defined as

$$L(\alpha, \hat{\alpha}) = \left(\frac{\hat{\alpha}}{\alpha}\right)^{\rho} - \rho \log\left(\frac{\hat{\alpha}}{\alpha}\right) - 1,$$

where $\rho \neq 0$. The constant ρ reflects the various shapes of GELF. It is important to note that, if $\rho < 0$, the negative error leads to more severe consequences than positive error and vice-versa. The Bayes estimator of α under GELF is

$$\hat{\alpha}_G = \left[E\left(\alpha^{-\rho}|\underline{y}\right)\right]^{-1/\rho}. \tag{20}$$

Additionally, for $\rho = -1$, the Bayes estimator in Equation (20) gets equivalent to the Bayes estimator under SELF; for $\rho = 1$, it reduces to the Bayes estimator with entropy loss function (ELF), and when $\rho = -2$, Bayes estimator under GELF coincide with Bayes estimator under precautionary loss function (PLF).

In our case, using likelihood function in (15) and prior distribution in (19), the un-normalized posterior distribution of α given data can be written as

$$\Pi(\alpha|\underline{y}) \propto \alpha^{n+b-1}\exp(-a\alpha)\prod_{i=1}^{n}\frac{y_i^{\alpha-1}(2+y_i)^{\alpha-1}}{(1+y_i)^{2\alpha+1}}. \tag{21}$$

Thus, under SELF and GELF, Bayes estimators of α are respectively given by

$$\hat{\alpha}_S = \int_0^{\infty} \alpha.\,\Pi(\alpha|\underline{y})d\alpha \quad \text{and} \quad \hat{\alpha}_G = \left[\int_0^{\infty} \alpha^{-\rho}.\,\Pi(\alpha|\underline{x})\,d\alpha\right]^{(-1/\rho)}.$$

The explicit expressions for Bayes estimators $\hat{\alpha}_S$ and $\hat{\alpha}_G$ are not possible to obtain. Therefore, we need some numerical approximation methods to compute the above integrals for α. In this context, we have utilized the Markov Chain Monte Carlo (MCMC) technique known as Metropolis-Hastings (M-H) algorithm (Metropolis and Ulam, 1949; Hastings, 1970). The main feature of this algorithm is that it allows us to simulate observations from the posterior distribution of the unknown parameter. After eliminating the sufficient burn-in-sample and testing for convergence of the draws to their target distribution, we make use of these values to obtain posterior sample-based Bayes estimators of the unknown parameter under various loss functions. Also, by ordering these generated values, we can obtain the HPD interval for the parameter α using Chen and Shao (1999) algorithm. We use the following steps of the M-H algorithm to obtain a Bayes estimator of α:

1. Initialize an arbitrary value of α as α_0 and set $i = 1$.

2. Generate α_i^* from the proposal distribution $q(\alpha_{i-1}, \alpha_i^*) = N(\hat{\alpha}_i,\, \sigma^2_{\hat{\alpha}_i})$ and u from Uniform $U(0, 1)$. Here, we use $\hat{\alpha}_i = \alpha_0$, and $\sigma^2_{\hat{\alpha}_i}$ is suitably chosen.

3. Calculate the M-H acceptance probability $P(\alpha_{i-1}, \alpha_i^*) = \min\left(\frac{\Pi(\alpha_i^*|\underline{x})}{\Pi(\alpha_{i-1}|\underline{x})} \times \frac{q(\alpha_i^*, \alpha_{i-1})}{q(\alpha_{i-1}, \alpha_i^*)}, 1\right)$, and if $u \leq P(\alpha_{i-1}, \alpha_i^*)$ then $\alpha_i = \alpha_i^*$, otherwise $\alpha_i = \alpha_{i-1}$.

4. Set $i = i + 1$.

5. Repeat steps 2-4, m times, and obtain α_i; $i = 1, 2, ..., m$.

6. Now, under SELF and GELF, Bayes estimators of α are respectively, obtained as $\hat{\alpha}_S = \frac{1}{m-m_0}\sum\limits_{i=m_0+1}^{m}\alpha_i$ and $\hat{\alpha}_G = \left[\frac{1}{m-m_0}\sum\limits_{i=m_0+1}^{m}\alpha_i^{-\rho}\right]^{(-1/\rho)}$, where m_0 is the burn-in period of the Markov Chain. Also, by setting ρ =1 and -2, we can obtain the Bayes estimators under ELF and PLF, respectively.

To compute the HPD interval for α, let $\alpha_{(m_0+1)} \leq \alpha_{(m_0+2)} \leq ... \leq \alpha_{(m)}$ denote the ordered values of $\alpha_{m_0+1}, \alpha_{m_0+2}, ..., \alpha_m$. Then, by Chen and Shao (1999) algorithm, the $100 \times (1-\gamma)\%$ HPD interval for α is $(\alpha_{(m_0+i^*)}, \alpha_{(m_0+i^*+[(1-\gamma)(m-m_0)])})$, where i^* is chosen so that, $\alpha_{(m_0+i^*+[(1-\gamma)(m-m_0)])} - \alpha_{(m_0+i^*)} = \min\limits_{m_0 \leq i \leq (m-m_0)-[(1-\gamma)(m-m_0)]}(\alpha_{(m_0+i+[(1-\gamma)(m-m_0)])} - \alpha_{(m_0+i)})$.

6. Simulation Study

In this sub-section, we observe the performance of different estimation techniques to estimate the unknown parameter of the proposed model. This assessment consists of the following steps:

1. Using Equation (6), generate 1000 samples of sizes n=15, 30, 50, and 100 from MTL distribution with $\alpha = 0.5, 1.5, 2.5,$ and 5.0.

2. Compute the ML, OLS, WLS, CVM, MPS, and Bayes (under IP and NIP with SELF, ELF, and PLF) estimates for the 1000 samples, say $\hat{\alpha}_\varphi^j; j = 1, 2, ..., 1000; \varphi$ = ML, OLS, WLS, CVM, MPS, and Bayes. Also, compute the 95% ACI and HPD intervals for the above-generated samples. It is worth mentioning that when using Bayesian estimation, we compute estimates for the parameter of MTL distribution under IP as Gamma prior and NIP with SELF, ELF, and PLF.

   Under Gamma IP, we calculate the values of hyper-parameters in such a way that the expectation of the associated prior density of the unknown parameter is equal to the real parametric value. In this estimation paradigm, we produced 200,000 MCMC draws for the parameter of the proposed distribution using the M-H algorithm, and we have omitted the first 10,000 samples as a burn-in period to remove the influence of starting values. Furthermore, to reduce auto-correlation between subsequent draws, we have saved every 50^{th} observation. To assess the convergence of generated chains, the Raftery-Lewis test (RLT) (Raftery and Lewis, 1992) and Heidelberger test (HT) (Heidelberger and Welch, 1981) are employed. Finally, we have computed the posterior quantities of interest using these posterior samples.

3. Compute the mean-squared error (MSE) and average bias (AB) for all point estimators, whereas calculate average upper confidence limit (AUCL), average lower confidence limit (ALCL), and average width (AW) for all interval estimators, where $MSE = \frac{1}{1000}\sum\limits_{j=1}^{1000}\left(\hat{\alpha}_\varphi^j - \alpha\right)^2$, $AB = \frac{1}{1000}\sum\limits_{j=1}^{1000}\left(\hat{\alpha}_\varphi^j - \hat{\alpha}\right)$, $AUCL = \frac{1}{1000}\sum\limits_{j=1}^{1000}UCL^j$, $ALCL = \frac{1}{1000}\sum\limits_{j=1}^{1000}LCL^j$, and $AW = \frac{1}{1000}\sum\limits_{j=1}^{1000}(UCL^j - LCL^j)$, here, UCL^j and LCL^j denotes the upper and lower confidence limit for the j^{th} sample, respectively.

4. The empirical results are shown in Tables 2-5.

The following key conclusions may be drawn from Tables 2-5:

- For all methods, the MSE decrease to zero as n tends to infinity. This shows the consistency of the estimators. Also, the AB decrease to zero as n becomes large.
- In Bayesian analysis, the value of RLT is near to 1 which provide a guarantee of convergence of generated chains. The same conclusion can be interpreted from the p-value of HT since its p-value is greater than 0.05.
- All the estimation procedures perform satisfactorily. However, in overall comparison, Bayes estimator with IP under PLF perform better in comparison to all other methods.
- In classical methods, the MPS estimator works superior to other classical estimators with respect to MSE.
- All classical point estimators except MPS over-estimate the parameter of the proposed model whereas Bayes estimator (under IP and NIP with SELF, ELF, and PLF) under-estimate the unknown parameter of the MTL distribution.
- We have also observed that as we increase the value of the parameter α, the MSE of all estimation procedures tends to increase i.e. the considered estimation methods is more sensitive to the large value of α.
- The AW of the ACI and HPD intervals decreases as we increase the sample size n. Also, HPD intervals with IP provides the shortest width in comparison to ACI and HPD intervals with NIP.

Table 2 Various classical point estimates for different values of α

Method	ML		OLS		MPS		CVM		WLS	
α	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
					n=15					
0.5	0.0358	0.0241	0.0264	0.0321	-0.0113	0.0191	0.0285	0.0301	0.0288	0.0325
1.5	0.1097	0.2108	0.0825	0.2750	-0.0320	0.1670	0.1028	0.2831	0.0755	0.2705
2.5	0.1664	0.5750	0.1115	0.7178	-0.0671	0.4628	0.1691	0.8122	0.1170	0.7404
5	0.3801	2.2921	0.2859	3.1165	-0.0927	1.7982	0.3066	3.1892	0.2956	3.2670
					n=30					
0.5	0.0190	0.0099	0.0147	0.0132	-0.0100	0.0087	0.0187	0.0132	0.0132	0.0128
1.5	0.0484	0.0888	0.0335	0.1147	-0.0376	0.0787	0.0526	0.1274	0.0431	0.1192
2.5	0.0821	0.2401	0.0622	0.3171	-0.0619	0.2126	0.0856	0.3183	0.0614	0.3207
5	0.2013	1.0339	0.1411	1.3243	-0.0876	0.8962	0.1748	1.3247	0.1007	1.2620
					n=50					
0.5	0.0088	0.0054	0.0057	0.0072	-0.0109	0.0050	0.0103	0.0073	0.0071	0.0072
1.5	0.0242	0.0489	0.0132	0.0628	-0.0349	0.0460	0.0334	0.0712	0.0264	0.0678
2.5	0.0574	0.1399	0.0424	0.1847	-0.0415	0.1283	0.0455	0.1822	0.0458	0.1858
5	0.1102	0.5639	0.0786	0.7273	-0.0873	0.5182	0.0798	0.7162	0.0733	0.7052
					n=100					
0.5	0.0057	0.0027	0.0037	0.0034	-0.0059	0.0026	0.0044	0.0035	0.0028	0.0033
1.5	0.0192	0.0247	0.0141	0.0320	-0.0157	0.0235	0.0140	0.0311	0.0136	0.0312
2.5	0.0242	0.0673	0.0128	0.0857	-0.0338	0.0650	0.0273	0.0833	0.0239	0.0854
5	0.0562	0.2540	0.0450	0.3435	-0.0602	0.2429	0.0470	0.3393	0.0503	0.3473

7. Application of MTL Distribution

In this part, we use three real datasets to demonstrate the relevance and superiority of the MTL distribution. These three datasets are from two distinct areas, with the first representing failure time

Table 3 Bayes estimates under NIP for different values of α

Method	Under SELF		Under ELF		Under PLF		Convergence test	
α	Bias	MSE	Bias	MSE	Bias	MSE	RLT	HT
				n=15				
0.5	-0.0312	0.0046	-0.0354	0.0046	-0.0292	0.0046	1.05	0.797
1.5	-0.0361	0.0016	-0.0719	0.0055	-0.0182	0.0006	0.991	0.218
2.5	-0.0235	0.0007	-0.0346	0.0013	-0.0180	0.0005	1.01	0.433
5	-0.1651	0.0300	-0.2652	0.0732	-0.1168	0.0164	0.97	0.667
				n=30				
0.5	-0.0236	0.0027	-0.0276	0.0027	-0.0217	0.0027	0.991	0.595
1.5	-0.0292	0.0011	-0.0564	0.0034	-0.0155	0.0004	1.01	0.344
2.5	-0.0230	0.0007	-0.0341	0.0013	-0.0175	0.0004	0.991	0.116
5	-0.1487	0.0244	-0.2479	0.0639	-0.1008	0.0124	1.01	0.825
				n=50				
0.5	-0.0212	0.0021	-0.0246	0.0021	-0.0195	0.0021	0.951	0.473
1.5	-0.0273	0.0009	-0.0547	0.0032	-0.0136	0.0004	0.951	0.253
2.5	-0.0212	0.0006	-0.0323	0.0012	-0.0157	0.0004	0.991	0.629
5	-0.1118	0.0151	-0.2110	0.0474	-0.0640	0.0067	0.97	0.839
				n=100				
0.5	-0.0168	0.0012	-0.0199	0.0012	-0.0153	0.0012	1.02	0.402
1.5	-0.0272	0.0010	-0.0545	0.0032	-0.0135	0.0004	0.991	0.48
2.5	-0.0191	0.0005	-0.0280	0.0009	-0.0147	0.0003	1.03	0.706
5	-0.0878	0.0100	-0.1852	0.0368	-0.0406	0.0040	0.97	0.564

Table 4 Bayes estimates under IP for different values of α

Method	Under SELF		Under ELF		Under PLF		Convergence test	
α	bias	MSE	bias	MSE	bias	MSE	RLT	HT
				n=15				
0.5	-0.0261	0.0037	-0.0307	0.0038	-0.0238	0.0037	0.97	0.144
1.5	-0.0163	0.0004	-0.0281	0.0009	-0.0105	0.0002	1	0.799
2.5	-0.0234	0.0007	-0.0343	0.0013	-0.0180	0.0005	0.951	0.719
5	-0.2110	0.0515	-0.4587	0.2172	-0.0883	0.0148	1.03	0.932
				n=30				
0.5	-0.0234	0.0030	-0.0274	0.0031	-0.0214	0.0030	0.991	0.528
1.5	-0.0155	0.0003	-0.0272	0.0008	-0.0098	0.0002	1.08	0.284
2.5	-0.0221	0.0006	-0.0330	0.0012	-0.0167	0.0004	0.951	0.163
5	-0.2095	0.0508	-0.4567	0.2148	-0.0872	0.0148	1.05	0.257
				n=50				
0.5	-0.0210	0.0020	-0.0247	0.0020	-0.0191	0.0020	0.97	0.898
1.5	-0.0157	0.0003	-0.0274	0.0009	-0.0100	0.0002	0.991	0.444
2.5	-0.0231	0.0010	-0.0558	0.0036	-0.0067	0.0006	0.97	0.611
5	-0.2036	0.0475	-0.4518	0.2096	-0.0809	0.0128	1.05	0.692
				n=100				
0.5	-0.0141	0.0011	-0.0168	0.0012	-0.0128	0.0011	0.991	0.976
1.5	-0.0148	0.0003	-0.0264	0.0008	-0.0090	0.0002	0.991	0.712
2.5	-0.0222	0.0009	-0.0553	0.0035	-0.0057	0.0004	0.97	0.823
5	-0.1926	0.0434	-0.4396	0.1993	-0.0706	0.0114	1.05	0.899

Table 5 Classical and Bayesian confidence intervals for different values of α

Method	ACI			NIP			IP		
α	AUCL	ALCL	AW	AUCL	ALCL	AW	AUCL	ALCL	AW
				n=15					
0.5	0.2114	0.6447	0.4333	0.3816	0.6300	0.2484	0.3736	0.6387	0.2650
1.5	0.6087	1.8560	1.2473	1.1223	1.8545	0.7321	1.2727	1.7350	0.4622
2.5	1.3051	3.9795	2.6744	2.1652	2.7496	0.5844	2.1800	2.7640	0.5839
5	2.2180	6.7630	4.5450	3.4000	6.0702	2.6701	3.0742	6.7182	3.6440
				n=30					
0.5	0.2822	0.5968	0.3145	0.3545	0.6161	0.2615	0.3925	0.6209	0.2283
1.5	0.8870	1.8757	0.9886	1.1502	1.7929	0.6427	1.2449	1.7347	0.4898
2.5	1.8825	3.9807	2.0981	2.0847	2.7473	0.6626	2.2115	2.8971	0.6855
5	2.7937	5.9074	3.1136	3.5714	6.3575	2.7860	3.1077	6.6407	3.5330
				n=50					
0.5	0.3235	0.5717	0.2481	0.3753	0.6097	0.2343	0.3434	0.6133	0.2699
1.5	1.1645	2.0576	0.8931	1.1517	1.8058	0.6540	1.2164	1.6987	0.4822
2.5	1.8487	3.2667	1.4179	2.1866	2.7695	0.5829	2.0664	2.9952	0.9288
5	3.1754	5.6108	2.4354	3.6786	6.3249	2.6463	3.0398	6.7480	3.7081
				n=100					
0.5	0.3912	0.5820	0.1907	0.3390	0.6607	0.3216	0.3625	0.6329	0.2704
1.5	1.2729	1.8936	0.6206	1.1569	1.7815	0.6245	1.1796	1.6760	0.4964
2.5	2.1689	3.2264	1.0575	2.2066	2.7626	0.5560	2.0241	2.9709	0.9468
5	3.5239	5.2421	1.7181	3.7037	5.9785	2.2747	3.1623	6.7689	3.6066

and the last two containing daily COVID-19 cases in India and France, respectively. The fitting capability of the proposed model has been compared to that of various well-known conventional and recently developed models. Table 6 has a list of the competing models.

Table 6 The competitive models

Model	Parameter(s)	Abbreviation	Author(s)
Burr-Hatke Exponential	λ	BHE	Yadav et al. (2021)
Gamma	α, β	G	-
Generalized Lindley	λ, α	GL	Nadarajah et al. (2011)
Inverse-Gamma	α, β	IG	-
Inverse-Lindley	θ	IL	Sharma et al. (2015)
Inverted NadarajahHaghighi	λ, α	INH	Tahir et al. (2018)
Inverse-Weibull	α, β	IW	-
Inverted Topp-Leone	θ	ITL	Hassan et al. (2020)
Inverse-xgamma	θ	IXG	Yadav et al. (2021)
Lindley	θ	L	Lindley (1958)
Power Lindley	α, β	PL	Mazucheli et al. (2013)
Weibull	α, β	W	-
Weighted-Lindley	c, θ	WL	Ghitany et al. (2011)

For comparison purposes, the estimation of the fitted models has been done through ML estimation. The model comparison has been done on the basis of LL, Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Bayesian information criterion (BIC), and Kolmogorov-Smirnov (KS) statistics with the associated p-value. Here, the lower value of these criteria and the higher p-value indicates the best fit. All required computations have been done using open-source software R.

Dataset I: This dataset consists of the number of successive failures for the air conditioning system of each member in a fleet of 13 Boeing 720 jet airplanes (Cordeiro and Lemonte, 2011). This dataset has been divided by 100 for fitting easiness. For this data, the fit of the proposed model is compared with G, GL, IG, IL, INH, IW, ITL, IXG, L, PL, W, and WL models. Table 7 shows the ML estimates of the parameters (standard error abbreviated by SE written in parentheses) as well as the values of the LL, AIC, BIC, CAIC, and KS statistics with its p-value. Table 7 shows that the proposed model has the lowest -LL, AIC, BIC, CAIC, and KS statistics, as well as the greatest p-value; hence, the MTL distribution is superior to several other competitive models for this dataset. Figure 3 (first panel) portrays the empirical vs fitted CDF for the proposed model under dataset I. This figure also announces that the MTL distribution closely follows the pattern of the real data.

Table 7 The goodness-of-fit statistics for various fitted models under dataset I

Model	ML Estimate (SE)	-LL	AIC	BIC	CAIC	KS	P-value
MTL	1.2656(0.0923)	170.255	342.5092	345.7457	342.5307	0.0381	0.9481
G	0.9049(0.0814), 0.9829(0.1162)	171.84	347.6807	354.1536	347.7456	0.0702	0.3109
GL	1.3483(0.1115), 0.8094(0.0789)	175.202	354.4036	360.8765	354.4685	0.0865	0.1197
IG	0.6874(0.0603), 0.1374(0.0170)	206.597	417.1936	423.6665	417.2585	0.1310	0.0031
IL	0.3483(0.0182)	267.59	537.1803	540.4168	537.2018	0.2829	<0.0001
INH	0.4811(0.0414), 0.9296(0.1900)	182.099	368.1986	374.6715	368.2635	0.0780	0.2023
IW	0.2549(0.0264), 0.7466(0.0373)	195.647	395.2935	401.7664	395.3584	0.0971	0.0577
ITL	4.1429(0.3019)	187.407	376.8143	380.0508	376.8358	0.1501	0.0004
IXG	0.4350(0.0211)	292.652	587.3039	590.5403	587.3254	0.3191	<0.0001
L	1.5173(0.0864)	177.638	357.2759	360.5123	357.2974	0.1106	0.0199
PL	1.5963(0.0934), 0.8428(0.0462)	172.264	348.5288	355.0017	348.5937	0.0601	0.5046
W	0.9108(0.0503), 0.8776(0.0743)	170.979	345.9583	352.4312	346.0232	0.0572	0.5699
WL	1.2960(0.1212), 0.8090(0.0784)	175.109	354.2181	360.691	354.283	0.0849	0.1326

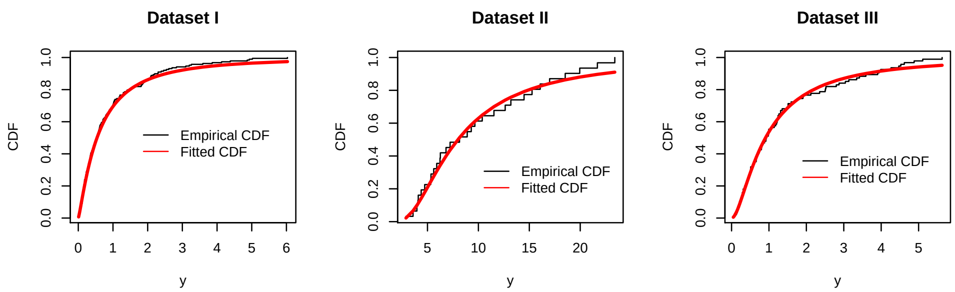

Figure 3 The empirical vs fitted CDFs under datasets I, II and III

Dataset II: In the second application, we consider the daily new cases of COVID-19 in India. The data is available at https://www.worldometers.info/coronavirus/country/india/ and contains the daily new cases between 16 March 2021 to 16 April 2021. This dataset has been divided by 10000 to make fitting easier. The fitting of the MTL distribution to this COVID data is compared to the BHE, G, GL, IL, INH, ITL, IXG, L, PL, W, and WL models. Table 8 displays the ML estimates of the parameters (SE between parentheses) as well as the values of the LL, AIC, BIC, CAIC, and KS statistics with their p-value. Table 8 indicates that the proposed model has the lowest -LL, AIC, BIC, CAIC, and KS statistics, as well as the highest p-value; as a result, the MTL distribution outperforms other competing models for this dataset. The empirical versus fitted CDF for the proposed model under dataset II is depicted in Figure 3 (second panel). This figure also shows that the MTL distribution closely resembles the real data pattern.

Dataset III: Here, we consider the daily new cases of COVID-19 in France between 5 August 2020 to 5 November 2020. The data is available at https://www.worldometers.info/coronavirus /country/france/. This dataset has been divided by 10000 for convenience of the fitting. Under this COVID data, the fitting of MTL distribution is compared with BHE, GL, IG, IL, INH, IW, IXG, L, PL, W, and WL models. The ML estimates of the parameters (SE between parentheses) as well as the values of the LL, AIC, BIC, CAIC, and KS statistics with their p-value are shown in Table 9. The values of -LL, AIC, BIC, CAIC and KS statistics are the lowest for the suggested model

Table 8 The goodness-of-fit statistics for various fitted models under dataset II

Model	ML Estimate (SE)	-LL	AIC	BIC	CAIC	KS	P-value
MTL	55.676(9.9997)	93.7815	189.5629	190.9969	189.7009	0.0893	0.9467
BHE	0.0552(0.1108)	103.759	209.5174	210.9514	209.6554	0.2831	0.0108
G	3.1179(0.7535), 0.3153(0.7535)	93.8	191.6	194.4679	192.0285	0.129	0.6337
GL	0.2648(0.0437), 2.1436(0.6717)	93.9187	191.8375	194.7054	192.266	0.1309	0.6162
IL	7.8937(1.279)	101.71	205.4199	206.8539	205.5578	0.2593	0.025
INH	14.7742(6.842), 0.3258(0.1251)	96.9953	197.9905	200.8585	198.4191	0.2028	0.1353
ITL	0.6169(0.1108)	117.874	237.7481	239.182	237.886	0.4094	<0.0001
IXG	8.48(1.3980)	102.735	207.4706	208.9046	207.6086	0.2726	0.0158
L	0.1863(0.0238)	96.7713	195.5426	196.9766	195.6805	0.1662	0.3216
PL	0.0879(0.0353),1.3171(0.1560)	94.4391	192.8782	195.7462	193.3068	0.1308	0.6167
W	1.8423(0.2544),11.2075(1.1701)	94.6521	193.3042	196.1722	193.7328	0.1294	0.6305
WL	0.3309(0.0818),2.3935(0.7345)	93.9305	191.8609	194.7289	192.2895	0.1322	0.6033

(Table 9), and the p-value is the highest, therefore the MTL distribution is superior to numerous other competing models for this dataset. The empirical vs fitted CDF for the proposed model under this data is depicted in Figure 3 (third panel) and it can concludes that MTL distribution closely reflects real dataset pattern.

Table 9 The goodness-of-fit statistics for various fitted models under dataset III

Model	ML Estimate (SE)	-LL	AIC	BIC	CAIC	KS	P-value
MTL	2.1588(0.2226)	127.002	256.0039	258.5472	256.0474	0.0484	0.98
BHE	0.4081(0.0469)	128.559	259.1182	261.6615	259.1617	0.0836	0.527
GL	1.0573(0.1155), 1.0289(0.1499)	128.106	260.2122	265.2988	260.3441	0.0977	0.3301
IG	1.0117(0.1300), 0.5113(0.0840)	136.972	277.9438	283.0304	278.0757	0.1145	0.17
IL	0.7878(0.0604)	142.876	287.7516	290.2949	287.795	0.1534	0.0238
INH	0.6806(0.1033), 1.0063(0.3021)	134.036	272.0716	277.1582	272.2035	0.1116	0.1915
IW	0.5266(0.0620), 0.9300(0.0674)	136.452	276.9039	281.9905	277.0358	0.1045	0.2557
IXG	0.9745(0.0704)	147.127	296.253	298.7963	296.2965	0.1734	0.0069
L	1.0415(0.0814)	128.125	258.2508	260.7941	258.2943	0.0962	0.3489
PL	1.0655(0.0966), 0.9643(0.0729)	128.007	260.0141	265.1007	260.146	0.0888	0.4483
W	1.0862(0.0857), 1.4793(0.0857)	127.117	258.2336	263.3202	258.3654	0.0861	0.488
WL	1.0648(0.1388), 1.0312(0.1497)	128.103	260.2053	265.2919	260.3372	0.098	0.3266

7.1. Various classical and Bayes estimates for datasets I, II and III

In this sub-section, we compute the estimate of the unknown parameter of MTL distribution by other considered classical and Bayesian methods. We also obtain the 95% ACI and HPD intervals for α under datasets I, II, and III.

Table 10 consists of OLS, WLS, MPS, and CVM estimate with their SEs and 95% ACI for α. To compare different methods, the KS statistics with associated p-values for all methods are also provided in Table 10. In the Bayesian framework, as we have no prior information regarding the considered datasets, therefore, we obtain the Bayes estimates under NIP. Using the same approach as we have done in the simulation part, we compute the Bayes estimates and posterior standard error (PSE) for α under NIP with SELF, ELF, and PLF. We also find the 95% HPD interval with NIP. These point and interval estimates can be viewed in Table 11. This table also provides the KS statistics and its p-value to choose the best method used for estimation. The value of RLT and HT in Table 11 confirms the convergence of the generated chains.

From Table 10 and 11, we can easily observe that all estimation methods perform quite satisfactorily as the KS statistics and its p-value are nearly equal.

Table 10 Classical estimates for datasets I, II, and III

Datasets	Methods	Estimate	SE	KS	P-value	ACI
Dataset I	OLS	1.2197	0.2284	0.0371	0.9995	[0.7720, 1.6673]
	WLS	1.2197	0.0003	0.0371	0.9995	
	MPS	1.2212	0.2284	0.0371	0.9995	
	CVM	1.686	1.2197	0.0678	0.9491	
Dataset II	OLS	53.82	26.32	0.0967	0.9991	[2.2328, 105.4072]
	WLS	53.821	0.286	0.0967	0.9991	
	MPS	54.15	21.22	0.0967	0.9991	
	CVM	52.67	21.22	0.0967	0.9991	
Dataset III	OLS	2.16	0.5886	0.0531	0.9994	[1.0063, 3.3136]
	WLS	2.16001	0.001925	0.0531	0.9994	
	MPS	2.1645	0.5881	0.0531	0.9994	
	CVM	2.109	2.126	0.0705	0.9850	

Table 11 Bayes estimates with NIP under SELF, ELF, and PLF for datasets I, II, III

	Bayes	Estimate	PSE	KS	P-value	HPD Interval	RLT	HT
Dataset I	SELF	1.2555	0.0667	0.0372	0.9995	[1.1193, 1.3873]	0.99	0.429
	ELF	1.2520		0.0372	0.9995			
	PLF	1.2573		0.0372	0.9995			
Dataset II	SELF	54.4163	5.8149	0.0967	0.9991	[45.0065, 63.9250]	1.04	0.96
	ELF	53.7987		0.0967	0.9991			
	PLF	54.7260		0.0967	0.9991			
Dataset III	SELF	1.9878	0.1306	0.0531	0.9994	[1.7359, 2.2429]	1.06	0.146
	ELF	1.9792		0.0531	0.9994			
	PLF	1.9921		0.0531	0.9994			

8. Conclusions

Here, we have proposed a new continuous model called Modified Topp-Leone distribution. The density and hazard rate functions of the proposed model include a variety of shapes that permit it to capture a wide range of real data. We have derived its various important distributional properties. An impressive feature of the order statistics of the MTL distribution is that the distribution of the maximum order statistics is also the MTL distribution which enables the proposed model to analyze the reliability characteristics of parallel systems.

The unknown parameter of the proposed model is estimated under the classical and Bayesian framework. In classical point estimation, we have used the method of maximum likelihood, ordinary and weighted least squares, Cramer-Von-Mises, and maximum product spacings estimation. The maximum likelihood estimator of the model's parameter is in a nice closed form which is difficult to find in many well-known models. The asymptotic distribution of the maximum likelihood estimator is also derived and it is used to develop the asymptotic confidence interval. In Bayesian estimation, we have used Gamma (as informative prior) and non-informative prior under squared error, entropy, and precautionary loss functions to obtain the Bayes estimator of the unknown parameter. The highest posterior density interval of the parameter is also obtained. An extensive simulation study is presented to the assessment of the different estimation procedures. The simulation study concludes that all estimation methods perform satisfactorily, but Bayesian estimation outperforms the classical methods. Finally, three real datasets are used to showcase the practicality of the MTL distribution. Hence, we can conclude that the suggested model may be used as an alternative model to some well-known existing models to analyze data generated from various domains.

A future plan of action might be an examination of the censored data using the proposed model. We may also investigate the load share model where the components' failure times follow the MTL distribution. The stress-strength parameter may be examined using various types of censored data. In addition, the suggested model can be discretized, or a bivariate extension of the MTL distribution can be developed.

Acknowledgements

The authors would like to thank the editor and reviewers for their thoughtful comments and efforts towards improving our manuscript. The last author is also thankful to the Department of Science and Technology, India for providing financial aid for the research work under the Inspire Fellowship Program vide letter number DST/INSPIRE Fellowship/2017/IF170038.

References

Al-Zahrani B, Alshomrani A. Inference on stress-strength reliability from Topp-Leone distributions. J King Abdulaziz Univ. 2012; 24(1):73-88.

Barco KV, Mazucheli J, Janeiro V. The inverse power Lindley distribution. Commun Stat Simulat. 2017; 46(8):6308-6323.

Bayoud H. Estimating the shape parameter of the Topp-Leone distribution based on Type I censored samples. Appl. Math. 2015; 2(42):219-230.

Bonferroni C. Elmenti di Statistica Generale [Elements of general statistics]. Firenze: Libreria Seber. 1930.

Calabria R, Pulcini G. Point estimation under asymmetric loss functions for left-truncated exponential samples. Commun Stat Theory. 1996; 25(3): 585-600.

Chaubey YP, Zhang R. An extension of Chens family of survival distributions with bathtub shape or increasing hazard rate function. Commun Stat Theory. 2015; 44(19): 4049-64.

Chen MH, Shao QM. Monte Carlo estimation of Bayesian credible and HPD intervals. J Comput Graph Stat. 1999; 8(1):69-92.

Cheng RCH, Amin, NAK. Maximum product-of-spacings estimation with applications to the log-normal distribution. University of Wales IST, Math Report, 1979; 79-1.

Cheng RC, Amin NA. Estimating parameters in continuous univariate distributions with a shifted origin. J Roy Stat Soc B Meth. 1983; 45(3): 394-403.

Chesneau C, Sharma VK, Bakouch HS. Extended ToppLeone family of distributions as an alternative to beta and Kumaraswamy type distributions: Application to glycosaminoglycans concentration level in urine. Int J Biomath. 2021; 14(02):2050088.

Choudhary N, Tyagi A, Singh B. A Flexible Bathtub-Shaped Failure Time Model: Properties and Associated Inference. Statistica. 2021; 81(1): 65-92.

Cordeiro GM, Lemonte AJ. The β-BirnbaumSaunders distribution: An improved distribution for fatigue life modelling. Comput Stat Data An. 2011; 55(3): 1445-1461.

Ekhosuehi N, Opone F. A three parameter generalized Lindley distribution: properties and application. Statistica. 2018 Dec 21; 78(3): 233-249.

El-Morshedy M, Alshammari FS, Tyagi A, Elbatal I, Hamed YS, and Eliwa MS. Bayesian and frequentist inferences on a type I half-logistic odd Weibull generator with applications in engineering. Entropy. 2021; 23(4): 446.

Genc AI. Estimation of P (X> Y) with ToppLeone distribution. J Stat Comput Sim. 2013; 83(2): 326-439.

Mazucheli J, Ghitany ME, Louzada F. Power Lindley distribution: Different methods of estimations and their applications to survival times data. J Appl Statis Sci. 2013; 21(2): 135-144.

Ghitany ME, Alqallaf F, Al-Mutairi DK, Husain HA. A two-parameter weighted Lindley distribution and its applications to survival data. Math Comput Simulat. 2011; 81(6): 1190-1201.

Goel R, Singh B. Estimation of P (Y< X) for modified Weibull distribution under progressive Type-II censoring. Life Cycle Reliab Saf Eng. 2020; 9(3): 227-240.

Gupta RD, Kundu D. Theory & methods: Generalized exponential distributions. Aust NZ J Stat. 1999; 41(2):173-188.

Bidram H, Behboodian J, Towhidi M. A new generalized exponential geometric distribution. Commun Stat Theory. 2013; 42(3): 528-542.

Hassan AS, Elgarhy M, Ragab R. Statistical properties and estimation of inverted Topp-Leone distribution. J Stat Appl Probab. 2020; 9: 319-331.

Hastings WK. Monte Carlo sampling methods using Markov chains and their applications. BIometrika. 1970; 57(1): 97-109.

Heidelberger P, Welch PD. A spectral method for confidence interval generation and run length control in simulations. Commun ACM. 1981; 24(4): 233-245.

Ikechukwu AF, Eghwerido JT, Emmanuel RF. The Type II Topp-Leone Generalized Power Ishita Distribution with Properties and Applications. Thail Stat. 2021; 19(3): 472-483.

Keller AZ and Kamath ARR. Alternate reliability models for mechanical systems. Proceeding of the 3rd International Conference on Reliability and Maintainability. 1982, p. 411-415.

Kenney JF and Keeping ES. Mathematics of statistics. part 1, 3rd ed. Princeton, New Jersey; 1962.

Lehmann EL, Shaffer JP. Inverted distributions. In Selected Works of EL Lehmann 2012 (pp. 833-836). Springer, Boston, MA.

Lin CT, Duran BS, Lewis TO. Inverted gamma as a life distribution. Microelectron Reliab. 1989; 29(4): 619-626.

Lindley DV. Fiducial distributions and Bayes' theorem. J Roy Stat Soc B Met. 1958; 20(2): 102-107.

Lorenz MO. Methods of measuring the concentration of wealth. Publ Am Stat Assoc. 1905; 9(70): 209-219.

Macdonald PD. Comments and queries comment on an estimation procedure for mixtures of distributions by choi and bulgren. J Roy Stat Soc B Met. 1971; 33(2):326-329.

Metropolis N, Ulam, S. The Monte Carlo method. J Am Stat Assoc. 1949; 44: 335-341.

Moors JJ. A quantile alternative for kurtosis. J Roy Stat Soc D-Sta. 1988; 37(1): 25-32.

Nadarajah S, Haghighi F. An extension of the exponential distribution. Statistics. 2011; 45(6): 543-558.

Nadarajah S, Kotz S. Moments of some J-shaped distributions. J Appl Stat. 2003; 30(3): 311-317.

Nadarajah S, Kotz S. The beta exponential distribution. Reliab Eng Syst Saf. 2006; 91(6): 689-697.

Nadarajah S, Bakouch HS, Tahmasbi R. A generalized Lindley distribution. Sankhya B. 2011; 73(2): 331-359.

Raftery, A.E. and Lewis, S.M. One long run with diagnostics: Implementation strategies for Markov chain Monte Carlo. Stat Sci. 1992; 7: 493-497.

Ranneby B. The maximum spacing method. An estimation method related to the maximum likelihood method. Scand J Stat. 1984; 11(2): 93-112.

Rnyi A. On measures of entropy and information. InProceedings of the Fourth Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics 1961 (pp. 547-561). University of California Press.

Reyad HM, Othman SA. The Topp-Leone Burr-XII distribution: properties and applications. Brit J Math Comput Sci. 2017; 21(5): 1-5.

Robert C. The Bayesian choice: from decision-theoretic foundations to computational implementation. 2007; Springer Science & Business Media.

Shaked, M, Shanthikumar, JG. Stochastic orders. 2007; New York: Wiley.

Sharma VK. ToppLeone normal distribution with application to increasing failure rate data. J Stat Comput. Sim. 2018; 88(9):1782-1803.

Sharma VK, Singh SK, Singh U, Agiwal V. The inverse Lindley distribution: a stress-strength reliability model with application to head and neck cancer data. J Ind Prod Eng. 2015; 32(3): 162-173.

Sheikh AK, Ahmad M, Ali Z. Some remarks on the hazard functions of the inverted distributions. Reliability engineering. 1987 Jan 1; 19(4):255-61.

Swain JJ, Venkatraman S and Wilson JR. Least-squares estimation of distribution functions in Johnson's translation system.J Stat Comput Sim. 1988; 29(4): 271-297.

Tahir MH, Cordeiro GM, Ali S, Dey S, Manzoor A. The inverted NadarajahHaghighi distribution: estimation methods and applications. J Stat Comput Sim. 2018; 88(14): 2775-2798.

Topp CW, Leone FC. A family of J-shaped frequency functions. J Am Stat Assoc. 1955; 50(269): 209-219.

Tyagi, S., Kumar, S., Pandey, A., Saha, M., & Bagariya, H. Power xgamma distribution: Properties and its applications to cancer data. Int J Stat Reliab Eng. 2022; 9(1): 51-60. https://ijsreg.com/article/view/power-power-power-power.

Voda VG. On the inverse Rayleigh distributed random variable. Rep Stat App Res. 1972; 19(4): 13-21.

Yadav AS, Altun E, Yousof HM. BurrHatke Exponential Distribution: A Decreasing Failure Rate Model, Statistical Inference and Applications. Ann Data Sci. 2021; 8(2): 241-260.

Yadav AS, Maiti SS, Saha M. The inverse xgamma distribution: statistical properties and different methods of estimation. Ann Data Sci. 2021; 8(2): 275-293.

Zenga M. Inequality curve and inequality index based on the ratios between lower and upper arithmetic means. Stat Appl. 2007; 5(1): 3-27.