Thailand Statistician
January 2025; 23(1): 97-114
http://statassoc.or.th
Contributed paper

An Alternative Version of Half-Logistic Distribution: Properties, Estimation and Application

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Received: 15 March 2022
Revised: 23 July 2022
Accepted: 4 September 2022

Abstract

In this paper, we derived a new version of half-logistic distribution by using standard half-logistic distribution and a one-parameter family introduced by Zhao et al. (2020). The new model's statistical properties were determined mathematically. The beauty and the novelty of the new model are that it was proved that it has three more shapes of hazard rate function than the existing one in the literature, which has only an increasing shape. The new model parameter estimation efficiency was checked by simulated data sets and different classical estimation methods. Also, we showed numerically that the behavior of our proposed model parameter estimation is better than the existing one in the literature. A real data set was analyzed, and it was discovered that the proposed model outperforms the half-logistic distribution and other competing models for fitting this data. Finally, we showed that the exponentiated version of our proposed model is better than the exponentiated half-logistic distribution.

Keywords: Moments, entropy, order statistics, AndersonDarling estimation, real data.

1. Introduction

Whether or not a probability distribution is flexible can affect whether or not it is used to model real-world phenomena. A probability distribution's flexibility may make it easier to work with randomly generated data sets from various random samples. Rather than transforming the existing data set, it is preferable to use probability distributions that fit the available data set. As a result, many efforts have recently been made to ensure that the current standard theoretical distributions are changed and developed to achieve higher flexibility in modeling real-world data sets.

To expand the current standard distribution, different methods could be put into use. For example, the flexibility of a distribution can be increased using generalization, which involves using the accessible generalized family of distributions. When a distribution is generalized, extra shape parameter(s) from the family of distributions used would have been added. The generalization of classical distributions is an ancient practice, and it is important as many other practical problems in statistics. These generalizations began with the introduction of an added location, scale, or shape parameters to the original model. This branch of statistics has received considerable attention, and many general distribution classes have been derived in recent years. Recently, many authors have been interested in introducing new lifetime distributions for fitting real lifetime data obtained by a generalization of well-known distributions in the literature. Among them, e.g., Teamah et al. (2020a); Aljohani et al.

(2021); Wang et al. (2021); Afify et al. (2020); Almongy et al. (2021); Tung et al. (2021); Teamah et al. (2020b,c, 2021).

Zhao et al. (2020) presented a heavy tailed family for generating a continuous distributions, for more details about heavy tailed distributions see Foss et al. (2011). As you can see, the cumulative distribution function (CDF) and probability density function (PDF) of this family are defined as follows, respectively.

$$G(x) = 1 - \left[\frac{1-F(x)}{1-(1-a)F(x)}\right]^a, \quad a>0, x\in\mathbb{R}, \tag{1}$$

$$g(x) = \frac{a^2\, f(x)\, (1-F(x))^{a-1}}{\left[1-(1-a)F(x)\right]^{a+1}}. \tag{2}$$

The half-logistic (HL) distribution was studied by Balakrishnan (1985), the CDF and the PDF of the HL distribution, respectively, are given by

$$F(x) = \frac{1-e^{-ax}}{e^{-ax}+1}, \quad a>0, x>0, \tag{3}$$

$$f(x) = \frac{2ae^{ax}}{\left(e^{ax}+1\right)^2}, \tag{4}$$

when $a = 1$, we have standard half-logistic (SHL) distribution.

Many extension of HL model presented in recent years such as Kumaraswamy half-logistic distribution by Usman et al. (2017), type I generalized half logistic distribution by Olapade (2014), power half-logistic distribution by Krishnarani (2016), generalized half logistic distribution by Muhammad and Liu (2019), exponential half-logistic model by Rao et al. (2013), transmuted half-logistic distribution by Samuel and Kehinde (2019), a new two parameter lifetime distribution by Hashempour (2021), a new weighted half-logistic distribution by Hashempour and Alizade (2021), two-parameter lifetime distribution by Alizadeh et al. (2019) and half-logistic inverse Rayleigh by Almarashi et al. (2020).

Our proposed in this paper is to introduce a relatively new model of one parameter half-logistic distribution, which we hope to be used than the half-logistic distribution that exists in the literature. Also, we propose to show that the newly presented model is the best when generalized it is compared with the generalized half-logistic in the literature with the same way of generalization.

The remaining parts of the paper are arranged as the following. In Section 2, we derived the proposed new version along with studying possible shapes of the hazard function. In Section 3, important statistical properties of the proposed model were deduced. Estimating methods for the proposed model were described in Section 4, while the behavior of model parameters estimated by these methods was introduced in Section 5. In Section 6, the Real data set was analyzed, showing the superiority of the proposed model over other compared models. In 7, we showed that the generalization of our proposed model has more flexibility than the generalization of half-logistic (4) with the same method of generalization. The conclusions are presented in Section 7.

2. Formulation of the New Version of Half-Logistic Distribution

In this section, we introduced a novel half-logistic distribution known as the flexible half-logistic (FHL) distribution, which is obtained by substituting the CDF in Equation (1) to get the CDF of the FHL model as follows

$$F(x;a) = 1 - 2^a\left(a\left(e^x-1\right)+2\right)^{-a}, \quad x>0, \tag{5}$$

and its PDF is defined as follows

$$f(x;a) = 2^a a^2 e^x \left(a\left(e^x-1\right)+2\right)^{-a-1}, \tag{6}$$

where $a > 0$ is a shape parameter.

The hazard rate function (HZF) of FHL distribution is defined as follows

$$h(x;a) = \frac{a^2 e^x}{a\left(e^x - 1\right) + 2},$$

by using Glaser (1980) lemma, we studied its behavior as follows.

- If $0 < a < 2$, $\delta'(x,a) = -\frac{(a-2)a(a+1)e^x}{(a(e^x-1)+2)^2} > 0$, it is an non-decreasing function,
- If $a = 2$, $\delta'(x,a) = -\frac{(a-2)a(a+1)e^x}{(a(e^x-1)+2)^2} = 0$, it is a constant function,
- If $a > 2$, $\delta'(x,a) = -\frac{(a-2)a(a+1)e^x}{(a(e^x-1)+2)^2} < 0, \forall\, x > 0$, it is a decreasing function,

where $\delta(x,a) = \frac{-f'(x,a)}{f(x,a)}$. We can see that the proposed model presents different shapes of HZF compared with HL distribution (3) which has only increasing shape of HZF by using the same lemma. Then, we can say that our new version of half-logistic model in (6) has more flexibility than old version of half-logistic model in (4) according to having more shapes of HZF as shown in Figure 1 (right panel).

Figure 1 (left panel) displays a graphical illustration of the suggested model's PDF for different a values. The PDF of the FHL model might be right-skewed, unimodal, symmetric, or decreasing density function, as seen in this figure. Figure 1 (right panel) also shows the determined shapes of the proposed model's HZF.

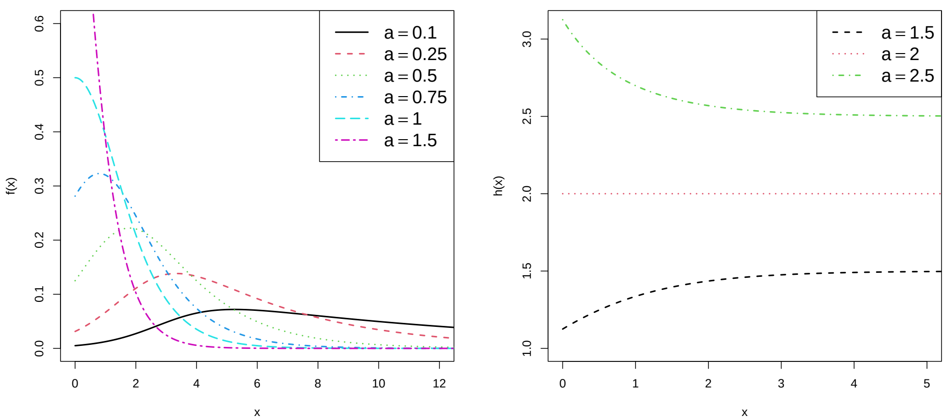

Figure 1 Plots PDF and HZF of FHL model for different parametric values

3. Statistical Inference for FHL Model

The suggested model's statistical properties are deduced in this section, and all of these properties are derived mathematically.

3.1. Mode and quantile function

For determining FHL model mode, we calculate the first derivative of the PDF in (6) as follows:

$$\frac{df(x;a)}{dx} = -2^a a^2 e^x \left(a^2 e^x + a - 2\right)\left(a\left(e^x - 1\right) + 2\right)^{-a-2},$$

after equating the previous equation to zero, we have the FHL distirbution mode as follows

$$x_0 = \log\left(\frac{2-a}{a^2}\right), \forall a < 1.$$

The FHL model quantile function (QF) is defined as the follows

$$Q(u) = \log\left[\frac{2(1-u)^{-1/a} + a - 2}{a}\right]. \tag{7}$$

3.2. Moments

The FHL distribution's r^{th} ordinary moments defined as follows

$$\mu_r' = \int_0^\infty x^r f(x;a)dx = r! 2^a a^{-a-1} {}_{r+1}F_r\left(a_1,\ldots,a_{r+1}; a_1+1,\ldots,a_r+1; \frac{a-2}{a}\right),$$

where ${}_{r+1}F_r\left(a_1,\ldots,a_{r+1}; a_1+1,\ldots,a_r+1; \frac{a-2}{a}\right)$ is generalized hyper function and $a_i = a$, $i = 1,\ldots,r+1$.

The coefficients of skewness (β_1) and kurtosis (β_2) of FHL model are, respectively, defined as follows

$$\beta_1 = \frac{2^{2-a}\left(3a^{2a}\,{}_4F_3 - 3\,2^a a^a\,{}_2F_1\,{}_3F_2 + 4^a\,{}_2F_1^3\right)^2}{\left(2a^a\,{}_3F_2 - 2^a\,{}_2F_1^2\right)^3},$$

$$\beta_2 = -\frac{3\,2^{-a}\left\{8^a\,{}_2F_1^4 - 4a^a\left[2a^a\left(a^a\,{}_5F_4 - 2^a\,{}_2F_1\,{}_4F_3\right) + 4^a\,{}_2F_1^2\,{}_3F_2\right]\right\}}{\left(2^a\,{}_2F_1^2 - 2a^a\,{}_3F_2\right)^2},$$

where ${}_{r+1}F_r = {}_{r+1}F_r\left(a_1,\ldots,a_{r+1}; a_1+1,\ldots,a_r+1; \frac{a-2}{a}\right)$, $a_i = a$, $i = 1,\ldots,r+1$, $\mu_2 = \mu_2' - (\mu_1')^2$, $\mu_3 = 2(\mu_1')^3 - 3\mu_2'\mu_1' + \mu_3'$ and $\mu_4 = -3(\mu_1')^4 + 6\mu_2'(\mu_1')^2 - 4\mu_3'\mu_1' + \mu_4'$.

The moment generating function (MGF) of the proposed model is given by

$$M(t) = E(e^{tX}) = 2^a a^{-a-1}\sum_{m=0}^{\infty} t^m\,{}_{m+1}F_m\left(a,\ldots,a; a+1,\ldots,a+1; \frac{a-2}{a}\right).$$

3.3. Incomplete moments

The FHL distribution's r^{th} incomplete moments defined as follows

$$\Phi_r(t) = \int_0^t x^r f(x;a)dx = 2^a a^{-a+1}\sum_{k=0}^{\infty}\binom{k+a}{a}\left(\frac{a-2}{2}\right)^k (k+a)^{-r-1}\,\Gamma\left[r, t(k+a)\right],$$

where $\Gamma\left[r, t(k+a)\right]$ is lower incomplete gamma function.

It is used to determine mean residual life and mean waiting time, respectively, as follows $m_1(t) = [1-\Phi_1(t)]/S(t) - t$ and $M_1(t) = t - \Phi_1(t)/F(t)$. Also, Bonferroni and Lorenz curves can de determine by $\Phi_r(t)$, respectively, as follows $L(p) = \Phi_1(t)/\mu_1'$ and $B(p) = \Phi_1(x_p)/(p\mu_1')$.

3.4. Entropies

The continuous Rnyi, Tsallis, and Shannon entropies of the FHL distribution are calculated in this subsection. The Rnyi entropy, represented by $R_X(k)$, Tsallis entropy, denoted by $T_X(k)$, order

k entropies, and Shannon entropy, denoted by H_X, can be calculated for the FHL distribution as follows

$$\begin{aligned} R_X(k) &= \frac{1}{1-k}\log\int_0^\infty f^k(x;a)dx = \frac{1}{1-k}\log\left\{2^{ak}(a-2)^{-ak}a^k B_{\frac{a-2}{a}}[ak,1-(a+1)k]\right\}, \\ T_X(k) &= \frac{1}{1-k}\left[\int_{x=0}^\infty f^k(x;a)dx - 1\right] \\ &= \frac{1}{k-1}\left\{1-2^{ak}(a-2)^{-ak}a^k B_{\frac{a-2}{a}}[ak,1-(a+1)k]\right\}, \quad k>0, k\neq 1, \\ H_X &= -\int f(x)\log f(x)dx \\ &= -\frac{2^a\left\{a^{-a}\,{}_2F_1\left(a,a;a+1;\frac{a-2}{a}\right)-2^{-a}[a+a\log(2)-2a\log(a)+1]\right\}}{a}, \end{aligned}$$

where $B_z(a,b)=\int_0^z t^{a-1}(1-t)^{b-1}dt$ and ${}_2F_1\left(a,a;a+1;\frac{a-2}{a}\right)$ is hyper geometric function.

3.5. Order statistics

The i^{th} order statistic's PDF and CDF for the FHL distribution are, respectively, defined as follows

$$\begin{aligned} f_{i:n}(x) &= \frac{n!}{(i-1)!(n-i)!}[F(x)]^{i-1}[1-F(x)]^{n-i}f(x) \\ &= \frac{a^2e^x n!\left\{1-\left[\frac{1}{2}a\left(e^x-1\right)+1\right]^{-a}\right\}^{i-1}\left(\frac{1}{2}a\left(e^x-1\right)+1\right)^{-a(-i+n+1)}}{\left[a\left(e^x-1\right)+2\right]\Gamma(i)\Gamma(-i+n+1)}, \\ F_{i:n}(x) &= \sum_{r=i}^n \binom{n}{r}[F(x)]^r[1-F(x)]^{n-r} = \binom{n}{i}\left\{1-\left[\frac{1}{2}a\left(e^x-1\right)+1\right]^{-a}\right\}^i \\ &\times\left(\frac{1}{2}a\left(e^x-1\right)+1\right)^{-a(n-i)}K, \end{aligned}$$

where $K={}_2F_1\left\{1,i-n;i+1;1-\left[\frac{1}{2}a\left(-1+e^x\right)+1\right]^a\right\}$ is hyper geometric function.

4. Classical Estimation Methods

The maximum likelihood estimation (MLE), Anderson-Darling estimation (ADE), Cramer-von Mises estimation (CVME), Maximum product of spacings estimation (MPSE), ordinary least-squares estimation (OLSE), percentile estimation (PCE), and weighted least squares estimation are all discussed in this section (WLSE).

The log-likelihood function of FHL model takes the following form

$$L=-(a+1)\sum_{i=1}^n\log\left[a\left(e^{x_i}-1\right)+2\right]+\sum_{i=1}^n x_i+n\log\left(2^a a^2\right). \tag{8}$$

By differentiating Equation (8) with respect to a and equating to zero, we have

$$\frac{dL}{da}=-(a+1)\sum_{i=1}^n\frac{e^{x_i}-1}{a\left(e^{x_i}-1\right)+2}-\sum_{i=1}^n\log\left[a\left(e^{x_i}-1\right)+2\right]+n\left[\frac{2}{a}+\log(2)\right]=0,$$

by solving the previous equations, we obtain estimator of the parameter a of the FHL model by the MLE.

By minimizing the following equation, we will obtain the LSE of the FHL parameter

$$O=\sum_{i=1}^n\left[F(x_{i:n})-\frac{i}{n+1}\right]^2=\sum_{i=1}^n\left[1-2^a\left(a\left(e^{x_{i:n}}-1\right)+2\right)^{-a}-\frac{i}{n+1}\right]^2,$$

where $x_{1:n}, x_{2:n}, \ldots, x_{2:n}$ is an order statistics of a random sample from the FHL distribution.

The WLSE of the FHL parameter a is calculated by minimizing the following equation:

$$W = \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[F(x_{i:n}) - \frac{i}{n+1}\right]^2$$
$$= \sum_{i=1}^{n} \frac{(n+1)^2(n+2)}{i(n-i+1)} \left[1 - 2^a \left(a\left(e^{x_{i:n}} - 1\right) + 2\right)^{-a} - \frac{i}{n+1}\right]^2.$$

The ADE of the FHL parameter a is obtained by minimizing the following equation:

$$A = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)[\log F(x_{i:n}) + \log S(x_{i:n})]$$
$$= -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left(\log\left[1 - 2^a \left(a\left(e^{x_{i:n}} - 1\right) + 2\right)^{-a}\right] + \log\left[2^a \left(a\left(e^{x_{i:n}} - 1\right) + 2\right)^{-a}\right]\right).$$

The CVME of the FHL parameter a is obtained by minimizing the following equation:

$$CV = \frac{1}{12n} + \sum_{i=1}^{n}\left[F(x_{i:n}) - \frac{2i-1}{2n}\right]^2 = \frac{1}{12n} + \sum_{i=1}^{n}\left[1 - 2^a \left(a\left(e^{x_{i:n}} - 1\right) + 2\right)^{-a} - \frac{2i-1}{2n}\right]^2.$$

The uniform spacings of a random sample from the FHL distribution is defined as follows

$$D_i = F(x_i) - F(x_{i-1}) = -2^a \left[a\left(e^{x_{i:n}} - 1\right) + 2\right]^{-a} + 2^a \left[a\left(e^{x_{i-1:n}} - 1\right) + 2\right]^{-a},$$

where D_i denotes to the uniform spacings, $F(x_0) = 0$, $F(x_{n+1} = 1)$ and $\sum_{i=1}^{n+1} D_i = 1$. We maximize the following equation in order to have MPS estimators for our proposed model

$$G = \frac{1}{n+1}\sum_{i=1}^{n+1}\log(D_i),$$

with respect to parameter a. The PCE of the FHL parameter is determined by minimizing the following equation

$$PCE = \sum_{i=1}^{n}[x_{i:n} - Q(p_i)]^2 = \sum_{i=1}^{n}\left\{x_{i:n} - \log\left[\frac{2(1-p_i)^{-1/a} + a - 2}{a}\right]\right\}^2, \;\; p_i = \frac{i}{n+1}.$$

5. Numerical Simulation

This section uses simulated data to demonstrate the performance of the provided estimate techniques for estimating the parameter of the FHL model. Based on Equation (7), different sample sizes ($n = 15, 35, 80, 150, 300$) and parameter values are evaluated to create $n = 1000$ random samples from the FHL distribution. Average estimates (AVEs), biases ($|BIAS|$), mean square errors (MSEs), mean relative errors (MREs), average length of percentile bootstrap confidence interval (LPBCI) and coverage probabilities (CP), $\sum Ranks$ (partial sum of the ranks) are calculated in Tables 7–10 in Appendix A by using `R` software. Each estimator's rank among all the estimators is indicated by a superscript.

From numerical values of simulation, we conclude the following

- The estimated parameter $\widehat{a}$ shows the property of consistency.
- For all estimate methods, the BIAS of The estimated parameter $\widehat{a}$ decreases as n increases.
- For all estimation methods, the MSE of The estimated parameter $\widehat{a}$ reduces as n increases.

- The MRE of The estimated parameter $\widehat{a}$ reduces as n increases for all estimation techniques.
- For all estimate methods, the LPBCI of The estimated parameter $\widehat{a}$ decreases as n increases.
- For all estimate methods, the CP of The estimated parameter $\widehat{a}$ increases as n increases.
- We infer from Table 1 that MPSE is the best (overall score of 31.0), followed by MLE (overall score of 39.5). So we advise researchers to use MPSE if they have data sets from our proposed model.

Table 1 Partial and overall ranks of all the methods of estimation of FHL distribution by various values of a

Parameter	n	MLE	ADE	CVME	MPSE	OLSE	PCE	WLSE
$a = 0.25$	15	4	2.5	6	1	7	2.5	5
	35	1	3	7	2	6	5	4
	80	1	2	6	3	4	7	5
	150	4	1	7	3	55	6	2
	300	1	4	5	2	6	7	3
$a = 0.75$	15	2	3	6	1	7	5	4
	35	1.5	5	4	1.5	7	3	6
	80	2	4	6.5	1	3	6.5	5
	150	1	5	6	2	7	4	3
	300	1	4	6	2	5	7	3
$a = 1.5$	15	3	5	6	1	7	4	2
	35	2	3	7	1	4.5	6	4.5
	80	1	5	3.5	2	7	6	3.5
	150	3	1.5	5	1.5	6	7	4
	300	1	4	5	2	6	7	3
$a = 3$	15	2	3	6	1	7	4	5
	35	2	3	5	1	7	5	5
	80	2.5	2.5	6.5	1	5	6.5	4
	150	2	4.5	4.5	1	6	7	3
	300	2.5	2.5	4.5	1	4.5	7	6
$\sum$ Ranks		39.5	67.5	112.5	31.0	117.0	112.5	80.0
Overall Rank		2	3	5.5	1	7	5.5	4

Now we want to see that is our proposed model's estimated parameter is more efficient than the estimated parameter of the HL model? According to the difficulty of having the value of our estimated parameter mathematically in explicit form by the MLE method, we will use the numerical solution to compare the estimated parameters of the two models. We will generate 100.000 random samples by different values of initial parameters from each model, and then we will compute BIAS, MSEs, and MREs for each model. From values of Table 2, we conclude that the estimated parameter of the FHL model is more efficient than the estimated parameter of the HL model according to having small values of BIAS, MSEs, and MREs. Then, we can say that our new version of the half-logistic model (6) has more flexibility than the old version of the half-logistic model (4) according to the results in Table 2.

Table 2 Simulation values of BIAS, MSE and MRE for FHL and HL models

		$a = 0.25$		$a = 0.75$		$a = 1.5$		$a = 3$	
n	Est.	FHL	HL	FHL	HL	FHL	HL	FHL	HL
15	BIAS	0.04305	0.04628	0.10855	0.13819	0.19616	0.27637	0.36292	0.55275
	MSE	0.00310	0.00384	0.01941	0.03435	0.06351	0.13741	0.21858	0.54966
	MRE	0.17220	0.18513	0.14473	0.18426	0.13077	0.18425	0.12097	0.18425
35	BIAS	0.02771	0.02900	0.07017	0.08707	0.12649	0.17412	0.23331	0.34824
	MSE	0.00124	0.00140	0.00793	0.01265	0.02578	0.05059	0.08797	0.20237
	MRE	0.11086	0.11600	0.09356	0.11609	0.08433	0.11608	0.07777	0.11608
80	BIAS	0.01831	0.01894	0.04624	0.05653	0.08326	0.11305	0.15339	0.22610
	MSE	0.00053	0.00058	0.00338	0.00514	0.01097	0.02056	0.03728	0.08223
	MRE	0.07324	0.07576	0.06165	0.07538	0.05551	0.07537	0.05113	0.07537
150	BIAS	0.01322	0.01370	0.03368	0.04114	0.06064	0.08226	0.11172	0.16452
	MSE	0.00028	0.00030	0.00179	0.00269	0.00580	0.01077	0.01969	0.04307
	MRE	0.05288	0.05481	0.04490	0.05485	0.04043	0.05484	0.03724	0.05484
300	BIAS	0.00940	0.00971	0.02370	0.02897	0.04267	0.05790	0.07858	0.11580
	MSE	0.00014	0.00015	0.00089	0.00132	0.00287	0.00529	0.00975	0.02115
	MRE	0.03760	0.03885	0.03160	0.03862	0.02844	0.03860	0.02619	0.03860

6. Modeling Real Data Sets

This section deals with discussing the flexibility of FHL distribution in fitting real data sets and comparing the FHL model with other competing distributions. The real data set consists of 116 observations presented in Table 3, and it represents the ozone level measurements in New York, MaySeptember 1973. It was obtained from the New York State Department of Conservation, and it was introduced by Nadarajah (2008).

Table 3 Numerical values of the real data set

20	18	85	21	21	23	9	85	34	110	16	78	13	30	24
49	135	64	23	32	44	14	11	44	73	1	71	46	11	13
32	10	7	118	7	76	79	48	18	28	16	14	91	77	37
24	96	12	97	23	16	23	20	7	19	22	21	31	14	63
41	97	28	11	39	78	36	21	28	168	61	18	115	18	52
6	84	89	13	20	32	35	39	9	80	59	45	20	65	37
45	108	36	50	73	30	27	23	64	13	82	23	8	12	14
66	47	122	9	40	29	59	4	44	35	16				

The real data set is utilized to show the FHL model's flexibility when compared to several well-known distributions like as Kumaraswamy half-logistic (KHL) Usman et al. (2017), type I generalized half logistic (TIGHL) Olapade (2014), power half-logistic (PHL) Krishnarani (2016), exponentiated half logistic (ExHL) Seo and Kang (2015), exponentiated generalized standardized half-logistic (ExGSHL) Kang and Seo (2011), half-logistic (HL), Poisson generalized half-logistic (PGHL) Muhammad and Liu (2019), generalized half-logistic Poisson (GHLP) Muhammad (2017), Poisson half-logistic (PHL) Abdel-Hamid (2016), exponential (E), Weibull (W), Lindley (L), Maxwell (M), Bilal (B) Abd-Elrahman (2013), Lomax (L), Frechet (F) and gamma (G) distributions. To compare the fitted competitive distributions different information criterion (IC) like as Akaike IC (AIC), the correct Akaike IC (CAIC), Bayesian IC (BIC), Hannan IC (HQIC), also we use Cramér–Von Mises (W), Anderson–Darling (A), and Kolmogorov–Smirnov (K-S) statistics with its corresponding p-value.

The following are the CDFs for the compared distribution

- KHL: $F(x)=1-\left(1-\left(\frac{1-e^{-cx}}{e^{-cx}+1}\right)^{\alpha}\right)^{\beta}$.
- TIGHL: $F(x)=1-2^{a}\left(e^{x}+1\right)^{-a}$.
- PHL: $F(x)=1-\frac{2}{e^{bx^{a}}+1}$.
- ExHL: $F(x)=\left(\frac{1-e^{-bx}}{e^{-bx}+1}\right)^{a}$.
- ExGSHL: $F(x)=\frac{\left(e^{-x}+1\right)^{a}-\left(2^{a}e^{-ax}\right)^{b}}{\left(e^{-bx}+1\right)^{ab}}$.
- HL: $F(x)=\frac{1-e^{-bx}}{e^{-bx}+1}$.
- PGHL: $F(x)=\frac{1-e^{-\beta\left(\frac{1-e^{-\alpha x}}{e^{-\alpha x}+1}\right)^{c}}}{1-e^{-\beta}}$.
- GHLP: $F(x)=\left(\frac{1-e^{-\frac{\beta\left(1-e^{-\alpha x}\right)}{e^{-\alpha x}+1}}}{1-e^{-\beta}}\right)^{c}$.
- PHL: $F(x)=\frac{e^{\frac{b\left(1-e^{-ax}\right)}{e^{-ax}+1}}-1}{e^{b}-1}$.
- E: $F(x)=1-e^{-ax}$.
- W: $F(x)=1-e^{-\left(\frac{x}{b}\right)^{a}}$.
- L: $F(x)=1-\frac{e^{\alpha(-x)}(\alpha+\alpha x+1)}{\alpha+1}$.
- M: $F(x)=1-\frac{2x\Gamma\left(\frac{3}{2},x^{2}\alpha\right)}{\sqrt{\pi}\sqrt{x^{2}}}$.
- B: $F(x)=1-e^{-\frac{2x}{\alpha}}\left(3-2e^{-\frac{x}{\alpha}}\right)$.
- L: $F(x)=1-\left(\frac{x}{\beta}+1\right)^{-\lambda}$.
- F: $F(x)=e^{-\left(\frac{a}{x}\right)^{b}}$.
- G: $F(x)=\frac{\gamma\left(\alpha,\frac{x}{\lambda}\right)}{\gamma(\alpha)}$.

For the presented data set, Table 4 shows parameter estimates (standard error) and $-widehatell$, as well as the comparative computed measures using MLE. In comparison to other competing distributions, the FHL model has a close fit to the modeled data set, as seen in this table.

Estimated PDF, CDF, SF, and P-P plots of the FHL model are illustrated in Figure 2. The proposed model estimated HRF and TTT plot is presented in Figure 3. The behavior of the log-likelihood function with estimated parameters is provided in Figure 4 (left panel), while the same figure (right panel) provides the existence and uniqueness of FHL estimated parameters. The values of $-widehatell$, W, A, K-S, and p-value of K-s for all studied estimate methods are shown in Table 5 and Figure 5 displays P-P plots with estimated PDFs of the FHL model by these methods.

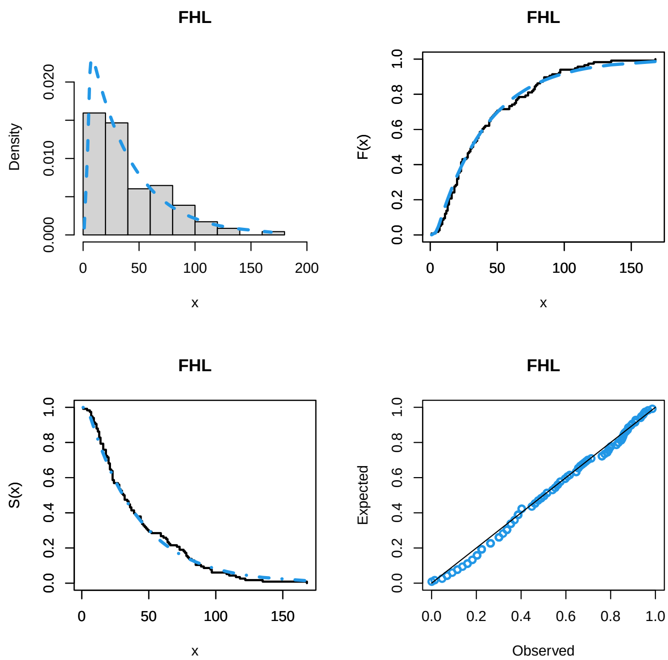

Figure 2 Histogram of estimated FHL PDF, CDF, SF and P-P plot for first data set

Table 4 Numerical analysis for the first data set

Distribution	$-\ell$	AIC	CAIC	BIC	HQIC	A	W	K-S (stat)	K-S (p-value)	Estimates
FHL	542.114	1086.23	1086.26	1088.98	1087.35	0.698298	0.0871639	0.0654361	0.703282	$\hat{a} = 0.026202\ (0.002375)$
KHL	543.415	1092.83	1093.04	1101.09	1096.18	1.13455	0.206988	0.101291	0.184899	$\hat{\alpha} = 1.28253\ (0.132883)$
										$\hat{\beta} = 1.63723\ (1.67605)$
										$\hat{c} = 0.026125\ (0.020159)$
TIGHL	548.347	1098.69	1098.73	1101.45	1099.81	2.61906	0.367572	0.125377	0.05214	$\hat{a} = 0.0241318\ (0.00224)$
PHL	544.359	1092.72	1092.82	1098.22	1094.95	1.14172	0.190753	0.0935367	0.262129	$\hat{a} = 1.12944\ (0.0851018)$
										$\hat{b} = 0.0193997\ (0.0071144)$
ExHL	543.263	1090.53	1090.63	1096.03	1092.76	1.11689	0.20232	0.102204	0.17712	$\hat{a} = 1.3201\ (0.166733)$
										$\hat{b} = 0.038388\ (0.003631)$
ExGSHL	546.891	1097.78	1097.89	1103.29	1100.02	2.99737	0.398379	0.231408	< 0.00001	$\hat{a} = 0.65519\ (0.402276)$
										$\hat{b} = 0.0368643\ (0.022387)$
HL	545.599	1093.2	1093.23	1095.95	1094.31	1.29767	0.155236	0.0902706	0.300948	$\hat{a} = 0.033331\ (0.002584)$
PGHL	542.133	1090.27	1090.48	1098.53	1093.62	0.765767	0.125789	0.0857119	0.361576	$\hat{\alpha} = 0.0314905\ (0.006735)$
										$\hat{\beta} = 1.40823\ (1.06475)$
										$\hat{c} = 1.50971\ (0.180032)$
GHLP	541.446	1088.89	1089.11	1097.15	1092.25	0.726815	0.103639	0.0795704	0.454747	$\hat{\alpha} = 0.029623\ (0.00685)$
										$\hat{\beta} = 1.987\ (1.11953)$
										$\hat{c} = 1.86154\ (0.359299)$
PHL	545.151	1094.3	1094.41	1099.81	1096.54	1.12672	0.158875	0.0790016	0.463993	$\hat{a} = 0.0374\ (0.004901)$
										$\hat{b} = 0.577841\ (0.585956)$
E	549.926	1101.85	1101.89	1104.61	1102.97	3.04885	0.436121	0.13497	0.029211	$\hat{a} = 0.023736\ (0.002203)$
W	542.61	1089.22	1089.33	1094.73	1091.46	0.90278	0.154566	0.089947	0.305	$\hat{a} = 1.34023\ (0.09542)$
										$\hat{b} = 46.0803\ (3.37544)$
L	542.256	1086.51	1086.55	1089.27	1087.63	1.23273	0.229929	0.108691	0.129002	$\hat{\alpha} = 0.046419\ (0.003049)$
M	600.346	1202.69	1202.73	1205.45	1203.81	33.0978	4.13697	0.3262	< 0.00001	$\hat{\alpha} = 0.000525\ (0.0000398)$
B	542.18	1086.36	1086.4	1089.11	1087.48	1.30116	0.238007	0.109667	0.122787	$\hat{\alpha} = 50.4317\ (3.37881)$
L	549.926	1103.85	1103.96	1109.36	1106.09	3.0488	0.436112	0.134969	0.029213	$\hat{\lambda} = 2.11193 \times 10^{8}\ (4.55042 \times 10^{11})$
										$\hat{\beta} = 8.89741 \times 10^{9}\ (1.91706 \times 10^{13})$
F	566.377	1136.75	1136.86	1142.26	1138.99	3.44706	0.447942	0.137225	0.025334	$\hat{a} = 19.4852\ (1.97517)$
										$\hat{b} = 0.973119\ (0.05755)$
G	541.538	1087.08	1087.18	1092.58	1089.31	0.737112	0.128594	0.087476	0.337223	$\hat{\alpha} = 1.69928\ (0.204974)$
										$\hat{\lambda} = 24.7925\ (3.47302)$

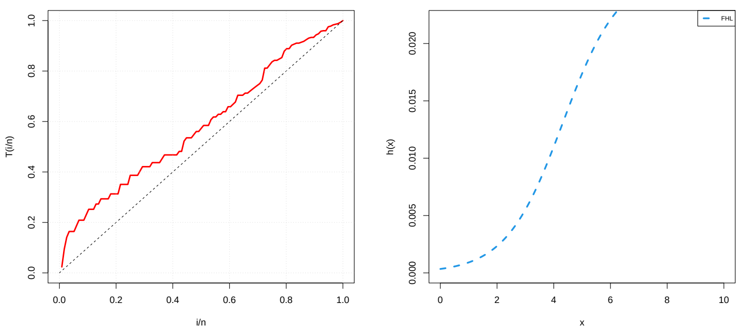

Figure 3 TTT plot and estimated HRF of FHL model for first data set

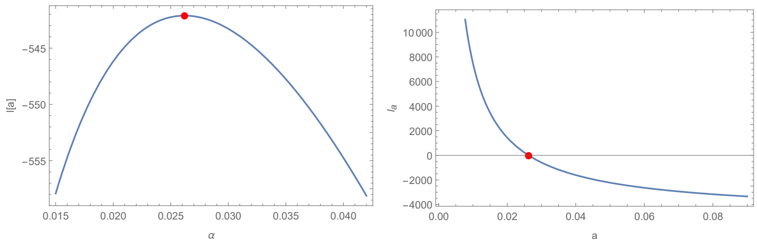

Figure 4 Plots of log-likelihood function with existence and uniqueness of estimated FHL parameter for first data set

Table 5 Numerical analysis for first data set by different estimation methods

	$\hat{a}$	$-L$	A	W	K-S (stat)	K-S (p-value)
MLE	0.026202	542.114	0.698298	0.0871639	0.0654361	0.703282
ADE	0.025014	542.243	0.585733	0.065682	0.0577828	0.833394
CVME	0.0250046	542.245	0.585741	0.0656804	0.057717	0.834411
MPSE	0.025855	542.245	0.585741	0.0656804	0.057717	0.834411
OLSE	0.024975	542.252	0.585863	0.0656941	0.0575265	0.837346
PCE	0.0270739	542.18	0.915945	0.127824	0.0718812	0.586657
WLSE	0.0250057	542.245	0.58574	0.0656804	0.0577247	0.834294

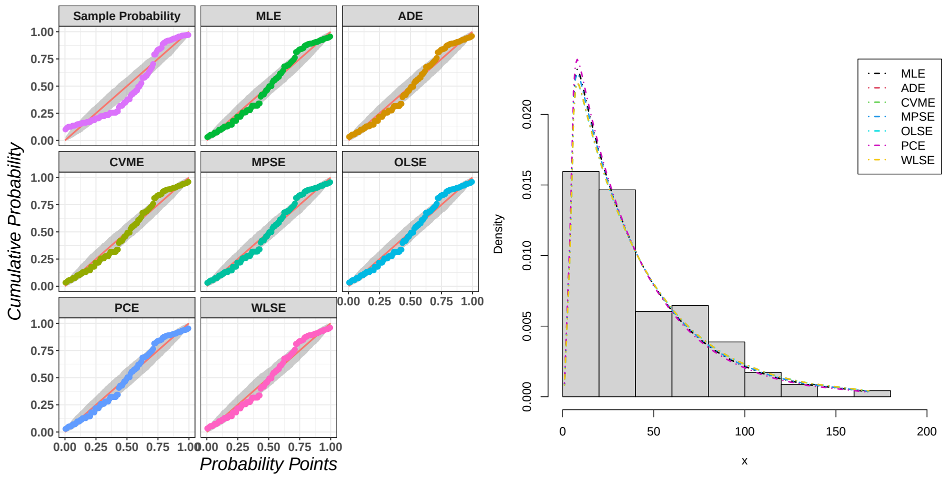

Figure 5 P-P plot and estimated PDFs of the FHL distribution for first data set

7. Two-parameter FHL distribution

In this section, we introduced an extension of half-logistic distribution by using the exponentiated family and FHL distribution to obtain our new version which called exponentiated FHL (ExFHL) distribution. Its PDF and CDF are, respectively, defined as follows

$$F(x) = \left(1 - 2^a \left(a\left(e^x - 1\right) + 2\right)^{-a}\right)^b, \tag{9}$$

$$f(x) = 2^a a^2 b e^x \left(a\left(e^x - 1\right) + 2\right)^{-a-1} \left(1 - 2^a \left(a\left(e^x - 1\right) + 2\right)^{-a}\right)^{b-1}. \tag{10}$$

Also, in this section, we will show the flexibility of the ExFHL distribution by comparing it with the exponentiated version of the half-logistic distribution introduced by Seo and Kang (2015) by using different data sets. We want to illustrate the flexibility of FHL distribution than half-logistic distribution when using the exponentiated generalized family of distributions on both of them.

The second data set represents 30 successive values for precipitation (in inches) in March for the Minneapolis/St. Paul area over a 30-year period, and it was studied by Seo and Kang (2015). The third data set was obtained from Lawless (2011), and it consists of 23 observations which represent the number of million revolutions before failure for each of the 23 ball bearings in the life test. The fourth data set was obtained from Lee and Wang (2003), and it represents the remission times (in months) of 128 bladder cancer patients.

Table 6 Values of numerical analysis for second, third and fourth data sets, respectively

Distribution	$-\ell$	AIC	CAIC	BIC	HQIC	A	W	K-S (stat)	K-S (p-value)	Estimates
ExFHL	-38.1483	80.2967	80.7411	83.0991	81.1932	0.111137	0.0148473	0.0560776	0.999983	$\hat{a}$ = 1.28649 (0.198947)
										$\hat{b}$ = 2.7274 (0.978382)
ExHL	-38.2149	80.4299	80.8743	83.2323	81.3264	0.11368	0.0150754	0.0567605	0.999977	$\hat{a}$ = 1.28559 (0.210454)
										$\hat{b}$ = 2.35058 (0.661399)
FHL	-41.9338	85.8677	86.0105	87.2689	86.3159	1.05747	0.154974	0.151452	0.496917	$\hat{a}$ = 0.842532 (0.102353)
ExFHL	-112.967	229.933	230.533	232.204	230.504	0.18462	0.0309302	0.103098	0.967391	$\hat{a}$ = 0.0314942 (0.00645279)
										$\hat{b}$ = 4.29743 (1.61974)
ExHL	-113.043	230.086	230.686	232.357	230.657	0.209571	0.0375849	0.114311	0.924597	$\hat{a}$ = 0.034831 (0.00625348)
										$\hat{b}$ = 3.40575 (1.21561)
FHL	-119.82	241.641	241.831	242.776	241.926	2.3234	0.431593	0.284333	0.048522	$\hat{a}$ = 0.0146368 (0.00302989)
ExFHL	-411.018	826.037	826.133	831.741	828.354	0.285811	0.0460255	0.0444091	0.962369	$\hat{a}$ = 0.0998192 (0.0144284)
										$\hat{b}$ = 0.570137 (0.0731405)
ExHL	-416.636	837.271	837.367	842.975	839.589	1.38102	0.255591	0.0948826	0.19938	$\hat{a}$ = 0.143968 (0.0143704)
										$\hat{b}$ = 0.952712 (0.109117)
FHL	-421.5	845.000	845.031	847.852	846.158	4.02184	0.514078	0.113881	0.0723003	$\hat{a}$ = 0.152504 (0.0113417)

From Table 6, we conclude that the ExFHL distribution performs well for fitting the three real data sets, respectively, than compared models. Also we can say the our relatively new version of half-logistic distribution is the best compared with half-logistic distribution when adding an additional parameter for both of them as shown in previous section.

8. Conclusions

We introduced in this paper a relatively new statistical distribution called a flexible half-logistic distribution which can be an alternative model to classical half-logistic distribution. The FHL model's statistical properties were derived mathematically. Seven classical estimation methods were considered to obtain point estimation for the unknown parameter of the proposed model. To compare the performance of various estimation methods, a simulation analysis was performed using the R software. A real data set was used to indicate the superiority of the proposed distribution. It was determined that the FHL model fits the data more effectively than most other competing distributions. The existence of MLEs is confirmed in Figure 4 (right panel), where the log-likelihood function crosses the x-axis at one point. The log-likelihood function is a decreasing function that meets the x-axis at just one point, as can be shown. Furthermore, Figure 4 (left panel) shows that the log-likelihood function has global maximum roots. Therefore, we can say that the log-likelihood function has unique roots, and they are global maximum at the same time. Also, we show that when we generalized the FHL distribution by exponentiated generalized family, it has more flexibility for fitting different data sets than exponentiated half-logistic distribution. Finally, we hope that the researchers will use the FHL model and its generalization version more than other models according to their flexibility which showed in our paper.

As a future work of this study, we aim to use different estimation methods such as Bayesian estimation under different types of loss functions. Also, we can apply our model to censored data sets. The proposed model can be used for analyzing and modeling data in other fields such as reliability engineering, medicine, economics, survival analyses, and life testing. Furthermore, a bivariate extension of the FHL distribution may also be studied.

Acknowledgments

We would like to thank the referees for his comments and suggestions on the manuscript.

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Appendix A

Table 7 Simulation values of AVEs, BIAS, MSE, MRE, LPBCI and CP for $a = 0.25$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	WLSE
15	AVES	0.2563	0.25537	0.26155	0.24785	0.25954	0.23442	0.25523
	BIAS	$0.04334^{\{3\}}$	$0.04032^{\{2\}}$	$0.04939^{\{6\}}$	$0.04006^{\{1\}}$	$0.05045^{\{7\}}$	$0.04873^{\{5\}}$	$0.04566^{\{4\}}$
	MSE	$0.00313^{\{3\}}$	$0.00269^{\{2\}}$	$0.00387^{\{6\}}$	$0.00265^{\{1\}}$	$0.00421^{\{7\}}$	$0.0035^{\{4\}}$	$0.00354^{\{5\}}$
	MRE	$0.17338^{\{3\}}$	$0.16129^{\{2\}}$	$0.19756^{\{6\}}$	$0.16024^{\{1\}}$	$0.20179^{\{7\}}$	$0.19493^{\{5\}}$	$0.18262^{\{4\}}$
	LPBCI	$0.21106^{\{3\}}$	$0.21398^{\{4\}}$	$0.22757^{\{7\}}$	$0.195^{\{1\}}$	$0.22621^{\{6\}}$	$0.19865^{\{2\}}$	$0.22058^{\{5\}}$
	CP	$0.932^{\{2\}}$	$0.96^{\{1\}}$	$0.922^{\{4.5\}}$	$0.93^{\{3\}}$	$0.91^{\{6\}}$	$0.876^{\{7\}}$	$0.922^{\{4.5\}}$
	$\sum Ranks$	$18^{\{4\}}$	$17^{\{2.5\}}$	$28.5^{\{6\}}$	$9^{\{1\}}$	$29^{\{7\}}$	$17^{\{2.5\}}$	$21.5^{\{5\}}$
35	AVES	0.25158	0.24888	0.25614	0.24963	0.25129	0.24118	0.25046
	BIAS	$0.02668^{\{1\}}$	$0.02777^{\{3\}}$	$0.03174^{\{7\}}$	$0.0277^{\{2\}}$	$0.02989^{\{5\}}$	$0.03123^{\{6\}}$	$0.02955^{\{4\}}$
	MSE	$0.00112^{\{1\}}$	$0.0012^{\{2\}}$	$0.00164^{\{7\}}$	$0.00126^{\{3\}}$	$0.00144^{\{5\}}$	$0.00148^{\{6\}}$	$0.00138^{\{4\}}$
	MRE	$0.10672^{\{1\}}$	$0.11107^{\{3\}}$	$0.12695^{\{7\}}$	$0.11082^{\{2\}}$	$0.11956^{\{5\}}$	$0.12493^{\{6\}}$	$0.11818^{\{4\}}$
	LPBCI	$0.13122^{\{2\}}$	$0.13468^{\{3\}}$	$0.1446^{\{7\}}$	$0.12854^{\{1\}}$	$0.14226^{\{6\}}$	$0.13649^{\{4\}}$	$0.1387^{\{5\}}$
	CP	$0.936^{\{3.5\}}$	$0.936^{\{3.5\}}$	$0.924^{\{5\}}$	$0.94^{\{1.5\}}$	$0.94^{\{1.5\}}$	$0.886^{\{7\}}$	$0.922^{\{6\}}$
	$\sum Ranks$	$9.5^{\{1\}}$	$15.5^{\{3\}}$	$31^{\{7\}}$	$14.5^{\{2\}}$	$27.5^{\{6\}}$	$23^{\{5\}}$	$19^{\{4\}}$
80	AVES	0.25237	0.24839	0.25063	0.248	0.25292	0.24425	0.24989
	BIAS	$0.01746^{\{1\}}$	$0.01898^{\{3\}}$	$0.02048^{\{6\}}$	$0.01937^{\{4\}}$	$0.01861^{\{2\}}$	$0.02205^{\{7\}}$	$0.01945^{\{5\}}$
	MSE	$0.00048^{\{1\}}$	$0.00056^{\{2.5\}}$	$0.00068^{\{6\}}$	$0.00057^{\{4\}}$	$0.00056^{\{2.5\}}$	$0.00074^{\{7\}}$	$0.00063^{\{5\}}$
	MRE	$0.06984^{\{1\}}$	$0.0759^{\{3\}}$	$0.08192^{\{6\}}$	$0.07749^{\{4\}}$	$0.07444^{\{2\}}$	$0.0882^{\{7\}}$	$0.07778^{\{5\}}$
	LPBCI	$0.08631^{\{2\}}$	$0.08955^{\{3\}}$	$0.09352^{\{5\}}$	$0.08381^{\{1\}}$	$0.09383^{\{6\}}$	$0.09394^{\{7\}}$	$0.09063^{\{4\}}$
	CP	$0.942^{\{2\}}$	$0.928^{\{3.5\}}$	$0.916^{\{6\}}$	$0.928^{\{3.5\}}$	$0.954^{\{1\}}$	$0.904^{\{7\}}$	$0.924^{\{5\}}$
	$\sum Ranks$	$11^{\{1\}}$	$16^{\{2\}}$	$25^{\{6\}}$	$17.5^{\{3\}}$	$19.5^{\{4\}}$	$29^{\{7\}}$	$22^{\{5\}}$
150	AVES	0.24992	0.25132	0.25173	0.24705	0.25055	0.24526	0.24995
	BIAS	$0.01393^{\{4\}}$	$0.01339^{\{1\}}$	$0.0142^{\{6\}}$	$0.01403^{\{5\}}$	$0.01362^{\{3\}}$	$0.01525^{\{7\}}$	$0.01341^{\{2\}}$
	MSE	$3e-04^{\{3\}}$	$0.00029^{\{1.5\}}$	$0.00033^{\{6\}}$	$0.00031^{\{4.5\}}$	$0.00031^{\{4.5\}}$	$0.00037^{\{7\}}$	$0.00029^{\{1.5\}}$
	MRE	$0.05571^{\{4\}}$	$0.05357^{\{1\}}$	$0.05681^{\{6\}}$	$0.05614^{\{5\}}$	$0.05449^{\{3\}}$	$0.06101^{\{7\}}$	$0.05364^{\{2\}}$
	LPBCI	$0.06275^{\{2\}}$	$0.06568^{\{4\}}$	$0.06873^{\{6\}}$	$0.06167^{\{1\}}$	$0.06821^{\{5\}}$	$0.06965^{\{7\}}$	$0.06545^{\{3\}}$
	CP	$0.92^{\{5\}}$	$0.928^{\{4\}}$	$0.936^{\{2\}}$	$0.906^{\{6\}}$	$0.94^{\{1\}}$	$0.9^{\{7\}}$	$0.934^{\{3\}}$
	$\sum Ranks$	$20^{\{4\}}$	$11.5^{\{1\}}$	$29^{\{7\}}$	$16.5^{\{3\}}$	$21.5^{\{5\}}$	$28^{\{6\}}$	$13.5^{\{2\}}$
300	AVES	0.25138	0.25126	0.24943	0.24949	0.25043	0.24725	0.24984
	BIAS	$0.00882^{\{1\}}$	$0.01028^{\{4.5\}}$	$0.01028^{\{4.5\}}$	$0.00896^{\{2\}}$	$0.01074^{\{6\}}$	$0.01097^{\{7\}}$	$0.00959^{\{3\}}$
	MSE	$0.00012^{\{1\}}$	$0.00017^{\{4.5\}}$	$0.00017^{\{4.5\}}$	$0.00013^{\{2\}}$	$0.00018^{\{6\}}$	$0.00019^{\{7\}}$	$0.00014^{\{3\}}$
	MRE	$0.03527^{\{1\}}$	$0.04112^{\{4\}}$	$0.04113^{\{5\}}$	$0.03583^{\{2\}}$	$0.04295^{\{6\}}$	$0.04388^{\{7\}}$	$0.03836^{\{3\}}$
	LPBCI	$0.04432^{\{2\}}$	$0.04586^{\{3\}}$	$0.04746^{\{5\}}$	$0.04385^{\{1\}}$	$0.04767^{\{6\}}$	$0.04969^{\{7\}}$	$0.04596^{\{4\}}$
	CP	$0.95^{\{1\}}$	$0.916^{\{5.5\}}$	$0.912^{\{7\}}$	$0.948^{\{2\}}$	$0.916^{\{5.5\}}$	$0.918^{\{4\}}$	$0.944^{\{3\}}$
	$\sum Ranks$	$12^{\{1\}}$	$18.5^{\{4\}}$	$20^{\{5\}}$	$13^{\{2\}}$	$26.5^{\{6\}}$	$32^{\{7\}}$	$18^{\{3\}}$

Table 8 Simulation values of AVEs, BIAS, MSE, MRE, LPBCI and CP for $a = 0.75$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	WLSE
15	AVES	0.77031	0.76596	0.7593	0.74192	0.76515	0.72318	0.77271
	BIAS	$0.10804^{\{3\}}$	$0.10575^{\{2\}}$	$0.10968^{\{5\}}$	$0.10302^{\{1\}}$	$0.11649^{\{6\}}$	$0.11838^{\{7\}}$	$0.10884^{\{4\}}$
	MSE	$0.01906^{\{2\}}$	$0.01957^{\{5\}}$	$0.01939^{\{4\}}$	$0.01748^{\{1\}}$	$0.02189^{\{6.5\}}$	$0.02189^{\{6.5\}}$	$0.01928^{\{3\}}$
	MRE	$0.14406^{\{3\}}$	$0.141^{\{2\}}$	$0.14624^{\{5\}}$	$0.13736^{\{1\}}$	$0.15533^{\{6\}}$	$0.15784^{\{7\}}$	$0.14512^{\{4\}}$
	LPBCI	$0.52746^{\{3\}}$	$0.53115^{\{4\}}$	$0.56122^{\{6\}}$	$0.49344^{\{1\}}$	$0.56603^{\{7\}}$	$0.50623^{\{2\}}$	$0.55352^{\{5\}}$
	CP	$0.934^{\{4.5\}}$	$0.938^{\{2\}}$	$0.936^{\{3\}}$	$0.926^{\{6\}}$	$0.934^{\{4.5\}}$	$0.906^{\{7\}}$	$0.948^{\{1\}}$
	$\sum Ranks$	$14.5^{\{2\}}$	$19^{\{3\}}$	$25^{\{6\}}$	$6^{\{1\}}$	$29^{\{7\}}$	$23.5^{\{5\}}$	$23^{\{4\}}$
35	AVES	0.7576	0.7592	0.75715	0.74138	0.75017	0.73261	0.75757
	BIAS	$0.06612^{\{1\}}$	$0.07406^{\{5\}}$	$0.07209^{\{3\}}$	$0.06903^{\{2\}}$	$0.07709^{\{7\}}$	$0.07327^{\{4\}}$	$0.07484^{\{6\}}$
	MSE	$0.00681^{\{1\}}$	$0.00873^{\{5\}}$	$0.00816^{\{3\}}$	$0.00754^{\{2\}}$	$0.00933^{\{6.5\}}$	$0.00847^{\{4\}}$	$0.00933^{\{6.5\}}$
	MRE	$0.08816^{\{1\}}$	$0.09875^{\{5\}}$	$0.09612^{\{3\}}$	$0.09204^{\{2\}}$	$0.10278^{\{7\}}$	$0.09769^{\{4\}}$	$0.09979^{\{6\}}$
	LPBCI	$0.33569^{\{2\}}$	$0.34253^{\{4\}}$	$0.35445^{\{7\}}$	$0.32117^{\{1\}}$	$0.35041^{\{6\}}$	$0.33773^{\{3\}}$	$0.34686^{\{5\}}$
	CP	$0.952^{\{1\}}$	$0.93^{\{4\}}$	$0.942^{\{2\}}$	$0.932^{\{3\}}$	$0.924^{\{6\}}$	$0.904^{\{7\}}$	$0.926^{\{5\}}$
	$\sum Ranks$	$12^{\{1.5\}}$	$23^{\{5\}}$	$22^{\{4\}}$	$12^{\{1.5\}}$	$28.5^{\{7\}}$	$16^{\{3\}}$	$26.5^{\{6\}}$
80	AVES	0.75567	0.75397	0.75661	0.74548	0.7475	0.73429	0.75474
	BIAS	$0.04872^{\{3\}}$	$0.05079^{\{6\}}$	$0.04922^{\{5\}}$	$0.04739^{\{2\}}$	$0.04536^{\{1\}}$	$0.05112^{\{7\}}$	$0.04907^{\{4\}}$
	MSE	$0.00371^{\{3\}}$	$0.00404^{\{6\}}$	$0.00391^{\{4\}}$	$0.00358^{\{2\}}$	$0.00326^{\{1\}}$	$0.00415^{\{7\}}$	$0.00396^{\{5\}}$
	MRE	$0.06496^{\{3\}}$	$0.06772^{\{6\}}$	$0.06562^{\{5\}}$	$0.06318^{\{2\}}$	$0.06049^{\{1\}}$	$0.06815^{\{7\}}$	$0.06542^{\{4\}}$
	LPBCI	$0.21869^{\{2\}}$	$0.22452^{\{3\}}$	$0.23201^{\{7\}}$	$0.21218^{\{1\}}$	$0.2302^{\{6\}}$	$0.22782^{\{4\}}$	$0.22785^{\{5\}}$
	CP	$0.918^{\{4.5\}}$	$0.914^{\{6\}}$	$0.92^{\{3\}}$	$0.918^{\{4.5\}}$	$0.944^{\{1\}}$	$0.87^{\{7\}}$	$0.924^{\{2\}}$
	$\sum Ranks$	$14.5^{\{2\}}$	$23^{\{4\}}$	$26^{\{6.5\}}$	$10.5^{\{1\}}$	$16^{\{3\}}$	$26^{\{6.5\}}$	$24^{\{5\}}$
150	AVES	0.75473	0.74906	0.75284	0.74621	0.74779	0.74258	0.74751
	BIAS	$0.03309^{\{1\}}$	$0.03509^{\{6\}}$	$0.03422^{\{4\}}$	$0.03365^{\{3\}}$	$0.03695^{\{7\}}$	$0.03489^{\{5\}}$	$0.03364^{\{2\}}$
	MSE	$0.00172^{\{1\}}$	$0.00198^{\{6\}}$	$0.00185^{\{4\}}$	$0.00178^{\{3\}}$	$0.00217^{\{7\}}$	$0.00197^{\{5\}}$	$0.00173^{\{2\}}$
	MRE	$0.04412^{\{1\}}$	$0.04678^{\{6\}}$	$0.04563^{\{4\}}$	$0.04486^{\{3\}}$	$0.04927^{\{7\}}$	$0.04652^{\{5\}}$	$0.04485^{\{2\}}$
	LPBCI	$0.1592^{\{2\}}$	$0.16184^{\{3\}}$	$0.16923^{\{6\}}$	$0.15793^{\{1\}}$	$0.16809^{\{5\}}$	$0.1701^{\{7\}}$	$0.16284^{\{4\}}$
	CP	$0.934^{\{3\}}$	$0.924^{\{5\}}$	$0.95^{\{1\}}$	$0.92^{\{6\}}$	$0.928^{\{4\}}$	$0.912^{\{7\}}$	$0.944^{\{2\}}$
	$\sum Ranks$	$10^{\{1\}}$	$24^{\{5\}}$	$25^{\{6\}}$	$12^{\{2\}}$	$30^{\{7\}}$	$23^{\{4\}}$	$16^{\{3\}}$
300	AVES	0.75081	0.74982	0.74982	0.74969	0.75131	0.74242	0.75309
	BIAS	$0.02413^{\{1\}}$	$0.02483^{\{5\}}$	$0.02577^{\{6\}}$	$0.02425^{\{2\}}$	$0.02462^{\{4\}}$	$0.0263^{\{7\}}$	$0.02458^{\{3\}}$
	MSE	$0.00094^{\{2.5\}}$	$0.00093^{\{1\}}$	$0.00105^{\{6\}}$	$0.00095^{\{4\}}$	$0.00099^{\{5\}}$	$0.00107^{\{7\}}$	$0.00094^{\{2.5\}}$
	MRE	$0.03218^{\{1\}}$	$0.0331^{\{5\}}$	$0.03436^{\{6\}}$	$0.03234^{\{2\}}$	$0.03283^{\{4\}}$	$0.03506^{\{7\}}$	$0.03277^{\{3\}}$
	LPBCI	$0.11183^{\{1.5\}}$	$0.11528^{\{3\}}$	$0.1182^{\{5\}}$	$0.11183^{\{1.5\}}$	$0.11969^{\{6\}}$	$0.1209^{\{7\}}$	$0.11539^{\{4\}}$
	CP	$0.922^{\{6\}}$	$0.924^{\{3.5\}}$	$0.924^{\{3.5\}}$	$0.924^{\{3.5\}}$	$0.938^{\{1\}}$	$0.914^{\{7\}}$	$0.924^{\{3.5\}}$
	$\sum Ranks$	$8^{\{1\}}$	$18.5^{\{4\}}$	$27.5^{\{6\}}$	$14^{\{2\}}$	$26^{\{5\}}$	$29^{\{7\}}$	$17^{\{3\}}$

Table 9 Simulation values of AVEs, BIAS, MSE, MRE, LPBCI and CP for $a = 1.5$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	WLSE
15	AVES	1.52638	1.52705	1.53818	1.48902	1.53693	1.46781	1.51363
	BIAS	$0.19562^{\{2\}}$	$0.21124^{\{5\}}$	$0.21528^{\{7\}}$	$0.19855^{\{3\}}$	$0.21407^{\{6\}}$	$0.20114^{\{4\}}$	$0.18606^{\{1\}}$
	MSE	$0.06732^{\{4\}}$	$0.07194^{\{5\}}$	$0.08045^{\{6\}}$	$0.06335^{\{2\}}$	$0.0843^{\{7\}}$	$0.06383^{\{3\}}$	$0.05914^{\{1\}}$
	MRE	$0.13041^{\{2\}}$	$0.14083^{\{5\}}$	$0.14352^{\{7\}}$	$0.13237^{\{3\}}$	$0.14272^{\{6\}}$	$0.13409^{\{4\}}$	$0.12404^{\{1\}}$
	LPBCI	$0.94608^{\{3\}}$	$0.9751^{\{4\}}$	$1.02916^{\{6\}}$	$0.89027^{\{1\}}$	$1.03614^{\{7\}}$	$0.92441^{\{2\}}$	$0.99299^{\{5\}}$
	CP	$0.922^{\{3\}}$	$0.908^{\{7\}}$	$0.912^{\{5.5\}}$	$0.912^{\{5.5\}}$	$0.926^{\{2\}}$	$0.914^{\{4\}}$	$0.948^{\{1\}}$
	$\sum Ranks$	$16^{\{3\}}$	$20^{\{5\}}$	$28.5^{\{6\}}$	$11.5^{\{1\}}$	$32^{\{7\}}$	$17^{\{4\}}$	$15^{\{2\}}$
35	AVES	1.51832	1.50392	1.51842	1.48129	1.50365	1.46181	1.5088
	BIAS	$0.12594^{\{1\}}$	$0.12994^{\{2\}}$	$0.14002^{\{7\}}$	$0.13105^{\{3\}}$	$0.13365^{\{5\}}$	$0.1371^{\{6\}}$	$0.13148^{\{4\}}$
	MSE	$0.02577^{\{1\}}$	$0.02757^{\{4\}}$	$0.03216^{\{7\}}$	$0.0271^{\{2\}}$	$0.02822^{\{5\}}$	$0.02871^{\{6\}}$	$0.02724^{\{3\}}$
	MRE	$0.08396^{\{1\}}$	$0.08662^{\{2\}}$	$0.09334^{\{7\}}$	$0.08737^{\{3\}}$	$0.0891^{\{5\}}$	$0.0914^{\{6\}}$	$0.08765^{\{4\}}$
	LPBCI	$0.60456^{\{2\}}$	$0.61663^{\{4\}}$	$0.65198^{\{7\}}$	$0.57937^{\{1\}}$	$0.64222^{\{6\}}$	$0.61512^{\{3\}}$	$0.62488^{\{5\}}$
	CP	$0.934^{\{2\}}$	$0.926^{\{3\}}$	$0.914^{\{4.5\}}$	$0.902^{\{7\}}$	$0.91^{\{6\}}$	$0.914^{\{4.5\}}$	$0.944^{\{1\}}$
	$\sum Ranks$	$11^{\{2\}}$	$17^{\{3\}}$	$31.5^{\{7\}}$	$10^{\{1\}}$	$23^{\{4.5\}}$	$24.5^{\{6\}}$	$23^{\{4.5\}}$
80	AVES	1.50922	1.50671	1.49444	1.48646	1.50001	1.46637	1.50505
	BIAS	$0.08247^{\{1\}}$	$0.08932^{\{5\}}$	$0.08681^{\{2\}}$	$0.08712^{\{3\}}$	$0.08995^{\{7\}}$	$0.08949^{\{6\}}$	$0.08796^{\{4\}}$
	MSE	$0.01094^{\{1\}}$	$0.0128^{\{7\}}$	$0.012^{\{3\}}$	$0.01156^{\{2\}}$	$0.01256^{\{6\}}$	$0.01225^{\{5\}}$	$0.01208^{\{4\}}$
	MRE	$0.05498^{\{1\}}$	$0.05955^{\{5\}}$	$0.05787^{\{2\}}$	$0.05808^{\{3\}}$	$0.05997^{\{7\}}$	$0.05966^{\{6\}}$	$0.05864^{\{4\}}$
	LPBCI	$0.39301^{\{2\}}$	$0.40538^{\{3\}}$	$0.41814^{\{6\}}$	$0.38286^{\{1\}}$	$0.42023^{\{7\}}$	$0.41437^{\{5\}}$	$0.40896^{\{4\}}$
	CP	$0.93^{\{2.5\}}$	$0.916^{\{6\}}$	$0.938^{\{1\}}$	$0.92^{\{5\}}$	$0.93^{\{2.5\}}$	$0.91^{\{7\}}$	$0.924^{\{4\}}$
	$\sum Ranks$	$10.5^{\{1\}}$	$22^{\{5\}}$	$20^{\{3.5\}}$	$12^{\{2\}}$	$32.5^{\{7\}}$	$23^{\{6\}}$	$20^{\{3.5\}}$
150	AVES	1.50737	1.50323	1.50637	1.49653	1.50289	1.48834	1.51345
	BIAS	$0.06121^{\{2\}}$	$0.06063^{\{1\}}$	$0.06416^{\{4\}}$	$0.06132^{\{3\}}$	$0.06549^{\{6\}}$	$0.06789^{\{7\}}$	$0.06491^{\{5\}}$
	MSE	$0.00602^{\{3\}}$	$0.00578^{\{1\}}$	$0.00647^{\{4\}}$	$0.00586^{\{2\}}$	$0.00682^{\{6\}}$	$0.00737^{\{7\}}$	$0.0066^{\{5\}}$
	MRE	$0.04081^{\{2\}}$	$0.04042^{\{1\}}$	$0.04278^{\{4\}}$	$0.04088^{\{3\}}$	$0.04366^{\{6\}}$	$0.04526^{\{7\}}$	$0.04328^{\{5\}}$
	LPBCI	$0.28607^{\{2\}}$	$0.29756^{\{4\}}$	$0.3084^{\{6\}}$	$0.28242^{\{1\}}$	$0.30631^{\{5\}}$	$0.30934^{\{7\}}$	$0.29713^{\{3\}}$
	CP	$0.938^{\{3\}}$	$0.942^{\{2\}}$	$0.946^{\{1\}}$	$0.934^{\{4\}}$	$0.926^{\{5\}}$	$0.908^{\{7\}}$	$0.922^{\{6\}}$
	$\sum Ranks$	$14^{\{3\}}$	$13^{\{1.5\}}$	$25^{\{5\}}$	$13^{\{1.5\}}$	$26^{\{6\}}$	$29^{\{7\}}$	$20^{\{4\}}$
300	AVES	1.49897	1.50238	1.50113	1.50149	1.50542	1.49371	1.49766
	BIAS	$0.04319^{\{1\}}$	$0.04392^{\{4\}}$	$0.0481^{\{6\}}$	$0.04347^{\{2\}}$	$0.04605^{\{5\}}$	$0.0495^{\{7\}}$	$0.04384^{\{3\}}$
	MSE	$0.00302^{\{2\}}$	$0.00313^{\{4\}}$	$0.00364^{\{6\}}$	$0.00289^{\{1\}}$	$0.00322^{\{5\}}$	$0.00386^{\{7\}}$	$0.00304^{\{3\}}$
	MRE	$0.02879^{\{1\}}$	$0.02928^{\{4\}}$	$0.03207^{\{6\}}$	$0.02898^{\{2\}}$	$0.0307^{\{5\}}$	$0.033^{\{7\}}$	$0.02922^{\{3\}}$
	LPBCI	$0.20103^{\{2\}}$	$0.20912^{\{4\}}$	$0.2167^{\{6\}}$	$0.19997^{\{1\}}$	$0.21551^{\{5\}}$	$0.22143^{\{7\}}$	$0.20745^{\{3\}}$
	CP	$0.928^{\{4\}}$	$0.924^{\{5\}}$	$0.916^{\{6\}}$	$0.942^{\{2\}}$	$0.95^{\{1\}}$	$0.908^{\{7\}}$	$0.934^{\{3\}}$
	$\sum Ranks$	$10^{\{1\}}$	$19^{\{4\}}$	$26^{\{5\}}$	$12^{\{2\}}$	$27^{\{6\}}$	$29^{\{7\}}$	$17^{\{3\}}$

Table 10 Simulation values of AVEs, BIAS, MSE, MRE, LPBCI and CP for $a = 3$

n	Est.	MLE	ADE	CVME	MPSE	OLSE	PCE	WLSE
15	AVES	3.06556	3.01987	3.01938	2.94486	3.07054	2.90196	3.06448
	BIAS	$0.33397^{\{1\}}$	$0.36775^{\{3\}}$	$0.38109^{\{5\}}$	$0.3574^{\{2\}}$	$0.40391^{\{7\}}$	$0.38442^{\{6\}}$	$0.38002^{\{4\}}$
	MSE	$0.17926^{\{1\}}$	$0.22175^{\{3\}}$	$0.24351^{\{6\}}$	$0.21679^{\{2\}}$	$0.2734^{\{7\}}$	$0.23368^{\{5\}}$	$0.22936^{\{4\}}$
	MRE	$0.11132^{\{1\}}$	$0.12258^{\{3\}}$	$0.12703^{\{5\}}$	$0.11913^{\{2\}}$	$0.13464^{\{7\}}$	$0.12814^{\{6\}}$	$0.12667^{\{4\}}$
	LPBCI	$1.78032^{\{3\}}$	$1.80013^{\{4\}}$	$1.9243^{\{6\}}$	$1.63049^{\{1\}}$	$1.95126^{\{7\}}$	$1.72906^{\{2\}}$	$1.89057^{\{5\}}$
	CP	$0.954^{\{1\}}$	$0.928^{\{5\}}$	$0.948^{\{2\}}$	$0.906^{\{6\}}$	$0.936^{\{3.5\}}$	$0.9^{\{7\}}$	$0.936^{\{3.5\}}$
	$\sum Ranks$	$13^{\{2\}}$	$16^{\{3\}}$	$28^{\{6\}}$	$9^{\{1\}}$	$32.5^{\{7\}}$	$20^{\{4\}}$	$21.5^{\{5\}}$
35	AVES	3.02888	2.99904	2.99606	2.96794	3.05036	2.89446	3.02547
	BIAS	$0.23523^{\{3\}}$	$0.23126^{\{2\}}$	$0.24247^{\{4\}}$	$0.22095^{\{1\}}$	$0.25475^{\{6\}}$	$0.26576^{\{7\}}$	$0.24849^{\{5\}}$
	MSE	$0.09021^{\{3\}}$	$0.08619^{\{2\}}$	$0.09188^{\{4\}}$	$0.07649^{\{1\}}$	$0.11389^{\{7\}}$	$0.10748^{\{6\}}$	$0.10436^{\{5\}}$
	MRE	$0.07841^{\{3\}}$	$0.07709^{\{2\}}$	$0.08082^{\{4\}}$	$0.07365^{\{1\}}$	$0.08492^{\{6\}}$	$0.08859^{\{7\}}$	$0.08283^{\{5\}}$
	LPBCI	$1.11062^{\{2\}}$	$1.15303^{\{3\}}$	$1.20384^{\{6\}}$	$1.07244^{\{1\}}$	$1.22492^{\{7\}}$	$1.16114^{\{4\}}$	$1.17587^{\{5\}}$
	CP	$0.926^{\{5\}}$	$0.94^{\{2\}}$	$0.952^{\{1\}}$	$0.934^{\{4\}}$	$0.912^{\{6\}}$	$0.874^{\{7\}}$	$0.936^{\{3\}}$
	$\sum Ranks$	$14^{\{2\}}$	$15^{\{3\}}$	$25^{\{5\}}$	$8^{\{1\}}$	$28^{\{7\}}$	$25^{\{5\}}$	$25^{\{5\}}$
80	AVES	3.01435	3.00629	3.01534	2.97316	3.00575	2.93688	3.01235
	BIAS	$0.15689^{\{4\}}$	$0.15394^{\{2\}}$	$0.16754^{\{6\}}$	$0.15385^{\{1\}}$	$0.15997^{\{5\}}$	$0.18667^{\{7\}}$	$0.15659^{\{3\}}$
	MSE	$0.03771^{\{2\}}$	$0.03735^{\{1\}}$	$0.04578^{\{6\}}$	$0.03805^{\{3\}}$	$0.04232^{\{5\}}$	$0.0515^{\{7\}}$	$0.04006^{\{4\}}$
	MRE	$0.0523^{\{4\}}$	$0.05131^{\{2\}}$	$0.05585^{\{6\}}$	$0.05128^{\{1\}}$	$0.05332^{\{5\}}$	$0.06222^{\{7\}}$	$0.0522^{\{3\}}$
	LPBCI	$0.72366^{\{2\}}$	$0.75779^{\{3\}}$	$0.79653^{\{5\}}$	$0.70472^{\{1\}}$	$0.79882^{\{6\}}$	$0.80659^{\{7\}}$	$0.76146^{\{4\}}$
	CP	$0.922^{\{5\}}$	$0.946^{\{1\}}$	$0.936^{\{2\}}$	$0.918^{\{6\}}$	$0.93^{\{4\}}$	$0.896^{\{7\}}$	$0.932^{\{3\}}$
	$\sum Ranks$	$15^{\{2.5\}}$	$15^{\{2.5\}}$	$29^{\{6.5\}}$	$8^{\{1\}}$	$25^{\{5\}}$	$29^{\{6.5\}}$	$19^{\{4\}}$
150	AVES	3.00171	3.0274	3.01298	2.97965	3.0088	2.96231	3.01348
	BIAS	$0.11625^{\{3\}}$	$0.1241^{\{6\}}$	$0.12197^{\{4\}}$	$0.10307^{\{1\}}$	$0.12317^{\{5\}}$	$0.13519^{\{7\}}$	$0.1147^{\{2\}}$
	MSE	$0.02121^{\{3\}}$	$0.02433^{\{5\}}$	$0.02472^{\{6\}}$	$0.0169^{\{1\}}$	$0.02336^{\{4\}}$	$0.02784^{\{7\}}$	$0.02045^{\{2\}}$
	MRE	$0.03875^{\{3\}}$	$0.04137^{\{6\}}$	$0.04066^{\{4\}}$	$0.03436^{\{1\}}$	$0.04106^{\{5\}}$	$0.04506^{\{7\}}$	$0.03823^{\{2\}}$
	LPBCI	$0.52404^{\{2\}}$	$0.55323^{\{3\}}$	$0.57355^{\{5\}}$	$0.51735^{\{1\}}$	$0.57793^{\{6\}}$	$0.60031^{\{7\}}$	$0.5586^{\{4\}}$
	CP	$0.914^{\{6\}}$	$0.916^{\{5\}}$	$0.928^{\{4\}}$	$0.938^{\{2\}}$	$0.936^{\{3\}}$	$0.902^{\{7\}}$	$0.944^{\{1\}}$
	$\sum Ranks$	$13^{\{2\}}$	$23^{\{4.5\}}$	$23^{\{4.5\}}$	$10^{\{1\}}$	$25^{\{6\}}$	$29^{\{7\}}$	$17^{\{3\}}$
300	AVES	3.00494	3.01183	2.99074	2.9849	2.99824	2.96768	3.00324
	BIAS	$0.07815^{\{2\}}$	$0.0824^{\{3\}}$	$0.08347^{\{5\}}$	$0.07731^{\{1\}}$	$0.08342^{\{4\}}$	$0.09862^{\{7\}}$	$0.08451^{\{6\}}$
	MSE	$0.00956^{\{2\}}$	$0.01093^{\{3\}}$	$0.01094^{\{4\}}$	$0.00943^{\{1\}}$	$0.01105^{\{5\}}$	$0.01522^{\{7\}}$	$0.01121^{\{6\}}$
	MRE	$0.02605^{\{2\}}$	$0.02747^{\{3\}}$	$0.02782^{\{5\}}$	$0.02577^{\{1\}}$	$0.02781^{\{4\}}$	$0.03287^{\{7\}}$	$0.02817^{\{6\}}$
	LPBCI	$0.37131^{\{2\}}$	$0.39^{\{4\}}$	$0.40609^{\{6\}}$	$0.36489^{\{1\}}$	$0.40553^{\{5\}}$	$0.43194^{\{7\}}$	$0.38752^{\{3\}}$
	CP	$0.942^{\{1\}}$	$0.922^{\{6\}}$	$0.934^{\{4\}}$	$0.924^{\{5\}}$	$0.938^{\{2\}}$	$0.89^{\{7\}}$	$0.936^{\{3\}}$
	$\sum Ranks$	$15^{\{2.5\}}$	$15^{\{2.5\}}$	$24^{\{4.5\}}$	$7^{\{1\}}$	$24^{\{4.5\}}$	$29^{\{7\}}$	$26^{\{6\}}$