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A Study on Bivariate Inverse Topp-Leone Model to Counter Heterogeneous Data: Properties, Dependence Studies, Classical and Bayesian Estimation

Shikhar Tyagi*
Department of Statistics and Data Science, Christ (Deemed to be University), Bangalore, India.
*Corresponding author; e-mail: shikhar1093tyagi@gmail.com

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Abstract

In probability and statistics, reliable modeling of bivariate continuous characteristics remains a real insurmountable consideration. During the analysis of bivariate data, we have to deal with heterogeneity that is present in data. Therefore, for dealing with such a scenario, we investigate a novel technique based on a Farlie-Gumbel-Morgenstern (FGM) copula and the inverse Topp-Leone (ITL) model in this study. The idea is to use the oscillating functionalities of the FGM copula and the flexibility of the ITL model to propose a serious bivariate solution for the modeling of bivariate lifetime phenomena to counter the heterogeneity present in data. Both theory and practice are developed. In particular, we determine the main functions related to the model, like the cumulative model function, probability density function, and various useful dependence measures for bivariate modeling. The model parameters are estimated using the maximum likelihood method and Bayesian framework of the Markov Chain Monte Carlo (MCMC) methodology. Following that, model comparison methods are used to compare models. To explain the findings and show that better models are recommended, the famous Drought and Burr data sets are used.

Keywords: Bivariate continuous model, copula, dependence, FGM, modeling, inference, inverse Topp-Leone, Bayesian, MCMC.

1. Introduction

Classical probability models are important throughout many domains of applied research, including reliability, economics, medical sciences, and other advanced disciplines. For assessing lifetime data, the gamma and exponential distributions are often used in probability distributions. In the literature, several extensions of the gamma and exponential distributions, as well as their mixtures, have been proposed and explored, and have been effectively used for modeling and understanding different lifespan phenomena (see Johnson et al. 1994; Sarhan and Kundu 2009; Sen et al. 2018). The classical distributions have constraints when dealing with a large range of real-world data, which motivates the development of new flexible distribution families. Various methods for creating bivariate distributions from conventional univariate distributions have been demonstrated in recent times.

Numerous distributions have been suggested for the study of bivariate lifetime data, which extend several prominent univariate distributions including exponential, Weibull, Pareto, gamma, log-normal, xgamma, inverse Lindley, Burr XII, and Teissier distributions. (see, for example, Gumbel 1960; Marshall and Olkin 1967; Sankaran and Nair 1993; Kundu and Gupta 2009; Sarhan et al. 2011; Abulebda et al. 2022; Abulebda et al. 2023; Tyagi et al. 2023; Tyagi 2024). The formation of bivariate distributions employing conditional and marginal distributions is a suitable strategy that has received a lot of attention in recent years. Several magnificent approaches for generating bivariate distributions through order statistics have recently been presented and researched, which contain both absolutely continuous and singular components and may be advantageous in circumstances when data ties exist. For some recent references, one can refer to Dolati et al. (2014), Mirhosseini et al. (2015), and Kundu and Gupta (2017). Copula models have lately been used to describe the dependency between random variables, in addition to current methodologies. A copula is a function that connects the marginals to the joint distribution and has been widely utilized in finance, biology, engineering, hydrology, and geophysics to explain dependency among random variables. On the unit interval [0,1], a copula is a multivariate distribution function with uniform one-dimensional margins. In this paper, we restrict our study to a bivariate copula. A formal definition of the bivariate copula is as follows:

A function $C:[0,1]\times[0,1]\rightarrow[0,1]$ is a bivariate copula if it satisfies the following properties:

i. For every $u,v\in[0,1]$

$$C(u,0)=0=C(0,v),\ \ C(u,1)=1 \text{ and } C(1,v)=v.$$

ii. For every $u_1,u_2,v_1,v_2\in[0,1]$ such that $u_1\le u_2$ and $v_1\le v_2$

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1)\ge 0.$$

Let X and Y be random variables with joint distribution function $F,$ and marginal F_1 and $F_2,$ respectively, then Sklar (1959) says that there exists a copula function C which connects marginals to the joint distribution via the relation $F(x,y)=P(X\le x,Y\le y)=C(F_1(x),F_2(y)).$ If X and Y are continuous, then the copula C is unique; otherwise, it is uniquely determined on $\text{Range}(F_1)\times\text{Range}(F_2).$ The associated joint density is $f(x,y)=c(F_1(x),F_2(y))f_1(x)f_2(y),$ where c is copula density. The copula approach provides a powerful tool for constructing a large class of multivariate distributions based on marginals from different families. Any joint distribution function may be represented through copula in which dependence structure and marginals are separately specified. For a good source on copulas, one may refer to Nelsen (2006) and Joe (2014). Copula methods could be a flexible approach for constructing a large class of bivariate lifetime distributions with the ability to cope with different kinds of data and perceive the two lifetimes of the same patient. For example, it may be of interest in the study of human organs associated with kidneys or eyes, and the times between the first and second hospitalization for a particular disease (see Rinne 2008; Bhattacharjee and Mishra 2016).

The aim of this paper is to introduce a new bivariate inverse Topp-Leone (BITL) model and explore its various statistical properties with an application in real data. This paper is organized as follows: In Section 2, we review some basics of the univariate ITL model. With the help of the univariate ITL model, we define a new family of BITL model using the FGM copula. In Section 3, we derive the expressions for joint survival function, joint hazard rate, and joint reversed hazard rate for the proposed BITL model. In Section 4, we present some concepts of dependence measures alike, orthant dependence, and hazard gradient function their important properties for the BITL model. In Sections 5 and 6 we estimate parameters of the BITL model using maximum likelihood estimation and Bayesian estimation paradigm as well as construct the confidence intervals for parameters under respective methods. Section 7, demonstrate data generation and several numerical experiments. Finally, an application to real data is demonstrated in Section 8. Essence and deliberation are done regarding the complete study in Section 9.

2. Bivariate Inverse Topp-Leone Model

Topp and Leone (1955) illustrated the Topp-Leone (TL) model with minimal support as a conceptual model in reliability assessments. The density function of the TL model is J-shaped, whereas the hazard function is bathtub-shaped. Numerous scholars have done groundbreaking disquisition due to the relevance of the TL model. Hassan et al. (2020) acquired an inverse modified form of the TL model specified on the IR+ domain, named the Inverted Topp-Leone model, due to the importance and relevance of inverted models with distribution function (DF), probability density function (pdf), and survival function:

$$F_X(x) = 1-(x+1)^{-2\xi}(2x+1)^{\xi}; x \in \mathrm{IR}^+, \xi \in \mathrm{IR}^+, \tag{1}$$

$$f_X(x) = 2\xi x(x+1)^{-(2\xi+1)}(2x+1)^{\xi-1}; x \in \mathrm{IR}^+, \xi \in \mathrm{IR}^+, \tag{2}$$

$$\phi_X(x) = (x+1)^{-2\xi}(2x+1)^{\xi}; x \in \mathrm{IR}^+, \xi \in \mathrm{IR}^+, \tag{3}$$

respectively. FGM copula is one of the most popular parametric families of copulas and has been widely used in literature due to its simple structure. Morgenstern (1956) proposed the FGM family and was later studied by Gumbel (1958, 1960) using normal and exponential marginals, respectively. Farlie (1960) extended this family and derived its correlation structure, hence termed the FGM family of distributions. The bivariate FGM copula is given by

$$C(u,v) = uv[1+\delta(1-u)(1-v)], \delta \in [-1,1]. \tag{4}$$

In order to achieve wider applications of the FGM copula in real applications, a large number of generalized FGM copulas have been proposed and studied in the literature. Some of the recent references include Amblard and Girard (2009) and Pathak and Vellaisamy (2016).

The bivariate distribution determined by FGM copula is

$$F(x,y) = F_1(x)F_2(y)[1+\delta(1-F_1(x))(1-F_2(y))]; \quad \delta \in [-1,1]. \tag{5}$$

A new family of BITL model via FGM copula is given by

$$F_{(X,Y)}(x,y) = \left(1-(x+1)^{-2\xi_1}(2x+1)^{\xi_1}\right)\left(1-(y+1)^{-2\xi_2}(2y+1)^{\xi_2}\right)\left(1+\delta\left(\left(\frac{2(2x+1)^{\xi_1}}{(x+1)^{2\xi_1}}\right)\left(\frac{2(2y+1)^{\xi_2}}{(y+1)^{2\xi_2}}\right)\right)\right). \tag{6}$$

A random vector (X,Y) is said to have a bivariate inverted Topp-Leone (BITL) model with parameters ξ_1, ξ_2 and δ if, its distribution function is given by (5), and is denoted by BITL (ξ_1, ξ_2, δ). This family includes a mixture of exponential and gamma distributions and may be useful in a wide class of real data. The joint density of the BITL model $f(x,y)$ defined in (5) is

$$f_{(X,Y)}(x,y) = 4\xi_1\xi_2 xy(x+1)^{-(2\xi_1+1)}(2x+1)^{\xi_1-1}(y+1)^{-(2\xi_2+1)}(2y+1)^{\xi_2-1}$$
$$\left(1+\delta\left(\left(\frac{2(2x+1)^{\xi_1}}{(x+1)^{2\xi_1}}-1\right)\left(\frac{2(2y+1)^{\xi_2}}{(y+1)^{2\xi_2}}-1\right)\right)\right). \tag{7}$$

3. Reliability Properties

Statistical properties are essential in influencing whether such a bivariate distribution can be implemented to a certain type of data. The bivariate model BITL established in this study is significant because it may be used to conduct an investigation of the reliability of a system consisting of two components. As a consequence, numerous reliability functions, such as the survival function, hazard function, reversed hazard rate, and conditional distribution must be constructed. The above-mentioned reliability characteristics for the bivariate distribution are derived in the ensuing subsections.

3.1. Survival function

There are several ways to construct the reliability function for the bivariate distribution; we prefer to use the copula approach to express the reliability function for the BITL model by using the marginal survival function $\phi(x)$ and $\phi(y)$ where X and Y the random variable and selection dependence structure.

Theorem 1. *The joint survival function for the copula is as follows*

$$\phi(x,y) = C(\phi(x),\phi(y)),$$

where the marginal survival function $u=\phi(x)$ *and* $v=\phi(y)$. *The reliability function of FGM-BITL based on Equation (8)*

$$\phi(x,y) = \frac{(2x+1)^{\xi_1}(2y+1)^{\xi_2}\left(\frac{\delta\left((2x+1)^{\xi_1}(2y+1)^{\xi_2}\right)}{(x+1)^{2\xi_1}(y+1)^{2\xi_2}}+1\right)}{(x+1)^{2\xi_1}(y+1)^{2\xi_2}}. \tag{8}$$

3.2. Hazard function

Theorem 2. *Let* (X,Y) *be a bivariate random vector with joint density* $f(x,y)$ *and survival function* $\phi(x,y) = P(X\in(x,+\infty), Y >\in (y,+\infty))$. *Then the bivariate hazard rate function is defined as*

$$H(x,y) = \frac{f(x,y)}{\phi(x,y)} = \frac{4\xi_1\xi_2 xy\left(\delta\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}-1\right)\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}-1\right)+1\right)}{(x+1)(2x+1)(y+1)(2y+1)\left(\delta(2x+1)^{\xi_1}(x+1)^{-2\xi_1}(y+1)^{-2\xi_2}(2y+1)^{\xi_2}+1\right)}.$$

3.3. Reversed hazard rate function

Theorem 3. *Let* (X,Y) *be a bivariate random vector with joint density* $f(x,y)$ *and distribution function* $F(x,y) = P(X\in(0,x), Y\in(0,y))$. *Then the bivariate reversed hazard rate function is defined as*

$$m(x,y) = \frac{f(x,y)}{F(x,y)}$$

$$m(x,y) = \frac{4\xi_1\xi_2 xy(2x+1)^{\xi_1-1}(2y+1)^{\xi_2-1}\left(\delta\left((x+1)^{2\xi_1}-2(2x+1)^{\xi_1}\right)\left((y+1)^{2\xi_2}-2(2y+1)^{\xi_2}\right)+(x+1)^{2\xi_1}(y+1)^{2\xi_2}\right)}{(x+1)(y+1)\left((x+1)^{2\xi_1}-(2x+1)^{\xi_1}\right)\left((y+1)^{2\xi_2}-(2y+1)^{\xi_2}\right)\left(\delta(2x+1)^{\xi_1}(2y+1)^{\xi_2}+(x+1)^{2\xi_1}(y+1)^{2\xi_2}\right)}.$$

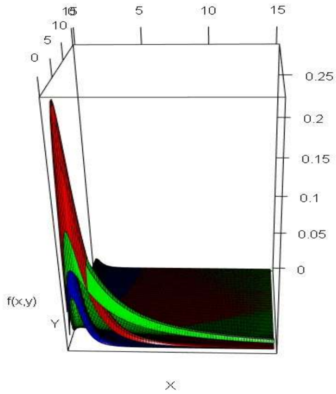

Figure 1 PDF BITL model

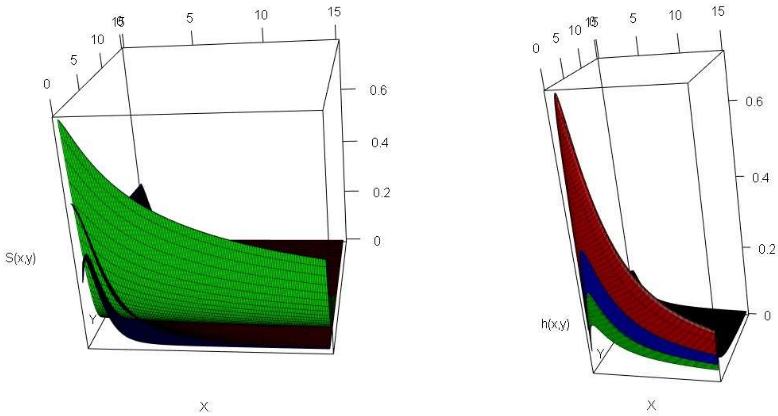

Figure 2 Survival BITL model and hazard BITL model

4. Constructive Dependence Measure

4.1. Orthant dependence

In the existing research, there are already several formulations of positive and negative dependency for multivariate distributions of varying degrees of strength; see, for example, Joe (1997). A random vector (X,Y) is said to be positive upper orthant dependent (PUOD) iff,

$$P(X \in (x,+\infty), Y \in (y,+\infty)) \geq P(X \in (x,+\infty))P(Y \in (y,+\infty)); \forall x, y \in \mathrm{IR}^{+}, \quad (9)$$

and negative upper orthant dependent (NUOD) iff,

$$P(X \in (x,+\infty), Y \in (y,+\infty)) \leq P(X \in (x,+\infty))P(Y \in (y,+\infty)); \forall x, y \in \mathrm{IR}^{+}. \quad (10)$$

Similarly, the second is; A random vector (X,Y) is said to be positive lower orthant dependent (PLOD) iff,

$$P(X \in (0,x), Y \in (0,y)) \geq P(X \in (0,x))P(Y \in (0,y)); \forall x, y \in \mathrm{IR}^{+}, \quad (11)$$

and negative upper orthant dependent (NLOD) iff,

$$P(X \in (0,x), Y \in (0,y)) \leq P(X \in (0,x))P(Y \in (0,y)); \forall x, y \in \mathrm{IR}^{+}. \quad (12)$$

We already have the joint survival function of the BITL model given in Equation (8) as well as the marginal survival function given in Equation (3). By using these equations, we can easily verify that (X,Y) satisfy (8). Using the joint DF of BITLD given in Equation (6) and the marginal DFs of X and Y, we can easily verify that (X,Y) satisfies (11). Therefore, the random vector (X,Y) is PUOD as well as PLOD if $\delta<0.$ Consequently, the random vector (X,Y) with BITLD is POD, if $\delta<0.$ Similarly, the random vector (X,Y) is NUOD as well as NLOD if $\delta>0.$ Thus, BITL satisfies both NUOD and NLOD, and hence, we can say that BITL is NOD.

4.2. Hazard gradient function

Consider a bivariate random vector (X,Y) with joint density $f(x,y)$ and survival function $\phi(x,y),$ then the hazard components are defined as (see Johnson and Kotz 1975)

$$\eta_1(x,y)=-\frac{\partial}{\partial x}\ln\phi(x,y),\quad \eta_2(x,y)=-\frac{\partial}{\partial y}\ln\phi(x,y).$$

The vector $(\eta_1(x,y),\eta_2(x,y))$ is termed the hazard gradient of a bivariate random vector $(X,Y).$ Note that $\eta_1(x,y)$ is the failure rate of X with given information $Y>y.$ Similarly, $\eta_2(x,y)$ is the failure rate of Y given $X>x.$ Hence, for the BITL model, the hazard gradient is in Proposition 1.

Proposition 1

$$\eta_1(x,y)=\frac{2\xi_1 x\left(2\delta(2x+1)^{\xi_1}(2y+1)^{\xi_2}+(x+1)^{2\xi_1}(y+1)^{2\xi_2}\right)}{(x+1)(2x+1)\left(\delta(2x+1)^{\xi_1}(2y+1)^{\xi_2}+(x+1)^{2\xi_1}(y+1)^{2\xi_2}\right)}. \tag{13}$$

$$\eta_2(x,y)=\frac{2\xi_2 y\left(2\delta(2x+1)^{\xi_1}(2y+1)^{\xi_2}+(x+1)^{2\xi_1}(y+1)^{2\xi_2}\right)}{(y+1)(2y+1)\left(\delta(2x+1)^{\xi_1}(2y+1)^{\xi_2}+(x+1)^{2\xi_1}(y+1)^{2\xi_2}\right)}. \tag{14}$$

5. Estimation Strategies

5.1. Maximum likelihood estimation

This section describes the estimation of the unknown parameters of the BITL model through the maximum likelihood method. Based on MLE, estimators are obtained by maximizing the loglikelihood function with respect to each parameter separately. Let consider $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$ be a bivariate random sample of size n from the BITL model. Then, the likelihood function is given as

$$L(\Xi,\delta)=4^n\xi_1^n\xi_2^n\prod_{i=1}^{n}\left(x_iy_i(x_i+1)^{-(2\xi_1+1)}(2x_i+1)^{\xi_1-1}(y_i+1)^{-(2\xi_2+1)}(2y_i+1)^{\xi_2-1}\right) \tag{15}$$
$$\prod_{i=1}^{n}\left(1+\delta\left(\left(\frac{2(2x_i+1)^{\xi_1}}{(x_i+1)^{2\xi_1}}-1\right)\left(\frac{2(2y_i+1)^{\xi_2}}{(y_i+1)^{2\xi_2}}-1\right)\right)\right),$$

where $\Xi\in(\xi_1,\xi_2).$

$$\log L(\Xi,\delta)=n\left(\log 4+\log\xi_1+\log\xi_2\right)-(1+2\xi_1)\sum_{i=1}^{n}\log(1+x_i)+(\xi_1-1)\sum_{i=1}^{n}\log(1+2x_i)$$
$$-(1+2\xi_2)\sum_{i=1}^{n}\log(1+y_i)+(\xi_2-1)\sum_{i=1}^{n}\log(1+2y_i)$$

$$+\sum_{i=1}^{n}\log\left(1+\delta\left(\left(\frac{2(2x_i+1)^{\xi_1}}{(x_i+1)^{2\xi_1}}-1\right)\left(\frac{2(2y_i+1)^{\xi_2}}{(y_i+1)^{2\xi_2}}-1\right)\right)\right).$$

$$\frac{\partial\ln L(\Xi,\delta)}{\partial\xi_1}=\frac{n}{\xi_1}-2\sum_{i=1}^{n}\log(1+x_i)+\sum_{i=1}^{n}\log(1+2x_i)\delta\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}-1\right)$$
$$+\frac{\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}\log(2x+1)-4(x+1)^{-2\xi_1}(2x+1)^{\xi_1}\log(x+1)\right)}{\delta\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}-1\right)\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}-1\right)+1}.$$

$$\frac{\partial\ln L(\Xi,\delta)}{\partial\xi_2}=\frac{n}{\xi_2}-2\sum_{i=1}^{n}\log(1+y_i)+\sum_{i=1}^{n}\log(1+2y_i)\delta\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}-1\right)$$
$$+\frac{\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}\log(2y+1)-4(y+1)^{-2\xi_2}(2y+1)^{\xi_2}\log(y+1)\right)}{\delta\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}-1\right)\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}-1\right)+1}.$$

$$\frac{\partial\ln L(\Xi,\delta)}{\partial\delta}=\frac{\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}-1\right)\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}-1\right)}{\delta\left(2(x+1)^{-2\xi_1}(2x+1)^{\xi_1}-1\right)\left(2(y+1)^{-2\xi_2}(2y+1)^{\xi_2}-1\right)+1}.$$

The MLE $(\hat{\xi}_1,\hat{\xi}_2,\hat{\delta})$ can be obtained by solving simultaneously the likelihood equations

$$\frac{\partial\ln L(\Xi,\delta)}{\partial\delta}|_{\delta=\hat{\delta}}=0,\ \frac{\partial\ln L(\Xi,\delta)}{\partial\xi_1}|_{\xi_1=\hat{\xi}_1}=0,\ \frac{\partial\ln L(\Xi,\delta)}{\partial\xi_2}|_{\xi_2=\hat{\xi}_2}=0.$$

Since the estimators based on likelihood equations are not in a close standard form. So, we perform the parameter estimation using a non-linear optimization algorithm through R software.

5.2. Bayesian estimation strategies via MCMC techniques

5.2.1. Methodology

In this section, the Bayesian paradigm for unknown parameters of both models is derived using left censoring in the case of both informative and flat priors. Three different loss functions are considered: the squared error loss function (SELF), modified (quadratic) squared error loss function (MQSELF), and precautionary loss function (PLF). The following is a brief description of these loss functions, priors, and credible intervals:

5.2.2. Square error loss function (SELF)

The loss function $L(\Xi,\hat{\Xi})=(\hat{\Xi}-\Xi)^2$ is called SELF, which is the simplest symmetric loss function. The Bayes estimator of Ξ under SELF is $\hat{\Xi}_{SELF}=E(\Xi\mid X,Y)$, with risk $Var(\Xi\mid X,Y)$ where the expectation and variance are taken with respect to posterior PDF. It was originally used in estimation problems when an unbiased estimator of Ξ is being considered. Another reason for SELF's popularity is its relationship to classical least squares theory. SELF is neither bound nor concave. The convexity is particularly distressing because large errors are severely penalized. The SELF gives equal weightage to overestimation and underestimation due to its symmetric nature, which is not always true. As a result, we consider two asymmetric loss functions, MQSELF and PLF.

5.2.3. Modified quadratic square error loss function (MQSELF)

The modified quadratic squared error loss function (MQSELF) is an alternative loss function of SELF with form,

$$L(\Xi,\hat{\Xi})=\left(\frac{(\hat{\Xi}-\Xi)}{\hat{\Xi}}\right)^2.$$

The Bayes estimator of Ξ under MQSELF is

$$\hat{\Xi}_{MQSELF}=\frac{E(\Xi^{-1}\mid X,Y)}{E(\Xi^{-2}\mid X,Y)},$$

with risk

$$R(\Xi,\hat{\Xi}_{MQSELF})=1-\frac{(E(\Xi^{-1}\mid X,Y))^2}{E(\Xi^{-2}\mid X,Y)},$$

where the expectation is taken with respect to posterior PDF.

5.2.4. Precautionary loss function (PLF)

Norstrom (1996) described an alternative asymmetric precautionary loss function with quadratic loss function as a special case. This loss function approaches infinity near the origin to prevent underestimation and thus given a conservative estimation. It is very useful when underestimation may lead to serious consequences. The PLF is defined as

$$L(\Xi,\hat{\Xi})=\frac{(\hat{\Xi}-\Xi)^2}{\hat{\Xi}}.$$

The Bayes estimator of Ξ under PLF is

$$\hat{\Xi}_{PLF}=\sqrt{E(\Xi^2\mid X,Y)},$$

with risk

$$R(\Xi,\hat{\Xi}_{PLF})=2\left[\sqrt{E(\Xi^2\mid X,Y)}-E(\Xi\mid X,Y)\right],$$

where the expectation is taken with respect to posterior PDF.

5.2.5. Flat prior

The choice of the prior distribution is often dependent on the type of prior information available to us. If we have little information or no information about the parameter, a flat prior should be used. A lot of practitioners earlier utilized flat priors (see Ibrahim et al. 2001; Santos and Achcar 2010). Under flat priors, we use the gamma distribution for baseline parameters $\hat{\Xi},\Xi$ and the uniform distribution for δ. That is, the considered priors PDFs are

$$g(\Xi)=\frac{1}{\nu_1^{\nu_2}\Gamma(\nu_2)}e^{-\frac{\Xi}{\nu_1}}\Xi^{\nu_2-1},\ g(\delta)=\frac{1}{b-a}.$$

Here, $\nu_1=\nu_2=0.0001, b=1,$ and $a=-1$.

5.2.6. Informative prior

In terms of informative priors, the hyper parameters are chosen in such a way that the expectation of the prior distribution of each unknown parameter equals the true value. This method has been used by several researchers, including Chacko and Mohan (2018).

This section studies the Bayesian estimation to achieve the estimates of BITL model parameters. As we see the maximum likelihood estimates (MLEs) method has crucial importance and is inappropriate when a high dimensional optimization problem is there. So, Bayesian estimation can be better to estimate the parameter than MLEs. In the BITL model, we have a three-dimensional optimization problem. In that scenario, it is not possible to compute posterior distribution in a close form. To apply the Bayesian approach, we consider the assumption of independence in the parameters (see Ibrahim et al. 2001; Santos and Achcar 2010). Under this assumption, the joint posterior density function of parameters for given variables and is obtained as

$$\pi(\Xi \mid X,Y) \propto L(\Xi \mid X,Y) \times g_1(\xi_1) g_2(\xi_2) g_3(\delta),$$

where $g_i(.)$ indicates the prior density function with known hyper parameters of the corresponding argument for parameters, and the likelihood function is L, defined in Section 5.1. We consider both informative and flat priors for better outcomes.

Due to the high-dimensions integration of joint posterior distributions, it is problematic to integrate out. So, we adopt the most popular MCMC technique. In the MCMC technique, the Metropolis-Hastings algorithm and Gibbs samplers have been used. Heidelberger-Welch test has been used to monitor the convergence of a Markov chain to a stationary distribution. For that, it has been considered that full conditional distributions can be obtained as proportional to the joint distribution of the parameter of the model. The full conditional distribution for the parameter ξ_1 is

$$\pi_1(\xi_1 \mid X,Y,\Xi-\xi_1) \ \propto \ L(\xi_1 \mid X,Y,\Xi-\xi_1) \cdot g_1(\xi_1).$$

Similarly, full conditional distributions for other parameters can be obtained.

6. Confidence Intervals

6.1. Asymptotic Confidence Intervals

Because although the MLEs of are not in compact structures, possessing accurate confidence intervals for is problematic. As a possible consequence, we can employ the asymptotic behavior of the maximum likelihood estimator to ascertain asymptotic confidence intervals (CIs) for the model parameters.

Because the specific sampling distributions of the MLEs cannot be derived explicitly, we employ a large sample theory to construct asymptotic confidence intervals for the model parameters. By using the general theory of MLEs, the asymptotic distribution of $(\Xi,\hat{\Xi})$ is $N_3(0,\zeta^{-1})$, where $\zeta(\Xi)$ is the Fisher's information matrix having elements as

$$\zeta\left(\Xi_{i,j}\right) = E\left[-\frac{\partial^2 \ln L}{\partial \Xi_i \partial \Xi_j}\right]; \forall i,j = 1,2,3$$

which may be numerically obtained. The Fisher information matrix $\zeta(\Xi)$ can be approximated by

$$\zeta(\Xi) = \begin{bmatrix} \frac{\partial^2 \ln L}{\partial \xi_1^2} & \frac{\partial^2 \ln L}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 \ln L}{\partial \xi_1 \partial \delta} \\ \frac{\partial^2 \ln L}{\partial \xi_1 \partial \xi_2} & \frac{\partial^2 \ln L}{\partial \xi_2^2} & \frac{\partial^2 \ln L}{\partial \xi_2 \partial \delta} \\ \frac{\partial^2 \ln L}{\partial \xi_1 \partial \delta} & \frac{\partial^2 \ln L}{\partial \xi_2 \partial \delta} & \frac{\partial^2 \ln L}{\partial \delta^2} \end{bmatrix}, \text{ and } \zeta\left(\hat{\Xi}\right) \approx \left[-\frac{\partial^2 \ln L}{\partial \Xi_i \partial \Xi_j}\Big|_{\Xi=\hat{\Xi}}\right].$$

6.2. Construction of highest posterior density credible interval

A credible interval is a range of values inside the region of a posterior probability distribution in Bayesian statistics. The $100\times(1-\eta)$ equal tail credible interval for appropriate posterior distribution can be determined as (Eberly and Casella 2003).

$$P(\xi_1 < L) = \int_{x\in(-\infty,L)} \pi(\xi_1 \mid X,Y) d\xi_1 = \frac{\eta}{2};\ \ P(\xi_1 > U) = \int_{x\in(U,\infty)} \pi(\xi_1 \mid X,Y) d\xi_1 = \frac{\eta}{2},$$

where $\pi(\xi_1 \mid X,Y)$ is the posterior density of ξ_1 and (L,U) are the lower and upper limit of the credible interval. Subsequently, we can find credible intervals for parameters ξ_2 and δ.

7. Simulation Analysis

7.1. Classical simulation

In this section, we report a simulation study for the BITL model derived using the FGM copula. First, we describe the random sample generation from the BITL model. We employ the conditional procedure for random sample generation which has been reported in Nelsen (2006). Let X and Y be a random sample having the BITL model determined by the FGM copula C. The copula C is a joint distribution of a bivariate vector (U,V) with marginals as uniform $U(0,1)$. The conditional distribution of the vector (U,V) is given as $P(V \le v \mid U = u) = \frac{\partial}{\partial u} C_1(u,v) = v[1+\delta(1-v)(1-2u)]$. Using the conditional distribution approach, random numbers (x,y) from the BITL can be generated using the following algorithm:

1) From uniform $U(0,1)$ generate two independent samples u and t.
2) Set $t = \frac{\partial}{\partial u} C(u,v)$ and solved for v.
3) Find $x = F^{-1}(u;\xi_1)$ and $y = F^{-1}(v;\xi_2)$, where F^{-1} is the inverse of ITL.
4) Finally, the desired random sample is (x,y).

For parameter estimation, we use maximum likelihood and Bayesian paradigm methods. A simulation study is carried out based on the following data generated from the BITL model. The value of the parameters ξ_1 and ξ_2 is chosen with different values of the copula parameter δ and different sizes of the sample $(n = 20, 50, 100)$, as shown for the following cases for the random variables generated from the BITL model:

Case 1: $(\xi_1 = 0.5, \xi_2 = 1.5, \delta = -0.1)$;

Case 2: $(\xi_1 = 1.5, \xi_2 = 0.5, \delta = 0.1)$;

Case 3: $(\xi_1 = 2, \xi_2 = 5, \delta = -0.6)$;

Case 4: $(\xi_1 = 5, \xi_2 = 2, \delta = 0.6)$.

The simulations in this study are repeated 1,00,000 times. The estimate of parameters by MLE methods along with the mean squared error (MSE) are summarized in Table 1. From the reported table, we conclude the following:

In the simulation study, if the sample size increases, the value of mean squared error decreases in the considered method i.e., MLE. In the simulation table, when the initial value of parameters increases, the corresponding mean square error increases for the small sample, and after that it decreases gradually by increasing the sample size as observed from the corresponding MSE for different values of the parameters. In general, the effect of marginal parameters has little effect on estimating the copula parameters as shown in the table. The simulation study was carried out using the R software (R 3.5.3).

7.2. Bayesian simulation

The main objective of the simulation study is to evaluate the performance of the Bayesian estimation procedure. For the simulation purpose, we have generated data (X, Y) from the BITL using the algorithm as explained in subsection (7.1). Since we do not have any information about the parameters model, we select prior distributions. We take the sample of different sizes 20, 50, and 100 and iterate the chain of Metropolis-Hasting algorithm and Gibbs sampling 1,00,000 times, neglecting the first 10,000 iterations to remove the effect of the initial values and to avoid the autocorrelation problems. The estimates of (Ξ, δ), the corresponding risk under different error LFs with informative and vague priors, HPD intervals, and the Heidelberger-Welch test are shown in Tables 3 and 4. From these tables, we can observe the following points:

1) The performance of Bayes estimates based on the informative priors is better than that of vague priors estimates.
2) As n increases, the risks of all Bayes estimates decrease.
3) Bayes estimates under SELF perform better in the aspect of risks.
4) As n increases, HPD intervals become narrow for all Bayes estimates.
5) The p-values of the Heidelberger-Welch test are large enough (> 0.05) to say that the chain reached stationary distribution.

Under various combinations of (Ξ, δ), the Bayesian approach with informative prior is found to be the best approach for point estimation.

8. Illustration of Real-Life Data

8.1. Drought data

To study the proposed model BITL and elucidate the MLE estimation procedure, we consider the drought data for (Panhandle) climate division of Nebraska state; the real drought data set is demonstrated for the 83 drought events in climate division (Panhandle), we got the data from Nadarajah (2009). The data comprises of the monthly modified Palmer Drought Severity Index (PDSI) for the period from January 1895 to December 2004. The PDSI is often used to measure droughts depending on recent precipitation and temperature; see Alley (1984) for details; when the PDSI is less than zero, then drought is said to have been occurring; see Yevjevich (1967). We applied the K-S test to check the goodness of fit. The goodness of fit was assessed using the K-S test. The fitted and empirical distribution lines are near enough in Figure 4 to indicate that the model fits the data effectively. Nadarajah (2009) used the bivariate Pareto model to analyze this data and discussed the estimation of three parameters. The bivariate data set x and y represent the drought duration and the non-drought duration, respectively. The model of BITL was determined by fitting the model to the observed values.

8.2. Burr data

There are 50 observations on the burr in this data collection. The first component has a hole diameter of 12 mm and a sheet thickness of 3.15 mm. The second component's hole diameter is 9 mm, and the sheet thickness is 2 mm. Two completely different computers create these two-component datasets. This data collection was used by Dasgupta (2011). Before even being processed, every data is multiplied by 10. These transitions will have no influence on our research and are entirely computational. To determine the goodness of fit, we used the K-S test. Figure 5 illustrates that the fitted and empirical distribution lines are near enough to indicate that the model fits the data effectively.

For both data sets, Tables 4 and 5 adduce the estimated values of BITL model parameters employing MLE and Bayesian paradigms. Table 6 gesticulates the result of various model selection criteria for the BITL model, such as AIC, BIC, AICc, and HQIC, and allows us to compare the BITL model to other models available in the literature for drought and Burr data sets. In aspects of AIC, BIC, AICc, and HQIC, we find that the BITL model outperforms the bivariate exponential and bivariate Weibull distributions for drought data, while the BITL model outperforms the bivariate generalized exponential under FGM and Clayton copula, bivariate inverse Lindley, bivariate Pareto, and bivariate Gumbel distributions for Burr data.

9. Essence and Deliberation

In this paper, we have introduced a new BITL model derived from the FGM copula whose univariate marginals follow the ITL model. We derive the expressions for survival function, conditional model, and some concepts related to reliability for the BITL model. Some dependence measures alike, orthant dependence, hazard gradient function measure of dependence are derived and is also studied. For the copula parameter δ, it has been seen that for $\delta < 0$, the BITL model exhibits POD property as well as for $\delta > 0$, the BITL model exhibits POD property, which is a powerful property of dependence. Parameters were estimated using two different methods namely MLE and Bayesian paradigm. Several numerical experiments are also reported in this study. Finally, an application to two real data shows that the BITL model works well and we anticipate that the BITL model may be useful in various piratical applications.

Table 1 Average estimate and MSE for the BITL model parameters under MLE

Parameter	EST	MSE	EST	MSE	EST	MSE
	$n = 20$		$n = 50$		$n = 100$	
$\xi_1(0.5)$	0.44994	0.03359	0.49124	0.00837	0.51038	0.00460
$\xi_2(1.5)$	1.63961	0.24579	1.55581	0.11947	1.55320	0.05628
$\delta(-0.1)$	–0.09708	0.04566	–0.12210	0.04269	–0.12190	0.04144
$\xi_1(1.5)$	1.64941	0.17353	1.45219	0.11205	1.51292	0.04259
$\xi_2(0.5)$	0.53654	0.01896	0.51677	0.01064	0.50434	0.00738
$\delta(0.1)$	0.07519	0.05621	0.11985	0.04653	0.11018	0.04459
$\xi_1(2)$	1.94253	0.3322	2.05297	0.17844	2.04146	0.08243
$\xi_2(5)$	4.83367	1.59106	5.22028	0.85143	4.98964	0.38134
$\delta(-0.6)$	–0.62794	0.03291	–0.59653	0.02988	–0.60825	0.02935
$\xi_1(5)$	5.24843	0.89470	4.98720	0.75524	4.99912	0.45281
$\xi_2(2)$	2.16860	0.27556	2.14914	0.19582	2.02598	0.07480
$\delta(0.6)$	0.59641	0.03518	0.59953	0.03201	0.59807	0.03089

Table 2 Average estimate and risk for the BITL parameters under non-informative prior

Parameter	SELF		MQSELF		PLF		LCL	UCL	H.B.
	EST	Risk	EST	Risk	EST	Risk			
				n = 20					
$\xi_1(0.5)$	0.42107	0.00908	0.37677	0.05580	0.43172	0.02130	0.23422	0.60198	0.448
$\xi_2(1.5)$	1.53472	0.10189	1.40187	0.04484	1.56756	0.06568	0.91744	2.10160	0.861
$\delta(-0.1)$	−0.09987	0.00068	−0.08296	0.08099	0.10398	0.40984	−0.14252	−0.05135	0.337
$\xi_1(1.5)$	1.65066	0.04755	1.59217	0.01810	1.51735	0.01381	1.23037	1.81192	0.625
$\xi_2(0.5)$	0.45284	0.00734	0.42265	0.03348	0.46087	0.01606	0.30173	0.60399	0.514
$\delta(0.1)$	0.09942	0.00090	0.07842	0.11979	0.10385	0.00885	0.05251	0.15960	0.491
$\xi_1(2)$	2.44687	0.07089	1.82431	0.01876	1.91073	0.03398	1.82135	2.66279	0.475
$\xi_2(5)$	6.04350	0.57561	4.33909	0.03998	5.58181	0.05708	4.42050	6.67669	0.886
$\delta(-0.6)$	−0.59628	0.0288	−0.48703	0.10227	0.61996	2.43249	−0.89707	−0.27508	0.253
$\xi_1(5)$	5.67176	0.97718	5.56884	0.04338	5.69730	0.19517	3.16635	6.93249	0.207
$\xi_2(2)$	1.79167	0.13556	1.75400	0.03831	1.80080	0.06663	1.26823	2.67085	0.580
$\delta(0.6)$	0.61615	0.01794	0.54971	0.05834	0.63054	0.02878	0.38023	0.84927	0.377
				n = 50					
$\xi_1(0.5)$	0.58105	0.00624	0.55919	0.01927	0.58640	0.01070	0.44830	0.75402	0.335
$\xi_2(1.5)$	1.41767	0.04037	1.35996	0.02086	1.43184	0.02834	1.02782	1.80458	0.701
$\delta(-0.1)$	−0.09676	0.00064	−0.08495	0.07254	0.10322	0.40617	−0.14623	−0.05757	0.311
$\xi_1(1.5)$	1.48602	0.02127	1.45784	0.00952	1.49316	0.01428	1.22031	2.05042	0.690
$\xi_2(0.5)$	0.55119	0.00673	0.52630	0.02320	0.55726	0.01214	0.41437	0.73473	0.728
$\delta(0.1)$	0.10044	0.00088	0.08026	0.11359	0.10474	0.00860	0.04492	0.15362	0.262
$\xi_1(2)$	1.89375	0.06463	2.37306	0.01656	2.46132	0.02889	1.91699	2.79402	0.188
$\xi_2(5)$	4.70126	0.84814	5.83884	0.01758	4.79061	0.17871	2.79050	6.36473	0.539
$\delta(-0.6)$	−0.62881	0.01611	−0.57190	0.04835	0.64150	2.54063	−0.84829	−0.40908	0.832
$\xi_1(5)$	5.20537	0.55873	4.54858	0.02105	5.25876	0.10679	3.65126	6.57042	0.325
$\xi_2(2)$	1.87603	0.07470	1.79371	0.02272	1.89584	0.03961	1.34756	2.42086	0.088
$\delta(0.6)$	0.59734	0.01443	0.54560	0.04575	0.6093	0.02445	0.35118	0.76468	0.737
				n = 100					
$\xi_1(0.5)$	0.54272	0.00312	0.53136	0.01054	0.54558	0.00573	0.44095	0.65700	0.982
$\xi_2(1.5)$	1.51035	0.02269	1.48039	0.01000	1.51784	0.01499	1.23924	1.79996	0.587
$\delta(-0.1)$	−0.10094	0.00062	−0.08741	0.07643	0.10000	0.39353	−0.14179	−0.05078	0.761
$\xi_1(1.5)$	1.51045	0.02091	1.48244	0.00940	1.66500	0.02868	1.23038	1.78819	0.838
$\xi_2(0.5)$	0.48965	0.00273	0.47856	0.01144	0.49244	0.00556	0.3783	0.57855	0.767
$\delta(0.1)$	0.10214	0.00086	0.08218	0.11120	0.10627	0.00827	0.04797	0.15669	0.529
$\xi_1(2)$	2.24066	0.05009	2.19633	0.00996	2.25181	0.02230	1.39123	2.39339	0.808
$\xi_2(5)$	5.55327	0.31777	5.43900	0.01039	6.09093	0.09487	4.69648	7.48499	0.912
$\delta(-0.6)$	−0.62945	0.01570	−0.57359	0.04752	0.64180	2.54250	−0.84895	−0.41285	0.918
$\xi_1(5)$	4.95800	0.29036	4.99059	0.00916	5.05558	0.05108	4.59938	6.61254	0.672
$\xi_2(2)$	2.01799	0.03279	1.87250	0.01072	2.05131	0.01825	1.43346	2.11660	0.784
$\delta(0.6)$	0.56235	0.01390	0.51283	0.04550	0.57458	0.02392	0.35183	0.79372	0.377

Table 3 Average estimate and risk for the BITL parameters under informative prior

Parameter	SELF		MQSELF		PLF		LCL	UCL	H.B.
	EST	Risk	EST	Risk	EST	Risk			
					$n = 20$				
$\xi_1(0.5)$	0.43249	0.00686	0.40417	0.03235	0.44035	0.01571	0.30116	0.58144	0.612
$\xi_2(1.5)$	1.60513	0.05361	1.53043	0.02456	1.62174	0.03323	1.18865	1.99847	0.601
$\delta(-0.1)$	–0.10609	0.00078	–0.08406	0.08948	0.10453	0.41051	–0.14691	–0.05360	0.255
$\xi_1(1.5)$	1.54243	0.0583	1.42217	0.02779	1.55468	0.03854	1.06090	1.93147	0.783
$\xi_2(0.5)$	0.52736	0.00798	0.48353	0.03302	0.53487	0.01503	0.36722	0.68984	0.551
$\delta(0.1)$	0.09680	0.00113	0.07207	0.14207	0.10690	0.01135	0.04035	0.15248	0.159
$\xi_1(2)$	2.22012	0.04563	2.17301	0.01118	2.23037	0.02396	1.79964	2.49816	0.568
$\xi_2(5)$	5.95027	0.66921	5.86365	0.0282	5.97126	0.13155	3.64046	6.72753	0.273
$\delta(-0.6)$	–0.58369	0.01211	–0.56601	0.03557	0.61755	2.45043	–0.79959	–0.42806	0.254
$\xi_1(5)$	5.42905	0.73468	5.15778	0.02927	5.49630	0.13449	3.85847	6.91938	0.504
$\xi_2(2)$	1.96534	0.05999	1.90552	0.01529	1.98054	0.03041	1.50254	2.39927	0.337
$\delta(0.6)$	0.56500	0.01251	0.52761	0.03685	0.57491	0.02066	0.40270	0.75940	0.150
					$n = 50$				
$\xi_1(0.5)$	0.46177	0.00461	0.44236	0.02122	0.46674	0.00994	0.32832	0.58832	0.823
$\xi_2(1.5)$	1.62633	0.03699	1.57672	0.01591	1.63767	0.02267	1.28794	1.99558	0.560
$\delta(-0.1)$	–0.09771	0.00062	–0.08455	0.07190	0.10081	0.39704	-0.13873	–0.05036	0.117
$\xi_1(1.5)$	1.51784	0.03796	1.46678	0.01727	1.53006	0.02451	1.14212	1.89147	0.286
$\xi_2(0.5)$	0.52129	0.00446	0.49433	0.01794	0.52549	0.00885	0.36995	0.62674	0.474
$\delta(0.1)$	0.10277	0.00092	0.07749	0.12006	0.10313	0.00915	0.05095	0.15784	0.371
$\xi_1(2)$	1.88604	0.04534	1.84002	0.01214	1.89802	0.0205	1.50542	2.25897	0.454
$\xi_2(5)$	5.35037	0.41773	4.77927	0.01556	5.38927	0.07779	4.22675	6.65756	0.774
$\delta(-0.6)$	–0.59271	0.01105	–0.55494	0.03269	0.60196	2.38933	–0.77121	-0.40705	0.297
$\xi_1(5)$	5.21843	0.20412	5.12841	0.00840	5.23349	0.04118	4.45937	5.97864	0.863
$\xi_2(2)$	1.97120	0.05172	1.95366	0.01326	1.97561	0.02550	1.73943	2.25417	0.337
$\delta(0.6)$	0.62428	0.01129	0.58554	0.03308	0.63302	0.01982	0.43793	0.79810	0.465
					$n = 100$				
$\xi_1(0.5)$	0.56471	0.00287	0.55451	0.00912	0.56725	0.00506	0.46440	0.67427	0.630
$\xi_2(1.5)$	1.48149	0.02339	1.45013	0.01067	1.48936	0.01574	1.19773	1.79059	0.943
$\delta(-0.1)$	–0.10073	0.00061	–0.09231	0.07161	0.10899	0.43016	–0.14961	–0.06307	0.276
$\xi_1(1.5)$	1.50301	0.03727	1.49107	0.01711	1.52228	0.02445	1.12757	1.86213	0.764
$\xi_2(0.5)$	0.50123	0.00440	0.50384	0.01717	0.50566	0.00841	0.40135	0.65058	0.465
$\delta(0.1)$	0.09856	0.00087	0.08194	0.11693	0.10248	0.00826	0.04312	0.15224	0.688
$\xi_1(2)$	1.93326	0.03340	1.89848	0.00909	1.94188	0.01724	1.60638	2.32037	0.491
$\xi_2(5)$	5.05434	0.25025	5.18832	0.00740	5.12012	0.04198	5.08042	6.91515	0.166
$\delta(-0.6)$	–0.60767	0.01000	–0.54959	0.02977	0.59219	2.35176	–0.75216	–0.40035	0.806
$\xi_1(5)$	4.94692	0.15733	4.86442	0.00586	4.96751	0.0301	4.07424	5.82884	0.456
$\xi_2(2)$	2.02179	0.01739	1.96920	0.00446	2.03454	0.00881	1.58973	2.42693	0.672
$\delta(0.6)$	0.60069	0.01098	0.55783	0.03270	0.61102	0.01747	0.40657	0.77996	0.971

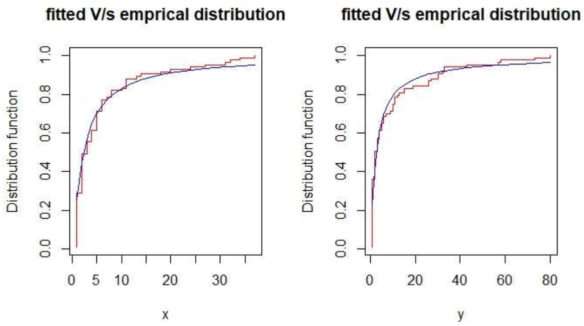

Figure 4 K-S Plots for the BITL model for drought data

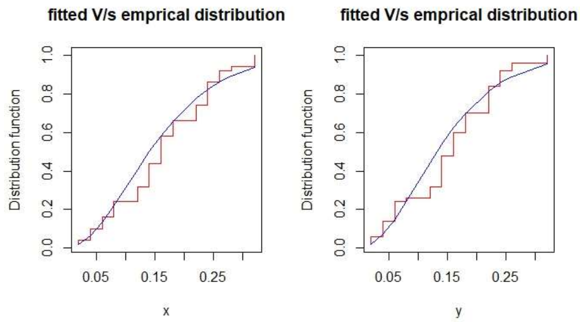

Figure 5 K-S Plots for the BITL model for Burr data

Table 4 MLE and Bayes estimates for the parameters of the BITL model for the drought data set

Estimator	Method	ξ_1	ξ_2	δ
Classical	MLE	1.0120	0.8920	0.4607
	SE	0.1108	0.0984	0.4037
NIP	SELF (Risk)	1.0424 (0.0083)	0.9132 (0.0051)	0.7896 (0.0067)
	MQSELF (Risk)	1.0264 (0.0073)	0.8954 (0.0063)	0.9463 (0.0080)
	PLF	1.0463 (0.0080)	0.9159 (0.0055)	0.7938 (0.0085)
	HPD Interval	(0.8573, 1.2208)	(0.7906, 1.0528)	(0.6505, 0.9268)
	Heidelberg test	0.4540	0.3810	0.8510
IP	SELF	1.0363 (0.0090)	0.9044 (0.0051)	0.7882 (0.0063)
	MQSELF (Risk)	1.0209 (0.0086)	0.8864 (0.0065)	0.7695 (0.0105)
	PLF	1.0406 (0.0086)	0.9072 (0.0056)	0.7921 (0.0080)
	HPD Interval	(0.8666, 1.2281)	(0.7615, 1.0350)	0.6516, 0.9220)
	Heidelberg test	0.6450	0.8950	0.5220

Table 5 MLE and Bayes estimates for the parameters of the BITL model for the Burr data set

Estimator	Method	ξ_1	ξ_2	δ
Classical	MLE	45.4087	50.8509	0.6622
	SE	5.9351	5.9316	0.4057
NIP	SELF (Risk)	44.7918	43.7812	45.0389
	MQSELF (Risk)	50.5703	49.5712	50.8126
	PLF	0.6839	0.6125	0.7006
	HPD Interval	(35.1194, 52.3941)	(41.8500, 59.5972)	(0.4191 0.9353)
	Heidelberg test	44.7918	43.7812	45.0389
IP	SELF	45.2191	44.4838	45.4002
	MQSELF (Risk)	50.0103	49.2706	50.1939
	PLF	0.6721	0.6002	0.6891
	HPD Interval	(38.1260, 53.1879)	(42.1901, 58.3946)	(0.4125, 0.9259)
	Heidelberg test	0.6470	0.2600	0.5850

Table 6 Model selection for the drought and Burr data sets

Data	Copula	Model	-LogL	AIC	BIC	AICc	HQIC
I	FGM	BITL BE	–477.0847	960.1694	967.4259	960.4732	963.0847
	FGM		503.3512	1012.702	1019.959	1013.006	1015.618
	FGM	BW	491.9845	993.969	1006.063	994.7482	998.8278
II	FGM	BITL	–110.6555	–215.311	–209.5749	–214.7893	–213.1267
	FGM	BGE	–106.936	–203.872	–194.312	–202.5084	–200.2315
	Clayton	BGE	–106.116	–202.231	–192.671	–200.8564	–198.5795
	FGM	BIL	–87.565	–169.130	–163.3939	–168.6083	–166.9457
	FGM	BP	–84.524	–163.048	–157.312	–162.5263	–160.8637
	FGM	BG	–84.453	–164.906	–161.082	–164.6507	–163.4498

Disclosure statement

No potential competing interest was reported by the author.

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