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Exponentiated Arctan-X Family of Distribution: Properties, Simulation and Applications to Insurance Data

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Abstract

A new family of distribution has been investigated in this paper. Exponentiated method has been used to generate the new family of distribution. The proposed family of distribution is known as the exponentiated arctan-X family of distributions which are immensely beneficial and play an important role in modelling data sets. Various properties like reliability analysis, moments, moment generating function, quantile function, median are studied. Exponential distribution has been taken as a special case for the specific purpose of the show of strength. To estimate the parameters of the exponentiated arctan exponential distribution, the frequentist approach, i.e., maximum likelihood estimation, is used. A rigorous Monte Carlo simulation analysis is used to determine the efficiency of the obtained estimators. A complete percentage point study has been discussed and presented. The exponentiated arctan exponential model is demonstrated using two life-time data sets. The proposed family of distribution is compared to well-known two, three, and four parameter competitors. For model comparison, we used the most precise tests used to know whether the exponentiated arctan-exponential distribution is more useful than competing models.

Keywords: Percentage points, maximum likelihood estimation method, Fisher's inverse matrix, confidence interval.

1. Introduction

Distribution theory takes uncertainties into account and provides a set of regulations for discussing financial and economic taking decision difficulties. Due to the importance of distribution theory, we were motivated to introduce a new distribution family based on the inverse trigonometric function. The T-X distribution are immensely beneficial and play an important role in modelling data sets. Let us suppose a random variable $T \in (a,b)$ for $-\infty \le a < b < \infty$ having a probability density function (pdf) $y(t)$ and $W[F(x)]$ be a function of a cumulative distribution function of the random variable X which satisfies some statistical conditions such as $W[F(x)] \in (a,b)$, $W[F(x)]$ is differentiable and monotonically non-decreasing and $W[F(x)] \to a$ as $x \to -\infty$ and

$W\left[F(x)\right] \to b$ as $x \to \infty$. Alzaatreh et al. (2014) defined the T-X family cdf by $G(x) = \int_a^{W[F(x)]} y(t)\,dt = Y\left\{W\left[F(x)\right]\right\}$, where $W\left[F(x)\right]$ satisfied all the conditions. The corresponding pdf of T-X family of distribution is $g(x) = \left\{\frac{d}{dx} W\left[F(x)\right]\right\} y\left\{W\left[F(x)\right]\right\}$.

The arctan-X family of distribution is proposed by Alkhairy et al. (2021) with pdf and corresponding cdf are given by

$$g_{\text{arctan}}(x;\theta) = \frac{4}{\pi} \frac{f(x;\theta)}{1 + F(x;\theta)^2}, \quad x \in \mathbb{R}, \tag{1}$$

$$G_{\text{arctan}}(x;\theta) = \frac{4}{\pi} \arctan F(x;\theta), \quad x \in \mathbb{R}\cdot \tag{2}$$

Here $f(x;\theta)$ and $F(x;\theta)$ are considered as the pdf and cdf of baseline random variable depending on the parameter vector $\theta \in \mathbb{R}$. Suppose x be a non-negative random variable which follows exponentiated distribution with the following pdf and cdf $g(x)$ and $G(x)$, respectively

$$g_\theta(x) = \theta\left[F(x)\right]^{\theta-1} f(x), \tag{3}$$

$$G_\theta(X) = \left[F(x)\right]^\theta. \tag{4}$$

Souza (2015) discussed some new trigonometric classes of probabilistic distributions. Some researchers have done works in new distribution using trigonometric function and their inverse. Here a table of chronological review has been added for the recent G-families based on the trigonometric functions and their inverses techniques. Some literature reviews of some recent trigonometric functions and G-families are shown in Table 1.

In this article, we propose and study the family of exponentiated arctan-X distributions (EAT-X). The main advantage of the EAT-X family is that practitioners will have a one-parameter class flexible enough to adapt to real data in applied fields. It can be a good alternative to other one, two, three or four parameters distribution. It may also perform better, in terms of model fit, than other classes of distributions in some practical situations that cannot always be guaranteed. Also, a full account of some of its mathematical properties is provided. We empirically prove that the special models of the EAT-X family can provide a better fit than the other competitive models generated by the aforementioned classes. The rest of the paper is outlined as follows.

In Section 2, we derive a very useful representation for the EAT-X density function. Section 3 includes some general mathematical properties of the proposed family including ordinary and incomplete moments, quantile function (qf), moment generating function (mgf) and entropies. One special model of this family is presented in Section 4 and some plots of their pdfs and hrfs are given. Maximum likelihood estimation of the model parameters is investigated in Section 5. The simulation study has taken placed in section 6. In Section 7, percentage points result of the proposed model are discussed. In Section 8, we perform two applications to real data sets to illustrate the potentiality of the special model of the proposed family. Finally, some concluding remarks are presented in Section 9.

Table 1 Literature reviews of some recent trigonometric functions and G-families

SN	Authors	Year	Contributions in distribution family
1	Souza et al.	2022	Sec-G class
2	Rahman M.	2021	Arcsine-G class
3	Thomas and Friday	2021	Teissier-G class
4	Chesneau et al.	2021	Distribution based on the arccosine
5	Muhammad et al.	2021	Exponentiated sine-G class
6	Jamal et al.	2021	Transmuted sin-G class
7	Ahmad et al.	2021	Exponential T-X class
8	Liang Tung et al.	2021	Arcsine-X class
9	Chaudhary and Kumar	2021	Arctan Lomax distribution
10	Muhammad et al.	2021	Extended cosine-G class
11	Souza et al.	2021	Tan-G class of probability distribution
12	He et al.	2020	Arcsine exponentiated-X class
13	Al-Babtain et al.	2020	Sine Topp-Leone-G class
14	Chesneau and Jamal	2020	Sine Kumaraswamy-G class
15	Chesneau et al.	2019	New class of probability distributions via cosine and sine functions
16	Mahmood et al.	2019	Sine-G class
17	Souza et al.	2019a	Sin-G class
18	Souza et al.	2019b	Cos-G class

2. Exponentiated Arctan-X (EAT-X) Family of Distribution

Here in this section, the derivation of pdf and cdf of exponentiated arctan-X family of distribution is discussed. Let us consider a random variable X that belongs to the Exponentiated Arctan-X family, then its cdf and pdf can be written in the following form

$$G_{e\arctan}\left(x;\theta,\Phi\right)=\left[\frac{4}{\pi}\arctan F\left(x;\Phi\right)\right]^{\theta},\ x\in\mathbb{R}. \tag{5}$$

$$g_{e\arctan}\left(x;\theta,\Phi\right)=\frac{4}{\pi}\frac{\theta\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta-1}f\left(x;\Phi\right)}{1+F\left(x;\Phi\right)^{2}},\ x\in\mathbb{R}. \tag{6}$$

Here $f\left(x;\Phi\right)$ and $F\left(x;\Phi\right)$ are considered as the pdf and cdf of baseline (or parent) random variable depending on the parameter vector $\Phi\in\mathbb{R}.$ The exponentiated parameter θ is the shape parameter of the proposed distribution. The complementary cdf (or survival function), instantaneous failure rate (or hazard rate function (hrf), retro hazard (or reversed hazard rate function), integrated hazard rate (or cumulative hazard rate function) can be written as below

$$S_{e\arctan}\left(x;\theta,\Phi\right)=1-\left[\frac{4}{\pi}\arctan F\left(x;\Phi\right)\right]^{\theta},\ x\in\mathbb{R}. \tag{7}$$

$$h_{e\arctan}\left(x;\theta,\Phi\right)=\frac{4}{\pi}\frac{\theta\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta-1}f\left(x;\Phi\right)}{\left\{1+F\left(x;\Phi\right)^{2}\right\}\left\{1-\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta}\right\}},\ x\in\mathbb{R}\cdot \tag{8}$$

$$r_{e\arctan}\left(x;\theta,\Phi\right)=\frac{4}{\pi}\frac{\theta f\left(x;\Phi\right)}{\left\{1+F\left(x;\Phi\right)^{2}\right\}\left\{\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]\right\}}, \tag{9}$$

$$H_{e\arctan}\left(x;\theta,\Phi\right)=-\log S_{e\arctan}\left(x;\theta,\Phi\right)=-\log\left\{1-\left[\frac{4}{\pi}\arctan F\left(x;\Phi\right)\right]^{\theta}\right\}.$$

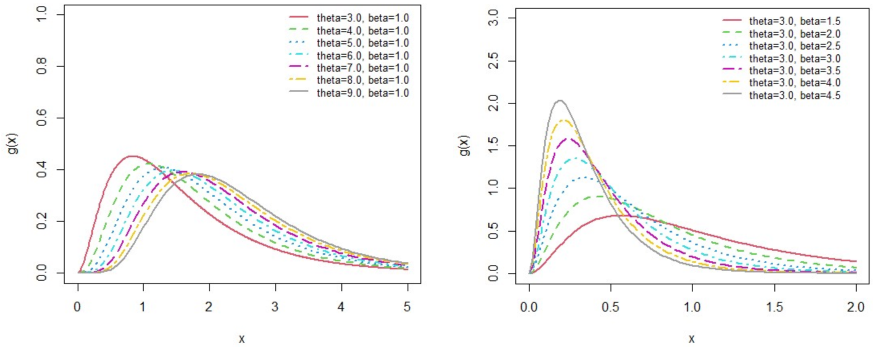

Figure 1 pdf plots of EAT-E distribution for varying the values parameters

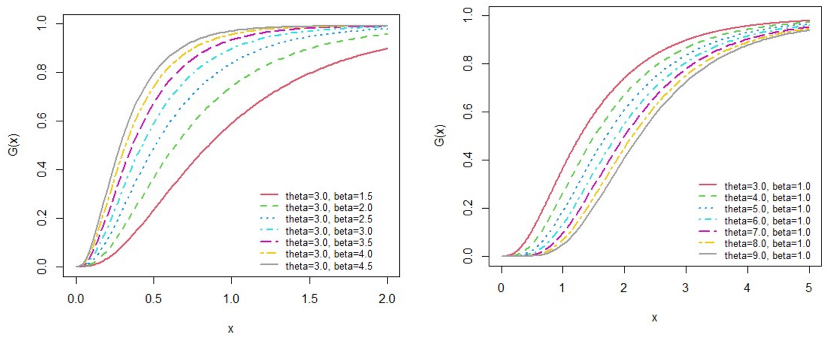

Figure 2 cdf plots of EAT-E distribution for varying the values parameters

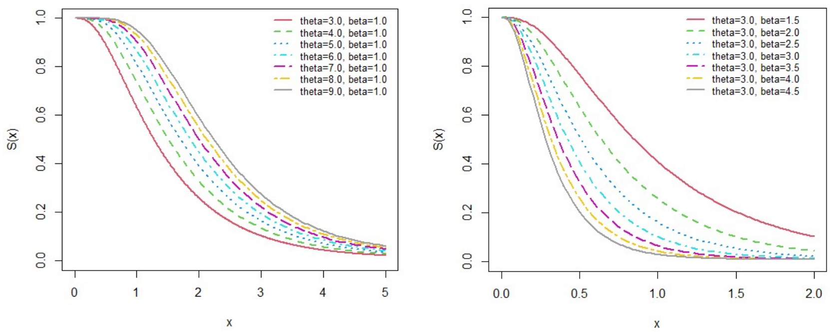

Figure 3 survival plots of EAT-E distribution for varying the values parameters

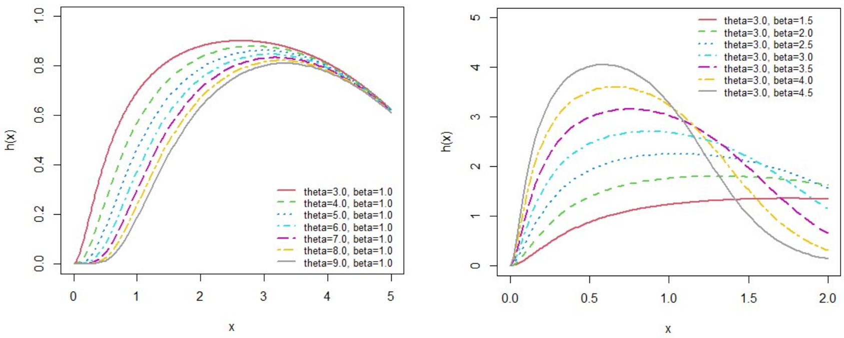

Figure 4 hazard rate plots of EAT-E distribution for varying the values parameters

3. Some Properties

3.1. Quantile function, median, Bowley skewness and Moors kurtosis

The quantile function (also known as the inverse cdf) of the exponentiated arctan-X family follows by inverting the exponentiated arctan-X distribution function. Let us consider $u \sim U(0,1)$, the u^{th} quantile function of EAT-X is defined as $Q_F(u)$ is the solution of $Q(u) > 0$. It may be written as follows in terms of the tangent trigonometric function as

$$x = Q_F(u) = G^{-1}(u) = F^{-1}\tan\left(\frac{\pi}{4}u^{\frac{1}{\theta}}\right), \tag{10}$$

where $u \in (0,1)$. The quantile function expression may be used to generate random numbers from EAT-X distributions. The median of the EAT-X family can be obtained by setting $u = 0.5$. The effects of the shape parameters on the skewness and kurtosis can be studied by using (10).

The Bowley skewness and Moors kurtosis can be formulated as

$$\text{Bowley Skewness}\,(S) = \frac{Q\left(\frac{3}{4}\right) + Q\left(\frac{1}{4}\right) - 2Q\left(\frac{1}{2}\right)}{Q\left(\frac{3}{4}\right) - Q\left(\frac{1}{4}\right)}, \tag{11}$$

$$\text{Moors kurtosis}\,(K) = \frac{Q\left(\frac{3}{8}\right) - Q\left(\frac{1}{8}\right) + Q\left(\frac{7}{8}\right) - Q\left(\frac{5}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}, \tag{12}$$

where $Q(.)$ represents the quantile function. When the distribution is symmetric, $S = 0$ and when the distribution is right (or left) skewed, $S > 0\,(\text{or}\, S < 0)$. As K increases, the tail of the distribution becomes heavier. These measures are less sensitive to outliers and they exist even for distributions without moments.

3.2. Ordinary and incomplete moments, moment generating function and mean deviation

In the field of actuarial science and financial science, moments are very important, particularly in applications. It gives the researcher a hand to get the key properties and characteristics of the proposed distribution under consideration. The r^{th} moment of the EAT-X family of distribution is given by

$$\mu_r' = \int_{-\infty}^{\infty} x^r g_{e\arctan}\left(x;\theta,\Phi\right)dx. \tag{13}$$

Using the pdf of EAT-X family of distribution in equation number (13), we have

$$\mu_r' = \int_{-\infty}^{\infty} x^r \frac{4}{\pi} \frac{\theta\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta-1} f\left(x;\Phi\right)}{1+F\left(x;\Phi\right)^2}dx. \tag{14}$$

Using the Taylor Series, we have

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}x^{2n+1}. \tag{15}$$

Let $x = F\left(x;\theta,\Phi\right),$ in (15), we have

$$\frac{1}{1+\left[F\left(x;\theta,\Phi\right)\right]^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[F\left(x;\theta,\Phi\right)\right]^{2n+1}. \tag{16}$$

By the aid of equation (16) and substituting in equation (14), we will have the following result

$$\mu_r' = \frac{4}{\pi}\theta\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta-1} x^r f\left(x;\Phi\right)\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[F\left(x;\theta,\Phi\right)\right]^{2n+1}. \tag{17}$$

The r^{th} moment of the EAT-X family of distribution is finally expressed as

$$\mu_r' = \frac{4}{\pi}\theta\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\Psi_{r,2n+1},$$

where $\Psi_{r,2n+1} = \int_{-\infty}^{\infty} x^r\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta-1} f\left(x;\Phi\right)\left[F\left(x;\theta,\Phi\right)\right]^{2n+1}dx.$

The mean can be obtained by setting $r=1$ in (17). The i^{th} incomplete moment is defined as $I\left(x,\theta,\Phi\right)$ and is given by

$$I\left(x,\theta,\Phi\right) = \int_{-\infty}^{t} x^i f\left(x,\theta,\Phi\right)dx = \frac{4}{\pi}\theta\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\Psi_{i,2n+1}, \tag{18}$$

where $\Psi_{i,2n+1} = \int_{-\infty}^{t} x^i\left[\frac{4}{\pi}\arctan F\left(x,\Phi\right)\right]^{\theta-1} f\left(x;\Phi\right)\left[F\left(x;\theta,\Phi\right)\right]^{2n+1}dx.$

For lifetime models, it is also of interest to obtain the mean deviations. For a random variable $X \sim EAT-X,$ the mean deviations about the mean and median can be expressed as follows

$$\varepsilon_1 = \int_0^{\infty}\left|x-\mu_1'\right| f_{EAT-X}\left(x,\theta,\Phi\right)dx = 2\mu_1' F\left(\mu_1'\right) - 2I_{(1)}\left(\mu_1'\right),$$

where $I_1\left(\mu_1\right)$ is the first incomplete moment of EAT-X family and $\varepsilon_2 = \int_0^{\infty}\left|x-Q\left(0.5\right)\right| f_{EAT-X}\left(x,\theta,\Phi\right)dx = \mu_1' - 2I_{(1)}\left(Q\left(0.5\right)\right).$ The moment-generating function and cumulant-generating function for the EAT-X family can be expressed in a general form as follows

$$M_x\left(t\right) = \frac{4}{\pi}\theta\sum_{r,n=0}^{\infty}\frac{(-1)^n}{\left(2n+1\right)r!}t^r\,\Psi_{r,2n+1},$$

$$\Phi_x(t) = M_x(it) = \frac{4}{\pi}\theta \sum_{r,n=0}^{\infty} \frac{(-1)^n}{(2n+1)r!}(it)^r \Psi_{r,2n+1}.$$

3.3. Reliability function for parallel and series systems

Let us Consider a system with $n*$ independent components, each component has the EAT-E family, the reliability of the parallel system (P) and reliability of the series system (S) are given by

$$R_p(x,\theta,\Phi) = \left[1 - \left\{\frac{4}{\pi}\arctan F(x;\Phi)\right\}\right]^{\theta n*}, \text{ and } R_s(x,\theta,\Phi) = \left[\left\{1 - \left(\frac{4}{\pi}\arctan F(x;\Phi)\right)\right\}^{\theta}\right]^{n*}.$$

3.4. Mean time to failure (MTTF), mean time between failure (MTBF) and availability (AvB)

MTTF, MTBF and AvB are reliability terms based on methods and procedures for lifecycle predictions for a product. MTTF, MTBF and AvB are ways of providing a numeric value based on a compilation of data to quantify a failure rate and the resulting time of expected performance.

If $X \sim EAT - G(\theta_1, \Phi_1)$ then the MTBF is given as

$$MTBF = \frac{-x}{\ln(1 - G(x, \theta_1, \Phi_1))}, \quad x > 0.$$

If $X \sim ExPo - G(\theta_2, \Phi_2)$ then the MTTF is given as

$$MTTF = E(X) = \mu_1' \,|\, (\theta_2, \Phi_2), \quad x > 0.$$

The AvB is consider the probability that the component is successful at time x, i.e.

$$AvB = MTTF \,/\, MTBF \;= -\mu_1' \,|\, (\theta_2, \Phi_2) \frac{\ln(1 - G(x, \theta_1, \Phi_1))}{x}.$$

3.5. Bonferroni and Lorenz curves

Bonferroni and Lorenz curves defined for a given probability π is given by

$$B(\pi) = I_1(q) / \pi\mu_1' \text{ and } L(\pi) = I_1(q) / \mu_1',$$

where $q = Q(\pi)$ is the quantile function of X at π.

4. Special EAT-X models

This section carries certain cases of the intended family of distributions by using different base cumulative distribution functions.

4.1. EAT-Exponential Distribution

Let us consider the cdf and pdf of Exponential distribution with positive parameter β given by $1 - e^{-\beta x}$ and $\beta e^{-\beta x}$ respectively with the random variable X. Considering that $F(x, \beta)$ is the cdf of the one-parameter Exponential distribution. The cdf of the EAT-E distribution, for $x > 0$, can be expressed as

$$G_{EAT-E}(x;\theta,\beta) = \left[\frac{4}{\pi}\arctan\left(1 - e^{-\beta x}\right)\right]^{\theta}, \quad x \in \mathbb{R}. \tag{19}$$

The corresponding pdf to the above cdf is given by

$$g_{EAT-E}\left(x;\theta,\beta\right)=\frac{4}{\pi}\theta\beta\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta-1}\frac{e^{-\beta x}}{1+\left(1-e^{-\beta x}\right)^{2}};\quad x\in\mathbb{R}. \tag{20}$$

The sf, hrf, inverted hrf, cumulative hrf are expressed as follows

$$S_{EAT-E}\left(x;\theta,\beta\right)=1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta},\quad x\in\mathbb{R},$$

$$h_{EAT-E}\left(x;\theta,\beta\right)=\frac{4}{\pi}\frac{\theta\beta\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta-1}e^{-\beta x}}{\left\{1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta}\right\}\left\{1+\left(1-e^{-\beta x}\right)^{2}\right\}},\quad x\in\mathbb{R},$$

$$r_{EAT-E}\left(x;\theta,\beta\right)=\frac{4}{\pi}\frac{\theta\beta e^{-\beta x}}{\left\{\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]\right\}\left\{1+\left(1-e^{-\beta x}\right)^{2}\right\}},\quad x\in\mathbb{R},$$

$$H_{EAT-E}\left(x;\theta,\beta\right)=-\log S_{EAT-E}\left(x;\theta,\beta\right)=-\log\left\{1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta}\right\},\quad x\in\mathbb{R}.$$

The residual lifetime and reverse residual life are calculated using the following equation of the EAT-E random variable (rv)

$$R_{(t)EAT-E}\left(x;\theta,\beta\right)=\frac{S_{EAT-E}\left(x+t;\theta,\beta\right)}{S_{EAT-E}\left(x;\theta,\beta\right)}=\frac{1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta(x+t)}\right)\right]^{\theta}}{1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta}},\quad x\in\mathbb{R},$$

$$\hat{R}_{(t)EAT-E}\left(x;\theta,\beta\right)=\frac{S_{EAT-E}\left(x;\theta,\beta\right)}{S_{EAT-E}\left(x+t;\theta,\beta\right)}=\frac{1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta}}{1-\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta(x+t)}\right)\right]^{\theta}},\quad x\in\mathbb{R}.$$

From Figure 1, the pdf can take various shapes depending on parameters value. The shape of the proposed distribution is closed to bell shape by increasing the shape parameter θ. Furthermore, the hrf can be increasing or unimodal-bell shape, which makes the distribution more flexible to fit different lifetime data sets shown in Figure 4. New contributed special cases of the exponentiated arctan-X family are shown in Table 2.

4.2. Quantile Function and Moments of EAT-E

The inverse cdf function is mostly employed in theoretical areas of distribution theory, such as the simulations and applicability. The simulation software uses a quantile function to create random samples. The quantile function of the EAT-E model as follows

$$x=Q_{F}\left(u\right)=G^{-1}\left(u\right)=\left\{-\frac{\log\left(1-\tan\left(\frac{\pi}{4}u^{\frac{1}{\theta}}\right)\right)}{\beta}\right\}. \tag{21}$$

where u is uniformly distributed from zero to one. The median, lower quartile, and upper quartile of EAT-E distribution can be obtained easily by using the quantile function by setting $u=\frac{1}{2},\frac{1}{4}$ and $\frac{3}{4}$

respectively. Moments are essential in statistical modelling, particularly in applications. The EAT-E distribution's r^{th} moment is defined as

$$\mu_r' = \frac{4}{\pi}\theta\beta x^r\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x}\right)\right]^{\theta-1} e^{-\beta x}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n+1}\left[\left(1-e^{-\beta x}\right)\right]^{2n+1}. \tag{22}$$

Table 2 New contributed special cases of the exponentiated arctan-X family

No.	Baseline mode	CDF form	Generated model	Support
1	Lindley	$\left[\frac{4}{\pi}\arctan\left\{1-\frac{e^{-\lambda x}\left(1+\lambda+\lambda x\right)}{1+\lambda}\right\}\right]^{\theta}$	EAT-Lindley	$x\in\mathbb{R}^*$
2	Rayleigh	$\left[\frac{4}{\pi}\arctan\left\{1-e^{-\lambda x^2}\right\}\right]^{\theta}$	EAT-Rayleigh	$x\in\mathbb{R}^*$
3	Gumbel	$\left[\frac{4}{\pi}\arctan\left\{e^{-e^{\frac{(x-\mu)}{\sigma}}}\right\}\right]^{\theta}$	EAT-Gumbel	$x\in\mathbb{R}$
4	Kumaraswamy	$\left[\frac{4}{\pi}\arctan\left\{1-\left(1-x^{\alpha}\right)^{\beta}\right\}\right]^{\theta}$	EAT-Kumaraswamy	$x\in[0,1]$
5	Weibull	$\left[\frac{4}{\pi}\arctan\left\{1-e^{-\lambda x^{\beta}}\right\}\right]^{\theta}$	EAT-Weibull	$x\in\mathbb{R}^*$
6	Frechet	$\left[\frac{4}{\pi}\arctan\left\{e^{-\lambda x-\beta}\right\}\right]^{\theta}$	EAT-Frechet	$x\in\mathbb{R}^*$
7	Log-logistic	$\left[\frac{4}{\pi}\arctan\left\{\frac{1}{1+\left(\frac{x}{\alpha}\right)^{-\beta}}\right\}\right]^{\theta}$	EAT-Log-logistic	$x\in\mathbb{R}^*$
8	Lomax	$\left[\frac{4}{\pi}\arctan\left\{1-\left(1+\alpha x\right)^{-\beta}\right\}\right]^{\theta}$	EAT-Lomax	$x\in\mathbb{R}^*$
9	Dagum	$\left[\frac{4}{\pi}\arctan\left\{1+\left(\frac{x}{\alpha}\right)^{-\beta}\right\}^{-p}\right]^{\theta}$	EAT-Dagum	$x\in\mathbb{R}^*$
10	Gamma	$\left[\frac{4}{\pi}\arctan\left\{\frac{1}{\Gamma(\alpha)}\gamma\left(\alpha,\beta\right)\right\}\right]^{\theta}$	EAT-Gamma	$x\in\mathbb{R}^*$

5. Classical Method of Estimation

Suppose that we have a random sample denoted as $X_1,\ldots,X_n$ that represents n independent random variables drawn from the EAT-X family that have the following observations $x_1,\ldots,x_n$, we can write the likelihood function for the EAT-X family is defined as follows:

$$L=\prod_{i=1}^{n} g\left(x_i;\theta,\beta\right) = \prod_{i=1}^{n}\frac{4}{\pi}\theta\beta\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta x_i}\right)\right]^{\theta-1}\frac{e^{-\beta x_i}}{1+\left(1-e^{-\beta x_i}\right)^2}.$$

We can express the log-likelihood function as below

$$l = n\log\left(\frac{4}{\pi}\right) + n\log\theta + n\log\beta + (\theta-1)\log\left[\frac{4}{\pi}\arctan\left\{1-e^{-\beta\sum_{i=1}^{n}x_i}\right\}\right] - \beta e^{\sum_{i=1}^{n}x_i} - \log\left[1+\left\{1-e^{-\beta\sum_{i=1}^{n}x_i}\right\}\right]^2.$$

Obtaining the partial derivate of the log-likelihood equation, we get

$$\frac{d}{d\theta}l = \frac{n}{\theta} + \log\left[\frac{4}{\pi}\arctan\left\{1-e^{-\beta\sum_{i=1}^{n}x_i}\right\}\right],$$

$$\frac{d}{d\beta}l = \frac{n}{\beta} + (\theta-1)\Psi_\beta - e^{\sum_{i=1}^{n}x_i} - \frac{2\beta\left[1+\left\{1-e^{-\beta\sum_{i=1}^{n}x_i}\right\}\right]e^{-\beta\sum_{i=1}^{n}x_i}\sum_{i=1}^{n}x_i}{\left[1+\left\{1-e^{-\beta\sum_{i=1}^{n}x_i}\right\}\right]^2},$$

where $\Psi_\beta = \frac{d}{d\beta}\left\{\log\left[\frac{4}{\pi}\arctan\left(1-e^{-\beta\sum_{i=1}^{n}x_i}\right)\right]\right\}$. By equating the first derivative to zero and trying to solve this equation numerically, we get the MLE estimator of parameters. But the equations are very complicated to solve numerically. We can find the estimates by the help of the R program directly (Adequacy model package). To obtain a confidence interval we use the asymptotic normality results. We have that, if $\hat{\alpha} = (\hat{\theta}, \hat{\beta})$ denotes the MLE of $\alpha = (\theta, \beta)$ we can state the results as follows:

$$\sqrt{n}(\hat{\alpha} - \alpha) \to N_2\left(0, I^{-1}(\alpha)\right),$$

where $I(\alpha)$ is Fisher's information matrix, i.e.,

$$I(\alpha) = -n\begin{pmatrix} E\left(\frac{\partial^2 \log L}{\partial\theta^2}\right) & E\left(\frac{\partial^2 \log L}{\partial\alpha\,\partial\beta}\right) \\ E\left(\frac{\partial^2 \log L}{\partial\theta\,\partial\beta}\right) & E\left(\frac{\partial^2 \log L}{\partial\beta^2}\right) \end{pmatrix}.$$

Since α being unknown, we estimate $I^{-1}(\alpha)$ by $I^{-1}(\hat{\alpha})$ and this can be used to obtain asymptotic confidence intervals for θ and β. The solution of the Fisher information matrix will yield asymptotic variance and covariance of the ML estimators for $(\hat{\theta}, \hat{\beta})$. The approximate $100(1-\alpha)\%$ confidence intervals for $(\hat{\theta}, \hat{\beta})$ respectively are $\hat{\theta} \pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\theta\theta}}{n}$ and $\hat{\beta} \pm Z_{\frac{\alpha}{2}}\frac{\sigma_{\beta\beta}}{n}$, where Z_α is the upper $100\alpha^{\text{th}}$ percentile of the standard normal distribution.

6. Monte-Carlo Simulation Study

We assess the effectiveness of the maximum likelihood estimation (MLE) method for estimating the exponentiated arctan-exponential distribution parameters using Monte Carlo simulation. A numerical evaluation of the performance of MLE of the EAT-E model is performed using nlminb () R-function with the argument method= "BFGS." The simulation study is conducted to investigate the average bias (AB) and mean square error (MSE) for the proposed model's parameters θ and β. We performed the simulation process by various samples and different values for the parameters. We generated the samples used in the simulation process from the quantile function of the EAT-E distribution. In order to generate accurate samples and to get perfect estimates, we made 1000 iterations using sample sizes $n = 20, 50, ..., 700$ and the parameter scenarios $\theta = 0.5$ and $\beta = 0.4$ in Set-I and $\theta = 1.0$ and $\beta = 1.5$ in Set-II. The MLEs are ascertained for each item of simulated data, say $(\hat{\theta}, \hat{\beta})$ for $i = 1, 2, ..., 700$ and the AB and MSEs of the parameters were computed by

$$AB = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha} - \alpha) \text{ and } MSE = \frac{1}{N}\sum_{i=1}^{N}(\hat{\alpha} - \alpha)^2, \ \alpha = (\theta, \beta).$$

Table 3 Monte Carlo simulation results for the EAT-E distribution using Set-I and Set-II values

Parameters	n	Set-I		Set-II	
		AB	MSE	AB	MSE
θ	20	1.448892	4.63285	0.2154406	0.1255961
	50	1.245194	3.384903	0.2002678	0.1058658
	100	1.187447	2.807057	0.1078564	0.0098746
	200	0.9307493	1.696894	0.0003468	0.0092025
	300	0.8171118	0.9171118	0.00009547	0.0025483
	500	0.7009665	0.714038	0.00000248	0.0009247
	700	0.0965487	0.5164793	−0.18585769	0.0000068
β	20	1.548892	4.932629	0.7154406	0.5910367
	50	1.345194	3.643942	0.7051978	0.5712349
	100	1.287447	3.054547	0.6813478	0.4924786
	200	1.030749	1.893005	0.5912478	0.4123579
	300	1.001564	1.574986	0.4125793	0.3917856
	500	1.000967	1.304231	0.2478962	0.2016791
	700	0.921678	1.0002495	0.1843792	0.1543971

Table 3 summarize the numerical findings of the MC simulation study. The average of the estimated parameters, as well as the AB and MSEs are evaluated. Furthermore, as the sample size increases, the AB and MSE decrease and so do the biases and MSE.

7. Percentage Points

A r.v. X is consider a continuous random variable by expecting random values in the interval (a, b) i.e., $a \le x \le b$ or more specifically it can assume any value like integral or fraction between certain limits. The number of expecting values are uncertain and infinite and for this reason assigning probability to each number is impossible. Therefore, in a continuous probability distribution we assign probabilities to intervals and not to individual values. For a given probability distribution, the specific value which a random variable X exceeds with a definite probability is called the percentage point of the distribution. In this part, a discussion of the percentage points of the proposed models EAT-E have been attempted. Percentage points of the proposed distribution have been computed at a number of different significance levels for different values of the parameters. The calculations in manual are very complicated, so computer programming R has used to calculate the values.

7.1. Percentage points of EAT-E model

Suppose $x_1, x_2, ..., x_n$ are n independent r.v. from EAT-E with pdf and cdf mentioned in the equation (19) and (20). The p^{th} percentile equation of EAT-E is represented as

$$G_x(x) = P(X \le x) = p, \quad X = G^{-1}(p) = Q(p).$$

$$X = Q_F(p) = G^{-1}(p) = \left\{ -\frac{\log\left(1 - \tan\left(\frac{\pi}{4} p^{\frac{1}{\theta}}\right)\right)}{\beta} \right\}. \tag{23}$$

The percent point function of the EAT-E does not exist in a simple closed form. The numeric computation is not possible in this case. We have used computer programming R to compute the different values for different points. Using the equation (7.1), we compute the percentage points of EAT-E for $p = 0.01, 0.05, 0.25, 0.50, 0.75, 0.90, 0.95, 0.99$ which has been tabulated in Tables 4 and 5. The parameters are varying different the values to compute the p-table in different cases.

Table 4 Percentage points of EAT-E for fixed values of parameter θ

p	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
θ					$\beta = 0.2$				
1	0.0015	0.0081	0.0165	0.0447	0.1079	0.2237	0.3950	0.5392	1.0049
2	0.0164	0.0393	0.0589	0.1079	0.1962	0.3381	0.5335	0.6991	1.0251
3	0.0377	0.0712	0.0969	0.1567	0.2579	0.4129	0.6232	0.8099	1.0426
4	0.0589	0.0995	0.1292	0.1962	0.3056	0.4693	0.6924	0.9044	1.0645
5	0.0786	0.1245	0.1572	0.2293	0.3447	0.5149	0.7504	0.9953	1.0821
6	0.0969	0.1468	0.1818	0.2579	0.3779	0.5537	0.8019	1.0928	1.1465
7	0.1137	0.1669	0.2037	0.2831	0.4068	0.5875	0.8495	1.2125	1.3254
8	0.1292	0.1852	0.2236	0.3056	0.4325	0.6178	0.8948	1.4039	1.5965
9	0.1437	0.2020	0.2417	0.3260	0.4556	0.6453	0.9389	1.6324	1.7845
10	0.1572	0.2176	0.2584	0.3447	0.4767	0.6706	0.9832	1.8071	2.0208

Table 5 Percentage points of EAT-E for fixed values of parameter β

p	0.01	0.05	0.10	0.25	0.50	0.75	0.90	0.95	0.99
β					$\theta = 1.5$				
0.1	0.0037	0.0114	0.0189	0.0392	0.0784	0.1442	0.2370	0.3147	0.6610
0.2	0.0075	0.0228	0.0377	0.0784	0.1567	0.2884	0.4740	0.6294	1.3221
0.3	0.0112	0.0342	0.0566	0.1176	0.2351	0.4326	0.7111	0.9441	1.9831
0.4	0.0149	0.0456	0.0754	0.1567	0.3135	0.5768	0.9481	1.2589	2.6441
0.5	0.0187	0.0569	0.0943	0.1959	0.3918	0.7211	1.1851	1.5736	3.3052
0.6	0.0224	0.0683	0.1132	0.2351	0.4702	0.8653	1.4221	1.8883	3.9662
0.7	0.0262	0.0797	0.1320	0.2743	0.5486	1.0095	1.6591	2.2029	4.6272
0.8	0.0299	0.0911	0.1509	0.3135	0.6269	1.1537	1.8961	2.5177	5.2883
0.9	0.0336	0.1025	0.1697	0.3527	0.7053	1.2979	2.1332	2.8324	5.9493
1.0	0.0374	0.1139	0.1886	0.3919	0.7837	1.4421	2.3702	3.1471	6.6103

Percentage points of proposed distribution has been presented in Tables 4 and 5. For chosen values of the parameters, different values of have been obtained at different significant levels using R Program. From Table 4, it is observed that percentage points increase as the parameter θ increases for the fixed values of β. From Table 5, it is observed that percentage points decreases when the value of parameter β decreases for fixed positive values of another parameter θ.

8. Practical Illustration Using the Insurance Data Set

In this section, we discuss the empirical importance of the EAT-E model utilizing two applications to complete real data. Certain analytical measures are used to identify the best fitting functionalities of the competitive distributions. In this regard, the Akaike Information Criterion (AIC), and Corrected Akaike Information Criterion (CAIC) values were used to select the most appropriate ones. Apart from discriminating tests, additional goodness-of-fit includes testing, like the Cramer-von Mises (W∗) distance value test, and the Kolmogorov-Smirnov (K-S) statistic with associating p-values, as well as the log-likelihood function, are also recorded. The best model has the lowest values of AIC and CAIC, as well as the W∗, and K-S tests. Furthermore, the model with the greatest p-values for the K-S statistics are applied to compare the competitive distributions. There are three data sets have been considered.

Data Set-1 (Insurance Data Set): This data set includes 58 observations and represents monthly unemployment insurance metrics from July 2008 to April 2013. It was reported by the Maryland state, USA, Department of Labour, Licensing, and Regulation. The data can be found at www.catalog.data.gov/dataset/unemployment-insurance-datajuly-2008-to-april-2013. In Table 6, the descriptive details of Data Set-1 are listed: 52, 33, 39, 50, 29, 52, 60, 32, 57, 64, 61, 64, 41, 36, 50, 53, 61, 68, 60, 50, 64, 57, 61, 59, 69, 70, 137, 170, 100, 90, 222, 109, 68, 63, 56, 90, 74, 95, 114, 133, 66, 75, 72, 54, 57, 52, 66, 69, 83, 44, 60, 80, 58, 80, 80, 52, 65, 73.

Table 6 Descriptive measure of the Data Set-1

N	Min	Median	Mode	Variance	Mean	Max
58	29	63.5	52	1065.698	70.67	222

From descriptive table of Data Set-1, we see that the data are right-skewed and leptokurtic. This is proved in a graphical display of the box plot in Figure 5. Figure 5 shows the TTT plot of this data set. It illustrates an increasing HRF plot. This section considered the proposed distribution's goodness-of-fit test results and some information criterion measures to those of some other well-known competing distributions, such as the Weibull, Kappa, Burr-XII, Beta-Weibull and Arctan-Weibull distributions.

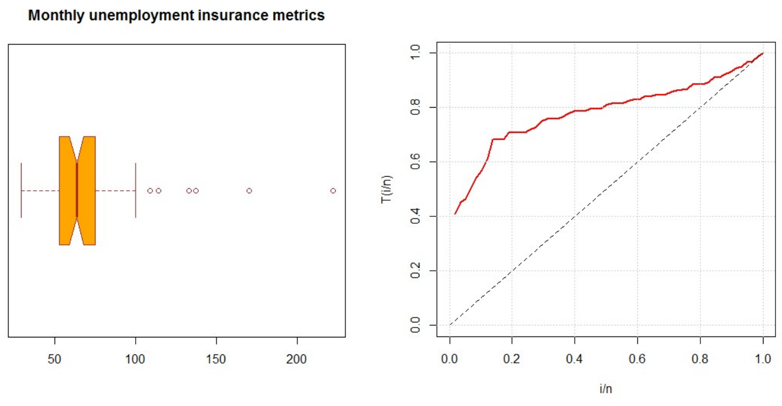

Figure 5 Box plot and TTT plot of the monthly unemployment insurance metrics

Table 7 MLEs and SEs of the proposed model parameters for Data Set-I

Models	ML Estimator with Standard Errors			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
EAT-E(θ,β)	-	0.045 (0.005)	-	19.251 (5.518)
AT-W(α,λ)	2.383 (0.204)	-	0.011 (0.001)	-
Weibull(α,β)	2.268 (0.196)	0.012 (0.001)	-	-
Burr-XII(α,β,λ)	7.151 (1.485)	0.225 (0.062)	36.285 (3.870)	-
Beta-Weibull$(\alpha,\beta,\lambda,\theta)$	6.754 (2.562)	0.128 (0.024)	1.164 (0.058)	10.050 (1.051)
Kappa(α,β)	0.745 (48.933)	77.672 (0.584)	-	-

Table 8 The AIC, CAIC, likelihood values, the goodness-of-fit tests, and the p-values for the insurance data

Models	$-2L$	AIC	CAIC	W*	K-S	p-value
EAT-E(θ,β)	**-272.290**	**548.581**	**552.701**	**0.049**	**0.156**	**550.186**
AT-W(α,λ)	−277.467	558.934	563.054	0.187	0.035	560.539
Weibull(α,β)	-279.439	562.878	566.483	0.184	0.039	564.483
Burr-XII(α,β,λ)	-278.724	563.449	569.630	0.295	0.001	565.856
Beta-W$(\alpha,\beta,\lambda,\theta)$	-282.961	573.922	582.164	0.293	0.001	577.132
Kappa(α,β)	-289.477	582.955	587.076	0.353	0.032	584.560

We observed that when compared to other distributions, the EAT-E model provided the greatest match and fitting because it has the smallest values of the measured analytical tools. Kolmogorov-Smirnov (K-S) tests for all competing distributions using the above Data Set-I. According to Table 8, the proposed EAT-E distribution has the lowest values in W*, and K-S tests, as well as the highest p-value. As a consequence, the suggested EAT-E distribution is selected as the best acceptable model among the competing distributions studied in this research.

Data Set-2: This data is about survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). In Table 9, the descriptive details of Data Set-2 are listed: 0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 1.07, 1.08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 9 Descriptive measures of the Data Set-2

N	Min	Median	Mode	Variance	Mean	Max
72	0.1	1.495	1.08	1.07	1.768	5.55

From descriptive table of Data Set-2, we see that the data are right-skewed and leptokurtic. This is proved in a graphical display of the box plot in Figure 6. Figure 6 shows the TTT plot of this data set. It illustrates an increasing HRF plot. This section considered the proposed distribution's goodness-of-fit test results and some information criterion measures to those of some other well-known competing generalized Exponential distributions, such as the exponential (Exp), moment exponential (ME), Marshall-Olkin exponential (MO-E), generalized Marshall-Olkin exponential (GMO-E), Kumaraswamy exponential (Kw-E) and Marshall-Olkin Kumaraswamy exponential (MOKw-E).

Table 10 MLEs and SEs of the proposed model parameters for Data Set-2

Models	ML Estimator with Standard Errors			
	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\lambda}$	$\hat{\theta}$
EAT-E(θ,β)	-	1.101 (0.129)	-	4.707 (0.950)
MOKW-E($\alpha,\beta,\lambda,\theta$)	0.008 (0.002)	0.099 (0.048)	2.716 (1.316)	1.986 (0.784)
KW-E(α,β,λ)	3.304 (1.106)	1.037 (0.764)	1.100 (0.614)	-
GMO-E(α,β,λ)	47.635 (44.901)	4.465 (1.327)	0.179 (0.070)	-
MO-E(α,β)	8.778 (3.555)	1.379 (0.193)	-	-
ME(α)	0.925 (0.077)	-	-	-
Exp(α)	0.540 (0.063)	-	-	-

Table 11 The AIC, CAIC, likelihood values, the goodness-of-fit tests, and the p-values for the survival data

Models	$-2L$	AIC	CAIC	W∗	K-S	p-value
EAT-E(θ,β)	**204.78**	**208.78**	**208.95**	**0.11**	**0.08**	**0.73**
MOKW-E($\alpha,\beta,\lambda,\theta$)	203.44	209.44	210.04	0.12	0.10	0.44
KW-E(α,β,λ)	203.42	209.42	209.77	0.11	0.08	0.50
GMO-E(α,β,λ)	204.54	210.54	210.89	0.16	0.09	0.51
MO-E(α,β)	204.36	210.36	210.53	0.17	0.10	0.43
ME(α)	204.40	210.40	210.45	0.25	0.14	0.13
EXP(α)	228.63	234.63	234.68	1.25	0.27	0.06

From the findings presented in the Table 11 on the basis of the lowest value different criteria like AIC, CAIC, W* and highest p-value of the KS statistics the EAT-E is found to be a better model than its recently introduced models Exp, ME, MO-E, GMO-E, Kw-E and MOKw-E for the Data Set-2 considered here.

9. Conclusion

The exponentiated arctan-X family of distribution is introduced here with various properties like survival function, hazard rate function, reverse hazard rate function, cumulative hazard rate function, quantile function, Median, Bowley skewness, Moors kurtosis, ordinary and incomplete moments, moment generating function, mean deviation, reliability function for parallel and series systems, Mean time to failure (MTTF), mean time between failure (MTBF), availability (AvB) and Bonferroni and Lorenz curves. The exponentiated arctan-exponential (EAT-E) distribution is defined as a subset of the family. The study developed the fundamental probability functions as well as some statistical properties of the sub model. The parameters of the EAT-E model are estimated using classical inference by the maximum likelihood estimation technique. A Monte Carlo Simulation study has been analysed. A detail of p-table has been studied. The proposed distribution is applied to insurance data set with a high degree of granularity. The proposed distribution is applied to two data set i.e., insurance data set and a survival data set with the high degree of granularity. The proposed family of distribution is compared to well-known two, three, and four parameter competitors. Four information criterion measures (AIC and CAIC) were used to make comparisons, as well as three goodness-of-fit measures ($W*$, and KS test statistics with corresponding p-values) and the likelihood function. The EAT-E distribution was compared to some well-known competitors, including arctan-Weibull, Weibull, Kappa, Burr-XII, and beta Weibull distributions using the Data Set-1. Then it is compared to Exp, ME, MO-E, GMO-E, Kw-E and MOKw-E for the Data Set-2 respectively. Using these metrics, it is discovered that the EAT-E model is very intriguing could be a good fit for analyzing high dimensional financial as well as survival data.

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Conflicts of Interest

The author declares no conflict of interest.

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