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Efficient Estimation for Ratio of Means Using Dual Auxiliary Information under Non-Response

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Abstract

The present research article deals with the problem of estimating the ratio of two populations with incomplete information due to the phenomenon of non-response. The main objective of this research is to develop new extensive classes of estimators for efficient estimation of the ratio of two population means, by introducing ranks of auxiliary character under both complete and incomplete information. To study the properties of the suggested classes of estimators, the first-order approximate bias and mean square error are obtained. The expressions of the minimum mean square error of the proposed estimators under specified conditions are derived. The efficiency in terms of mean square error of the proposed classes of estimators has been theoretically compared with conventional ratio, product, generalized, Khare and Sinha (2012a), and Sinha et al. (2022) estimators. An empirical study based on factual data sets has been carried out to analyze the efficiency of the estimators, and it has been observed that the proposed estimator outperforms all the existing relevant estimators, including the most efficient Sinha et al. (2022) estimator to the best of our knowledge so far.

Keywords: Ratio estimation, bias, mean square error, incomplete data

1. Introduction

Ratio estimation of two distinct population means plays an important role in various fields like medical, agricultural, engineering, socioeconomics, commerce, trade, etc., and it becomes more episodic when non-response occurs in sample surveys. Non-response is a common and blazing issue in sample surveys, as most of them are nowadays based on either online or computer-oriented. Särndal and Lundström (2005) and Bethlehem et al. (2011) had enlightened every aspect of non-response and explained how such non-response can increase a large potential bias in the estimation of parameters.

However, several authors including Singh (1965), Rao and Pareira (1968), Shah and Shah (1978), Upadhayaya and Singh (1985), Singh et al. (1994), Singh et al. (2009), Rawal (2011), Tailor and Lone (2014), Mehta and Tailor (2019), Ahuja et al. (2020) and Ahuja et al. (2021) proposed various estimators for estimating the ratio of two population means in simple random sampling under complete response available on study and auxiliary character(s). But in the case of non-response, authors like Khare and Pandey (2000), Khare and Sinha (2002, 2007, 2012b), Khare et al. (2013), Sinha (2014), Kumar and Patel (2015), Kumar and Kumar (2017), Sinha (2020) and Kumar et al. (2021) suggested improved estimators/families of estimators for estimation of ratio of two population means and studied their properties.

Let $V(=V_1,V_2,...V_N)$ be a population of size (N) consist with two non-overlapping strata of N_1 responding part of units and N_2 non-responding part of units. Suppose a sample (V_n) of n units has been drawn from (V) through simple random sampling without replacement method and it has been observed among these n units, n_1 units respond and n_2 units do not respond. The unknown stratum weights of responding $W_1\left(=\frac{N_1}{N}\right)$ and non-responding $W_2\left(=\frac{N_2}{N}\right)$ parts of the population under investigation are estimated by $w_1\left(=\frac{n_1}{n}\right)$ and $w_2\left(=\frac{n_2}{n}\right)$, respectively. To estimate the non-responding part, additional information has been collected on a simple random subsample without replacement of size $m\left(=\frac{n_2}{h};h>1\right)$ drawn from n_2 non-responding part of the population through own consultation or anyway. Let y_1,y_2 are the study characters and r_x is the rank of the units of auxiliary character x. On the available information of n_1 and m units and following the strategy of Hansen and Hurwitz (1946), the estimators $\bar{y}_1^*,\bar{y}_2^*,\bar{x}^*$, and $\bar{r}_x^*$ for the population means $\bar{Y}_1,\bar{Y}_2,\bar{X}$, and $\bar{R}_x$, respectively are given as

$$\bar{y}_1^*=w_1\bar{y}_{1(n_1)}+w_2\bar{y}_{1(m)},\ \bar{y}_2^*=w_1\bar{y}_{2(n_1)}+w_2\bar{y}_{2(m)},\ \bar{x}^*=w_1\bar{x}_{(n_1)}+w_2\bar{x}_{(m)},\ \text{and}\ \bar{r}_x^*=w_1\bar{r}_{x_{(n_1)}}+w_2\bar{r}_{x_{(m)}},$$

where $\left(\bar{y}_{1(n_1)},\bar{y}_{1(m)}\right)$, $\left(\bar{y}_{2(n_1)},\bar{y}_{2(m)}\right)$, $\left(\bar{x}_{(n_1)},\bar{x}_{(m)}\right)$, $\left(\bar{r}_{x_{(n_1)}},\bar{r}_{x_{(m)}}\right)$ are the sample means of the characters y_1,y_2,x,r_x, based on n_1 and m units respectively. The above estimators are unbiased and their variance up to the order n^{-1} are

$$V\left(\bar{y}_1^*\right)=\lambda S_{y_1}^2+\frac{W_2(h-1)}{n}S_{y_{1(m)}}^2,\ V\left(\bar{y}_2^*\right)=\lambda S_{y_2}^2+\frac{W_2(h-1)}{n}S_{y_{2(m)}}^2,$$

$$V\left(\bar{x}^*\right)=\lambda S_x^2+\frac{W_2(h-1)}{n}S_{x(m)}^2\ \text{and}\ V\left(\bar{r}_x^*\right)=\lambda S_{r_x}^2+\frac{W_2(h-1)}{n}S_{r_x(m)}^2,$$

where $\lambda=\frac{1}{n}-\frac{1}{N}$, $S_{y_1}^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{y}_{1i}-\bar{Y}_1\right)^2$, $S_{y_{1(m)}}^2=\frac{1}{N_2-1}\sum_{i=1}^{N_2}\left(\bar{y}_{1i}-\bar{Y}_{1(m)}\right)^2$,

$S_{y_2}^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{y}_{2i}-\bar{Y}_2\right)^2$, $S_{y_{2(m)}}^2=\frac{1}{N_2-1}\sum_{i=1}^{N_2}\left(\bar{y}_{2i}-\bar{Y}_{2(m)}\right)^2$, $S_x^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{x}_i-\bar{X}\right)^2$,

$S_{x(m)}^2=\frac{1}{N_2-1}\sum_{i=1}^{N_2}\left(\bar{x}_i-\bar{X}_{(m)}\right)^2$, $S_{r_x}^2=\frac{1}{N-1}\sum_{i=1}^{N}\left(\bar{r}_{x_i}-\bar{R}_x\right)^2$, $S_{r_x(m)}^2=\frac{1}{N_2-1}\sum_{i=1}^{N_2}\left(\bar{r}_{x_i}-\bar{R}_{x(m)}\right)^2$,

$\bar{Y}_1=\frac{1}{N}\sum_{i=1}^{N}\bar{y}_{1i}$, $\bar{Y}_{1(m)}=\frac{1}{N_2}\sum_{i=1}^{N_2}\bar{y}_{1i}$, $\bar{Y}_2=\frac{1}{N}\sum_{i=1}^{N}\bar{y}_{2i}$, $\bar{Y}_{2(m)}=\frac{1}{N_2}\sum_{i=1}^{N_2}\bar{y}_{2i}$, $\bar{X}=\frac{1}{N}\sum_{i=1}^{N}\bar{x}_i$,

$\bar{X}_{(m)}=\frac{1}{N_2}\sum_{i=1}^{N_2}\bar{x}_i$, $\bar{R}_x=\frac{1}{N}\sum_{i=1}^{N}\bar{r}_{x_i}$ and $\bar{R}_{x(m)}=\frac{1}{N_2}\sum_{i=1}^{N_2}\bar{r}_{x_i}$.

The conventional estimator for estimating the ratio of two population means $\left(\overline{R}=\frac{\overline{Y}_1}{\overline{Y}_2}\right)$ in presence of non-response is $\hat{R}=\frac{\overline{y}_1^*}{\overline{y}_2^*}$. Following Singh (1965, 67), the conventional ratio $\left(T_I^R\right)$, product $\left(T_I^P\right)$, regression $\left(T_I^{Reg}\right)$ and generalized $\left(T_I^G\right)$ estimators for estimating $\overline{R}$ when non-response occurs on both study and auxiliary characters are as follows:

$$T_I^R=\frac{\overline{y}_1^*}{\overline{y}_2^*}\frac{\overline{X}}{\overline{x}^*},\quad T_I^P=\frac{\overline{y}_1^*}{\overline{y}_2^*}\frac{\overline{x}^*}{\overline{X}},\quad T_I^{Reg}=\frac{\overline{y}_1^*}{\overline{y}_2^*}+\mu_1\left(\overline{X}-\overline{x}^*\right),\text{ and } T_I^G=\frac{\overline{y}_1^*}{\overline{y}_2^*}\left(\frac{\overline{x}^*}{\overline{X}}\right)^{\mu_2}.$$

Further, class of estimator (T_I^C) proposed by Khare and Sinha (2012a) for the non-response situation on both study and auxiliary characters is given by $T_I^C=f_1\left(\frac{\overline{y}_1^*}{\overline{y}_2^*},\frac{\overline{x}^*}{\overline{X}}\right)$.

Under the same situation, Sinha et al. (2022) suggested regression-cum-exponential estimators T_I^{RE} for estimating $\overline{R}$ as follows:

$$T_I^{RE}=\left[\varphi_1\frac{\overline{y}_1^*}{\overline{y}_2^*}+\psi_1\left(\overline{X}-\overline{x}^*\right)\right]\exp\left[\frac{\overline{X}-\overline{x}^*}{\overline{X}+\overline{x}^*}\right].$$

The mean square errors of these estimators up to the first order of approximation are as follows

$$MSE\left(\hat{R}\right)=\overline{R}^2\left(\frac{V\left(\overline{y}_1^*\right)}{\overline{Y}_1^2}+\frac{V\left(\overline{y}_2^*\right)}{\overline{Y}_2^2}-\frac{2C\left(\overline{y}_1^*,\overline{y}_2^*\right)}{\overline{Y}_1\overline{Y}_2}\right),\tag{1}$$

$$MSE\left(T_I^R\right)=MSE\left(\hat{R}\right)+\frac{\overline{R}^2}{\overline{X}^2}V\left(\overline{x}^*\right)-2\frac{\overline{R}^2}{\overline{X}}\left(\frac{C\left(\overline{y}_1^*,\overline{x}^*\right)}{\overline{Y}_1}-\frac{C\left(\overline{y}_2^*,\overline{x}^*\right)}{\overline{Y}_2}\right),\tag{2}$$

$$MSE\left(T_I^P\right)=MSE\left(\hat{R}\right)+\frac{\overline{R}^2}{\overline{X}^2}V\left(\overline{x}^*\right)+2\frac{\overline{R}^2}{\overline{X}}\left(\frac{C\left(\overline{y}_1^*,\overline{x}^*\right)}{\overline{Y}_1}-\frac{C\left(\overline{y}_2^*,\overline{x}^*\right)}{\overline{Y}_2}\right),\tag{3}$$

$$\left[MSE\left(T_I^{Reg}\right)\right]_{min}=MSE\left(\hat{R}\right)-\frac{\overline{R}^2}{V\left(\overline{x}^*\right)}\left(\frac{C\left(\overline{y}_1^*,\overline{x}^*\right)}{\overline{Y}_1}-\frac{C\left(\overline{y}_2^*,\overline{x}^*\right)}{\overline{Y}_2}\right)^2,\tag{4}$$

$$\left[MSE\left(T_I^{G}\right)\right]_{min}=MSE\left(\hat{R}\right)-\frac{\overline{R}^2}{V\left(\overline{x}^*\right)}\left(\frac{C\left(\overline{y}_1^*,\overline{x}^*\right)}{\overline{Y}_1}-\frac{C\left(\overline{y}_2^*,\overline{x}^*\right)}{\overline{Y}_2}\right)^2,\tag{5}$$

$$\left[MSE\left(T_I^{C}\right)\right]_{min}=MSE\left(\hat{R}\right)-\frac{\overline{R}^2}{V\left(\overline{x}^*\right)}\left(\frac{C\left(\overline{y}_1^*,\overline{x}^*\right)}{\overline{Y}_1}-\frac{C\left(\overline{y}_2^*,\overline{x}^*\right)}{\overline{Y}_2}\right)^2.\tag{6}$$

and

$$\left[MSE\left(T_I^{RE}\right)\right]_{min}=\left[MSE\left(T_I^{C}\right)\right]_{min}-\Delta_1^{(1)}-\Delta_2^{(1)}-\Delta_3^{(1)},\tag{7}$$

where

$$\Delta_1^{(1)}=\frac{\left[\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}}-\left(\frac{2C\left(\bar{y}_1^*,\bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2}-\frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right]^2}{\left(1+\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}^2}-\left(\frac{2C\left(\bar{y}_1^*,\bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2}-\frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right)},$$

$$\Delta_2^{(1)}=\frac{\frac{V\left(\bar{x}^*\right)}{\bar{X}^2}\left[\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}}+\frac{V\left(\bar{x}^*\right)\bar{R}^2}{16\bar{X}^2}\right]}{4\left(1+\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}^2}-\left(\frac{2C\left(\bar{y}_1^*,\bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2}-\frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right)}$$

and

$$\Delta_3^{(1)}=\frac{\left(\frac{2C\left(\bar{y}_1^*,\bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2}-\frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\left\{4\left[MSE\left(T_I^C\right)\right]_{min}-3\bar{R}^2\left(\frac{2C\left(\bar{y}_1^*,\bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2}-\frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)-\frac{3}{4}\frac{V\left(\bar{x}^*\right)\bar{R}^2}{\bar{X}^2}\right\}}{4\left(1+\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}^2}-\left(\frac{2C\left(\bar{y}_1^*,\bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2}-\frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right)},$$

where

$$C\left(\bar{y}_1^*,\bar{y}_2^*\right)=\lambda\rho_{y_1y_2}S_{y_1}S_{y_2}+\frac{W_2\left(h-1\right)}{n}\rho_{y_1y_2(m)}S_{y_1(m)}S_{y_2(m)},$$

$$C\left(\bar{y}_1^*,\bar{x}^*\right)=\lambda\rho_{y_1x}S_{y_1}S_{x}+\frac{W_2\left(h-1\right)}{n}\rho_{y_1x(m)}S_{y_1(m)}S_{x(m)},$$

and

$$C\left(\bar{y}_2^*,\bar{x}^*\right)=\lambda\rho_{y_2x}S_{y_2}S_{x}+\frac{W_2\left(h-1\right)}{n}\rho_{y_2x(m)}S_{y_2(m)}S_{x(m)}.$$

The correlation coefficients between (y_1,y_2), (y_1,x), (y_2,x) are respectively denoted by $\rho_{y_1y_2}$, ρ_{y_1x}, ρ_{y_2x} for the complete population whereas $\rho_{y_1y_2(m)}$, $\rho_{y_1x(m)}$, $\rho_{y_2x(m)}$ are the correlation coefficients between y_1 and y_2, y_1 and x, y_2 and x respectively for the non-responding units.

Rao (1986) discussed the practical situation where it is possible to assess the auxiliary characters to get the complete information however there may be a chance of non-response on the study characters which are highly correlated with auxiliary characters. Okafor and Lee (2000) and Sahoo et al. (2012) have brought up an example of a household survey where household size is considered as an auxiliary character for estimating family expenditure/income. In this case, complete information may be possible on the family size, while there is a possibility of non-response on the household expenditure / income.

Therefore, in such situation, the conventional ratio, product, regression, generalized and Sinha et al. (2022) estimators for estimating $\bar{R}$ when non-response occurs only on study characters are as follows

$$T_{II}^R=\frac{\bar{y}_1^*}{\bar{y}_2^*}\frac{\bar{X}}{\bar{x}}\rho_{y_1x},\quad T_{II}^P=\frac{\bar{y}_1^*}{\bar{y}_2^*}\frac{\bar{x}}{\bar{X}},\quad T_{II}^{Reg}=\frac{\bar{y}_1^*}{\bar{y}_2^*}+\mu_3\left(\bar{X}-\bar{x}\right),,\quad T_{II}^G=\frac{\bar{y}_1^*}{\bar{y}_2^*}\left(\frac{\bar{x}}{\bar{X}}\right)^{\mu_4},\quad T_{II}^C=f_2\left(\frac{\bar{y}_1^*}{\bar{y}_2^*},\frac{\bar{x}}{\bar{X}}\right),$$

and

$$T_{II}^{RE} = \left[\varphi_2 \frac{\bar{y}_1^*}{\bar{y}_2^*} + \psi_2\left(\bar{X} - \bar{x}\right)\right] \exp\left[\frac{\bar{X} - \bar{x}}{\bar{X} + \bar{x}}\right].$$

The mean square errors of these estimators up to the first order of approximation are as follows:

$$MSE\left(T_{II}^{R}\right) = MSE\left(\hat{R}\right) + \frac{\bar{R}^2}{\bar{X}^2} V\left(\bar{x}\right) - 2\frac{\bar{R}^2}{\bar{X}}\left(\frac{C\left(\bar{y}_1^*, \bar{x}\right)}{\bar{Y}_1} - \frac{C\left(\bar{y}_2^*, \bar{x}\right)}{\bar{Y}_2}\right), \tag{8}$$

$$MSE\left(T_{II}^{P}\right) = MSE\left(\hat{R}\right) + \frac{\bar{R}^2}{\bar{X}^2} V\left(\bar{x}\right) + 2\frac{\bar{R}^2}{\bar{X}}\left(\frac{C\left(\bar{y}_1^*, \bar{x}\right)}{\bar{Y}_1} - \frac{C\left(\bar{y}_2^*, \bar{x}\right)}{\bar{Y}_2}\right), \tag{9}$$

$$\left[MSE\left(T_{II}^{Reg}\right)\right]_{min} = MSE\left(\hat{R}\right) - \frac{\bar{R}^2}{V\left(\bar{x}\right)}\left(\frac{C\left(\bar{y}_1^*, \bar{x}\right)}{\bar{Y}_1} - \frac{C\left(\bar{y}_2^*, \bar{x}\right)}{\bar{Y}_2}\right)^2, \tag{10}$$

$$\left[MSE\left(T_{II}^{G}\right)\right]_{min} = MSE\left(\hat{R}\right) - \frac{\bar{R}^2}{V\left(\bar{x}\right)}\left(\frac{C\left(\bar{y}_1^*, \bar{x}\right)}{\bar{Y}_1} - \frac{C\left(\bar{y}_2^*, \bar{x}\right)}{\bar{Y}_2}\right)^2, \tag{11}$$

$$\left[MSE\left(T_{II}^{C}\right)\right]_{min} = MSE\left(\hat{R}\right) - \frac{\bar{R}^2}{V\left(\bar{x}\right)}\left(\frac{C\left(\bar{y}_1^*, \bar{x}\right)}{\bar{Y}_1} - \frac{C\left(\bar{y}_2^*, \bar{x}\right)}{\bar{Y}_2}\right)^2, \tag{12}$$

$$\left[MSE\left(T_{II}^{RE}\right)\right]_{min} = \left[MSE\left(T_{II}^{C}\right)\right]_{min} - \Delta_1^{(2)} - \Delta_2^{(2)} - \Delta_3^{(2)}, \tag{13}$$

where

$$\Delta_1^{(2)} = \frac{\left[\frac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}} - \left(\frac{2C\left(\bar{y}_1^*, \bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2} - \frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right]^2}{\left(1 + \frac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}^2} - \left(\frac{2C\left(\bar{y}_1^*, \bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2} - \frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right)},$$

$$\Delta_2^{(2)} = \frac{\frac{V\left(\bar{x}\right)}{\bar{X}^2}\left[\frac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}} + \frac{V\left(\bar{x}\right)\bar{R}^2}{16\bar{X}^2}\right]}{4\left(1 + \frac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}^2} - \left(\frac{2C\left(\bar{y}_1^*, \bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2} - \frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right)}$$

and

$$\Delta_3^{(2)} = \frac{\left(\frac{2C\left(\bar{y}_1^*, \bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2} - \frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\left\{4\left[MSE\left(T_{II}^{C}\right)\right]_{min} - 3\bar{R}^2\left(\frac{2C\left(\bar{y}_1^*, \bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2} - \frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right) - \frac{3}{4}\frac{V\left(\bar{x}\right)\bar{R}^2}{\bar{X}^2}\right\}}{4\left(1 + \frac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}^2} - \left(\frac{2C\left(\bar{y}_1^*, \bar{y}_2^*\right)}{\bar{Y}_1\bar{Y}_2} - \frac{2V\left(\bar{y}_2^*\right)}{\bar{Y}_2^2}\right)\right)},$$

where $V\left(\bar{x}\right) = \lambda S_x^2$, $C\left(\bar{y}_1^*, \bar{x}\right) = \lambda\rho_{y_1x}S_{y_1}S_x$ and $C\left(\bar{y}_2^*, \bar{x}\right) = \lambda\rho_{y_2x}S_{y_2}S_x$.

Based on the works described above, this research focuses on the inclusion of ratio and ordinal auxiliary variables to propose classes of estimators for estimating the ratio of two population means. In this article, the properties of the proposed classes of estimators are discussed comprehensively and its efficiency is compared theoretically as well as empirically with all the relevant preceding estimators in terms of mean square error.

2. Suggested Estimators

In the estimation of parameter, it has been observed that the utilized auxiliary character comprises with the additional property of attribute, which can be used for the improvisation of efficiency of the estimator. Therefore, in the light of this fact and motivated by the strategy of Sinha and Bharti (2020), wider classes of estimators for estimating the ratio of two means have been suggested using the known mean and rank of auxiliary character under two situations.

Situation I: When non-response occurs on both study and auxiliary characters, the suggested class of estimator for estimating the ratio of two population means in this situation is

$$T_I^{new} = \frac{\bar{y}_1^*}{\bar{y}_2^*} g_1(\theta_1, \eta_1); \;\; \theta_1 = \frac{\bar{x}^*}{\bar{X}}, \;\; \eta_1 = \frac{\bar{r}_x^*}{\bar{R}_x}, \tag{14}$$

such that $g_1(1,1) = 1$.

The function $g_1(\theta_1, \eta_1)$ is continuous, assumes positive value in a bounded subset $\mathfrak{D}_i$ containing the point $\mathfrak{T} = (1,1)$ on the real line and its partial derivatives are assumed to be continuous and bounded in $\mathfrak{D}_i$.

In order to calculate bias and mean square error of T_I^{new}, the following assumptions are considered

$$\frac{\bar{y}_1^* - \bar{Y}_1}{\bar{Y}_1} = \varepsilon_1, \;\; \frac{\bar{y}_2^* - \bar{Y}_2}{\bar{Y}_2} = \varepsilon_2, \;\; \frac{\bar{x}^* - \bar{X}}{\bar{X}} = \varepsilon_3, \;\; \frac{\bar{r}_x^* - \bar{R}_x}{\bar{R}_x} = \varepsilon_4, \tag{15}$$

such that $E(\varepsilon_1) = E(\varepsilon_2) = E(\varepsilon_3) = E(\varepsilon_4) = 0, \;\; E(\varepsilon_1^2) = \lambda C_{y_1}^2 + \frac{W_2(h-1)}{n} C_{y_{1(m)}}^2 = A(say),$

$$E(\varepsilon_2^2) = \lambda C_{y_2}^2 + \frac{W_2(h-1)}{n} C_{y_{2(m)}}^2 = B(say), \;\; E(\varepsilon_3^2) = \lambda C_x^2 + \frac{W_2(h-1)}{n} C_{x_{(m)}}^2 = F_1(say),$$

$$E(\varepsilon_4^2) = \lambda C_{r_x}^2 + \frac{W_2(h-1)}{n} C_{r_x(m)}^2 = G_1(say),$$

$$E(\varepsilon_1\varepsilon_2) = \lambda \rho_{y_1y_2} C_{y_1} C_{y_2} + \frac{W_2(h-1)}{n} \rho_{y_1y_2(m)} C_{y_1(m)} C_{y_2(m)} = H(say),$$

$$E(\varepsilon_1\varepsilon_3) = \lambda \rho_{y_1x} C_{y_1} C_x + \frac{W_2(h-1)}{n} \rho_{y_1x(m)} C_{y_1(m)} C_{x(m)} = J_1(say), \tag{16}$$

$$E(\varepsilon_1\varepsilon_4) = \lambda \rho_{y_1r_x} C_{y_1} C_{r_x} + \frac{W_2(h-1)}{n} \rho_{y_1r_x(m)} C_{y_1(m)} C_{r_x(m)} = K_1(say),$$

$$E(\varepsilon_2\varepsilon_3) = \lambda \rho_{y_2x} C_{y_2} C_x + \frac{W_2(h-1)}{n} \rho_{y_2x(m)} C_{y_2(m)} C_{x(m)} = L_1(say),$$

$$E(\varepsilon_2\varepsilon_4) = \lambda \rho_{y_2r_x} C_{y_2} C_{r_x} + \frac{W_2(h-1)}{n} \rho_{y_2r_x(m)} C_{y_2(m)} C_{r_x(m)} = M_1(say),$$

and

$$E\left(\varepsilon_3\varepsilon_4\right)=\lambda\rho_{xr_x}C_xC_{r_x}+\frac{W_2\left(h-1\right)}{n}\rho_{xr_x(m)}C_{x(m)}C_{r_x(m)}=P_1\left(say\right).$$

where

$$C_{y_1}=\frac{S_{y_1}}{\bar{Y}_1},\ C_{y_2}=\frac{S_{y_2}}{\bar{Y}_2},\ C_x=\frac{S_x}{\bar{X}},\ C_{r_x}=\frac{S_{r_x}}{\bar{R}_x},\ C_{y_1(m)}=\frac{S_{y_1(m)}}{\bar{Y}_1},$$

$$C_{y_2(m)}=\frac{S_{y_2(m)}}{\bar{Y}_2},\ C_{x(m)}=\frac{S_{x(m)}}{\bar{X}}\ \text{and}\ C_{r_x(m)}=\frac{S_{r_x(m)}}{\bar{R}_x}.$$

Expanding the function $g_1\left(\theta_1,\eta_1\right)$ by Taylor's series about the point $\mathfrak{T}$ and applying the condition $g_1\left(1,1\right)=1,$ the estimator given in (14) reduced to

$$T_I^{new}=\frac{\bar{y}_1^*}{\bar{y}_2^*}\left[g_1\left(1,1\right)+\left(\theta_1-1\right)g_1^{(1)}\left(1,1\right)+\left(\eta_1-1\right)g_1^{(2)}\left(1,1\right)+\frac{\left(\theta_1-1\right)^2}{2}g_1^{(11)}\left(\theta_1^*,\eta_1^*\right)\right.$$
$$\left.+\frac{\left(\eta_1-1\right)^2}{2}g_1^{(22)}\left(\theta_1^*,\eta_1^*\right)+\left(\theta_1-1\right)\left(\eta_1-1\right)g_1^{(12)}\left(\theta_1^*,\eta_1^*\right)\right],$$

where $\theta_1^*=\theta_1+\Delta_1\left(\theta_1-1\right),\ \eta_1^*=\eta_1+\Delta_2\left(\eta_1-1\right),\ 0<\Delta_1,\Delta_2<1.$ Using the approximations given in (15) and on simplifying, T_I^{new} becomes

$$T_I^{new}=\bar{R}\left[1+\varepsilon_3g_1^{(1)}\left(1,1\right)+\varepsilon_4g_1^{(2)}\left(1,1\right)+\frac{\varepsilon_3^2}{2}g_1^{(11)}\left(\theta_1^*,\eta_1^*\right)+\frac{\varepsilon_4^2}{2}g_1^{(22)}\left(\theta_1^*,\eta_1^*\right)+\varepsilon_3\varepsilon_4g_1^{(12)}\left(\theta_1^*,\eta_1^*\right)-\varepsilon_2\right.$$
$$\left.-\varepsilon_2\varepsilon_3g_1^{(1)}\left(1,1\right)-\varepsilon_2\varepsilon_4g_1^{(2)}\left(1,1\right)+\varepsilon_2^2+\varepsilon_1+\varepsilon_1\varepsilon_3g_1^{(1)}\left(1,1\right)+\varepsilon_1\varepsilon_4g_1^{(2)}\left(1,1\right)-\varepsilon_1\varepsilon_2\right]\tag{17}$$

Taking the expectation on both sides of (17), subtracting $\bar{R}$ from it and using the results given in (16) to get the bias of $T_I^{new},$ is given by

$$B\left(T_I^{new}\right)=\bar{R}\left[\frac{F_1}{2}g_1^{(11)}\left(\theta_1^*,\eta_1^*\right)+\frac{G_1}{2}g_1^{(22)}\left(\theta_1^*,\eta_1^*\right)+P_1g_1^{(12)}\left(\theta_1^*,\eta_1^*\right)-L_1g_1^{(1)}\left(1,1\right)\right.$$
$$\left.+B+J_1g_1^{(1)}\left(1,1\right)+K_1g_1^{(2)}\left(1,1\right)-H\right].$$

Subtracting $\bar{R}$ from (17), squaring it and after taking expectation we get the mean square error of T_I^{new} as follows:

$$MSE\left(T_I^{new}\right)=\bar{R}^2\left[F_1\left\{g_1^{(1)}\left(1,1\right)\right\}^2+G_1\left\{g_1^{(2)}\left(1,1\right)\right\}^2+B+A+2P_1g_1^{(1)}\left(1,1\right)g_1^{(2)}\left(1,1\right)\right.$$
$$\left.-2L_1g_1^{(1)}\left(1,1\right)+2J_1g_1^{(1)}\left(1,1\right)-2M_1g_1^{(2)}\left(1,1\right)+2K_1g_1^{(2)}\left(1,1\right)-2H\right].\tag{18}$$

Partially differentiate $MSE\left(T_I^{new}\right)$ with respect to $g_1^{(1)}\left(1,1\right)$ and $g_1^{(2)}\left(1,1\right)$ and equating them to zero to get optimum value of constants,

$$g_1^{(1)}\left(1,1\right)\Big|_{opt}=\frac{\left(K_1-M_1\right)P_1+\left(L_1-J_1\right)G_1}{F_1G_1-P_1^2},\ \ g_1^{(2)}\left(1,1\right)\Big|_{opt}=\frac{\left(M_1-K_1\right)F_1+\left(J_1-L_1\right)P_1}{F_1G_1-P_1^2}.$$

Substituting these values of constants in (18), the value of minimum mean square error is calculated as follows:

$$\left[MSE\left(T_I^{new}\right)\right]_{min} = \bar{R}^2\left(A+B-2H\right) - \frac{\bar{R}^2 G_1\left(L_1-J_1\right)^2}{F_1G_1-P_1^2} - \frac{\bar{R}^2 F_1\left(M_1-K_1\right)^2}{F_1G_1-P_1^2} - \frac{2\bar{R}^2 P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1-P_1^2}. \tag{19}$$

Situation II: When non-response occurs only on study characters, the suggested class of estimator for estimating the ratio of two population means in this situation is

$$T_{II}^{new} = \frac{\bar{y}_1^*}{\bar{y}_2^*} g_2\left(\theta_2,\eta_2\right);\ \theta_2 = \frac{\bar{x}}{\bar{X}},\ \eta_2 = \frac{\bar{r}_x}{\bar{R}_x}, \tag{20}$$

such that $g_2\left(1,1\right)=1$.

Continuous function $g_2\left(\theta_2,\eta_2\right)$ takes positive value in a bounded subset $\mathfrak{D}_j$ containing the point $\mathfrak{T}=\left(1,1\right)$ on the real line. The first and second order partial derivatives are assumed to be continuous and bounded in $\mathfrak{D}_j$.

To calculate bias and mean square error of T_{II}^{new}, the following assumptions are made in addition to the assumptions mentioned in (15)

$$\frac{\bar{x}-\bar{X}}{\bar{X}} = \varepsilon_5,\ \frac{\bar{r}_x-\bar{R}_x}{\bar{R}_x} = \varepsilon_6, \tag{21}$$

such that

$$E\left(\varepsilon_5\right)=E\left(\varepsilon_6\right)=0,\ E\left(\varepsilon_5^2\right)=\lambda C_x^2 = F_2\left(say\right),\ E\left(\varepsilon_6^2\right)=\lambda C_{r_x}^2 = G_2\left(say\right),$$
$$E\left(\varepsilon_1\varepsilon_5\right)=\lambda\rho_{y_1x}C_{y_1}C_x = J_2\left(say\right),\ E\left(\varepsilon_1\varepsilon_6\right)=\lambda\rho_{y_1r_x}C_{y_1}C_{r_x} = K_2\left(say\right),$$
$$E\left(\varepsilon_2\varepsilon_5\right)=\lambda\rho_{y_2x}C_{y_2}C_x = L_2\left(say\right),\ E\left(\varepsilon_2\varepsilon_6\right)=\lambda\rho_{y_2r_x}C_{y_2}C_{r_x} = M_2\left(say\right),\ \text{and}$$
$$E\left(\varepsilon_5\varepsilon_6\right)=\lambda\rho_{xr_x}C_xC_{r_x} = P_2\left(say\right). \tag{22}$$

Expanding the function $g_2\left(\theta_2,\eta_2\right)$ by Taylor's series about the point $\mathfrak{T}$ and applying the condition $g_2\left(1,1\right)=1$, the estimator given in (20) reduced to

$$T_{II}^{new} = \frac{\bar{y}_1^*}{\bar{y}_2^*}\Bigg[g_2\left(1,1\right)+\left(\theta_2-1\right)g_2^{(1)}\left(1,1\right)+\left(\eta_2-1\right)g_2^{(2)}\left(1,1\right)+\frac{\left(\theta_2-1\right)^2}{2}g_2^{(11)}\left(\theta_2^*,\eta_2^*\right)+\frac{\left(\eta_2-1\right)^2}{2}g_2^{(22)}\left(\theta_2^*,\eta_2^*\right)$$
$$+\left(\theta_2-1\right)\left(\eta_2-1\right)g_2^{(12)}\left(\theta_2^*,\eta_2^*\right)\Bigg],$$

where $\theta_2^* = \theta_2+\Delta_3\left(\theta_2-1\right),\ \eta_2^*=\eta_2+\Delta_4\left(\eta_2-1\right),\ 0<\Delta_3,\Delta_4<1$.

Using the approximations given in (21) and on simplifying, T_{II}^{new} converts in

$$T_{II}^{new} = \bar{R}\Bigg[1+\varepsilon_5 g_2^{(1)}\left(1,1\right)+\varepsilon_6 g_2^{(2)}\left(1,1\right)+\frac{\varepsilon_5^2}{2}g_2^{(11)}\left(\theta_2^*,\eta_2^*\right)+\frac{\varepsilon_6^2}{2}g_2^{(22)}\left(\theta_2^*,\eta_2^*\right)+\varepsilon_5\varepsilon_6 g_2^{(12)}\left(\theta_2^*,\eta_2^*\right)-\varepsilon_2$$
$$-\varepsilon_2\varepsilon_5 g_2^{(1)}\left(1,1\right)-\varepsilon_2\varepsilon_6 g_2^{(2)}\left(1,1\right)+\varepsilon_2^2+\varepsilon_1+\varepsilon_1\varepsilon_5 g_2^{(1)}\left(1,1\right)+\varepsilon_1\varepsilon_6 g_2^{(2)}\left(1,1\right)-\varepsilon_1\varepsilon_2\Bigg]. \tag{23}$$

Taking the expectation and subtracting $\bar{R}$ from both sides of (23), the bias of T_{II}^{new} using the results (22) is given as

$$B\left(T_{II}^{new}\right)=\bar{R}\left[\frac{F_2}{2}g_2^{(11)}\left(\theta_2^*,\eta_2^*\right)+\frac{G_2}{2}g_2^{(22)}\left(\theta_2^*,\eta_2^*\right)+P_2g_2^{(12)}\left(\theta_2^*,\eta_2^*\right)-L_2g_2^{(1)}\left(1,1\right)+B+J_2g_2^{(1)}\left(1,1\right)\right.$$
$$\left.+K_2g_2^{(2)}\left(1,1\right)-H\right].$$

Subtracting $\bar{R}$ from both sides of (23), squaring them and taking expectation, the mean square error of T_{II}^{new} is calculated as follows

$$MSE\left(T_{II}^{new}\right)=\bar{R}^2\left[F_2\left\{g_2^{(1)}\left(1,1\right)\right\}^2+G_2\left\{g_2^{(2)}\left(1,1\right)\right\}^2+B+A+2P_2g_2^{(1)}\left(1,1\right)g_2^{(2)}\left(1,1\right)-2L_2g_2^{(1)}\left(1,1\right)\right.$$
$$\left.+2J_2g_2^{(1)}\left(1,1\right)-2M_2g_2^{(2)}\left(1,1\right)+2K_2g_2^{(2)}\left(1,1\right)-2H\right]. \quad (24)$$

Partially differentiate $MSE\left(T_{II}^{new}\right)$ with respect to $g_2^{(1)}\left(1,1\right)$ and $g_2^{(2)}\left(1,1\right)$ and equating them to zero to get optimum value of constants.

$$g_2^{(1)}\left(1,1\right)\Big|_{opt}=\frac{\left(K_2-M_2\right)P_2+\left(L_2-J_2\right)G_2}{F_2G_2-P_2^2},\quad g_2^{(2)}\left(1,1\right)\Big|_{opt}=\frac{\left(M_2-K_2\right)F_2+\left(J_2-L_2\right)P_2}{F_2G_2-P_2^2}.$$

Substituting optimum values of $g_2^{(1)}\left(1,1\right)$ and $g_2^{(2)}\left(1,1\right)$ in (24) to obtain the value of minimum mean square error which is given as

$$\left[MSE\left(T_{II}^{new}\right)\right]_{min}=\bar{R}^2\left(A+B-2H\right)-\frac{\bar{R}^2G_2\left(L_2-J_2\right)^2}{F_2G_2-P_2^2}-\frac{\bar{R}^2F_2\left(M_2-K_2\right)^2}{F_2G_2-P_2^2}-\frac{2\bar{R}^2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2-P_2^2}. \quad (25)$$

***Remark*:** To obtain the minimum MSE of the proposed estimators, the optimum value of $g_i^{(1)}\left(1,1\right)$ and $g_i^{(2)}\left(1,1\right)$; $i=1,2$ should be calculated but as they involve the unknown population parameters, so in this case their estimates or the value from the prior data may be used which do not affect the efficiency of the estimators up to the first degree of approximation (see Koyuncu and Kadilar (2009)).

3. Efficiency Comparisons

Conditions are derived after comparing the mean square errors of the suggested class of estimators with relevant existing estimators to judge the efficiency.

For Situation I,

a) From (19) and (1), $\left[MSE\left(T_I^{new}\right)\right]_{min}\le MSE\left(\hat{R}\right)$ if

$$\frac{G_1\left(L_1-J_1\right)^2}{F_1G_1-P_1^2}+\frac{F_1\left(M_1-K_1\right)^2}{F_1G_1-P_1^2}+\frac{2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1-P_1^2}\ge 0.$$

b) From (19) and (2), $\left[MSE\left(T_I^{new}\right)\right]_{min}\le MSE\left(T_I^R\right)$ if

$$\frac{G_1\left(L_1-J_1\right)^2}{F_1G_1-P_1^2}+\frac{F_1\left(M_1-K_1\right)^2}{F_1G_1-P_1^2}+\frac{2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1-P_1^2}-F_1+2\left(J_1-L_1\right)\ge 0.$$

c) From (19) and (3), $\left[MSE\left(T_I^{new}\right)\right]_{min}\le MSE\left(T_I^P\right)$ if

$$\frac{G_1\left(L_1-J_1\right)^2}{F_1G_1-P_1^2}+\frac{F_1\left(M_1-K_1\right)^2}{F_1G_1-P_1^2}+\frac{2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1-P_1^2}-F_1-2\left(J_1-L_1\right)\ge 0.$$

d) From (19) and (4), $\left[MSE\left(T_I^{new}\right)\right]_{min} \le \left[MSE\left(T_I^{Reg}\right)\right]_{min}$ if

$$\frac{F_1\left(M_1-K_1\right)^2+\rho_{r_x}^2 G_1\left(L_1-J_1\right)^2-2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1\left(1-\rho_{r_x}^2\right)} \ge 0.$$

e) From (19) and (5), $\left[MSE\left(T_I^{new}\right)\right]_{min} \le \left[MSE\left(T_I^{G}\right)\right]_{min}$ if

$$\frac{F_1\left(M_1-K_1\right)^2+\rho_{r_x}^2 G_1\left(L_1-J_1\right)^2-2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1\left(1-\rho_{r_x}^2\right)} \ge 0.$$

f) From (19) and (6), $\left[MSE\left(T_I^{new}\right)\right]_{min} \le \left[MSE\left(T_I^{C}\right)\right]_{min}$ if

$$\frac{F_1\left(M_1-K_1\right)^2+\rho_{r_x}^2 G_1\left(L_1-J_1\right)^2-2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)}{F_1G_1\left(1-\rho_{r_x}^2\right)} \ge 0.$$

g) From (19) and (7), $\left[MSE\left(T_I^{new}\right)\right]_{min} \le \left[MSE\left(T_I^{RE}\right)\right]_{min}$ if

$$\frac{\bar{R}^2\left[F_1\left(M_1-K_1\right)^2+\rho_{r_x}^2 G_1\left(L_1-J_1\right)^2-2P_1\left(L_1-J_1\right)\left(K_1-M_1\right)\right]}{F_1G_1\left(1-\rho_{r_x}^2\right)}-\frac{\left\{\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}}-\left(2H-2B\right)\right\}^2}{1+\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}^2}-\left(2H-2B\right)}$$

$$-\frac{F_1\left\{\left[MSE\left(T_I^C\right)\right]_{min}+\frac{\bar{R}^2F_1}{16}\right\}}{4\left\{1+\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}^2}-\left(2H-2B\right)\right\}}-\frac{F_1\left\{\left[MSE\left(T_I^C\right)\right]_{min}+\frac{\bar{R}^2F_1}{16}\right\}}{4\left\{1+\frac{\left[MSE\left(T_I^C\right)\right]_{min}}{\bar{R}^2}-\left(2H-2B\right)\right\}} \ge 0$$

For Situation II,

a) From (25) and (1), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le MSE\left(\hat{R}\right)$ if

$$\frac{G_2\left(L_2-J_2\right)^2}{F_2G_2-P_2^2}+\frac{F_2\left(M_2-K_2\right)^2}{F_2G_2-P_2^2}+\frac{2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2-P_2^2} \ge 0.$$

b) From (25) and (8), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le MSE\left(T_{II}^{R}\right)$ if

$$\frac{G_2\left(L_2-J_2\right)^2}{F_2G_2-P_2^2}+\frac{F_2\left(M_2-K_2\right)^2}{F_2G_2-P_2^2}+\frac{2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2-P_2^2}-F_2+2\left(J_2-L_2\right) \ge 0.$$

c) From (25) and (9), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le MSE\left(T_{II}^{P}\right)$ if

$$\frac{G_2\left(L_2-J_2\right)^2}{F_2G_2-P_2^2}+\frac{F_2\left(M_2-K_2\right)^2}{F_2G_2-P_2^2}+\frac{2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2-P_2^2}-F_2-2\left(J_2-L_2\right) \ge 0.$$

d) From (25) and (10), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le \left[MSE\left(T_{II}^{Reg}\right)\right]_{min}$ if

$$\frac{F_2\left(M_2-K_2\right)^2+\rho_{r_x}^2 G_2\left(L_2-J_2\right)^2-2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2\left(1-\rho_{r_x}^2\right)} \ge 0.$$

e) From (25) and (11), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le \left[MSE\left(T_{II}^{G}\right)\right]_{min}$ if

$$\frac{F_2\left(M_2-K_2\right)^2+\rho_{r_x}^2 G_2\left(L_2-J_2\right)^2-2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2\left(1-\rho_{r_x}^2\right)} \ge 0.$$

f) From (25) and (12), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le \left[MSE\left(T_{II}^{C}\right)\right]_{min}$ if

$$\frac{F_2\left(M_2-K_2\right)^2+\rho_{r_x}^2 G_2\left(L_2-J_2\right)^2-2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)}{F_2G_2\left(1-\rho_{r_x}^2\right)} \ge 0.$$

g) From (25) and (13), $\left[MSE\left(T_{II}^{new}\right)\right]_{min} \le \left[MSE\left(T_{II}^{RE}\right)\right]_{min}$ if

$$\frac{\bar{R}^2\left[F_2\left(M_2-K_2\right)^2+\rho_{r_x}^2 G_2\left(L_2-J_2\right)^2-2P_2\left(L_2-J_2\right)\left(K_2-M_2\right)\right]}{F_2G_2\left(1-\rho_{r_x}^2\right)}-\frac{\left\{\dfrac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}}-\left(2H-2B\right)\right\}^2}{1+\dfrac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}^2}-\left(2H-2B\right)}$$

$$-\frac{F_2\left\{\left[MSE\left(T_{II}^{C}\right)\right]_{min}+\dfrac{\bar{R}^2F_2}{16}\right\}}{4\left\{1+\dfrac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}^2}-\left(2H-2B\right)\right\}}-\frac{\left(2H-2B\right)\left\{4\left[MSE\left(T_{II}^{C}\right)\right]_{min}-3\bar{R}^2\left(2H-2B\right)-\dfrac{3}{4}\bar{R}^2F_2\right\}}{4\left\{1+\dfrac{\left[MSE\left(T_{II}^{C}\right)\right]_{min}}{\bar{R}^2}-\left(2H-2B\right)\right\}} \ge 0.$$

4. Empirical Work

To appraise the results and comparative performance of suggested estimators with relevant estimators, an empirical work has been done by using two different real data sets. The variables and parameters used in both the data sets are derived from the ICMR study by the Department of Pediatrics, Banaras Hindu University during 1983-1984 on the growth of the upper socio-economic group of 95 school going children from Varanasi (Khare and Sinha 2007). The upper 25% (i.e. 27 children) of the population are considered non-responders. The description of variables and parameters for both data sets are as follows:

Data I: In this data set, the variables under study are the children's height y_1 and the children's weight y_2 while to evaluate $\left(\bar{R}=\dfrac{\bar{Y}_1}{\bar{Y}_2}\right)$, the children's middle arm circumference and its rank are assumed to be auxiliary variables x and r_x, respectively. The parameters for this data set are as follows:

$N=95$	$n=27$	$N_2=24$	
$\bar{Y}_1=115.95$	$\bar{Y}_2=19.4968$	$\bar{X}=16.7968$	$\bar{R}_x=48$
$S_{y_1}=5.96670$	$S_{y_2}=3.04398$	$S_x=1.45306$	$S_{r_x}=27.30268$
$S_{y_1(2)}=5.10430$	$S_{y_2(2)}=2.35421$	$S_{x(2)}=1.19667$	$S_{r_x(2)}=27.05416$
$\rho_{y_1y_2}=0.713$	$\rho_{y_1x}=0.364$	$\rho_{y_1r_x}=0.346$	$\rho_{y_2x}=0.797$
$\rho_{y_2r_x}=0.650$	$\rho_{xr_x}=0.885$	$\rho_{y_1y_2(2)}=0.678$	$\rho_{y_1x(2)}=0.244$

$\rho_{y_1r_x(2)} = 0.352$ $\rho_{y_2x(2)} = 0.757$ $\rho_{y_2r_x(2)} = 0.760$ $\rho_{xr_x(2)} = 0.962$

Data II: In this data set, the variables under study are the children's height y_1 and the children's weight y_2 while to evaluate $\left(\bar{R} = \frac{\bar{Y}_1}{\bar{Y}_2}\right)$, the children's chest circumference and its rank are assumed to be auxiliary variables x and r_x, respectively. The parameters used in this data set are as follows:

$N = 95$ $n = 27$ $N_2 = 24$

$\bar{Y}_1 = 115.95$ $\bar{Y}_2 = 19.4968$ $\bar{X} = 55.8611$ $\bar{R}_x = 48$

$S_{y_1} = 5.96670$ $S_{y_2} = 3.04398$ $S_x = 3.27366$ $S_{r_x} = 27.48917$

$S_{y_1(2)} = 5.10430$ $S_{y_2(2)} = 2.35421$ $S_{x(2)} = 2.51483$ $S_{r_x(2)} = 25.32771$

$\rho_{y_1y_2} = 0.713$ $\rho_{y_1x} = 0.620$ $\rho_{y_1r_x} = 0.623$ $\rho_{y_2x} = 0.846$

$\rho_{y_2r_x} = 0.722$ $\rho_{xr_x} = 0.930$ $\rho_{y_1y_2(2)} = 0.678$ $\rho_{y_1x(2)} = 0.401$

$\rho_{y_1r_x(2)} = 0.479$ $\rho_{y_2x(2)} = 0.729$ $\rho_{y_2r_x(2)} = 0.718$ $\rho_{xr_x(2)} = 0.974.$

The percentage relative efficiency (PRE) of suggested class of estimators and relevant existing estimators with respect to $\hat{R}$ has been calculated by using formula

$$PRE = \frac{MSE\left(\hat{R}\right)}{MSE\left(T_i^{new/exist}\right)} \times 100 .$$

Mean square errors (MSE) and percentage relative efficiency of the estimators have been given in Tables 1 and 3 for situation I and in Tables 2 and 4 for situation II respectively for different sub-sampling fractions.

Table 1. MSE and PRE of estimators when non-response occurs on both study and auxiliary characters for Data I

Estimator	MSE (PRE)		
	$h = 4$	$h = 3$	$h = 2$
$\hat{R}$	0.0238385 (100%)	0.0207578 (100%)	0.017677 (100%)
T_I^R	0.0644517(36.99%)	0.0558883(37.14%)	0.0473249(3735%)
T_I^P	0.00733644(324.93%)	0.00637947(325.38%)	0.0054225(326.00%)
T_I^{Reg}	0.00692644(344.17%)	0.00599354(346.34%)	0.00505858(349.45%)
T_I^G	0.00692644(344.17%)	0.00599354(346.34%)	0.00505858(349.45%)
T_I^C	0.00692644(344.17%)	0.00599354(346.34%)	0.00505858(349.45%)
T_I^{RE}	0.00699381(340.85%)	0.00604479(343.40%)	0.00509592(346.89%)
T_I^{new}	0.00650139(366.67%)	0.00557099(372.60%)	0.0046376(381.17%)

Table 2. MSE and PRE of estimators when non-response occurs on study characters only for Data I

Estimator	MSE (PRE)		
	$h=4$	$h=3$	$h=2$
$\hat{R}$	0.0238385(100%)	0.0207578(100%)	0.0176771(100%)
T_{II}^{R}	0.0480036(49.66%)	0.0449229(46.21%)	0.0418422(42.25%)
T_{II}^{P}	0.0137075(173.91%)	0.0106269(195.33%)	0.00754621(234.25%)
T_{II}^{Reg}	0.0133621(178.40%)	0.0102814(201.90%)	0.00720076(245.49%)
T_{II}^{G}	0.0133621(178.40%)	0.0102814(201.90%)	0.00720076(245.49%)
T_{II}^{C}	0.0133621(178.40%)	0.0102814(201.90%)	0.00720076(245.49%)
T_{II}^{RE}	0.0134341(177.45%)	0.0103355(200.84%)	0.0072394(244.18%)
T_{II}^{new}	0.0129408(184.21%)	0.00986017(210.52%)	0.0067795(260.74%)

Table 3. MSE and PRE of estimators when non-response occurs on both study and auxiliary characters for Data II

Estimator	MSE (PRE)		
	$h=4$	$h=3$	$h=2$
$\hat{R}$	0.0238385(100%)	0.0207578(100%)	0.0176771(100%)
T_{I}^{R}	0.0463701(51.41%)	0.0405222(51.22%)	0.0346743(50.98%)
T_{I}^{P}	0.0117711(202.52%)	0.0101164(205.19%)	0.00846176(208.91%)
T_{I}^{Reg}	0.00953877(249.91%)	0.00809056(256.57%)	0.0066402(266.21%)
T_{I}^{G}	0.00953877(249.91%)	0.00809056(256.57%)	0.0066402(266.21%)
T_{I}^{C}	0.00953877(249.91%)	0.00809056(256.57%)	0.0066402(266.21%)
T_{I}^{RE}	0.00961003(248.06%)	0.00814466(254.86%)	0.0066795(264.65%)
T_{I}^{new}	0.00833477(286.04%)	0.00693342(299.39%)	0.00552482(319.96%)

Table 4. MSE and PRE of estimators when non-response occurs on study characters only for Data II

Estimator	MSE (PRE)		
	$h=4$	$h=3$	$h=2$
$\hat{R}$	0.0238385(100%)	0.0207578(100%)	0.0176771(100%)
T_{II}^{R}	0.0380684(62.62%)	0.0349877(59.33%)	0.031907(55.40%)
T_{II}^{P}	0.0160491(148.53%)	0.0129684(160.06%)	0.00988776(178.78%)
T_{II}^{Reg}	0.0144284(165.22%)	0.0113477(182.92%)	0.00826702(213.83%)
T_{II}^{G}	0.0144284(165.22%)	0.0113477(182.92%)	0.00826702(213.83%)
T_{II}^{C}	0.0144284(165.22%)	0.0113477(182.92%)	0.00826702(213.83%)
T_{II}^{RE}	0.0145021(164.38%)	0.0114034(182.03%)	0.00830706(212.80%)
T_{II}^{new}	0.0133436(178.65%)	0.010263(202.26%)	0.00718229(246.12%)

5. Conclusions

In the present study, the problem of estimating ratio of two population means has been dealt with wider classes of estimators using mean and rank of auxiliary variable for two different situations. The empirical work based on real data sets has been made in support of the theoretical results derived in it. The MSE and PRE shown in Tables 1 and 3 for situation I reveals that the suggested class of estimators $\left(T_I^{new}\right)$ achieves significant efficiency than the conventional ratio, product, regression, generalized and other relevant existing estimators at every level of sub-sampling fraction $\left(1/h\right)$.

Similarly, Tables 2 and 4 for situation II confirms that the suggested class of estimators is more efficient than all the suggested estimators in its category at different sub-sampling fraction $\left(1/h\right)$ and it is also evident from Tables 1 to 4 that the mean square error of the estimators is decreasing with increasing the value of sub-sampling fraction $\left(1/h\right)$, which confirms the principle of sub-sampling. Therefore, on the basis of theoretical and empirical studies, the proposed classes of estimators may be recommended over other relevant existing estimators to improvise the efficiency of the estimate.

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