Thailand Statistician
April 2025; 23(2): 279-297
http://statassoc.or.th
Contributed paper

Bayesian Inference for Nadarajah-Haghighi Distribution Under Progressively Type-II Censored Data

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Received: 17 April 2022
Revised: 1 October 2022
Accepted: 27 November 2022

Abstract

In this article, the problem of estimating parameters, reliability function, and hazard function of the Nadarajah-Haghighi distribution under progressively type-II censored samples are studied. The maximum likelihood and Bayesian estimators under squared error, LINEX, and general entropy loss functions are derived for parameters and some survival time parameters namely reliability and hazard functions. We used Lindley's approximation to obtain the Bayesian estimators. Asymptotic confidence intervals, for unknown parameters, are constructed using the observed Fisher information matrix. A numerical example using the real data set is presented to illustrate the proposed methods. Monte Carlo simulation study is conducted to compare the performance of the estimators in terms of their mean square error. The Monte Carlo simulation analysis shows that in most cases, the Bayesian method have a better performance than the standard maximum likelihood method.

Keywords: Maximum likelihood estimation, Bayes estimation, reliability function, progressively order statistics, hazard function.

1. Introduction

In lifetime studies we are often faced with limitations such as time and cost. There are several ways to meet these restrictions. One of these methods is the use of censored data.In many life-testing experiments and reliability analyses (medical, biological, or industrial applications), there are situations in which experiments are terminated before failure times of all items are observed. Some experimental units are removed from experimentation before failure of all items is observed. For example, individuals in some clinical trial, may drop out of the study for personal reasons in the middle of the trial or in an industrial experiment, some of the units may break accidentally. In such cases, we have complete information only on part of the sample. On all the units which have not failed, we have only partial information about the sample. Data obtained from such experiments are called censored data. Censoring is common in life-testing work and reliability studies because of time limits and other restrictions on data collection. There are several types of censored data.

The readers can find some studies about censored data in Sindhu and Hussain (2020), Zhang and Gui (2019), Sindhu et al. (2016), Lodhi et al. (2021) and Sindhu et al. (2017). In this paper, we consider an advance scheme of censoring called progressive type-II censoring which expanded by Balakrishnan and Aggarwala (2000). Under this scheme, n units are placed on the lifetime test at time zero and only m units observed until end of test. Immediately after the first failure at time x_1, R_1 of the $n-1$ surviving units are randomly withdrawn from the test. The remaining $n-1-R_1$

units continue to work. At the time x_2 the second failure has occurred and R_2 of the $n-2-R_1$ surviving units are censored, and so on. Finally, at the time of the m-th failure, all of the remaining $R_m = n-m-R_1-\cdots-R_{m-1}$ units are censored and the experiment terminates. Then $x_1,\ldots,x_m$ are values of $X_{1:m:n},\ldots,X_{m:m:n}$ whose called the progressively order statistics (POS). Figure 1 shows a perspective for this scheme.

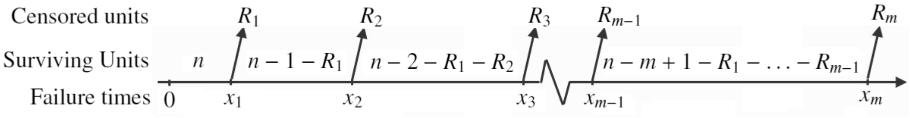

Figure 1 A perspective for a POS data

Let $(X_{1:m:n},\ldots,X_{m:m:n})$ be a POS sample with scheme $(R_1,\ldots,R_m)$ and $\mathbf{x}=(x_1,\ldots,x_m)$ be its corresponding observation. The associated likelihood function (LF) of $\mathbf{x}$ is given by Balakrishnan and Aggarwala (2000), as

$$L(F;\mathbf{x}) = c_p \prod_{i=1}^{m} f(x_i)\,[1-F(x_i)]^{R_i}, \quad x_1<\cdots<x_m, \tag{1}$$

where $c_p = n(n-1-R_1)\cdots(n-m+1-R_1-\cdots-R_{m-1})$ is the normalized constant and $f(.)$ and $F(.)$ are the probability density function (PDF) and cumulative distribution function (CDF) of random variable X, respectively. In this paper, we use the Nadarajah-Haghighi distribution (extension of the exponential distribution) CDF as the baseline for the discussed scheme and our focus is to obtain the Bayesian estimations of the parameters, reliability, and hazard function of Nadarajah-Haghighi distribution.

A random variable X has the Nadarajah-Haghighi (NH) distribution with parameters α and λ if its CDF is given by

$$F(\alpha,\lambda) = 1-e^{1-(1+\lambda x)^{\alpha}}, \quad x>0,\ \alpha,\lambda>0, \tag{2}$$

where α and λ are scale and shape parameters, respectively. The corresponding PDF to (2) is as follows:

$$f(\alpha,\lambda) = \alpha\lambda(1+\lambda x)^{\alpha-1}e^{1-(1+\lambda x)^{\alpha}}, \quad x>0,\ \alpha,\lambda>0. \tag{3}$$

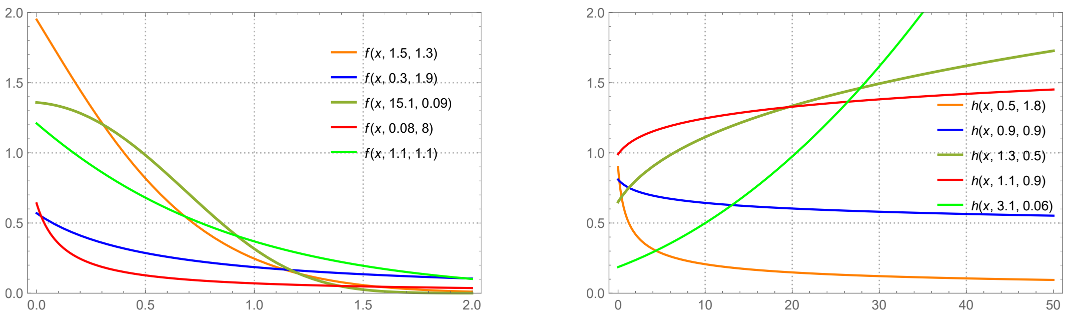

Figure 2 The PDF (left panel) and hazard rate function (right panel) of NH distribution for different parameters values.

Also, the hazard rate (HR) function and reliability function of the NH distribution are given by

$$H(x) = \alpha\lambda(1+\lambda x)^{\alpha-1}, \qquad R(x) = e^{1-(1+\lambda x)^{\alpha}}.$$

The NH distribution is introduced by Nadarajah and Haghighi (2011) as an alternative to the gamma, Weibull, and exponentiated exponential distributions in lifetime studies. This is an extension of exponential distribution as a special case with $\alpha = 1$. The different shapes of the PDF and HR of NH distribution are illustrated in Figure 2 to some values of several parameters. Figure 2 show that the PDF of NH distribution can be right-skewed and HR function of NH distribution can be increasing, decreasing, and constant depending on the parameter values.

The Weibull distribution is useful in analyzing and modeling the lifetime, particularly when the data are censored, which is very common in most life testing experiments. From Singh *et al.* (2014), Nadarajah and Haghighi (2011), and Kumar *et al.* (2017) it is observed that the NH distribution is a good alternative for the Weibull distribution and provides a better fit than it for several life data such as daily rainfall data, failure times of the air conditioning system of an airplane data and bladder cancer data. So, the NH distribution and, consequently, the results of the present study can be used effectively in analyzing some lifetime data in the field of biological and medical applications, especially in the different censoring schemes.

For NH distribution, Singh *et al.* (2014) considered different estimation procedures (classical and Bayesian) under progressive type-II censored data with binomial removals. Sana and Faizan (2019) considered the Bayes estimation based on upper record values. They derived different Bayes estimates under squared error, balanced squared error, and general entropy loss functions by using Jeffreys' prior information. Selim (2018) made a statistical inference about the two unknown parameters of NH distribution based on record values. Kumar *et al.* (2017) establish recurrence relations for the single and product moments of order statistics from the extended exponential distribution. Kumar and Dey (2017) derived the exact explicit expressions as well as recurrence relations for the single, product, and conditional moments of generalized order statistics from the extended exponential (EE) distribution. MirMostafaee *et al.* (2016) established the exact explicit expressions as well as several recurrence relations for the single and product moments of record values from the NH distribution. In addition, they discussed confidence intervals for the unknown parameters and prediction intervals for future records. Kumar *et al.* (2019) derived the recurrence relations for the single and product moments based on progressively type-II right censored order statistics for the EE distribution. In addition, they obtained maximum likelihood estimations of EE parameters under progressively type-II right-censored order statistics. For different methods of estimation based on NH distribution and inference about other lifetime distributions, the readers are referred to Sindhu et al. (2021), Almarashi et al. (2022), Ashour et al. (2020), Sindhu and Atangana (2021), Alhussain and Ahmed (2020), and Sindhu et al. (2020).

The contents of this paper are organized as follows. LF of POS data is obtained in Section 2. Also, the maximum likelihood (ML) estimators of parameters, reliability, and hazard rate functions are derived in section 2. In section 3 the asymptotic confidence interval estimations of the parameters are obtained. Section 4 is played the Bayes estimation under some loss functions and Lindley's approximation method is discussed to evaluate these Bayes estimators. In section 5 we fit the NH distribution to a real data. In Section 6 we make the simulation study to investigate the precision of the estimators obtained in Sections 3 and 4. Finally, in the last section, we provide some conclusions.

2. Maximum Likelihood Estimation

In this section, we find the maximum likelihood estimators (MLEs) for the NH distribution, based on the POS sample. Let $(X_{1:m:n}, \ldots, X_{m:m:n})$ be a POS sample withdrawn NH distribution with scheme $(R_1, \ldots, R_m)$ and $\mathbf{x} = (x_1, \ldots, x_m)$ be its corresponding observation. Substitution of (2) and (3) in (1), the associated LF is given by

$$\begin{aligned} L(\alpha, \lambda; \mathbf{x}) &\propto \prod_{i=1}^{m} \alpha\lambda \left(1+\lambda x_i\right)^{\alpha-1} e^{1-(1+\lambda x_i)^{\alpha}} e^{R_i[1-(1+\lambda x_i)^{\alpha}]} \\ &\propto \alpha^m \lambda^m e^{-\sum_{i=1}^{m}(R_i+1)(1+\lambda x_i)^{\alpha}} e^{(\alpha-1)\sum_{i=1}^{m}\log(1+\lambda x_i)}. \end{aligned} \tag{4}$$

The associated log-likelihood function is given by

$$\ell(\alpha, \lambda; \mathbf{x}) \propto m\log(\alpha) + m\log(\lambda) + (\alpha-1)\sum_{i=1}^{m}\log\left(1+\lambda x_i\right) - \sum_{i=1}^{m}(R_i+1)\left(1+\lambda x_i.\right)^{\alpha}$$

The normal equations to estimate the unknown parameters may obtained by equating the first partial derivatives $\ell(\alpha, \lambda; \mathbf{x})$ with respect to α and λ to zero. The corresponding likelihood equations for α and λ are obtained as

$$\frac{m}{\alpha} + \sum_{i=1}^{m}\log\left(1+\lambda x_i\right) - \sum_{i=1}^{m}(R_i+1)\left(1+\lambda x_i\right)^{\alpha}\log\left(1+\lambda x_i\right) = 0, \tag{5}$$

and

$$\frac{m}{\lambda} + (\alpha-1)\sum_{i=1}^{m}\frac{x_i}{1+\lambda x_i} - \alpha\sum_{i=1}^{m}(R_i+1)x_i\left(1+\lambda x_i\right)^{(\alpha-1)} = 0. \tag{6}$$

The MLEs of parameters α and λ, say $\hat{\alpha}$ and $\hat{\lambda}$, respectively, are simultaneous solutions of Equations (5) and (6). These equations do not yield explicit estimators for α and λ, and hence (5) and (6) must be solved by numerical methods in order to obtain the MLE of the α and λ. Furthermore, by the invariant property of the MLE, the MLEs of the reliability function $R(t)$ and HR function $H(t)$ are given by

$$\hat{R}(t) = e^{1-(1+\hat{\lambda}t)^{\hat{\alpha}}}, \tag{7}$$

and

$$\hat{H}(t) = \hat{\alpha}\hat{\lambda}(1+\hat{\lambda}t)^{\hat{\alpha}-1}, \tag{8}$$

respectively.

3. Asymptotic Confidence Interval Estimation

Let $\boldsymbol{\mu} = (\alpha, \lambda)$ be the parameter vector of NH distribution. Also, $\ell(\alpha, \lambda; \mathbf{x})$ be the Log likelihood function of NH distribution. Then, the Fisher information matrix of the parameters $\boldsymbol{\mu}$ is given by

$$I(\boldsymbol{\mu}) = E\begin{bmatrix} -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\lambda^2} \end{bmatrix},$$

where

$$\begin{aligned}
\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha^2} &= -\frac{m}{\alpha^2} - \sum_{i=1}^{m}\left(R_i+1\right)\left(1+\lambda x_i\right)^{\alpha}\left(\log\left(1+\lambda x_i\right)\right)^2, \\
\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\lambda^2} &= -\frac{m}{\lambda^2} - (\alpha-1)\sum_{i=1}^{m}\frac{x_i^2}{(1+\lambda x_i)^2} + \alpha(\alpha-1)\sum_{i=1}^{m}(R_i+1)x_i^2(1+\lambda x_i)^{\alpha-2}, \\
\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha\partial\lambda} &= \sum_{i=1}^{m}\frac{x_i}{1+\lambda x_i} - \sum_{i=1}^{m}(R_i+1)x_i(1+\lambda x_i)^{\alpha-1}\left[1+\alpha\log(1+\lambda x_i)\right].
\end{aligned}$$

If we denote the MLE of $\boldsymbol{\mu} = (\alpha, \lambda)$ by $\hat{\boldsymbol{\mu}} = (\hat{\alpha}, \hat{\lambda})$, the observed information matrix is then given by

$$I(\hat{\boldsymbol{\mu}}) = \begin{bmatrix} -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha^2} & -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha\partial\lambda} \\ -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\lambda\partial\alpha} & -\frac{\partial^2\ell(\alpha,\lambda;\mathbf{x})}{\partial\lambda^2} \end{bmatrix}_{\boldsymbol{\mu}=\hat{\boldsymbol{\mu}}},$$

and hence the variance covariance matrix of MLEs is given by

$$I^{-1}(\hat{\boldsymbol{\mu}}) = \begin{bmatrix} \hat{V}ar(\hat{\alpha}) & \hat{C}ov(\hat{\alpha}, \hat{\lambda}) \\ \\ \hat{C}ov(\hat{\lambda}, \hat{\alpha}) & \hat{V}ar(\hat{\lambda}) \end{bmatrix}.$$

The approximate $(1-\zeta)100\%$ confidence intervals for the parameters α and λ are

$$\hat{\alpha} \pm z_{\frac{\zeta}{2}} \sqrt{\hat{V}ar(\hat{\alpha})}, \qquad \hat{\lambda} \pm z_{\frac{\zeta}{2}} \sqrt{\hat{V}ar(\hat{\lambda})},$$

where $z_{\frac{\zeta}{2}}$ is the $\frac{\zeta}{2}$ quantile of the standard normal distribution.

4. Bayes Estimation

4.1. Loss functions, prior and posterior distribution

In this section, we obtain Bayes estimators of the unknown parameters α, λ, reliability function $R(t)$, and HR function $H(t)$ under symmetric as well as asymmetric loss functions such as squared error (SE), LINEX and general entropy (GE) loss functions. One symmetric loss function is the SE which is defined as $L_{SE}(\mu, \tilde{\mu}) = (\tilde{\mu} - \mu)^2$, where $\tilde{\mu}$ is a Bayes estimator of μ. The Bayes estimators under the SE loss function are the mean of the posterior distribution. One asymmetric loss function is the LINEX, due to Zellner (1986).

The LINEX loss function can be expressed as $L_L(\mu, \tilde{\mu}) = e^{\xi(\tilde{\mu}-\mu)} - \xi(\tilde{\mu} - \mu) - 1, \xi \neq 0$, where $\tilde{\mu}$ is a Bayes estimator of μ and, it is obtained from its posterior distribution as $\tilde{\mu} = -\frac{1}{\xi} \log\left(E\left[e^{-\xi\mu}|\mathbf{X}\right]\right)$, where $\xi \neq 0$ is an arbitrary real number. Another useful asymmetric loss function is the GE, proposed by Calabria and Pulcini (1996), defined as $L_{GE}(\mu, \tilde{\mu}) \propto \left(\frac{\tilde{\mu}}{\mu}\right)^{\tau} - \tau \log\left(\frac{\tilde{\mu}}{\mu}\right) - 1$, where $\tau \neq 0$ is an arbitrary real number. The Bayes estimator of μ with respect to GE loss function obtained from its posterior distribution as $\tilde{\mu} = \left(E\left[\mu^{-\tau}|\mathbf{X}\right]\right)^{-\frac{1}{\tau}}$. Suppose that the unknown parameters α and λ are distributed independently gamma distribution as

$$\pi(\alpha) = \frac{b^a}{\Gamma(a)} \alpha^{a-1} e^{-b\alpha}, \alpha > 0, a, b > 0, \quad \pi(\lambda) = \frac{d^c}{\Gamma(c)} \lambda^{c-1} e^{-d\lambda}, \lambda > 0, c, d > 0.$$

The joint prior distribution of parameters α and λ can be written as

$$\pi(\alpha, \lambda) \propto \pi(\alpha)\pi(\lambda) \propto \alpha^{a-1} e^{-b\alpha} \lambda^{c-1} e^{-d\lambda}. \tag{9}$$

From (4) and (9), it follows that the joint posterior distribution of α and λ is given by

$$\pi(\alpha, \lambda|\mathbf{X}) = \frac{L(\alpha, \lambda|\mathbf{X})\pi(\alpha, \lambda)}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\mathbf{X})\pi(\alpha, \lambda) d\alpha d\lambda}.$$

Therefore, the Bayes estimators of parameters α, λ, reliability function $R(t)$, and HR function $H(t)$ under the SE loss function are given by

$$\hat{\alpha}_{SE} = E\left[\alpha|\mathbf{X}\right] = \frac{\int_0^\infty \int_0^\infty \alpha L(\alpha, \lambda|\mathbf{X})\pi(\alpha, \lambda) d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\mathbf{X})\pi(\alpha, \lambda) d\alpha d\lambda}, \tag{10}$$

$$\hat{\lambda}_{SE} = E\left[\lambda|\mathbf{X}\right] = \frac{\int_0^\infty \int_0^\infty \lambda L(\alpha, \lambda|\mathbf{X})\pi(\alpha, \lambda) d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha, \lambda|\mathbf{X})\pi(\alpha, \lambda) d\alpha d\lambda}, \tag{11}$$

$$\hat{R}(t)_{SE} = E\left[R(t)|\mathbf{X}\right] = \frac{\int_0^\infty \int_0^\infty R(t)L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}, \tag{12}$$

and

$$\hat{H}(t)_{SE} = E\left[H(t)|\mathbf{X}\right] = \frac{\int_0^\infty \int_0^\infty H(t)L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}. \tag{13}$$

Furthermore, the Bayes estimators of parameters α, λ, reliability function $R(t)$, and HR function $H(t)$ under the LINEX loss function are given by

$$\hat{\alpha}_L = \frac{-1}{\xi}\log\left[E\left(e^{-\xi\alpha}|\mathbf{X}\right)\right] = \frac{-1}{\xi}\log\left[\frac{\int_0^\infty \int_0^\infty e^{-\xi\alpha}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right], \tag{14}$$

$$\hat{\lambda}_L = -\frac{1}{\xi}\log\left[E\left(e^{-\xi\lambda}|\mathbf{X}\right)\right] = -\frac{1}{\xi}\log\left[\frac{\int_0^\infty \int_0^\infty e^{-\xi\lambda}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right], \tag{15}$$

$$\hat{R}(t)_L = \frac{-1}{\xi}\log\left[E\left(e^{-\xi R(t)}|\mathbf{X}\right)\right] = \frac{-1}{\xi}\log\left[\frac{\int_0^\infty \int_0^\infty e^{-\xi R(t)}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right], \tag{16}$$

and

$$\hat{H}(t)_L = \frac{-1}{\xi}\log\left[E\left(e^{-\xi H(t)}|\mathbf{X}\right)\right] = \frac{-1}{\xi}\log\left[\frac{\int_0^\infty \int_0^\infty e^{-\xi H(t)}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right]. \tag{17}$$

Finally, the Bayes estimators of parameters α, λ, reliability function $R(t)$ and HR function $H(t)$ under the GE loss function are given by

$$\hat{\alpha}_{GE} = \left[E\left(\alpha^{-\tau}|\mathbf{X}\right)\right]^{-\frac{1}{\tau}} = \left(\frac{\int_0^\infty \int_0^\infty \alpha^{-\tau}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right)^{-\frac{1}{\tau}}, \tag{18}$$

$$\hat{\lambda}_{GE} = \left[E\left(\lambda^{-\tau}|\mathbf{X}\right)\right]^{-\frac{1}{\tau}} = \left(\frac{\int_0^\infty \int_0^\infty \lambda^{-\tau}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right)^{-\frac{1}{\tau}}, \tag{19}$$

$$\hat{R}(t)_{GE} = \left[E\left(R(t)^{-\tau}|\mathbf{X}\right)\right]^{-\frac{1}{\tau}} = \left(\frac{\int_0^\infty \int_0^\infty R(t)^{-\tau}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right)^{-\frac{1}{\tau}}, \tag{20}$$

and

$$\hat{H}(t)_{GE} = \left[E\left(H(t)^{-\tau}|\mathbf{X}\right)\right]^{-\frac{1}{\tau}} = \left(\frac{\int_0^\infty \int_0^\infty H(t)^{-\tau}L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)d\alpha d\lambda}\right)^{-\frac{1}{\tau}}. \tag{21}$$

From (10)-(21), we observe that the Bayes estimators are of the form of ratio of two integrals, which cannot be simplified into a closed form. However, using the developed approach by Lindley (1980), one can approximate these Bayes estimators into forms containing no integrals.

4.2. Lindley approximation

From the previous subsection, we have observed that all Bayes estimators of any function of parameters α and λ, say $\eta(\alpha,\lambda)$, are in the form of ratio of two integrals, which the simplified closed forms are not available. For evaluation of the ratio of two integrals, Lindley (1980) gave an approximation method. To illustrate this method, consider the ratio of integral defined as:

$$G(\mathbf{X}) = \frac{\int_0^\infty \int_0^\infty \eta(\alpha,\lambda) L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)\, d\alpha\, d\lambda}{\int_0^\infty \int_0^\infty L(\alpha,\lambda|\mathbf{X})\pi(\alpha,\lambda)\, d\alpha\, d\lambda}. \tag{22}$$

Applying Lindley's approximation, the integral fraction of $G(\mathbf{X})$ as defined in (22) can be approximated as

$$\begin{aligned} G(\mathbf{X}) \;=\;& \eta(\hat\alpha,\hat\lambda) + \frac{1}{2}\Big[(\hat\eta_{11} + 2\hat\eta_1\hat\nu_1)\hat\sigma_{11} + (\hat\eta_{12} + 2\hat\eta_1\hat\nu_2)\hat\sigma_{12} \\ &+ (\hat\eta_{21} + 2\hat\eta_2\hat\nu_1)\hat\sigma_{21} + \hat\sigma_{22}(\hat\eta_{22} + 2\hat\eta_2\hat\nu_2) \\ &+ (\hat\eta_1\hat\sigma_{11} + \hat\eta_2\hat\sigma_{12})L_1(\hat\ell,\hat\sigma) + (\hat\eta_1\hat\sigma_{21} + \hat\eta_2\hat\sigma_{22})L_2(\hat\ell,\hat\sigma)\Big], \end{aligned} \tag{23}$$

where $\hat\alpha$ and $\hat\lambda$ are the MLE of α and λ, respectively. Also $\hat\sigma_{ij}$ is the (i,j)-th element of the observed variance-covariance matrix $I^{-1}(\hat\mu)$ as computed in section 3. Other expressions are given by

$$\nu = \log\left[\pi(\alpha,\lambda)\right], \nu_1 = \frac{\partial\nu}{\partial\alpha} = \frac{a-1}{\alpha} - b, \nu_2 = \frac{\partial\nu}{\partial\lambda} = \frac{c-1}{\lambda} - d,$$

$$\begin{cases} L_1(\hat\ell,\hat\sigma) = \hat\ell_{30}\hat\sigma_{11} + 2\hat\ell_{21}\hat\sigma_{12} + \hat\ell_{12}\hat\sigma_{22}, \\ L_2(\hat\ell,\hat\sigma) = \hat\ell_{21}\hat\sigma_{11} + 2\hat\ell_{12}\hat\sigma_{12} + \hat\ell_{03}\hat\sigma_{22}. \\ \eta_1 = \frac{\partial\eta}{\partial\alpha}, \eta_2 = \frac{\partial\eta}{\partial\lambda}, \eta_{11} = \frac{\partial^2\eta}{\partial\alpha^2}, \eta_{12} = \frac{\partial^2\eta}{\partial\alpha\partial\lambda}, \eta_{21} = \frac{\partial^2\eta}{\partial\lambda\partial\alpha}, \eta_{22} = \frac{\partial^2\eta}{\partial\lambda^2}, \end{cases} \tag{24}$$

$$\begin{cases} \ell_{30} = \frac{\partial^3\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha^3} = \frac{2m}{\alpha^3} - \sum_{i=1}^m (R_i+1)(1+\lambda x_i)^\alpha \left(\log(1+\lambda x_i)\right)^3, \\ \ell_{03} = \frac{\partial^3\ell(\alpha,\lambda;\mathbf{x})}{\partial\lambda^3} = \frac{2m}{\lambda^3} + (\alpha-1)\sum_{i=1}^m \dfrac{2x_i^3}{(1+\lambda x_i)^3} - \alpha(\alpha-1)(\alpha-2)\sum_{i=1}^m \dfrac{(R_i+1)x_i^3}{(1+\lambda x_i)^{3-\alpha}}, \\ \ell_{12} = \frac{\partial^3\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha\partial\lambda^2} = \sum_{i=1}^m \dfrac{(R_i+1)x_i^2}{(1+\lambda x_i)^{2-\alpha}}\Big[1 - 2\alpha + (1-\alpha)\alpha\log(1+\lambda x_i)\Big] - \sum_{i=1}^m \dfrac{x_i^2}{(1+\lambda x_i)^2}, \\ \ell_{21} = \frac{\partial^3\ell(\alpha,\lambda;\mathbf{x})}{\partial\alpha^2\partial\lambda} = -\sum_{i=1}^m (R_i+1)x_i(1+\lambda x_i)^{\alpha-1}\left(\log(1+\lambda x_i)\right)\left[2 + \alpha(\log(1+\lambda x_i))\right]. \end{cases}$$

4.3. Approximate Bayes estimators under SE loss function

In Equation (22) if we take $\eta(\alpha,\lambda) = \alpha$, then $\eta_1 = 1, \eta_2 = \eta_{11} = \eta_{12} = \eta_{21} = \eta_{22} = 0$. By substituting above relations in Equations (23) and (10) respectively, the approximate Bayes estimator $\hat\alpha_{SE}$ of α uunder SE loss function is obtained as

$$\hat\alpha_{SE} = \hat\alpha + \left(\frac{a-1}{\hat\alpha} - b\right)\hat\sigma_{11} + \left(\frac{c-1}{\hat\lambda} - d\right)\hat\sigma_{12} + \frac{1}{2}\hat\sigma_{11}L_1(\hat\ell,\hat\sigma) + \frac{1}{2}\hat\sigma_{21}L_2(\hat\ell,\hat\sigma). \tag{25}$$

In Equation (22) if we take $\eta(\alpha,\lambda) = \lambda$, then $\eta_2 = 1, \eta_1 = \eta_{11} = \eta_{12} = \eta_{21} = \eta_{22} = 0$. By substituting above relations in Equations (23) and (11) respectively, the approximate Bayes estimator $\hat\lambda_{SE}$ of λ uunder SE loss function is obtained as

$$\hat\lambda_{SE} = \hat\lambda + \left(\frac{a-1}{\hat\alpha} - b\right)\hat\sigma_{21} + \left(\frac{c-1}{\hat\lambda} - d\right)\hat\sigma_{22} + \frac{1}{2}\hat\sigma_{12}L_1(\hat\ell,\hat\sigma) + \frac{1}{2}\hat\sigma_{22}L_2(\hat\ell,\hat\sigma). \tag{26}$$

For obtaining the approximate Bayes estimator of $R(t)$ under SE loss function, by setting $\eta(\alpha,\lambda) = R(t) = e^{1-(1+\lambda)^{\alpha}}$ in Equation (22) we have,

$$\begin{cases} \eta_1 = -e^{1-(1+\lambda t)^{\alpha}}(1+\lambda t)^{\alpha}\log(1+\lambda t), \\ \eta_2 = -\alpha t e^{1-(1+\lambda t)^{\alpha}}(1+\lambda t)^{\alpha-1}, \\ \eta_{11} = e^{1-(1+\lambda t)^{\alpha}}(1+\lambda t)^{\alpha}\left((1+\lambda t)^{\alpha}-1\right)[\log(1+\lambda t)]^2, \\ \eta_{12} = \frac{\partial^2\eta}{\partial\alpha\partial\lambda} = e^{1-(1+\lambda t)^{\alpha}}(1+\lambda t)^{\alpha-1}\left[\log(1+\lambda t)\right] t\big[\alpha\left((1+\lambda t)^{\alpha}-1\right)-1\big], \\ \eta_{22} = e^{1-(1+\lambda t)^{\alpha}}(1+\lambda t)^{\alpha-2}t^2\alpha\big[\alpha\left((1+\lambda t)^{\alpha}-1\right)+1\big]. \end{cases}$$

Now, by substituting above relations in Equations (23) and (12) the approximate Bayes estimator of $\hat{R}(t)_{SE}$ under SE loss function is obtained as

$$\begin{aligned} \hat{R}(t)_{SE} \quad = \quad & e^{1-(1+\hat{\lambda}t)^{\hat{\alpha}}} + \frac{1}{2}\Big[(\hat{\eta}_{11}+2\hat{\eta}_1\hat{\nu}_1)\hat{\sigma}_{11} + (\hat{\eta}_{12}+2\hat{\eta}_1\hat{\nu}_2)\hat{\sigma}_{12} \\ & +(\hat{\eta}_{21}+2\hat{\eta}_2\hat{\nu}_1)\hat{\sigma}_{21} + (\hat{\eta}_{22}+2\hat{\eta}_2\hat{\nu}_2)\hat{\sigma}_{22} \\ & +(\hat{\eta}_1\hat{\sigma}_{11}+\hat{\eta}_2\hat{\sigma}_{12})L_1(\hat{\ell},\hat{\sigma}) + (\hat{\eta}_1\hat{\sigma}_{21}+\hat{\eta}_2\hat{\sigma}_{22})L_2(\hat{\ell},\hat{\sigma})\Big]. \end{aligned} \tag{27}$$

Similarly, for obtaining the approximate Bayes estimator of $H(t)$ under SE loss function, by setting $\eta(\alpha,\lambda) = H(t) = \alpha\lambda\,(1+\lambda t)^{\alpha-1}$, in Equation (22) we have,

$$\begin{cases} \eta_1 = \lambda(1+\lambda t)^{\alpha-1}(\alpha\log(1+\lambda t)+1), \\ \eta_2 = \alpha(1+\lambda t)^{\alpha-2}(\alpha\lambda t+1), \\ \eta_{11} = \lambda(1+\lambda t)^{\alpha-1}\log(1+\lambda t)\left(\alpha\log(1+\lambda t)+2\right), \\ \eta_{12} = (1+\lambda t)^{\alpha-2}\left(2\alpha\lambda t+\alpha(\alpha\lambda t+1)\log(1+\lambda t)+1\right), \\ \eta_{22} = (\alpha-1)\alpha t(1+\lambda t)^{\alpha-3}(\alpha\lambda t+2). \end{cases}$$

Now, by substituting above relations in Equations (23) and (13) the approximate Bayes estimator of HR function under SE loss function, noted by $\hat{H}(t)_{SE}$ is obtained as

$$\begin{aligned} \hat{H}(t)_{SE} \quad = \quad & \hat{\alpha}\hat{\lambda}\left(1+\hat{\lambda}t\right)^{\hat{\alpha}-1} + \frac{1}{2}\Big[(\hat{\eta}_{11}+2\hat{\eta}_1\hat{\nu}_1)\hat{\sigma}_{11} + (\hat{\eta}_{12}+2\hat{\eta}_1\hat{\nu}_2)\hat{\sigma}_{12} \\ & +(\hat{\eta}_{21}+2\hat{\eta}_2\hat{\nu}_1)\hat{\sigma}_{21} + (\hat{\eta}_{22}+2\hat{\eta}_2\hat{\nu}_2)\hat{\sigma}_{22} \\ & +(\hat{\eta}_1\hat{\sigma}_{11}+\hat{\eta}_2\hat{\sigma}_{12})L_1(\hat{\ell},\hat{\sigma}) + (\hat{\eta}_1\hat{\sigma}_{21}+\hat{\eta}_2\hat{\sigma}_{22})L_2(\hat{\ell},\hat{\sigma})\Big]. \end{aligned} \tag{28}$$

4.4. Approximate Bayes estimators under LINEX loss function

In Equation (22) if $\eta(\alpha,\lambda) = e^{-\xi\alpha}$, then $\eta_1 = -\xi e^{-\xi\alpha}, \eta_{11} = \xi^2 e^{-\xi\alpha}, \eta_2 = \eta_{12} = \eta_{21} = \eta_{22} = 0$. By substituting above relations in Equations (23) and (14) respectively, the approximate Bayes estimator $\hat{\alpha}_L$ of α under LINEX loss function is obtained as

$$\begin{aligned} \hat{\alpha}_L \quad = \quad & \hat{\alpha} - \frac{1}{\xi}\log\Bigg(1+\frac{\xi}{2}\Bigg[\left(\xi-2\left(\frac{a-1}{\hat{\alpha}}-b\right)\right)\hat{\sigma}_{11} - 2\left(\frac{c-1}{\hat{\lambda}}-d\right)\hat{\sigma}_{12} \\ & -\hat{\sigma}_{11}L_1(\hat{\ell},\hat{\sigma}) - \hat{\sigma}_{21}L_2(\hat{\ell},\hat{\sigma})\Bigg]\Bigg). \end{aligned} \tag{29}$$

In Equation (22) if $\eta(\alpha,\lambda) = e^{-\xi\lambda}$, then $\eta_2 = -\xi e^{-\xi\lambda}, \eta_{22} = \xi^2 e^{-\xi\lambda}, \eta_1 = \eta_{12} = \eta_{21} = \eta_{22} = 0$. So by substituting above relations in Equations (23) and (15) respectively, the approximate Bayes estimator $\hat{\lambda}_L$ of λ under LINEX loss function is obtained as

$$\begin{aligned} \hat{\lambda}_L \quad = \quad & \hat{\lambda} - \frac{1}{\xi}\log\Bigg(1+\frac{\xi}{2}\Bigg[\left(\xi-2\left(\frac{c-1}{\hat{\lambda}}-d\right)\right)\hat{\sigma}_{22} - 2\left(\frac{a-1}{\hat{\alpha}}-b\right)\hat{\sigma}_{21} \\ & -\hat{\sigma}_{12}L_1(\hat{\ell},\hat{\sigma}) - \hat{\sigma}_{22}L_2(\hat{\ell},\hat{\sigma})\Bigg]\Bigg). \end{aligned} \tag{30}$$

For obtaining the approximate Bayes estimator of $R(t)$ under LINEX loss function, in Equation (22) assume that $\eta(\alpha,\lambda) = e^{-\xi R(t)} = e^{-\xi e^{1-(1+\lambda t)^{\alpha}}}$. Then from (24) we can write

$$\begin{cases} \eta_1 = \xi(1+\lambda t)^{\alpha} \log(1+\lambda t) e^{[-\xi e^{1-(1+\lambda t)^{\alpha}} - (1+\lambda t)^{\alpha}+1]}, \\ \eta_2 = \alpha\xi t(1+\lambda t)^{\alpha-1} e^{[-\xi e^{1-(1+\lambda t)^{\alpha}} - (1+\lambda t)^{\alpha}+1]}, \\ \eta_{11} = \left[\left((1+\lambda t)^{\alpha}-1\right)e^{(1+\lambda t)^{\alpha}} - e\xi(1+\lambda t)^{\alpha}\right]\left(-e^{[-\xi e^{1-(1+\lambda t)^{\alpha}} - 2(1+\lambda t)^{\alpha}+1]}\right) \\ \times\xi(1+\lambda t)^{\alpha}\left[\log(1+\lambda t)\right]^2, \\ \eta_{12} = \eta_{21} = \left[\alpha(1+\lambda t)^{\alpha}\log(1+\lambda t)\left(e^{(1+\lambda t)^{\alpha}} - e\xi\right) - e^{(1+\lambda t)^{\alpha}}\left[\alpha\log(1+\lambda t)+1\right]\right] \\ \times\xi t(1+\lambda t)^{\alpha-1}\left(-e^{-\xi e^{1-(1+\lambda t)^{\alpha}} - 2(1+\lambda t)^{\alpha}+1}\right), \\ \eta_{22} = \left[e^{(1+\lambda t)^{\alpha}}\left(\alpha\left[(1+\lambda t)^{\alpha}-1\right]+1\right) - e\alpha\xi(1+\lambda t)^{\alpha}\right]\left(-e^{-\xi e^{1-(1+\lambda t)^{\alpha}} - 2(1+\lambda t)^{\alpha}+1}\right) \\ \qquad \times\alpha\xi t^2(1+\lambda t)^{\alpha-2}. \end{cases}$$

Now, by substituting above relations in Equations (23) and (16) respectively, the approximate Bayes estimator $\hat{R}(t)_L$ is obtained as

$$\begin{aligned} \hat{R}(t)_L \;=\; -\frac{1}{\xi}\log\Bigg(e^{-\xi e^{1-(1+\hat{\lambda} t)^{\hat{\alpha}}}} &+ \frac{1}{2}\Big[(\hat{\eta}_{11}+2\hat{\eta}_1\hat{\nu}_1)\hat{\sigma}_{11} + (\hat{\eta}_{12}+2\hat{\eta}_1\hat{\nu}_2)\hat{\sigma}_{12} \\ &+(\hat{\eta}_{21}+2\hat{\eta}_2\hat{\nu}_1)\hat{\sigma}_{21} + (\hat{\eta}_{22}+2\hat{\eta}_2\hat{\nu}_2)\hat{\sigma}_{22} \\ &+(\hat{\eta}_1\hat{\sigma}_{11}+\hat{\eta}_2\hat{\sigma}_{12})L_1(\hat{\ell},\hat{\sigma}) + (\hat{\eta}_1\hat{\sigma}_{21}+\hat{\eta}_2\hat{\sigma}_{22})L_2(\hat{\ell},\hat{\sigma})\Big]\Bigg). \end{aligned} \tag{31}$$

Similarly, for obtaining the approximate Bayes estimator $\hat{H}(t)_L$ of HR function $H(t)$ under the LINEX loss function, in Equation (22) assume that $\eta(\alpha,\lambda) = e^{-\xi H(t)} = e^{-\xi\alpha\lambda(1+\lambda t)^{\alpha-1}}$. Then from (24), we easily write following.

$$\begin{cases} \eta_1 = \alpha\lambda\xi(1+\lambda t)^{\alpha-1}\log(1+\lambda t)\left(-e^{-\alpha\lambda\xi(1+\lambda t)^{\alpha-1}}\right), \\ \eta_2 = \alpha\xi(1+\lambda t)^{\alpha-2}(1+\alpha\lambda t)\left(-e^{-\alpha\lambda\xi(1+\lambda t)^{\alpha-1}}\right), \\ \eta_{11} = \alpha\lambda\xi(1+\lambda t)^{\alpha-2}\big[\log(1+\lambda t)\big]^2 e^{-\alpha\lambda\xi(1+\lambda t)^{\alpha-1}}\big(\alpha\lambda\xi(1+\lambda t)^{\alpha} - \lambda t - 1\big), \\ \eta_{12} = \eta_{21} = \Big(-t\lambda(1+\lambda t) - (1+\alpha\lambda t)\log(1+\lambda t)\big[-\alpha\lambda\xi(1+\lambda t)^{\alpha}+1+\lambda t\big]\Big) \\ \qquad\qquad \times\alpha\xi(1+\lambda t)^{\alpha-3}e^{-\alpha\lambda\xi(1+\lambda t)^{\alpha-1}}, \\ \eta_{22} = \alpha\xi(1+\lambda t)^{\alpha-4}e^{-\alpha\lambda\xi(1+\lambda t)^{\alpha-1}}\Big[\alpha\xi(\lambda t)^{\alpha}(1+\alpha\lambda t)^2 - (\alpha-1)t(1+\lambda t)(2+\alpha\lambda t)\Big]. \end{cases}$$

By substituting above relations in Equations (23) and (17) respectively, the approximate Bayes estimator $\hat{H}(t)_L$ is obtained as

$$\begin{aligned} \hat{H}(t)_L \;=\; -\frac{1}{\xi}\log\Bigg(e^{-\xi\hat{\alpha}\hat{\lambda}(1+\hat{\lambda} t)^{\hat{\alpha}-1}} &+ \frac{1}{2}\Big[(\hat{\eta}_{11}+2\hat{\eta}_1\hat{\nu}_1)\hat{\sigma}_{11} + (\hat{\eta}_{12}+2\hat{\eta}_1\hat{\nu}_2)\hat{\sigma}_{12} \\ &+(\hat{\eta}_{21}+2\hat{\eta}_2\hat{\nu}_1)\hat{\sigma}_{21} + (\hat{\eta}_{22}+2\hat{\eta}_2\hat{\nu}_2)\hat{\sigma}_{22} \\ &+(\hat{\eta}_1\hat{\sigma}_{11}+\hat{\eta}_2\hat{\sigma}_{12})L_1(\hat{\ell},\hat{\sigma}) + (\hat{\eta}_1\hat{\sigma}_{21}+\hat{\eta}_2\hat{\sigma}_{22})L_2(\hat{\ell},\hat{\sigma})\Big]\Bigg). \end{aligned} \tag{32}$$

4.5. Approximate Bayes estimators under GE loss function

In Equation (22) if $\eta(\alpha,\lambda) = \alpha^{-\tau}$, then , $\eta_1 = -\tau\alpha^{-(\tau+1)}, \eta_{11} = \tau(\tau+1)\alpha^{-(\tau+2)}, \eta_2 = \eta_{12} = \eta_{21} = \eta_{22} = 0$. So by substituting above relations in Equations (23) and (18) respectively, the approximate Bayes estimator $\hat{\alpha}_{LA-GE}$ of α under GE loss function is obtained as

$$\hat{\alpha}_{GE} = \hat{\alpha}\left\{1+\frac{\tau}{2\hat{\alpha}}\left[\left(\frac{\tau+1}{\hat{\alpha}}-2\left[\frac{a-1}{\hat{\alpha}}-b\right]\right)\hat{\sigma}_{11}-2\left(\frac{c-1}{\hat{\lambda}}-d\right)\hat{\sigma}_{12}\right.\right.$$
$$\left.\left.-\hat{\sigma}_{11}L_1(\hat{\ell},\hat{\sigma})-\hat{\sigma}_{21}L_2(\hat{\ell},\hat{\sigma})\right]\right\}^{-\frac{1}{\tau}}. \quad (33)$$

In Equation (22) if $\eta(\alpha,\lambda)=\lambda^{-\tau}$, then $\eta_2=-\tau\lambda^{-(\tau+1)}, \eta_{22}=\tau(\tau+1)\lambda^{-(\tau+2)}, \eta_1=\eta_{12}=\eta_{21}=\eta_{11}=0$. By substituting above relations in Equations (23) and (19) respectively, the approximate Bayes estimator $\hat{\lambda}_{GE}$ of λ under the GE loss function is obtained as

$$\hat{\lambda}_{GE} = \hat{\lambda}\left\{1+\frac{\tau}{2\hat{\lambda}}\left[\left(\frac{\tau+1}{\hat{\lambda}}-2\left[\frac{c-1}{\hat{\lambda}}-d\right]\right)\hat{\sigma}_{22}-2\left(\frac{a-1}{\hat{\alpha}}-b\right)\hat{\sigma}_{21}\right.\right.$$
$$\left.\left.-\hat{\sigma}_{12}L_1(\hat{\ell},\hat{\sigma})-\hat{\sigma}_{22}L_2(\hat{\ell},\hat{\sigma})\right]\right\}^{-\frac{1}{\tau}}. \quad (34)$$

For obtaining the approximate Bayes estimator of $R(t)$ under GE loss function, in Equation (22) assume that $\eta(\alpha,\lambda)=R(t)^{-\tau}=\left(e^{1-(1+\lambda t)^{\alpha}}\right)^{-\tau}$. Then from (24) we can write

$$\begin{cases} \eta_1=\tau(1+\lambda t)^{\alpha}\log(1+\lambda t)\left(e^{1-(1+\lambda t)^{\alpha}}\right)^{-\tau}, \\ \eta_2=\alpha\tau t(1+\lambda t)^{\alpha-1}\left(e^{1-(1+\lambda t)^{\alpha}}\right)^{-\tau}, \\ \eta_{11}=\tau(1+\lambda t)^{\alpha}\left[\log(1+\lambda t)\right]^2\left(e^{1-(1+\lambda t)^{\alpha}}\right)^{-\tau}\left(\tau(1+\lambda t)^{\alpha}+1\right), \\ \eta_{12}=\eta_{21}=\tau t(\lambda t+1)^{\alpha-1}\left(e^{1-(\lambda t+1)^{\alpha}}\right)^{-\tau}\Big(\left[\alpha\log(\lambda t+1)\right]\left(\tau(1+\lambda t)^{\alpha}+1\right)+1\Big), \\ \eta_{22}=\alpha\tau t^2(\lambda t+1)^{\alpha-2}\left(e^{1-(\lambda t+1)^{\alpha}}\right)^{-\tau}\left[\alpha\tau(\lambda t+1)^{\alpha}+\alpha-1\right]. \end{cases}$$

Now, by substituting above relations in Equations (23), and (20) respectively, the approximate Bayes estimator $\hat{R}(t)_{GE}$ is obtained as

$$\hat{R}(t)_{GE} = \left\{\left(e^{1-(1+\hat{\lambda}t)^{\hat{\alpha}}}\right)^{-\tau}+\frac{1}{2}\Big[(\hat{\eta}_{11}+2\hat{\eta}_1\hat{\nu}_1)\hat{\sigma}_{11}+(\hat{\eta}_{12}+2\hat{\eta}_1\hat{\nu}_2)\hat{\sigma}_{12}\right.$$
$$+(\hat{\eta}_{21}+2\hat{\eta}_2\hat{\nu}_1)\hat{\sigma}_{21}+(\hat{\eta}_{22}+2\hat{\eta}_2\hat{\nu}_2)\hat{\sigma}_{22}$$
$$\left.+(\hat{\eta}_1\hat{\sigma}_{11}+\hat{\eta}_2\hat{\sigma}_{12})L_1(\hat{\ell},\hat{\sigma})+(\hat{\eta}_1\hat{\sigma}_{21}+\hat{\eta}_2\hat{\sigma}_{22})L_2(\hat{\ell},\hat{\sigma})\Big]\right\}^{-\frac{1}{\tau}}. \quad (35)$$

Similarly, the approximate Bayes estimator $\hat{H}(t)_{GE}$ of HR function under the GE loss function is obtained by setting $\eta(\alpha,\lambda)=H(t)^{-\tau}=\left(\alpha\lambda\left(1+\lambda t\right)^{\alpha-1}\right)^{-\tau}$ in Equation (22). Then from (24) we can write

$$\begin{cases} \eta_1=-\frac{\tau\left(\alpha\lambda(1+\lambda t)^{\alpha-1}\right)^{-\tau}\left(\alpha\log(1+\lambda t)+1\right)}{\alpha}, \\ \eta_2=-\frac{\tau(1+\alpha\lambda t)\left(\alpha\lambda(1+\lambda t)^{\alpha-1}\right)^{-\tau}}{\lambda(1+\lambda t)}, \\ \eta_{11}=\frac{\tau\left(\alpha\lambda(1+\lambda t)^{\alpha-1}\right)^{-\tau}\Big(1+\tau+\alpha\tau\log(1+\lambda t)\left(2+\alpha\log(1+\lambda t)\right)\Big)}{\alpha^2}, \\ \eta_{12}=\tau(1+\lambda t)^{\alpha-2}\left(\alpha\lambda(1+\lambda t)^{\alpha-1}\right)^{-(\tau+1)}\left[\tau+\alpha\lambda(\tau-1)t+\alpha\tau(1+\alpha\lambda t)\log(\lambda t+1)\right], \\ \eta_{22}=\frac{\tau\left(\alpha\lambda(1+\lambda t)^{\alpha-1}\right)^{-\tau}\left[\tau(1+\alpha\lambda t)^2+\lambda t(2+\alpha\lambda t)+1\right]}{\lambda^2(1+\lambda t)^2}. \end{cases}$$

Now, by substituting above relations in Equations (23) and (21) respectively, the approximate Bayes estimator $\hat{H}(t)_{GE}$ is obtained as

$$\begin{aligned}\hat{H}(t)_{GE} \;=\; & \Bigg[\Big(\hat{\alpha}\hat{\lambda}\big[1+\hat{\lambda}t\big]^{\hat{\alpha}-1}\Big)^{-\tau} + \frac{1}{2}\Big\{(\hat{\eta}_{11}+2\hat{\eta}_1\hat{\nu}_1)\hat{\sigma}_{11} + (\hat{\eta}_{12}+2\hat{\eta}_1\hat{\nu}_2)\hat{\sigma}_{12} \\ & + (\hat{\eta}_{21}+2\hat{\eta}_2\hat{\nu}_1)\hat{\sigma}_{21} + (\hat{\eta}_{22}+2\hat{\eta}_2\hat{\nu}_2)\hat{\sigma}_{22} \\ & + (\hat{\eta}_1\hat{\sigma}_{11}+\hat{\eta}_2\hat{\sigma}_{12})L_1(\hat{\ell},\hat{\sigma}) + (\hat{\eta}_1\hat{\sigma}_{21}+\hat{\eta}_2\hat{\sigma}_{22})L_2(\hat{\ell},\hat{\sigma})\Big\}\Bigg]^{-\frac{1}{\tau}}. \end{aligned} \tag{36}$$

5. Real Data Analysis

In this section, we analyze a data set due to Lee and Wang (2003) for the purpose of illustration. This data set represents remission times (in months) of a random sample of 128 bladder cancer patients. For this data set, 128 failures are observed. Kumar *et al.* (2019) showed that NH distribution is an appropriate model for this data set. So, we can assume that this data set comes from NH distribution and accordingly, we make inferences on the unknown parameters α, λ, reliability function $R(t)$, and HR function $H(t)$. We used $m = 88$ artificial progressive type-II censoring sample from $n = 128$ bladder cancer patients generated by Alhussain and Ahmed (2020).

The 88 generated data are as follows:

0.08, 0.2, 0.4, 0.5, 0.51, 0.81, 0.9, 1.05, 1.19, 1.26, 1.35, 1.4, 2.02, 2.02, 2.07, 2.26, 2.46, 2.64, 2.69, 2.75, 3.02, 3.31, 3.57, 3.64, 3.7, 3.82, 3.88, 4.18, 4.26, 4.33, 4.34, 4.5, 5.09, 5.17, 5.32, 5.34, 5.49, 5.62, 5.85, 6.25, 6.54, 6.93, 6.94, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 8.26, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 12.02, 12.03, 12.07, 12.63, 13.11, 13.8, 14.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.36, 19.13, 20.28, 22.69, 23.63, 25.74, 25.82, 26.31, 46.12, 79.05,

with $R_1 = R_2 = R_3 = R_4 = 10$ and $R_i = 0$ for $i = 5, \cdots, 88$. Table 1 provides the MLE and Bayes estimates of α, λ, $R(t = 1)$ and $H(t = 1)$ relative to SE, LINEX ($\xi = -2$) and GE ($\tau = -1.5$) loss functions for this sample, when the hyperparameters $a = 150, b = 0.2, c = 160, d = 0.2$ are used for prior parameters. The parameter estimates, estimates of the reliability function R(t) and HR function H(t), and 95% asymptotic confidence intervals for parameters α and λ are presented in Table 1. Figure 3 is a contour plot of the log-likelihood function of these data. The plot indicates that the maximum of the log-likelihood function is at ($\alpha = 0.9720, \beta = 0.1099$), as showed in the Table 1. Figure 3 indicates that the estimate of the survival function of NH distribution is very close to the Kaplan-Meier estimator. Therefore, the NH distribution provides a closer fit to the empirical survivals for the bladder cancer data set.

Table 1 Estimates α, λ, $R(t)$ and $H(t)$ for the bladder cancer data set based on the NH distribution

parameters	MLE	Bayes Estimates under the loss function			95% ACI
		SE	LI	GE	
α	0.9720	3.2012	0.0472	2.6267	(0.8458, 1.0982)
λ	0.1099	0.0665	0.0647	0.0611	(0.0859, 0.1339)
$R(t)$	0.8987	0.706	0.7090	0.6938	
$H(t)$	0.1065	0.3330	0.2934	0.2770	

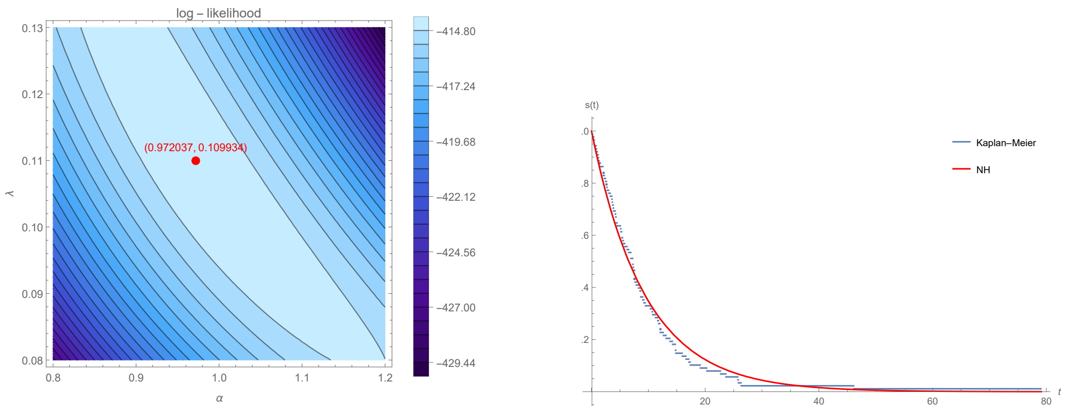

Figure 3 Contour plot of the log-likelihood function(left panel) and Kaplan-Meier and fitted survival function of NH distribution(right panel) considering the data from the bladder cancer patients

6. Simulation Study

In order to compare the estimators of parameters, reliability and HR functions of NH distribution, Monte Carlo simulations were performed utilizing 2000 progressively type-II censored samples. The performance of all estimators (Bayes and MLE) for parameters α and λ, reliability and HR functions is compared numerically in terms of their mean square error (MSE) values for different combinations of n, m and different censoring schemes. All of the computations were performed using Mathematica 10 software. Program code is available if needed. Applying the algorithm of Balakrishnan and Aggarwala (2000), we used the following steps to generate a progressive type-II censored sample from the NH distribution, and then, we get the estimations of the parameters.

1. Simulate m independent exponential random variables $Z_1, Z_2, ..., Z_m$. This can be done using inverse transformation $Z_i = -ln(1-U_i)$ where $(U_1, \cdots, U_m)$ are independent standard uniform random variables.

2. Set $X_i = \frac{Z_1}{n} + \frac{Z_2}{n-R_1-1} + \frac{Z_3}{n-R_1-R_2-2} + \cdots + \frac{Z_i}{n-R_1-R_2-\cdots-R_{i-1}-i+1}$ for $i = 1, 2, ..., m$. Then $(X_1, \cdots, X_m)$ is a progressively type-II censored sample from the standard exponential distribution.

3. Let $Y_i = F^{-1}(1-\exp(-X_i))$, for $i = 1, 2, ..., m$, where $F^{-1}(.)$ is the inverse CDF of the NH distribution. Then $Y_1, \cdots, Y_m$ is the required progressively type-II censored sample from the NH distribution function $F(.)$.

4. The MLEs of the parameters α and λ were obtained by iteratively solving Equations (5) and (6) using the *NMaximize* function of Mathematica 10. Also, we compute the MLEs of the reliability function $R(t)$ and HR function $H(t)$, by (7) and (8), respectively.

5. The approximate Bayes estimates $\hat{\alpha}_{SE}$, $\hat{\lambda}_{SE}$, $\hat{R}(t)_{SE}$ and $\hat{H}(t)_{SE}$ based on the SE loss function, are computed from (25), (26), (27) and (28).

6. The approximate Bayes estimates $\hat{\alpha}_{LL}$, $\hat{\lambda}_L$, $\hat{R}(t)_L$ and $\hat{H}(t)_L$ based on the LINEX loss function, are computed from (29), (30), (31) and (32).

7. The approximate Bayes estimates $\hat{\alpha}_{GE}$, $\hat{\lambda}_{GE}$, $\hat{R}(t)_{GE}$ and $\hat{H}(t)_{GE}$ based on the GE loss function, are computed from (33), (34), (35) and (36).

8. Above steps are repeated by 2000 times. Then we estimate the mean and then, we compute the MSE as

$$MSE = \frac{1}{2000}\sum_{i=1}^{2000}\left(s_i(\hat{\varrho}) - s(\varrho)\right)^2,$$

where $s(\varrho)$ is the true value and $s_i(\hat{\varrho})$ is the i-th estimate of $s(\varrho)$ evaluated at $\hat{\varrho}$. In all of the above cases the prior parameters chosen as $a = b = c = d = 0.1$, which yield the generated values of $\alpha = 0.05$, $\lambda = 0.35$, $R(t = 1) = 0.984995$ and $H(t = 1) = 0.0131589$ as the true values. In Table 2, the censoring scheme (5*4, 25*0) means (4, 4, 4, 4, 4, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0). In Table 4 the estimated lower bound of asymptotic confidence intervals for parameter λ in most cases is negative values. Since $\lambda > 0$, we get all of the negative values as $0^* = 0$. The simulation results are summarized in Tables 3-6.

7. Conclusions

In this article we have considered the inference of the NH distribution based on progressively type-II censored samples. We discussed the ML and Bayes estimators for parameters α and λ, reliability function, and HR function. Asymptotic confidence intervals are also obtained for parameters. The Bayes estimators are discussed under symmetric loss function SE and asymmetric loss functions LINEX and GE. We used Lindley's approximations to compute the approximate Bayes estimates. A simulation study was conducted to examine the performance of the different estimators under different sample sizes and different censoring schemes. In addition, the bladder cancer data set from Lee and Wang (2003) were analyzed by using the NH distribution.

Based on simulation results, we observe that in most cases, the MSEs for all of the estimators (MLE and different Bayes estimators) decrease when the sample size n and effective sample size m increases. It is observed from Table 3 that the approximate Bayes estimate of α under the GE loss function ($\hat{\alpha}_{GE}$ with $\tau = 0.5$) is better than the other estimates in most cases (the average of approximate Bayes estimates of α under the GE loss function ($\hat{\alpha}_{GE}$ with $\tau = 0.5$) are close to the true parameters and it has the smallest estimated MSEs as compared with the other estimates). From Table 4, it is clear that the approximate Bayes estimator of parameter λ, under the LINEX loss function ($\hat{\lambda}_L$ with $\xi = 2$)) are close to the true parameters and it has the smallest estimated MSEs as compared with other estimates (approximate Bayes estimates under SE and GE loss functions and MLEs). From Table 5, it is clear that the MLEs of reliability function $R(t)$, is close to the true parameter and it has the smallest estimated MSEs as compared with the other estimates. From Table 6, it is clear that the approximate Bayes estimator of HR function $H(t)$, under the GE loss function ($\hat{H}(t)_{GE}, \tau = 0.5$) is close to the true parameter and it has the smallest estimated MSEs as compared with the other estimates. It is observed from Tables 3 and 4, the average length of asymptotic confidence intervals is decreased as the sample size is increased. The results of the application of the NH distribution on remission times in bladder cancer patients in the presence of progressive type censoring are interested since they will enable the use of the methods presented in this paper in various application issues such as cancer data modeling. The current work can also be extended to other censored data such as balanced joint progressive type-II censoring, generalized order statistics, and progressive sequential order statistics.

Acknowledgements

The authors are grateful to two anonymous reviewers for their valuable suggestions which have improved the manuscript substantially.

Table 2 Different sampling schemes

Different sampling schemes (D)	n	m	R_i
1	50	30	(5*4,25*0)
2	50	30	(12*0,5*4,13*0)
3	50	30	(4,2,2,22*0,4,4,2,2,0)
4	50	30	(29*0,20)
5	50	40	(5*2,35*0)
6	50	40	(17*0,10*1,13*0)
7	50	40	(30*0,10*1)
8	50	40	(5,37*0,5,0)
9	50	50	(50*0)
10	80	50	(5*6,45*0)
11	80	50	(22*0,6*5,22*0)
12	80	50	(44*0,6*5)
13	80	50	(15,48*0,15)
14	80	65	(21*0,5*2,5,38*0)
15	80	65	(60*0,5*3)
16	80	65	(64*0,15)
17	80	80	(80*0)
18	110	80	(10,10,5,5,76*0)
19	110	80	(19*0,10*3,51*0)
20	110	80	(70*0,10*3)
21	110	80	(10,41*0,10,36*0,10)
22	110	95	(5*1,80*0,10*1)
23	110	95	(93*0,7,8)
24	110	95	(32*0,5*3,58*0)
25	110	110	(110*0)
26	150	50	(20*0,5*20,25*0)
27	150	50	(40*0,10*10)
28	150	50	(50,48*0,50)
29	150	100	(28*0,5*10,62*0)
30	150	100	(99*0,50)
31	150	100	(50*1,50*0)
32	150	150	(150*0)

Table 3 The ML and approximate Bayes estimators of α when $\alpha = 0.05, \lambda = 0.35, a = b = c = d = 0.1$

D	$\hat{\alpha}$, MSE	$95\% ACI$	$\hat{\alpha}_{SE}$, MSE	$\hat{\alpha}_L((\xi = -2))$,MSE	$\hat{\alpha}_L(\xi = 2)$, MSE	$\hat{\alpha}_{GE}(\tau = -0.5)$, MSE	$\hat{\alpha}_{GE}(\tau = 0.5)$, MSE
1	0.0521, 5.41×10^{-5}	(0.0413, 0.0629)	0.0511, 5.09×10^{-5}	0.0516, 5.14×10^{-5}	0.0511, 5.07×10^{-5}	0.0509, 5.03×10^{-5}	0.0505, 4.99×10^{-5}
2	0.0526, 7.06×10^{-5}	(0.0406, 0.0646)	0.0515, 9.88×10^{-5}	0.0519, 8.79×10^{-5}	0.0515, 1.09×10^{-4}	0.0513, 1.28×10^{-4}	0.0593, 1.31×10^{-4}
3	0.0523, 7.59×10^{-5}	(0.0395, 0.0651)	0.0509, 7.23×10^{-5}	0.0516, 7.19×10^{-5}	0.0509, 7.20×10^{-5}	0.0507, 7.20×10^{-5}	0.0502, 8.46×10^{-5}
4	0.0528, 8.78×10^{-5}	(0.0386, 0.0669)	0.0510, 8.00×10^{-5}	0.0521, 8.11×10^{-5}	0.0509, 7.96×10^{-5}	0.0507, 7.89×10^{-5}	0.0501, $7.95{*}10^{-5}$
5	0.0516, 3.99×10^{-5}	(0.0140,0.0608)	0.0507, 3.66×10^{-5}	0.0513, 3.88×10^{-5}	0.0507, 3.64×10^{-5}	0.0506, 3.61×10^{-5}	0.0503, 3.53×10^{-5}
6	0.0516, 4.10×10^{-5}	(0.0419, 0.0613)	0.0507, 4.09×10^{-5}	0.0513, 4.33×10^{-5}	0.0507, 4.13×10^{-5}	0.0505, 4.32×10^{-5}	0.0505, 2.03×10^{-4}
7	0.0575, 4.97×10^{-5}	(0.0413, 0.0621)	0.0507, 5.40×10^{-5}	0.0513, 5.33×10^{-5}	0.0507, 5.55×10^{-5}	0.0506, 5.95×10^{-5}	0.0508, 1.02×10^{-3}
8	0.0517, 4.34×10^{-5}	(0.0417, 0.0617)	0.0508, 4.23×10^{-5}	0.0514, 4.16×10^{-5}	0.0508, 4.22×10^{-5}	0.0506, 4.21×10^{-5}	0.0503, 4.26×10^{-5}
9	0.0512, 2.82×10^{-5}	(0.0430, 0.0593)	0.0505, 2.59×10^{-5}	0.0510, 2.80×10^{-5}	0.0504, 2.58×10^{-5}	0.0503, 2.57×10^{-5}	0.0501, 2.53×10^{-5}
10	0.0513, 2.76×10^{-5}	(0.0431, 0.0594)	0.0507, 2.57×10^{-5}	0.0510, 2.68×10^{-5}	0.0506, 2.56×10^{-5}	0.0505, 5.54×10^{-5}	0.0503 , 2.49×10^{-5}
11	0.0514, 3.31×10^{-5}	(0.0425, 0.0604)	0.0507, 5.21×10^{-5}	0.0514, 6.36×10^{-5}	0.0507, 5.53×10^{-5}	0.0507, 1.02×10^{-3}	0.0503, 3.13×10^{-5}
12	0.0516, 4.65×10^{-5}	(0.0411, 0.0620)	0.0505, 4.28×10^{-5}	0.0513, 4.49×10^{-5}	0.0505, 4.26×10^{-5}	0.0503, 4.23×10^{-5}	0.0500, 4.17×10^{-5}
13	0.0513, 3.66×10^{-5}	(0.0417, 0.0609)	0.0504, 3.35×10^{-5}	0.0510, 3.57×10^{-5}	0.0504, 3.38×10^{-5}	0.0503, 3.31×10^{-5}	0.0500, 3.25×10^{-5}
14	0.0509, 2.14×10^{-5}	(0.0436, 0.0582)	0.0504, 2.00×10^{-5}	0.0508, 2.12×10^{-5}	0.0503, 1.99×10^{-5}	0.0503, 1.98×10^{-5}	0.0501, 1.96×10^{-5}
15	0.0511, 2.86×10^{-5}	(0.0430, 0.0593)	0.0505, 2.65×10^{-5}	0.0510, 2.82×10^{-5}	0.0504, 2.64×10^{-5}	0.0503, 2.62×10^{-5}	0.0501, 2.58×10^{-5}
16	0.0512, 2.94×10^{-5}	(0.0430, 0.0594)	0.0505, 2.71×10^{-5}	0.0511, 2.90×10^{-5}	0.0505, 2.70×10^{-5}	0.0504, 2.68×10^{-5}	0.0502, 2.64×10^{-5}
17	0.0506, 1.53×10^{-5}	(0.0442, 0.0570)	0.0502, 1.45×10^{-5}	0.0505, 1.52×10^{-5}	0.0501, 1.45×10^{-5}	0.0501, 1.44×10^{-5}	0.0499, 1.43×10^{-5}
18	0.0507, 1.65×10^{-5}	(0.0444, 0.0571)	0.0503, 1.57×10^{-5}	0.0506, 1.63×10^{-5}	0.0503, 1.56×10^{-5}	0.0503, 1.56×10^{-5}	0.0501, 1.54×10^{-5}
19	0.0508, 1.83×10^{-5}	(0.0442, 0.0574)	0.0504, 1.74×10^{-5}	0.0507, 1.81×10^{-5}	0.0503, 1.73×10^{-5}	0.0503, 1.73×10^{-5}	0.0501, 1.71×10^{-5}
20	0.0509, 2.35×10^{-5}	(0.0433, 0.0586)	0.0503, 2.19×10^{-5}	0.0508, 2.32×10^{-5}	0.0503, 2.18×10^{-5}	0.0502, 2.17×10^{-5}	0.0500, 2.14×10^{-5}
21	0.0508, 1.95×10^{-5}	(0.0437, 0.0579)	0.0503, 1.83×10^{-5}	0.0507, 1.92×10^{-5}	0.0502, 1.83×10^{-5}	0.0502, 1.82×10^{-5}	0.0500, 1.80×10^{-5}
22	0.0507, 1.54×10^{-5}	(0.0445, 0.0570)	0.0503, 1.46×10^{-5}	0.0507, 1.52×10^{-5}	0.0503, 1.45×10^{-5}	0.0503, 1.45×10^{-5}	0.0501, 1.43×10^{-5}
23	0.0505, 1.54×10^{-5}	(0.0441,0.0570)	0.0501, 1.47×10^{-5}	0.0505, 1.53×10^{-5}	0.0501, 1.47×10^{-5}	0.0500, 1.47×10^{-5}	0.0499, 1.45×10^{-5}
24	0.0506, 1.42×10^{-5}	(0.0447, 0.0566)	0.0502, 1.35×10^{-5}	0.0506, 1.41×10^{-5}	0.0502, 1.35×10^{-5}	0.0502, 1.34×10^{-5}	0.0500, 1.33×10^{-5}
25	0.0505, 1.17×10^{-5}	(0.0451,0.0559)	0.0502, 1.12×10^{-5}	0.0504, 1.16×10^{-5}	0.0501, 1.12×10^{-5}	0.0501, 1.12×10^{-5}	0.0500, 1.11×10^{-5}
26	0.0515, 4.32×10^{-5}	(0.0419, 0.0611)	0.0506, 3.94×10^{-5}	0.0512, 4.21×10^{-5}	0.0505, 3.92×10^{-5}	0.0504, 3.89×10^{-5}	0.0501, 3.81×10^{-5}
27	0.0519, 7.03×10^{-5}	(0.0388, 0.0651)	0.0502, 5.94×10^{-5}	0.0515, 6.73×10^{-5}	0.0521, 5.90×10^{-5}	0.0499, 5.82×10^{-5}	0.0494, 5.66×10^{-5}
28	0.0566, 5.38×10^{-5}	(0.0403,0.0630)	0.0504, 4.82×10^{-5}	0.0512, 515×10^{-5}	0.0504, 4.79×10^{-5}	0.0502, 4.74×10^{-5}	0.0498, 4.63×10^{-5}
29	0.0506, 1.47×10^{-5}	(0.0447,0.0565)	0.0502, 1.41×10^{-5}	0.0505, 1.46×10^{-5}	0.0502, 1.41×10^{-5}	0.0502, 1.40×10^{-5}	0.0501, 1.39×10^{-5}
30	0.0508, 2.00×10^{-5}	(0.0437,0.0579)	0.0503, 1.88×10^{-5}	0.0507, 1.98×10^{-5}	0.0503, 1.88×10^{-5}	0.0502, 1.87×10^{-5}	0.0500, 1.85×10^{-5}
31	0.0505, 1.33×10^{-5}	(0.0447,0.0564)	0.0502, 1.28×10^{-5}	0.0505, 1.32×10^{-5}	0.0502, 1.27×10^{-5}	0.0501, 1.27×10^{-5}	0.0500, 1.26×10^{-5}
32	0.0504, 8.10×10^{-6}	(0.0458,0.0550)	0.0501, 7.84×10^{-6}	0.0503, 8.08×10^{-6}	0.0501, 7.84×10^{-6}	0.0501, 7.82×10^{-6}	0.0500, 7.77×10^{-6}

Table 4 The ML and approximate Bayes estimators of λ when $\alpha = 0.05, \lambda = 0.35, a = b = c = d = 0.1$

D	$\hat{\lambda}$, MSE	$95\%ACI$	$\hat{\lambda}_{SE}$, MSE	$\hat{\lambda}_L\ (\xi = -2)$, MSE	$\hat{\lambda}_L\ (\xi = 2)$, MSE	$\hat{\lambda}_{GE}\ (\tau = -0.5)$, MSE	$\hat{\lambda}_{GE}\ (\tau = 0.5)$, MSE
1	0.4756, 0.3618	(0*, 1.2472)	0.7608, 0.9324	0.7960, 1.0195	0.4575, 0.1211	0.6634, 0.6788	0.4258, 0.2524
2	0.4406, 0.2136	(0*,1.1421)	0.7285, 0.6821	0.7499, 0.7040	0.4569, 0.0820	0.6442, 0.4761	0.4186, 0.1490
3	0.4955, 0.4883	(0*,1.3328)	0.8054, 1.0625	0.8389, 1.2886	0.4827, 0.1754	0.6983, 0.6759	0.4384, 0.1970
4	0.4550, 0.3593	(0*,1.2001)	0.7547, 0.8292	0.7767, 0.9778	0.4760, 0.1489	0.6648, 0.5495	0.4296, 0.1709
5	0.4989, 0.3637	(0*,1.2903)	0.7890, 0.9361	0.8290, 1.0456	0.4734, 0.1207	0.6886, 0.6259	0.4476, 0.2000
6	0.4605, 0.2972	(0*,1.1919)	0.7402, 0.7742	0.7713, 0.8694	0.5623, 0.1091	0.6501, 0.5126	0.4252, 0.1578
7	0.4815, 0.3757	(0*,1.2476)	0.7686, 0.8283	0.8064, 1.0186	0.4771, 0.1524	0.6790, 0.5636	0.4450, 0.1811
8	0.4710, 0.4339	(0*,1.2347)	0.7460, 0.8451	0.7887, 1.0972	0.4684, 0.1766	0.6535, 0.5500	0.4229, 0.1774
9	0.4695, 0.3183	(0*,1.2004)	0.7372, 0.7478	0.7777, 0.8985	0.4659, 0.1176	0.6483, 0.5015	0.4314, 0.1706
10	0.4300, 0.1434	(0*,0.9884)	0.6029, 0.3249	0.6574, 0.4234	0.4341, 0.0660	0.5420, 0.2393	0.4023, 0.1073
11	0.4112, 0.1373	(0*, 0.9242)	0.5799, 0.3050	0.6234 , 0.3837	0.4374, 0.0714	0.5278, 0.2259	0.4021, 0.1039
12	0.4207, 0.1388	(0*,0.9565)	0.6071, 0.3307	0.6485, 0.4062	0.4503, 0.0759	0.5528, 0.2498	0.4169, 0.1135
13	0.4382, 0.1855	(0*,1.0328)	0.6407, 0.4411	0.6863, 0.5350	0.4536, 0.0803	0.5750, 0.3102	0.4170, 0.1225
14	0.4149, 0.1316	(0*,0.9231)	0.5761, 0.2982	0.6218, 0.3688	0.4356, 0.0688	0.5245, 0.2197	0.4030, 0.1017
15	0.4325, 0.1600	(0*,0.9700)	0.6068, 0.3511	0.6543, 0.4441	0.4492, 0.0792	0.5521, 0.2659	0.4209, 0.1262
16	0.4214, 0.1289	(0*,0.9408)	0.5910, 0.2973	0.6369, 0.3736	0.4434, 0.0706	0.5386, 0.2265	0.4125, 0.1083
17	0.4109, 0.1079	(0*,0.9074)	0.5662, 0.2380	0.6135, 0.3114	0.4338, 0.0622	0.5165, 0.1822	0.3992, 0.0907
18	0.4228, 0.1084	(0*,0.8960)	0.5516, 0.2081	0.6031, 0.2870	0.4364, 0.0614	0.5061, 0.1603	0.4044, 0.0851
19	0.3944, 0.0697	(0*,0.8034)	0.5083, 0.1370	0.5492, 0.1839	0.4234, 0.0503	0.4725, 0.1094	0.3902, 0.0629
20	0.4049, 0.0916	(0*,0.8339)	0.5282, 0.1783	0.5708, 0.2371	0.4327, 0.0601	0.4900, 0.1412	0.4016, 0.0796
21	0.4042, 0.0901	(0*,0.8439)	0.5311, 0.1825	0.5748, 0.2416	0.4295, 0.0572	0.4909, 0.1436	0.3978, 0.0786
22	0.3886, 0.0689	(0*,0.7961)	0.5018, 0.1348	0.5428, 0.1812	0.4165, 0.0498	0.4654, 0.1075	0.3828, 0.0622
23	0.4007, 0.0860	(0*, 0.8195)	0.5176, 0.1651	0.5601, 0.2207	0.4257, 0.0574	0.4806, 0.1322	0.3957, 0.0765
24	0.3933, 0.0719	(0*, 0.7989)	0.5045, 0.1379	0.5458, 0.1855	0.4199, 0.0518	0.4690, 0.1110	0.3881, 0.0657
25	0.3997, 0.0725	(0*, 0.8103)	0.5106, 0.1399	0.5532, 0.1891	0.4241, 0.0520	0.4747, 0.1120	0.3933, 0.0652
26	0.3798, 0.0571	(0.0217, 0.7380)	0.4789, 0.1079	0.5121, 0.1414	0.4140, 0.0464	0.4505, 0.0887	0.3845, 0.0550
27	0.3986, 0.0753	(0*, 0.8061)	0.5305, 0.1624	0.5645, 0.2040	0.4449, 0.0598	0.4973, 0.1327	0.4147, 0.0766
28	0.4039, 0.0964	(0*, 0.8055)	0.5472, 0.2120	0.5884, 0.2710	0.4342, 0.0605	0.5046, 0.1658	0.4020, 0.0858
29	0.3889, 0.0530	(0.0431, 0.7348)	0.4727, 0.0910	0.5073, 0.1225	0.4145, 0.0435	0.4462, 0.0760	0.3871, 0.0502
30	0.3777, 0.0527	(0.0322, 0.7232)	0.4661, 0.0918	0.4996, 0.1228	0.4077, 0.0429	0.4391, 0.0764	0.3780, 0.0501
31	0.3909, 0.0516	(0.0385, 0.7433)	0.4758, 0.0899	0.5116, 0.1217	0.4156, 0.0426	0.4483, 0.0744	0.3874, 0.0483
32	0.3828, 0.0435	(0.0472, 0.7183)	0.4612, 0.0739	0.4946, 0.0998	0.4080, 0.0378	0.4356, 0.0616	0.3794, 0.0411

Table 5 The ML and approximate Bayes estimators of $R(t=1)=0.984995$ when $\alpha=0.05, \lambda=0.35, a=b=c=d=0.1$

D	$\hat{R}(t)$, MSE	$\hat{R}(t)_{SE}$, MSE	$\hat{R}(t)_L(\xi=-2)$, MSE	$\hat{R}(t)_L(\xi=2)$, MSE	$\hat{R}(t)_{GE}(\tau=-0.5)$, MSE	$\hat{R}(t)_{GE}(\tau=0.5)$, MSE
1	$0.9830, 1.73\times10^{-4}$	$0.9781, 2.45\times10^{-4}$	$0.9789, 2.11\times10^{-4}$	$0.9786, 2.25\times10^{-4}$	$0.9781, 2.47\times10^{-4}$	$0.9780, 2.50\times10^{-4}$
2	$0.9835, 1.33\times10^{-4}$	$0.9784, 2.10\times10^{-4}$	$0.9791, 1.80\times10^{-4}$	$0.9788, 1.92\times10^{-4}$	$0.9783, 2.12\times10^{-4}$	$0.9782, 2.15\times10^{-4}$
3	$0.9838, 1.79\times10^{-4}$	$0.9778, 2.42\times10^{-4}$	$0.9787, 2.00\times10^{-4}$	$0.9784, 2.15\times10^{-4}$	$0.9777, 2.44\times10^{-4}$	$0.9776, 2.48\times10^{-4}$
4	$0.9835, 1.42\times10^{-4}$	$0.9784, 2.11\times10^{-4}$	$0.9793, 1.75\times10^{-4}$	$0.9790, 1.87\times10^{-4}$	$0.9783, 2.13\times10^{-4}$	$0.9783, 2.16\times10^{-4}$
5	$0.9824, 1.80\times10^{-4}$	$0.9774, 2.55\times10^{-4}$	$0.9782, 2.22\times10^{-4}$	$0.9778, 2.39\times10^{-4}$	$0.9774, 2.57\times10^{-4}$	$0.9773, 2.62\times10^{-4}$
6	$0.9832, 1.53\times10^{-4}$	$0.9781, 2.27\times10^{-4}$	$0.9789, 1.96\times10^{-4}$	$0.9786, 2.10\times10^{-4}$	$0.9781, 2.29\times10^{-4}$	$0.9780, 2.32\times10^{-4}$
7	$0.9828, 1.60\times10^{-4}$	$0.9777, 2.37\times10^{-4}$	$0.9785, 2.04\times10^{-4}$	$0.9782, 2.18\times10^{-4}$	$0.9777, 2.39\times10^{-4}$	$0.9776, 2.42\times10^{-4}$
8	$0.9833, 1.65\times10^{-4}$	$0.9782, 2.29\times10^{-4}$	$0.9790, 1.97\times10^{-4}$	$0.9787, 2.12\times10^{-4}$	$0.9782, 2.31\times10^{-4}$	$0.9781, 2.34\times10^{-4}$
9	$0.9831, 1.58\times10^{-4}$	$0.9781, 2.28\times10^{-4}$	$0.9788, 2.01\times10^{-4}$	$0.9784, 2.14\times10^{-4}$	$0.9781, 2.29\times10^{-4}$	$0.9780, 2.33\times10^{-4}$
10	$0.9834, 1.07\times10^{-4}$	$0.9801, 1.44\times10^{-4}$	$0.9805, 1.31\times10^{-4}$	$0.9803, 1.38\times10^{-4}$	$0.9800, 1.45\times10^{-4}$	$0.9800, 1.47\times10^{-4}$
11	$0.9840, 8.99\times10^{-5}$	$0.9805, 1.24\times10^{-4}$	$0.9810, 1.12\times10^{-4}$	$0.9808, 1.17\times10^{-4}$	$0.9805, 1.25\times10^{-4}$	$0.9805, 1.26\times10^{-4}$
12	$0.9838, 9.15\times10^{-5}$	$0.9802, 1.30\times10^{-4}$	$0.9807, 1.16\times10^{-4}$	$0.9806, 1.21\times10^{-4}$	$0.9802, 1.31\times10^{-4}$	$0.9801, 1.32\times10^{-4}$
13	$0.9834, 1.06\times10^{-4}$	$0.9796, 1.48\times10^{-4}$	$0.9802, 1.31\times10^{-4}$	$0.9799, 1.38\times10^{-4}$	$0.9796, 1.51\times10^{-4}$	$0.9795, 1.51\times10^{-4}$
14	$0.9839, 9.22\times10^{-5}$	$0.9806, 1.26\times10^{-4}$	$0.9810, 1.16\times10^{-4}$	$0.9808, 1.21\times10^{-4}$	$0.9805, 1.27\times10^{-4}$	$0.9805, 1.29\times10^{-4}$
15	$0.9835, 9.98\times10^{-5}$	$0.9801, 1.37\times10^{-4}$	$0.9805, 1.24\times10^{-4}$	$0.9803, 1.30\times10^{-4}$	$0.9800, 1.38\times10^{-4}$	$0.9800, 1.39\times10^{-4}$
16	$0.9837, 9.50\times10^{-5}$	$0.9803, 1.32\times10^{-4}$	$0.9807, 1.20\times10^{-4}$	$0.9805, 1.26\times10^{-4}$	$0.9803, 1.33\times10^{-4}$	$0.9802, 1.34\times10^{-4}$
17	$0.9839, 8.73\times10^{-5}$	$0.9806, 1.21\times10^{-4}$	$0.9809, 1.12\times10^{-4}$	$0.9808, 1.17\times10^{-4}$	$0.9806, 1.22\times10^{-4}$	$0.9805, 1.23\times10^{-4}$
18	$0.9833, 7.97\times10^{-5}$	$0.9806, 1.05\times10^{-4}$	$0.9810, 9.77\times10^{-5}$	$0.9808, 1.01\times10^{-4}$	$0.9806, 1.05\times10^{-4}$	$0.9806, 1.06\times10^{-4}$
19	$0.9841, 6.07\times10^{-5}$	$0.9815, 8.13\times10^{-5}$	$0.9818, 7.61\times10^{-5}$	$0.9817, 7.87\times10^{-5}$	$0.9815, 8.16\times10^{-5}$	$0.9815, 8.23\times10^{-5}$
20	$0.9839, 6.96\times10^{-5}$	$0.9813, 9.18\times10^{-5}$	$0.9817, 8.44\times10^{-5}$	$0.9815, 8.74\times10^{-5}$	$0.9813, 9.17\times10^{-5}$	$0.9813, 9.25\times10^{-5}$
21	$0.9839, 7.21\times10^{-5}$	$0.9812, 9.60\times10^{-5}$	$0.9816, 8.89\times10^{-5}$	$0.9814, 9.23\times10^{-5}$	$0.9812, 9.64\times10^{-5}$	$0.9812, 9.73\times10^{-5}$
22	$0.9843, 6.24\times10^{-5}$	$0.9818, 8.24\times10^{-5}$	$0.9821, 7.72\times10^{-5}$	$0.9819, 7.99\times10^{-5}$	$0.9818, 8.27\times10^{-5}$	$0.9817, 8.34\times10^{-5}$
23	$0.9841, 6.93\times10^{-5}$	$0.9815, 9.04\times10^{-5}$	$0.9818, 8.45\times10^{-5}$	$0.9817, 8.74\times10^{-5}$	$0.9815, 9.08\times10^{-5}$	$0.9815, 9.16\times10^{-5}$
24	$0.9842, 6.43\times10^{-5}$	$0.9817, 8.43\times10^{-5}$	$0.9820, 7.94\times10^{-5}$	$0.9818, 8.21\times10^{-5}$	$0.9817, 8.47\times10^{-5}$	$0.9817, 8.54\times10^{-5}$
25	$0.9840, 6.49\times10^{-5}$	$0.9815, 8.62\times10^{-5}$	$0.9818, 8.13\times10^{-5}$	$0.9816, 8.41\times10^{-5}$	$0.9815, 8.66\times10^{-5}$	$0.9814, 8.73\times10^{-5}$
26	$0.9845, 4.73\times10^{-5}$	$0.9825, 5.97\times10^{-5}$	$0.9828, 5.53\times10^{-5}$	$0.9827, 5.68\times10^{-5}$	$0.9825, 5.99\times10^{-5}$	$0.9824, 6.03\times10^{-5}$
27	$0.9841, 5.14\times10^{-5}$	$0.9818, 6.75\times10^{-5}$	$0.9823, 5.97\times10^{-5}$	$0.9822, 6.14\times10^{-5}$	$0.9818, 6.77\times10^{-5}$	$0.9817, 6.82\times10^{-5}$
28	$0.9841, 6.79\times10^{-5}$	$0.9814, 8.97\times10^{-5}$	$0.9819, 8.00\times10^{-5}$	$0.9818, 8.30\times10^{-5}$	$0.9814, 9.01\times10^{-5}$	$0.9814, 9.09\times10^{-5}$
29	$0.9842, 4.78\times10^{-5}$	$0.9823, 6.02\times10^{-5}$	$0.9825, 5.72\times10^{-5}$	$0.9824, 5.87\times10^{-5}$	$0.9822, 6.04\times10^{-5}$	$0.9822, 6.08\times10^{-5}$
30	$0.9846, 4.59\times10^{-5}$	$0.9826, 5.72\times10^{-5}$	$0.9828, 5.38\times10^{-5}$	$0.9828, 5.51\times10^{-5}$	$0.9826, 5.74\times10^{-5}$	$0.9826, 5.77\times10^{-5}$
31	$0.9841, 4.81\times10^{-5}$	$0.9822, 6.10\times10^{-5}$	$0.9824, 5.79\times10^{-5}$	$0.9823, 5.95\times10^{-5}$	$0.9821, 6.12\times10^{-5}$	$0.9821, 6.16\times10^{-5}$
32	$0.9843, 4.38\times10^{-5}$	$0.9824, 5.55\times10^{-5}$	$0.9826, 5.31\times10^{-5}$	$0.9825, 5.45\times10^{-5}$	$0.9824, 5.57\times10^{-5}$	$0.9824, 5.60\times10^{-5}$

Table 6 The ML and approximate Bayes estimators of $H(t=1)=0.0131589$ when $\alpha=0.05, \lambda=0.35, a=b=c=d=0.1$

D	$\hat{H}(t)$, MSE	$\hat{H}(t)_{SE}$, MSE	$\hat{H}(t)_L(\xi=-2)$, MSE	$\hat{H}(t)_L(\xi=2)$, MSE	$\hat{H}(t)_{GE}(\tau=-0.5)$, MSE	$\hat{H}(t)_{GE}(\tau=0.5)$, MSE
1	$0.0136, 6.92\times10^{-5}$	$0.0159, 6.81\times10^{-5}$	$0.0159, 6.92\times10^{-5}$	$0.0158, 6.71\times10^{-5}$	$0.0147, 5.91\times10^{-5}$	$0.0123, 5.25\times10^{-5}$
2	$0.0134, 5.84\times10^{-5}$	$0.0159, 6.08\times10^{-5}$	$0.0159, 6.18\times10^{-5}$	$0.0158, 5.99\times10^{-5}$	$0.0148, 5.22\times10^{-5}$	$0.0125, 4.56\times10^{-5}$
3	$0.0136, 6.82\times10^{-5}$	$0.0160, 6.58\times10^{-5}$	$0.0161, 6.68\times10^{-5}$	$0.0160, 6.48\times10^{-5}$	$0.0149, 5.68\times10^{-5}$	$0.0125, 5.07\times10^{-5}$
4	$0.0133, 6.00\times10^{-5}$	$0.0158, 6.13\times10^{-5}$	$0.0159, 6.23\times10^{-5}$	$0.0158, 6.04\times10^{-5}$	$0.0148, 5.31\times10^{-5}$	$0.0125, 4.68\times10^{-5}$
5	$0.1401, 7.06 \times 10^{-5}$	$0.0163, 7.00 \times 10^{-5}$	$0.0163, 7.11 \times 10^{-5}$	$0.0162, 6.90 \times 10^{-5}$	$0.0152, 6.08 \times 10^{-5}$	$0.0128, 5.35 \times 10^{-5}$
6	$0.0135, 6.36 \times 10^{-5}$	$0.0160, 6.49 \times 10^{-5}$	$0.0160, 6.59 \times 10^{-5}$	$0.0159, 6.39 \times 10^{-5}$	$0.0149, 5.62 \times 10^{-5}$	$0.0126, 4.95 \times 10^{-5}$
7	$0.0137, 6.52 \times 10^{-5}$	$0.0162, 6.69 \times 10^{-5}$	$0.0162, 6.78 \times 10^{-5}$	$0.0161, 6.59 \times 10^{-5}$	$0.0151, 5.79 \times 10^{-5}$	$0.0128, 5.05 \times 10^{-5}$
8	$0.0134, 6.51 \times 10^{-5}$	$0.0159, 6.42 \times 10^{-5}$	$0.0160, 6.51 \times 10^{-5}$	$0.0158, 6.32 \times 10^{-5}$	$0.0148, 5.57 \times 10^{-5}$	$0.0124, 5.03 \times 10^{-5}$
9	$0.0135, 6.39 \times 10^{-5}$	$0.0160, 6.56 \times 10^{-5}$	$0.0161, 6.66 \times 10^{-5}$	$0.0160, 6.47 \times 10^{-5}$	$0.0150, 5.70 \times 10^{-5}$	$0.0126, 5.02 \times 10^{-5}$
10	$0.0134, 4.18 \times 10^{-5}$	$0.0152, 4.34 \times 10^{-5}$	$0.0152, 4.39 \times 10^{-5}$	$0.0152, 4.30 \times 10^{-5}$	$0.0145, 3.93 \times 10^{-5}$	$0.0129, 3.54 \times 10^{-5}$
11	$0.0133, 4.14 \times 10^{-5}$	$0.0151, 4.27 \times 10^{-5}$	$0.0151, 4.31 \times 10^{-5}$	$0.0150, 4.22 \times 10^{-5}$	$0.0143, 3.87 \times 10^{-5}$	$0.0128, 3.53 \times 10^{-5}$
12	$0.0134, 4.18 \times 10^{-5}$	$0.0152, 4.34 \times 10^{-5}$	$0.0152, 4.39 \times 10^{-5}$	$0.0152, 4.30 \times 10^{-5}$	$0.0145, 3.93 \times 10^{-5}$	$0.0129, 3.54 \times 10^{-5}$
13	$0.0136, 4.57 \times 10^{-5}$	$0.0155, 4.67 \times 10^{-5}$	$0.0155, 4.72 \times 10^{-5}$	$0.0455, 4.61 \times 10^{-5}$	$0.0146, 4.15 \times 10^{-5}$	$0.0128, 3.71 \times 10^{-5}$
14	$0.0133, 4.22 \times 10^{-5}$	$0.0150, 4.35 \times 10^{-5}$	$0.0151, 4.40 \times 10^{-5}$	$0.0150, 4.31 \times 10^{-5}$	$0.0143, 3.96 \times 10^{-5}$	$0.0128, 3.62 \times 10^{-5}$
15	$0.0135, 4.34 \times 10^{-5}$	$0.0153, 4.49 \times 10^{-5}$	$0.0154, 4.53 \times 10^{-5}$	$0.0153, 4.44 \times 10^{-5}$	$0.0146, 4.07 \times 10^{-5}$	$0.0130, 3.67 \times 10^{-5}$
16	$0.0134, 4.37 \times 10^{-5}$	$0.0152, 4.53 \times 10^{-5}$	$0.0152, 4.58 \times 10^{-5}$	$0.0151, 4.49 \times 10^{-5}$	$0.0144, 4.12 \times 10^{-5}$	$0.0129, 3.75 \times 10^{-5}$
17	$0.0133, 4.11 \times 10^{-5}$	$0.0151, 4.27 \times 10^{-5}$	$0.0151, 4.32 \times 10^{-5}$	$0.0150, 4.23 \times 10^{-5}$	$0.0143, 3.88 \times 10^{-5}$	$0.0128, 3.54 \times 10^{-5}$
18	$0.0138, 3.64 \times 10^{-5}$	$0.0152, 3.76 \times 10^{-5}$	$0.0152, 3.80 \times 10^{-5}$	$0.0151, 3.72 \times 10^{-5}$	$0.0145, 3.43 \times 10^{-5}$	$0.0131, 3.09 \times 10^{-5}$
19	$0.0134, 2.99 \times 10^{-5}$	$0.0147, 3.15 \times 10^{-5}$	$0.0148, 3.18 \times 10^{-5}$	$0.0147, 3.12 \times 10^{-5}$	$0.0142, 2.91 \times 10^{-5}$	$0.0130, 2.67 \times 10^{-5}$
20	$0.0134, 3.28 \times 10^{-5}$	$0.0148, 3.38 \times 10^{-5}$	$0.0148, 3.41 \times 10^{-5}$	$0.0148, 3.35 \times 10^{-5}$	$0.0142, 3.14 \times 10^{-5}$	$0.0131, 2.90 \times 10^{-5}$
21	$0.0134, 3.39 \times 10^{-5}$	$0.0148, 3.50 \times 10^{-5}$	$0.0149, 3.53 \times 10^{-5}$	$0.0148, 3.47 \times 10^{-5}$	$0.0142, 3.24 \times 10^{-5}$	$0.0130, 2.98 \times 10^{-5}$
22	$0.0132, 3.12 \times 10^{-5}$	$0.0145, 3.23 \times 10^{-5}$	$0.0146, 3.26 \times 10^{-5}$	$0.0145, 3.20 \times 10^{-5}$	$0.0140, 3.00 \times 10^{-5}$	$0.0128, 2.80 \times 10^{-5}$
23	$0.0134, 3.30 \times 10^{-5}$	$0.0147, 3.40 \times 10^{-5}$	$0.0147, 3.42 \times 10^{-5}$	$0.0146, 3.37 \times 10^{-5}$	$0.0141, 3.17 \times 10^{-5}$	$0.0129, 2.95 \times 10^{-5}$
24	$0.0132, 3.15 \times 10^{-5}$	$0.0146, 3.26 \times 10^{-5}$	$0.0146, 3.29 \times 10^{-5}$	$0.0146, 3.24 \times 10^{-5}$	$0.0140, 3.04 \times 10^{-5}$	$0.0129, 2.84 \times 10^{-5}$
25	$0.0134, 3.19 \times 10^{-5}$	$0.0147, 3.34 \times 10^{-5}$	$0.0148, 3.37 \times 10^{-5}$	$0.0147, 3.31 \times 10^{-5}$	$0.0142, 3.10 \times 10^{-5}$	$0.0130, 2.85 \times 10^{-5}$
26	$0.0131, 2.40 \times 10^{-5}$	$0.0142, 2.44 \times 10^{-5}$	$0.0142, 2.45 \times 10^{-5}$	$0.0142, 2.42 \times 10^{-5}$	$0.0138, 2.31 \times 10^{-5}$	$0.0129, 2.18 \times 10^{-5}$
27	$0.0134, 2.46 \times 10^{-5}$	$0.0145, 2.49 \times 10^{-5}$	$0.0146, 2.50 \times 10^{-5}$	$0.0145, 2.47 \times 10^{-5}$	$0.0141, 2.33 \times 10^{-5}$	$0.0132, 2.17 \times 10^{-5}$
28	$0.0133, 3.20 \times 10^{-5}$	$0.0146, 3.19 \times 10^{-5}$	$0.0146, 3.22 \times 10^{-5}$	$0.0146, 3.17 \times 10^{-5}$	$0.0140, 2.96 \times 10^{-5}$	$0.0128, 2.76 \times 10^{-5}$
29	$0.0134, 3.39 \times 10^{-5}$	$0.0144, 2.48 \times 10^{-5}$	$0.0144, 2.50 \times 10^{-5}$	$0.0144, 2.47 \times 10^{-5}$	$0.0140, 2.34 \times 10^{-5}$	$0.0131, 2.19 \times 10^{-5}$
30	$0.0131, 2.31 \times 10^{-5}$	$0.0142, 2.35 \times 10^{-5}$	$0.0142, 2.37 \times 10^{-5}$	$0.0141, 2.34 \times 10^{-5}$	$0.0137, 2.23 \times 10^{-5}$	$0.0129, 2.12 \times 10^{-5}$
31	$0.0134, 2.42 \times 10^{-5}$	$0.0145, 2.52 \times 10^{-5}$	$0.0145, 2.54 \times 10^{-5}$	$0.0145, 2.51 \times 10^{-5}$	$0.0140, 2.37 \times 10^{-5}$	$0.0131, 2.21 \times 10^{-5}$
32	$0.0133, 2.27 \times 10^{-5}$	$0.0143, 2.38 \times 10^{-5}$	$0.0144, 2.39 \times 10^{-5}$	$0.0143, 2.36 \times 10^{-5}$	$0.0139, 2.24 \times 10^{-5}$	$0.0130, 2.09 \times 10^{-5}$

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