Thailand Statistician
April 2025; 23(2): 331-354
http://statassoc.or.th
Contributed paper

On Some Aspects of Exponential Intervened Geometric Distribution

Jayakumar Kuttan Pillai and Rehana Chettiparambil Joshy*
Department of Statistics, University of Calicut, Calicut, Kerala, India.
*Corresponding author; e-mail: rehanacjoshy1992@gmail.com

Received: 21 February 2021
Revised: 2 December 2022
Accepted: 31 December 2022

Abstract

In this paper, we study different aspects of Exponential Intervened Geometric (EIG) distribution. EIG distribution arises as the distribution of random minimum and is a generalization of extended exponential distribution. The shape properties of the probability density function and hazard rate function of EIG are studied, along with structural properties such as moments, moment generating function, skewness and kurtosis, mean deviation about mean and median. Expression for various reliability measures corresponding to EIG distribution are derived along with stochastic ordering property. Expression for quantiles are obtained and random number generation is discussed. The distributions of order statistics are derived and limit distributions of sample extrema are obtained. Four characterizations of EIG distribution are proved. The parameters of EIG are estimated through the method of maximum likelihood (ML) and a simulation study is conducted to show the performance of ML estimates. The existence and uniqueness of ML estimates are proved. The EIG model is fitted to a real data set and is showed that the model performs better as compared to ten competitive models. Also, the adequacy of the model for the data set is established using parametric bootstrap approach.

Keywords: characterizations, limit distribution of extremes, maximum likelihood, parametric bootstrap, stochastic ordering.

1. Introduction

Recently there has been growing interest in developing new distributions that have capability of modeling real data sets more appropriately as compared to existing models (see, Dey et al. (2019), Lemonte (2013), Lemonte (2014) and Nadarajah et al. (2013)). Intervened type distributions have found many applications in several areas such as epidemiological studies, life testing problems etc. In epidemiological study, like cholera cases, various preventive actions are taken by health service agency. The information regarding the effect of such actions taken by health service agency can be obtained by Intervened Poisson distribution (IPD) considered by Shanmugam (1985). An advantage of the IPD is that it provides information on how effective various preventive actions taken by health service agents, where Poisson fails. The IPD is applicable in several areas such as reliability analysis, queuing problems, epidemiological studies, etc. In life testing experiments, during the observational period, the failed units are either replaced by new units or rebuilt. This kind of replacement changes the reliability of a system as only some of its components have longer life. Quality engineer is always interested in improving the quality and hence he keeps on making changes in the incidence of defective items in the remaining observational period. A manager of supermarket, for instance, might decide to provide additional assistance at a service counter to speed up its service rate, and it is of interest to study the impact of such decision on the queuing mechanism. IPD and Intervened Geometric distribution (IGD) provide stochastic models to study the effect of such actions as they are closer to real life situations.

The intervened type distributions such as intervened Poisson distribution, intervened Geometric distribution and modified intervened Geometric distribution (MIGD) has been studied by several authors. For example, see Shanmugam (1985), Shanmugam (1992), Huang and Fung (1989), Scollinik (2006), Dhanavanthan (1998), Dhanavanthan (2000), Kumar and Sreejakumari (2016), etc.

Jayakumar and Sankaran (2019) introduced a new family of distributions using zero truncated power series distribution. In this article, we consider a special sub-model in the new family of distributions, generated through compound intervened geometric distribution called Exponential Intervened Geometric (EIG) distribution and is a generalization of extended exponential distribution studied in Marshall and Olkin (1997) and Adamidis and Loukas (1998). We consider exponential distribution as the base distribution due to its simplicity and popularity in life testing problems. In Section 2, we discuss Intervened Geometric compounded family of distributions. In Section 3, Exponential Intervened Geometric distribution is introduced. The shape properties of pdf and hazard rate are proved, along with the compounding property. In Section 4, we derive various properties of the EIG distribution, such as moments, moment generating function (mgf), quantile function, random number generation, skewness, kurtosis, mean deviation, Bonferroni curve and Lorenz curve. Also, some reliability properties of the new model are discussed in Section 5. The distribution of order statistics is investigated in Section 6. Various characterizations of EIG distribution are obtained in Section 7. In Section 8, stochastic ordering property is proved for EIG random variables. Limiting Distributions of Sample Extremes are obtained in section 9. The estimates of the model parameters are obtained using maximum likelihood method in Section 10. The existence and uniqueness of maximum likelihood estimates (MLEs) are proved. Simulation studies are carried out to show the performance of MLEs in Section 11. In Section 12, we analyze a real data set to illustrate the use of the proposed distribution.

2. Intervened Geometric Compounded Family of Distributions

Let Y be the number of trials performed for some life testing experiment. Since the event $Y = 0$ is not observable, we consider a zero truncated geometric distribution for a positive integer valued random variable Y with probability function

$$P(Y = y) = (1-\theta)\theta^{y};\ y = 1, 2, 3, ...,\ 0 < \theta < 1,$$

where θ is interpreted as an incidence parameter. If failed unit is replaced by a new unit or rebuilt, it is reasonable to assume that θ changes. We assume that θ changes to $\rho\theta$ for $0 < \rho < \frac{1}{\theta} < \infty$ where ρ is an intervention parameter.

Let Z be the number of trials after some over hauling or servicing the mechanism. Hence Z will have a geometric distribution with parameter $\rho\theta$, $0 < \rho\theta < 1$ and Y and Z are stochastically independent. Assuming that $X = Y + Z$ represents the total number of trials, then the probability function of X is given by

$$\begin{aligned} P(X = x) &= \sum_{l=0}^{x-l} P(Y = x-l)P(Z = l|Y = x-l), \\ &= \frac{(1-\theta)(1-\rho\theta)}{(1-\rho)}(1-\rho^{x})\theta^{x-1}. \end{aligned}$$

Now the probability generating function of X is

$$\begin{aligned} J(s) = E(s^{X}) &= \sum_{x=1}^{\infty} \frac{(1-\theta)(1-\rho\theta)}{(1-\rho)}(1-\rho^{x})\theta^{x-1}, \\ &= \frac{(1-\theta)(1-\rho\theta)s}{(1-\theta s)(1-\rho\theta s)}. \end{aligned}$$

Therefore,

$$J(\bar{F}(x)) = \frac{(1-\theta)(1-\rho\theta)\bar{F}(x)}{(1-\theta\bar{F}(x))(1-\rho\theta\bar{F}(x))}.$$

The corresponding cumulative distribution function (cdf) is given by

$$G(x) = 1 - \frac{(1-\theta)(1-\rho\theta)\bar{F}(x)}{(1-\theta\bar{F}(x))(1-\rho\theta\bar{F}(x))}. \tag{1}$$

The survival function is given by

$$\bar{G}(x) = \frac{(1-\theta)(1-\rho\theta)\bar{F}(x)}{(1-\theta\bar{F}(x))(1-\rho\theta\bar{F}(x))}. \tag{2}$$

The probability density function (pdf) is given by

$$g(x) = \frac{(1-\theta)(1-\rho\theta)f(x)(1-\rho\theta^2\bar{F}^2(x))}{[(1-\theta\bar{F}(x))(1-\rho\theta\bar{F}(x))]^2};\ 0<\theta<1,\ 0<\rho<\frac{1}{\theta}. \tag{3}$$

3. Exponential Intervened Geometric Distribution

Here, we study one member of intervened geometric compounded family of distributions namely Exponential Intervened Geometric (EIG) distribution in detail.

Let $X \sim$ Exponential(λ) distribution, $\lambda > 0$. Then $\bar{F}(x) = e^{-\lambda x}$. Hence from (2) the survival function of the new family of distributions is given by

$$\bar{G}(x) = \frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}, \tag{4}$$

where $0 < \theta < 1$, $0 \leq \rho < \frac{1}{\theta} < \infty$, $\lambda > 0$; $x > 0$.

3.1. Probability density function

The pdf of the new distribution is given by

$$g(x;\rho,\theta,\lambda) = \frac{(1-\theta)(1-\rho\theta)(1-\rho\theta^2 e^{-2\lambda x})\lambda e^{-\lambda x}}{[(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})]^2}, \tag{5}$$

where $\lambda > 0$, $0 < \theta < 1$, $0 \leq \rho < \frac{1}{\theta} < \infty$. We refer to this new distribution as EIG with parameters ρ, θ and λ. When $\rho = 0$ this distribution reduces to exponential geometric distribution introduced and studied in Adamidis and Loukas (1998).

3.2. Compounding property

Compounding of distributions gives a method for deriving new families of distributions in terms of the existing models. Let $\bar{H}(x|\delta), -\infty < x < \infty$, be the conditional survival function of a continuous random variable X given a continuous random variable Δ. Let Δ follows a distribution with the pdf $m(\delta)$. A distribution with the survival function

$$\bar{H}(x) = \int_{-\infty}^{\infty} \bar{H}(x|\delta)m(\delta)\,d\delta, \quad -\infty < x < \infty,$$

is called a compound distribution with mixing density $m(\delta)$. The following theorem shows that EIG distribution can be expressed as compound distribution.

Theorem 1 *Let the conditional survival function of a continuous random variable X given $\Delta = \delta$ be expressed as*

$$\bar{H}(x|\delta) = exp(\delta[1 - \frac{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}{(1-\theta)(1-\rho\theta)e^{-\lambda x}}]),\ x > 0;$$

where $\delta, \lambda > 0, 0 < \theta < 1, 0 < \rho < \frac{1}{\theta}$. Let Δ follows an Exponential distribution with the pdf $m(\delta) = e^{-\delta};\ \delta > 0$. Then the compound distribution of X is EIG distribution.

Proof:
For all $x > 0$, the unconditional survival function of X is given by

$$\begin{aligned}\bar{H}(x) &= \int \bar{H}(x|\delta)m(\delta)d\delta,\\ &= \int_0^\infty e^{\delta[1-\frac{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}{(1-\theta)(1-\rho\theta)e^{-\lambda x}}]}e^{-\delta}d\delta,\\ &= \int_0^\infty e^{\delta[-\frac{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}{(1-\theta)(1-\rho\theta)e^{-\lambda x}}]}d\delta,\\ &= \frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})},\end{aligned}$$

which is the survival function of a random variable with EIG distribution.

3.3. Hazard rate

The hazard rate of EIG distribution is given by

$$h(x;\rho,\theta,\lambda) = \frac{\lambda(1-\rho\theta^2 e^{-2\lambda x})}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}. \quad (6)$$

Proposition 1 *If $\bar{F}$ is a mixture of exponential survival function, then it has a decreasing hazard rate.*

For proof, (see in Marshall and Olkin , 2007, p. 117).

Theorem 2 *The hazard rate of EIG distribution is always decreasing.*

Proof:
According to Theorem 1, EIG distribution can be written as a mixture of exponential survival function with mean 1. Also by Proposition 1, if $\bar{F}$ is a mixture of exponential survival function, then it has a decreasing hazard rate. Therefore EIG distribution has decreasing hazard rate.

Proposition 2 *Suppose that $F(0) = 0$ and F has a decreasing hazard rate, then F has a density except possibly for positive mass at the origin. There is a version f of the density that is decreasing and satisfies $f(x) > 0,\ \forall x > 0$.*

For proof, (see in Marshall and Olkin , 2007, p. 117).

Theorem 3 *The density of EIG distribution is always decreasing.*

Proof:
Proof follows easily from Theorem 2 and Proposition 2.

4. Statistical Properties

4.1. Moments

Theorem 4 *If X be a random variable having pdf in* (5)*, then the r^{th} moment about origin of X is*

$$E(X^r) = \frac{r\Gamma r(1-\theta)(1-\rho\theta)}{\theta\lambda^r(1-\rho)}\left[\sum_{k=1}^{\infty}\frac{\theta^k}{k^r} - \sum_{k=1}^{\infty}\frac{(\rho\theta)^k}{k^r}\right].$$

Proof:

$$E(X^r) = \int_0^{\infty} rx^{r-1}\bar{G}(x)dx = \int_0^{\infty} rx^{r-1}\frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}dx.$$

Using Partial fraction

$$\frac{1}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})} = \frac{1}{(1-\rho)(1-\theta e^{-\lambda x})} - \frac{\rho}{(1-\rho)(1-\rho\theta e^{-\lambda x})}.$$

$$\begin{aligned}
E(X^r) &= \frac{r(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[\int_0^{\infty}\frac{e^{-\lambda x}x^{r-1}}{(1-\theta e^{-\lambda x})}dx - \rho\int_0^{\infty}\frac{e^{-\lambda x}x^{r-1}}{(1-\rho\theta e^{-\lambda x})}dx\right],\\
&= \frac{r(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[\int_0^{\infty}\frac{x^{r-1}}{(e^{\lambda x}-\theta)}dx - \rho\int_0^{\infty}\frac{x^{r-1}}{(e^{\lambda x}-\rho\theta)}dx\right],\\
&= \frac{r(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[\frac{\Gamma r}{\theta\lambda^r}\sum_{k=1}^{\infty}\frac{\theta^k}{k^r} - \frac{\rho\Gamma r}{\rho\theta\lambda^r}\sum_{k=0}^{\infty}\frac{(\rho\theta)^k}{k^r}\right].
\end{aligned}$$

Since,

$$\int_0^{\infty}\frac{x^{p-1}}{(e^{rx}-q)}dx = \frac{\Gamma p}{qr^p}\sum_{k=1}^{\infty}\frac{q^k}{k^p},$$ (see in Gradshteyn and Ryzhik , 2007, p. 354, 1039).

4.2. Mean and variance

The mean and variance of EIG are respectively given by

$$\begin{aligned}
E(X) &= \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\ln\left[\frac{1-\rho\theta}{1-\theta}\right],\\
V(X) &= E(X^2) - [E(X)]^2,\\
E(X^2) &= \frac{2(1-\theta)(1-\rho\theta)}{\theta\lambda^2(1-\rho)}\left[\sum_{k=1}^{\infty}\frac{\theta^k}{k^2} - \sum_{k=1}^{\infty}\frac{(\rho\theta)^k}{k^2}\right].
\end{aligned}$$

Table 1 Mean,variance,skewness and kurtosis of EIG distribution for different values of θ

(ρ,λ)	θ	.3	.4	.5	.6	.7
$\rho = .5\lambda = 1$	mean	0.7701522	0.690437	0.6081977	0.5223081	0.4307772
$\rho = .5\lambda = 1$	variance	0.7562818	0.6670369	0.5738593	0.4765803	0.3746623
$\rho = .5\lambda = 1$	skewness	3.346367	3.031433	2.665296	2.247382	1.777472
$\rho = .5\lambda = 1$	kurtosis	8.833981	10.34129	12.37098	15.26324	19.76921

From Tables 1, 2 and 3, it can be seen that when λ, ρ and θ increasing, mean and variance are decreasing.

Table 2 Mean,variance,skewness and kurtosis of EIG distribution for different values of ρ

(θ,λ)	ρ	.5	.7	.9	1.1	1.3
$\theta=.5\lambda=1$	mean	0.6081977	.5684559	0.524206	0.4741223	0.4161208
$\theta=.5\lambda=1$	variance	0.5738593	0.5246078	0.471465	0.413462	0.3491419
$\theta=.5\lambda=1$	skewness	2.665296	2.442578	2.198364	1.927617	1.623032
$\theta=.5\lambda=1$	kurtosis	12.37098	13.65685	15.33012	17.6219	21.00731

Table 3 Mean,variance,skewness and kurtosis of EIG distribution for different values of λ

(θ,ρ)	λ	1	2	3	4	5
$\theta=.5\rho=.5$	mean	0.6081977	0.3040988	0.2027326	0.1520494	0.1216395
$\theta=.5\rho=.5$	variance	0.5738593	0.1434648	0.06376214	0.0358662	0.02295437
$\theta=.5\rho=.5$	skewness	2.665296	.166581	0.03290489	0.01041131	0.004264474
$\theta=.5\rho=.5$	kurtosis	12.37098	12.37098	12.37098	12.37098	12.37098

4.3. Moment generating function

Theorem 5 *If $X\sim EIG(\rho,\theta,\lambda)$, then the moment generating function of X is*

$$M_X(t)=(1-\theta)(1-\rho\theta)\sum_{n=1}^{\infty}\frac{\theta^{n-1}}{n!}\sum_{y=0}^{n-1}\binom{n}{y}\Gamma(n-y+1)\Gamma(y+1)\rho^y\left[\frac{n\lambda}{n\lambda-t}\right],$$

provided $n\lambda>t$.

Proof:
If $X_{(1)}=Min(X_1,X_2,...,X_N)$, where $N\sim IG$ and each X_i are independent and identically distributed as Exponential(λ) and X_i is independent of N, then $X_{(1)}\sim EIG$.

$$\begin{aligned}M_{X_{(1)}}(t)&=\int_0^{\infty}e^{tx}f_{X_{(1)}}(t)dx,\\&=\int_0^{\infty}e^{tx}\sum_{n=1}^{\infty}P(N=n)g_{X_{(1)}}(x)dx,\\&=\sum_{n=1}^{\infty}P(N=n)\int_0^{\infty}e^{tx}g_{X_{(1)}}(x)dx,\end{aligned}$$

where $g_{X_{(1)}}$ is the pdf of $Y=Min(X_1,X_2,...,X_n)$.

We know that since $N\sim IG$,

$$P(N=n)=(1-\theta)(1-\rho\theta)\frac{\theta^{n-1}}{n!}\sum_{y=0}^{n-1}\binom{n}{y}\Gamma(n-y+1)\Gamma(y+1)\rho^y.$$

Also $\quad g_{X_{(1)}}(x)=ng(x)\left[1-G(x)\right]^{n-1}$, where $g(x)$ is the pdf of X_i.

$g_{X_{(1)}}(x)=n\lambda e^{-n\lambda x}$.

$$M_{X_{(1)}}(t) = \left[\sum_{n=1}^{\infty}(1-\theta)(1-\rho\theta)\frac{\theta^{n-1}}{n!}\sum_{y=0}^{n-1}\binom{n}{y}\Gamma(n-y+1)\Gamma(y+1)\rho^y\right]\int_0^{\infty} e^{tx} n\lambda e^{-n\lambda x}dx$$

$$= (1-\theta)(1-\rho\theta)\sum_{n=1}^{\infty}\frac{\theta^{n-1}}{n!}\sum_{y=0}^{n-1}\binom{n}{y}\Gamma(n-y+1)\Gamma(y+1)\rho^y\left[\frac{n\lambda}{n\lambda-t}\right],$$

provided $n\lambda > t$.

4.4. Quantile function and random number generation

For a non-negative continuous random variable X that follows the EIG distribution, the quantile function x_p is given by

$$x_p = \frac{-1}{\lambda}log\left[\frac{-(p\rho\theta + p\theta - \rho\theta^2 - 1) - \sqrt{(p\rho\theta + p\theta - \rho\theta^2 - 1)^2 - 4\rho\theta^2(1-p)^2}}{2\rho\theta^2(1-p)}\right].$$

In particular, the median is

$$x_{\frac{1}{2}} = \frac{-1}{\lambda}log\left[\frac{-(\frac{\theta(\rho+1)}{2} - \rho\theta^2 - 1) - \sqrt{(\frac{\theta(\rho+1)}{2} - \rho\theta^2 - 1)^2 - \rho\theta^2}}{\rho\theta^2}\right].$$

The random number generation from X that has EIG distribution can be done using the following relation

$$\frac{(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})} = u, \text{ where } u \sim U(0,1).$$

Thus

$$x = \frac{-1}{\lambda}log\left[\frac{-(u\rho\theta + u\theta - \rho\theta^2 - 1) - \sqrt{(u\rho\theta + u\theta - \rho\theta^2 - 1)^2 - 4\rho\theta^2(1-u)^2}}{2\rho\theta^2(1-u)}\right].$$

One can use this to generate random numbers from EIG distribution when the parameters ρ, θ, λ are known.

4.5. Mean deviation about mean and median

The mean deviation of X about the mean μ

$$D_1(\mu) = \int_0^{\infty} |x-\mu|\, g(x)dx = 2\mu G(\mu) - 2m(\mu),$$

where $m(z) = \int_0^z xg(x)dx$ and $G(.)$ denote the proposed cdf.

The mean deviation of X about the median M

$$D_2(M) = \mu - 2\int_M^{\infty} xg(x)dx = \mu - 2m(M).$$

Theorem 6 *For the EIG(λ, ρ, θ) distribution, $m(z)$ is given by*

$$m(z) = \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[\lambda z\left(\frac{1}{(1-\rho\theta e^{-\lambda z})} - \frac{1}{(1-\theta e^{-\lambda z})}\right) - log\left(\frac{e^{-\lambda z}}{1-\theta e^{-\lambda z}}\right)\right.$$
$$\left. - log\left(\frac{e^{-\lambda z}}{1-\rho\theta e^{-\lambda z}}\right) + log\left(\frac{1-\rho\theta}{1-\theta}\right)\right].$$

Proof:

$$\begin{aligned}
m(z) &= \int_0^z xg(x)dx \\
&= \int_0^z x\frac{(1-\theta)(1-\rho\theta)(1-\rho\theta^2 e^{-2\lambda x})\lambda e^{-\lambda x}}{[(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})]^2} \\
&= \frac{\lambda(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[\int_0^z \frac{xe^{-\lambda x}}{(1-\theta e^{-\lambda x})^2}dx - \rho\int_0^z \frac{xe^{-\lambda x}}{(1-\rho\theta e^{-\lambda x})^2}dx\right] \\
&= \frac{(1-\theta)(1-\rho\theta)}{\theta(1-\rho)}\left[\int_{1-\theta}^{1-\theta e^{-\lambda z}} \frac{\frac{-1}{\lambda}log\left(\frac{1-u_1}{\theta}\right)}{u_1^2}du_1 - \int_{1-\rho\theta}^{1-\rho\theta e^{-\lambda x}} \frac{\frac{-1}{\lambda}log\left(\frac{1-u_2}{\rho\theta}\right)}{u_2^2}du_2\right]
\end{aligned}$$

where $u_1 = 1-\theta e^{-\lambda x}$ and $u_2 = 1-\rho\theta e^{-\lambda x}$.

Hence

$$m(z) = \frac{(1-\theta)(1-\rho\theta)}{\lambda(1-\rho)}\left[\int_1^{e^{-\lambda z}} \frac{logw_1}{(1-\theta w_1)^2}dw_1 - \rho\int_1^{e^{-\lambda z}} \frac{logw_2}{(1-\rho\theta w_2)^2}dw_2\right],$$

where $w_1 = \frac{1-u_1}{\theta}$ and $w_2 = \frac{1-u_2}{\rho\theta}$.

Also we have the result

$$\int \frac{logx}{(a+bx)^2}dx = \frac{-logx}{b(a+bx)} + \frac{1}{ab}log\left(log\frac{x}{a+bx}\right),$$

(see in Gradshteyn and Ryzhik , 2007, p.239). Using this

$$\begin{aligned}
m(z) =& \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[\lambda z\left(\frac{1}{(1-\rho\theta e^{-\lambda z})} - \frac{1}{(1-\theta e^{-\lambda z})}\right) - log\left(\frac{e^{-\lambda z}}{1-\theta e^{-\lambda z}}\right)\right. \\
&\left. - log\left(\frac{e^{-\lambda z}}{1-\rho\theta e^{-\lambda z}}\right) + log\left(\frac{1-\rho\theta}{1-\theta}\right)\right].
\end{aligned}$$

4.6. Bonferroni and Lorenz curves

The Bonferroni curve $B[F(x)]$ for the EIG distribution is defined by

$$\begin{aligned}
B[G(x)] =& \frac{1}{E(X)G(x)}\int_0^x yg(y)dy, \\
=& \frac{1}{\mu G(x)}\frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[\lambda x\left(\frac{1}{(1-\rho\theta e^{-\lambda x})} - \frac{1}{(1-\theta e^{-\lambda x})}\right)\right. \\
&\left. - log\left(\frac{e^{-\lambda x}}{1-\theta e^{-\lambda x}}\right) - log\left(\frac{e^{-\lambda x}}{1-\rho\theta e^{-\lambda x}}\right) + log\left(\frac{1-\rho\theta}{1-\theta}\right)\right],
\end{aligned}$$

where μ is the mean and $G(.)$ is the cdf of EIG distribution.

Also, the Lorenz curve of $G(.)$ that follows EIG distribution is the graph of

$$L[G(x)] = B[G(x)]G(x).$$

Theorem 7 *The scaled total time on test transform of EIG distribution is*

$$S[G(t)] = \frac{1}{E(X)}\int_0^t \bar{G}(x)dx = \frac{1}{E(X)}\frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[log\left(\frac{(1-\theta e^{-\lambda t})}{(1-\rho\theta e^{-\lambda t})}\frac{(1-\rho\theta)}{(1-\theta)}\right)\right].$$

Proof:

$$\int_0^t \bar{G}(x)dx = \int_0^t \frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}dx,$$
$$= \frac{(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[\int_0^t \frac{e^{-\lambda x}}{(1-\theta e^{-\lambda x})}dx - \rho\int_0^t \frac{e^{-\lambda x}}{(1-\theta e^{-\lambda x})}dx\right],$$
$$= \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[\int_{1-\theta}^{1-\theta e^{-\lambda t}} \frac{1}{u_1}du_1 - \int_{1-\rho\theta}^{1-\rho\theta e^{-\lambda t}} \frac{1}{u_2}du_2\right],$$
$$= \frac{(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[log\left(\frac{(1-\theta e^{-\lambda t})}{(1-\rho\theta e^{-\lambda t})}\right) + log\left(\frac{1-\rho\theta}{1-\theta}\right)\right],$$
$$= \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[log\left(\frac{(1-\theta e^{-\lambda t})}{(1-\rho\theta e^{-\lambda t})}\frac{(1-\rho\theta)}{(1-\theta)}\right)\right].$$

5. Reliability Measures of EIG

5.1. Mean inactivity and strong mean inactivity time functions

Let X be a lifetime random variable with distribution function $G(.)$. Then the mean inactivity time (MIT) and strong mean inactivity time (SMIT) are defined by

$$\zeta_{MIT}(t) = \frac{1}{G(t)}\int_0^t G(x)dx; t > 0 \text{ and } \vartheta_{SMIT}(t) = \frac{1}{G(t)}\int_0^t 2xG(x)dx; t > 0.$$

The next two theorems give expressions of MIT and SMIT for the EIG distribution.

Theorem 8 *The MIT function of a lifetime random variable X with EIG distribution is*

$$\zeta_{MIT}(t) = \frac{1}{G(t)}\left[t - \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left(log\left[\frac{(1-\theta e^{-\lambda t})}{(1-\rho\theta e^{-\lambda t})}\frac{(1-\rho\theta)}{(1-\theta)}\right]\right)\right]; t > 0,$$

where $G(.)$ is the cdf of the EIG distribution.

Theorem 9 *The SMIT function of a lifetime random variable X with EIG distribution is*

$$\vartheta_{SMIT}(t) = \frac{1}{G(t)}\Bigg[t^2 - \frac{2(1-\theta)(1-\rho\theta)}{\lambda^2(1-\rho)}\Bigg(\frac{\lambda t}{\theta}\log\left(\frac{1-\theta e^{-\lambda t}}{1-\rho\theta e^{-\lambda t}}\right)$$
$$+ e^{-\lambda t}\left(\rho\Phi(\rho\theta e^{-\lambda t},2,1) - \Phi(\theta e^{-\lambda t},2,1)\right) + \Phi(\theta,2,1) - \rho\Phi(\rho\theta,2,1)\Bigg)\Bigg].$$

Proof:

$$\int_0^t 2xG(x)dx = \int_0^t 2x(1-\bar{G}(x))dx,$$
$$= t^2 - 2(1-\theta)(1-\rho\theta)\int_0^t \frac{xe^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}dx,$$
$$= t^2 - \frac{2(1-\theta)(1-\rho\theta)}{(1-\rho)}\left[\int_0^t \frac{xe^{-\lambda x}}{(1-\theta e^{-\lambda x})}dx - \rho\int_0^t \frac{xe^{-\lambda x}}{(1-\rho\theta e^{-\lambda x})}dx\right],$$
$$= t^2 - \frac{2(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\Bigg[\int_{1-\theta}^{1-\theta e^{-\lambda t}} \frac{\frac{-1}{\lambda}\log\left(\frac{1-u_1}{\theta}\right)}{u_1^2}du_1$$
$$- \int_{1-\rho\theta}^{1-\rho\theta e^{-\lambda t}} \frac{\frac{-1}{\lambda}\log\left(\frac{1-u_2}{\rho\theta}\right)}{u_2^2}du_2\Bigg],$$

$$\int_0^t 2xG(x)dx = t^2 - \frac{2(1-\theta)(1-\rho\theta)}{\lambda^2(1-\rho)}\left[\int_1^{e^{-\lambda t}} \frac{\log w_1}{1-\theta w_1}dw_1 - \rho\int_1^{e^{-\lambda t}} \frac{\log w_2}{1-\rho\theta w_2}dw_2\right].$$

We have a result $\int \frac{\log x}{(a+bx)}dx = \frac{1}{b}\log x\log(a+bx) - \frac{1}{b}\int \frac{\log(a+bx)}{x}dx$.

Also $\displaystyle\int \frac{\log(a+bx)}{x}dx = \log a\log x + \frac{bx}{a}\Phi(\frac{-bx}{a}, 2, 1)$,

where Φ is known as the Lerch function.

$$\Phi(z, s, v) = \sum_{n=0}^{\infty}(v+n)^{-s}z^n, \ \ |z| < 1; v \neq 0, -1, ...$$

(see in Gradshteyn and Ryzhik , 2007, p. 239, 1039).

Using this result we have

$$\vartheta_{SMIT}(t) = \frac{1}{G(t)}\Bigg[t^2 - \frac{2(1-\theta)(1-\rho\theta)}{\lambda^2(1-\rho)}\Bigg(\frac{\lambda t}{\theta}\log\left(\frac{1-\theta e^{-\lambda t}}{1-\rho\theta e^{-\lambda t}}\right)$$
$$+ e^{-\lambda t}\left(\rho\Phi(\rho\theta e^{-\lambda t}, 2, 1) - \Phi(\theta e^{-\lambda t}, 2, 1)\right) + \Phi(\theta, 2, 1) - \rho\Phi(\rho\theta, 2, 1)\Bigg)\Bigg].$$

5.2. Residual life and reversed residual life functions

5.2.1 Residual lifetime function

The residual life is the period from time t until the time of failure and defined by the conditional random variable $R_{(t)} = X - t|X > t, t > 0$.

Theorem 10 *Survival function of the residual lifetime $R_{(t)}$ for the EIG distribution is*

$$\bar{G}_{R_{(t)}}(x) = \frac{e^{-\lambda x}(1-\theta e^{-\lambda t})(1-\rho\theta e^{-\lambda t})}{(1-\theta e^{-\lambda(x+t)})(1-\rho\theta e^{-\lambda(x+t)})}.$$

Proof:
The proof follows from the identity $\bar{G}_{R_{(t)}}(x) = \frac{\bar{G}(x+t)}{\bar{G}(t)}$, where $\bar{G}(.)$ is the survival function of EIG distribution.

Corollary 1 *The pdf and hazard rate function of $R_{(t)}$ are respectively given as*

$$f_{R_{(t)}}(x) = \frac{(1-\theta e^{-\lambda t})(1-\rho\theta e^{-\lambda t})}{(1-\rho)}\lambda e^{-\lambda x}\left[\frac{1}{(1-\theta e^{-\lambda(x+t)})^2} - \frac{\rho}{(1-\rho\theta e^{-\lambda(x+t)})^2}\right]$$

and

$$h_{R_{(t)}}(x) = \frac{\lambda}{(1-\rho)}\left[\frac{(1-\rho\theta e^{-\lambda(x+t)})}{(1-\theta e^{-\lambda(x+t)}} - \frac{\rho(1-\theta e^{-\lambda(x+t)})}{(1-\rho\theta e^{-\lambda(x+t)})}\right].$$

Theorem 11 *The mean of $R_{(t)}$ for the EIG distribution is*

$$E(R_{(t)}) = \frac{1}{\bar{G}(t)}\int_t^{\infty}(x-t)f(x)dx = \frac{1}{\bar{G}(t)}\left[E(X) - \int_0^t xf(x)dx\right] - t,$$

where

$$\int_0^t xf(x)dx = \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\Bigg[\lambda t\left(\frac{1}{(1-\rho\theta e^{-\lambda t})} - \frac{1}{(1-\theta e^{-\lambda t})}\right) - log\left(\frac{e^{-\lambda t}}{1-\theta e^{-\lambda t}}\right)$$
$$- log\left(\frac{e^{-\lambda t}}{1-\rho\theta e^{-\lambda t}}\right) + log\left(\frac{1-\rho\theta}{1-\theta}\right)\Bigg].$$

5.2.2 Reversed residual life function

The reversed residual life is the time elapsed from the failure of a component given that its life $X \leq t$ and defined as the conditional random variable $\bar{R}_{(t)} = t - X|X \leq t$.

Theorem 12 *The survival function of the reversed residual lifetime $\bar{R}_{(t)}$ for the EIG distribution is*

$$\bar{G}_{\bar{R}_{(t)}}(x) = \frac{(1-e^{-\lambda(t-x)})(1-\rho\theta^2 e^{-\lambda(t-x)})}{(1-\theta e^{-\lambda(t-x)})(1-\rho\theta e^{-\lambda(t-x)})} \frac{(1-\theta e^{-\lambda t})(1-\rho\theta e^{-\lambda t})}{(1-e^{-\lambda t})(1-\rho\theta^2 e^{-\lambda t})}.$$

Corollary 2 *The pdf and hazard rate function of $\bar{R}_{(t)}$ are respectively given as*

$$f_{\bar{R}_{(t)}}(x) = \frac{(1-\theta e^{-\lambda t})(1-\rho\theta e^{-\lambda t})}{(1-e^{-\lambda t})(1-\rho\theta^2 e^{-\lambda t})} \frac{\lambda e^{-\lambda(t-x)}((1+\rho\theta^2)-\theta(1+\rho))\left[1-\rho\theta^2 e^{-2\lambda(t-x)}\right]}{\left[(1-\theta e^{-\lambda(t-x)})(1-\rho\theta e^{-\lambda(t-x)})\right]^2}$$

and

$$h_{\bar{R}_{(t)}}(x) = \frac{\lambda e^{-\lambda(t-x)}((1+\rho\theta^2)-\theta(1+\rho))\left[1-\rho\theta^2 e^{-2\lambda(t-x)}\right]}{(1-\theta e^{-\lambda(t-x)})(1-\rho\theta e^{-\lambda(t-x)})(1-e^{-\lambda(t-x)})(1-\rho\theta^2 e^{-\lambda(t-x)})}.$$

Theorem 13 *The mean of $\bar{R}_{(t)}$ for the EIG distribution is*

$$E(\bar{R}_{(t)}) = \frac{1}{G(t)}\int_0^t (t-x)f(x)dx = t - \frac{m(t)}{G(t)},$$

$$\textit{where } m(t) = \frac{(1-\theta)(1-\rho\theta)}{\theta\lambda(1-\rho)}\left[\lambda t\left(\frac{1}{(1-\rho\theta e^{-\lambda t})} - \frac{1}{(1-\theta e^{-\lambda t})}\right) - log\left(\frac{e^{-\lambda t}}{1-\theta e^{-\lambda t}}\right) - log\left(\frac{e^{-\lambda t}}{1-\rho\theta e^{-\lambda t}}\right) + log\left(\frac{1-\rho\theta}{1-\theta}\right)\right].$$

6. Order Statistics

Let $X_1, X_2, ..., X_n$ be a random sample of size n, from the EIG distribution and $X_{(1)}, X_{(2)}, ..., X_{(n)}$ denote the corresponding order statistics. It is well known that the pdf $g_r(x)$ of r^{th} (for $r = 1, 2, ..., n$) order statistics $X_{(r)}$, when the population cdf $G(x)$ is given by

$$G_{(r)}(x) = \sum_{j=r}^{n}\binom{n}{j}G^j(x)\left[1-G(x)\right]^{n-j},$$
$$= \sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l G^{j+1}(x).$$

Note that we can express the cdf of EIG distribution as

$$G(x) = 1 - \frac{(1-\theta)(1-\rho\theta)}{\theta}\left[\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(\lambda(k-1)x)^m}{m!}\left(\frac{1}{\rho\theta}-\frac{1}{\theta}\right)^{k-1}\right].$$

Hence

$$G_{(r)}(x) = \sum_{j=r}^{n}\sum_{l=0}^{n-j}\binom{n}{j}\binom{n-j}{l}(-1)^l \left(1 - \frac{(1-\theta)(1-\rho\theta)}{\theta}\left[\sum_{k=1}^{\infty}\sum_{m=0}^{\infty}\frac{(\lambda(k-1)x)^m}{m!}\left(\frac{1}{\rho\theta}-\frac{1}{\theta}\right)^{k-1}\right]\right)^{j+1}.$$

7. Characterization Results

This section deals with the characterizations of the EIG distribution based on hazard function, reverse hazard function and conditional expectation of certain function of the random variable. These characterizations employ theorems of Glanzel and Hamedani (2001) and Hamedani and Safavimanesh (2017).

7.1. Characterization based on hazard function

Theorem 14 *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable. The pdf of X is* (5) *if and only if its hazard function* $h_G(x)$ *satisfies the differential equation*

$$h'_G(x) - \frac{2\lambda\rho\theta^2 e^{-2\lambda x}}{(1-\rho\theta^2 e^{-2\lambda x})} h_G(x) = -\frac{\lambda^2\theta e^{-\lambda x}(1-\rho\theta^2 e^{-2\lambda x})\left(\rho+1-2\rho\theta e^{-\lambda x}\right)}{\left[(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})\right]^2}; x > 0,$$

with boundary condition $\lim_{x\to 0} h_G(x) = \frac{\lambda(1-\rho\theta^2)}{(1-\theta)(1-\rho\theta)}$.

Proof:

If X has pdf (5), then clearly the above differential equation holds.

Now, if the differential equation holds, then we have to show that the solution is the hazard function $h_G(x)$ of EIG distribution. Consider the differential equation

$$h'_G(x) - \frac{2\lambda\rho\theta^2 e^{-2\lambda x}}{(1-\rho\theta^2 e^{-2\lambda x})} h_G(x) = -\frac{\lambda^2\theta e^{-\lambda x}(1-\rho\theta^2 e^{-2\lambda x})\left(\rho+1-2\rho\theta e^{-\lambda x}\right)}{\left[(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})\right]^2}; x > 0.$$

The solution of the differential equation $h'_G(x) + a(x)h_G(x) = f(x)$ is

$$h_G(x) = \frac{\int u(x)f(x)dx + c}{u(x)}, \text{ where } u(x) = e^{\int a(x)dx} \text{ is the integrating factor.}$$

Here integrating factor, $u(x) = \frac{1}{(1-\rho\theta^2 e^{-2\lambda x})}$. Then

$$\int u(x)f(x)dx = \lambda^2\theta \int \frac{e^{-\lambda x}(2\rho\theta e^{-\lambda x} - (\rho+1))}{\left[(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})\right]^2} dx.$$

Using partial fraction

$$\int u(x)f(x)dx = \lambda^2\theta \left[\frac{1}{(\rho-1)} \int \frac{e^{-\lambda x}}{(1-\theta e^{-\lambda x})^2} dx - \frac{\rho^2}{(\rho-1)} \int \frac{e^{-\lambda x}}{(1-\rho\theta e^{-\lambda x})^2} dx\right],$$
$$= \frac{\lambda}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}.$$

Then the solution is

$$h_G(x) = \frac{\lambda(1-\rho\theta^2 e^{-2\lambda x})}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}, \text{ which is the hazard function of EIG distribution.}$$

7.2. Characterization in terms of the reverse (or reversed) hazard function

Theorem 15 *Let* $X : \Omega \to (0, \infty)$ *be a continuous random variable. The pdf of X is* (5) *if and only if its reverse hazard function* $r_G(x)$ *satisfies the following differential equation*

$$r'_G(x) - \frac{(3\lambda\rho\theta^2 e^{-2\lambda x} - \lambda)}{(1-\rho\theta^2 e^{-2\lambda x})} r_G(x) = -\frac{\lambda(1-\theta)(1-\rho\theta)(e^{-\lambda x} - \rho\theta^2 e^{-3\lambda x})}{\left[(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})\right]^2}$$
$$\Big[(1-\theta e^{-\lambda x})(1-e^{-\lambda x})\left(\lambda\rho\theta(1+\theta)e^{-\lambda x} - 2\lambda\rho^2\theta^3 e^{-2\lambda x}\right)$$
$$+ (1-\rho\theta e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})\left(\lambda(1+\theta)e^{-\lambda x} - 2\lambda\theta e^{-2\lambda x}\right)\Big],$$

with the boundary condition $\lim_{x\to 0} r_G(x) = 0$.

Proof:
If X has pdf (5), then clearly the above differential equation holds.
Now, if the differential equation holds, then we have to show that the solution is the reverse hazard function $r_G(x)$ of EIG distribution. Consider the above differential equation.

Here integrating factor, $u(x) = \dfrac{1}{(e^{-\lambda x} - \rho\theta^2 e^{-3\lambda x})}$.

$$\begin{aligned}\text{Then} \int u(x)f(x)dx =&\lambda(1-\theta)(1-\rho\theta)\left[\int \frac{\lambda\theta^2 e^{-\lambda x}}{(1-\theta)(1-\rho)(1-\rho\theta)(1-\theta e^{-\lambda x})^2}dx\right. \\ &\left. -\int \frac{\lambda e^{-\lambda x}}{(1-\theta)(1-\rho\theta)(1-\rho\theta^2)(1-e^{-\lambda x})^2}dx\right], \\ =&\frac{\lambda(1-\theta)(1-\rho\theta)}{(1-\theta e^{-\lambda x})(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}.\end{aligned}$$

Then the solution is,

$$r_G(x) = \frac{\lambda(1-\theta)(1-\rho\theta)e^{-\lambda x}(1-\rho\theta^2 e^{-2\lambda x})}{(1-\theta e^{-\lambda x})(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})(1-\rho\theta e^{-\lambda x})},$$

which is the reverse hazard function of EIG distribution.

7.3. Characterization based on the conditional expectation of certain function of the random variable

Theorem 16 *Let* $X : \Omega \to (0,\infty)$ *be a continuous random variable with cdf* $G(.)$. *Let* $\psi(x)$ *be a differentiable function on* $(0,\infty)$ *such that*

$$\psi(x) = \delta\left[1-\left(\frac{(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}\right)^{\frac{1}{\delta-1}}\right]^{-1}, \text{ where } \delta > 1.$$

Then $E\left[(\psi(x))^{\delta}|X \le x\right] = \delta(\psi(x))^{\delta-1};\ x\epsilon(0,\infty),$ if and only if $X \sim EIG$ distribution.

Proof: We have $E\left[(\psi(x))^{\delta}|X \le x\right] = \delta(\psi(x))^{\delta-1}$.

$$\int_a^x (\psi(u))^{\delta} g(u)du = \delta(\psi(x))^{\delta-1}G(x)$$

Taking derivatives on both sides of the above equation

$$\begin{aligned}(\psi(x))^{\delta}g(x) &= \delta\left[(\delta-1)\psi'(x)(\psi(x))^{\delta-2}G(x) + (\psi(x))^{\delta-1}g(x)\right], \\ \frac{g(x)}{G(x)} &= \delta(\delta-1)\frac{\psi'(x)}{\psi(x)(\psi(x)-\delta)}.\end{aligned}$$

Using partial fraction

$$\frac{g(x)}{G(x)} = (\delta-1)\left[-\frac{\psi'(x)}{\psi(x)} + \frac{\psi'(x)}{\psi(x)-\delta}\right].$$

Integrating both sides of the above equation and applying the limit $x \to \infty$

$$G(x) = \left[1 - \frac{\delta}{\psi(x)}\right]^{\delta-1}.$$

We have $\psi(x) = \delta\left[1 - \left(\frac{(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}\right)^{\frac{1}{\delta-1}}\right]^{-1}$, where $\delta > 1$.

Then $G(x) = \frac{(1-e^{-\lambda x})(1-\rho\theta^2 e^{-\lambda x})}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}$.

That is, X follows EIG distribution.

Theorem 17 *Let $X : \Omega \to (0,\infty)$ be a continuous random variable with cdf $G(.)$. Let $\phi(x)$ be a differentiable function on $(0,\infty)$ such that*

$$\phi(x) = \delta\left[1 + \left(\frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}\right)^{\frac{1}{\delta-1}}\right]^{-1}, \text{ where } \delta > 1.$$

Then $E\left[(\phi(x))^{\delta}|X \geq x\right] = \delta(\phi(x))^{\delta-1}; x\epsilon(0,\infty)$, if and only if $X \sim EIG$ distribution.

Proof:

We have $E\left[(\phi(x))^{\delta}|X \geq x\right] = \delta(\phi(x))^{\delta-1}$.

$$\int_x^{\infty} (\phi(u))^{\delta} g(u)du = \delta(\phi(x))^{\delta-1}\bar{G}(x)$$

Taking derivatives on both sides of the above equation

$$-(\phi(x))^{\delta} g(x) = \delta\left[(\delta-1)\phi'(x)(\phi(x))^{\delta-2}\bar{G}(x) - (\phi(x))^{\delta-1}g(x)\right].$$

$$\frac{g(x)}{\bar{G}(x)} = \delta(\delta-1)\frac{\phi'(x)}{\phi(x)(\delta-\phi(x))}.$$

Using partial fraction

$$\frac{g(x)}{\bar{G}(x)} = (\delta-1)\left[-\frac{\phi'(x)}{\phi(x)} + \frac{\phi'(x)}{\delta-\phi(x)}\right].$$

Integrating both sides of the above equation from 0 to x, we get

$$\bar{G}(x) = \left[-1 + \frac{\delta}{\phi(x)}\right]^{\delta-1}.$$

We have $\phi(x) = \delta\left[1 + \left(\frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}\right)^{\frac{1}{\delta-1}}\right]^{-1}$, where $\delta > 1$.

Then $\bar{G}(x) = \frac{(1-\theta)(1-\rho\theta)e^{-\lambda x}}{(1-\theta e^{-\lambda x})(1-\rho\theta e^{-\lambda x})}$.

That is, X follows EIG distribution.

8. Stochastic Ordering

Stochastic ordering of positive continuous random variables is an important tool for judging the comparative behavior. A random variable X is said to be smaller than a random variable Y in the following contexts:

- stochastic order $(X \leq_{st} Y)$ if $F_X(x) \geq F_Y(x)$ for all x ;
- hazard rate order $(X \leq_{hr} Y)$ if $h_X(x) \geq h_Y(x)$ for all x ;
- mean residual life order $(X \leq_{mrl} Y)$ if $K_X(x) \leq K_Y(x)$ for all x ;
- likelihood ratio order $(X \leq_{lr} Y)$ if $\frac{f_X(x)}{f_Y(x)}$ decreasing in x.

The four stochastic orders defined above are related to each other have the following implications

$$(X \leq_{lr} Y) \Rightarrow (X \leq_{hr} Y) \Rightarrow (X \leq_{mrl} Y).$$

Also $(X \leq_{hr} Y) \Rightarrow (X \leq_{st} Y)$.

Theorem 18 *Let $X \sim EIG(\theta, \rho_1, \lambda)$ and $Y \sim EIG(\theta, \rho_2, \lambda)$. If $\rho_1 < \rho_2$ and $\theta_2 < \theta < \theta_1$, then $X \geq_{lr} Y$ and hence $(X \geq_{hr} Y)$,$(X \geq_{mrl} Y)$, $(X \geq_{st} Y)$.*

Proof:
The density ratio is given by

$$\frac{g_x(x)}{g_Y(x)} = \left[\frac{1-\rho_1\theta}{1-\rho_2\theta}\right]\left[\frac{1-\rho_1\theta^2 e^{-2\lambda x}}{1-\rho_2\theta^2 e^{-2\lambda x}}\right]\left[\frac{1-\rho_2\theta e^{-\lambda x}}{1-\rho_1\theta e^{-\lambda x}}\right]^2.$$

It follows that

$$\begin{aligned}\frac{d}{dx}\log\frac{g_X(x)}{g_Y(x)} =& 2\lambda\theta e^{-\lambda x}\left(\frac{\rho_2}{1-\rho_2\theta e^{-\lambda x}} - \frac{\rho_2\theta e^{-\lambda x}}{1-\rho_2\theta^2 e^{-2\lambda x}}\right)\\ &- 2\lambda\theta e^{-\lambda x}\left(\frac{\rho_1}{1-\rho_1\theta e^{-\lambda x}} - \frac{\rho_1\theta e^{-\lambda x}}{1-\rho_1\theta^2 e^{-2\lambda x}}\right).\end{aligned}$$

Also we can show that

$$\left(\frac{\rho_1}{1-\rho_1\theta e^{-\lambda x}} - \frac{\rho_1\theta e^{-\lambda x}}{1-\rho_1\theta^2 e^{-2\lambda x}}\right) < \left(\frac{\rho_2}{1-\rho_2\theta e^{-\lambda x}} - \frac{\rho_2\theta e^{-\lambda x}}{1-\rho_2\theta^2 e^{-2\lambda x}}\right).$$

Then $\frac{d}{dx}\log\frac{g_X(x)}{g_Y(x)} > 0$, which implies that $(X \geq_{lr} Y)$ and hence the remaining statements follow from the above implication, which completes the proof.

9. Limiting Distributions of Sample Extremes

Let $X_1, X_2, ..., X_n$ be a random sample of size n from an absolutely continuous distribution with pdf $g(x)$ and cdf $G(x)$. Limiting distributions of sample minima $X_{(1)} = min(X_1, X_2, ..., X_n)$ and maxima $X_{(n)} = max(X_1, X_2, ..., X_n)$ can be derived by using the asymptotic results for $X_{(1)}$ and $X_{(n)}$ given in Arnold et al. (1992) and Kotz and Nadarajah (2000).

For the minimum $X_{(1)}$ we have

$$\lim_{n\to\infty} P(X_{(1)} \leq a_n^* + b_n^* x) = 1 - e^{-x^c}; x > 0, c > 0,$$

of Weibull type, where $a_n^* = G^{-1}(0)$ and $b_n^* = G^{-1}(\frac{1}{n}) - G^{-1}(0)$, if and only if $G^{-1}(0)$ is finite and for all $x > 0$ and $c > 0$,

$$\lim_{\epsilon\to 0^+}\frac{G(G^{-1}(0)+\epsilon t)}{G(G^{-1}(0)+\epsilon)} = x^c.$$

For the maximum $X_{(n)}$ we have

$$\lim_{n\to\infty} P(X_{(n)} \le a_n + b_n x) = e^{(-e^{-t})}; -\infty < x < \infty,$$

of Extreme value type, where $a_n = G^{-1}(1-\frac{1}{n})$ and $b_n = [nf(a_n)]^{-1}$ if $\lim_{x\to G^{-1}(1)} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = 0.$

The following theorem gives the limiting distributions of the smallest and largest order statistics from EIG distribution.

Theorem 19 *Let $X_{(1)}$ and $X_{(n)}$ be representing the smallest and largest order statistics from $EIG(\rho,\theta,\lambda)$ distribution. Then*

1. $\lim_{n\to\infty} P(X_{(1)} \le a_n^* + b_n^* x) = 1 - e^{-x^c}; x > 0, c > 0,$
   where $a_n^* = G^{-1}(0)$ and $b_n^* = G^{-1}(\frac{1}{n}) - G^{-1}(0)$
   and $G^{-1}(.)$ is the quantile function of EIG distribution.

2. $\lim_{n\to\infty} P(X_{(n)} \le a_n + b_n x) = e^{(-e^{-t})}; -\infty < x < \infty,$
   where $a_n = G^{-1}(1-\frac{1}{n})$ and $b_n = [nf(a_n)]^{-1}$.

   Here $G^{-1}(.)$ and $g(.)$ are the quantile and pdf of EIG distribution.

Proof:

1. For EIG distribution $G^{-1}(0) = -\frac{1}{\lambda}\log\left(\frac{1}{\rho\theta^2}\right)$ is finite.

   By using L Hospitals rule, we have $\displaystyle\lim_{\epsilon\to 0^+} \frac{G(G^{-1}(0)+\epsilon t)}{G(G^{-1}(0)+\epsilon)} = t.$

   Hence statement 1 holds.

2. For the EIG distribution, we have

$$\lim_{x\to G^{-1}(1)} \frac{d}{dx}\left(\frac{1}{h(x)}\right) = \frac{1}{\lambda}\lim_{x\to\infty} \frac{1}{(1-\rho\theta^2 e^{-2\lambda x})^2}\Big[(1-\rho\theta^2 e^{-2\lambda x})\left(\lambda\theta(1+\rho)e^{-\lambda x} - 2\lambda\rho\theta^2 e^{-2\lambda x}\right) - (2\lambda\rho\theta^2 e^{-2\lambda x})\left(1-\theta(1+\rho)e^{-\lambda x} + \rho\theta^2 e^{-2\lambda x}\right)\Big] = 0.$$

   Hence the statement 2 also holds.

10. Estimation

Let $X_1, X_2, ..., X_n$ be a random sample with observed values $x_1, x_2, ..., x_n$ from EIG distribution with parameters θ, ρ and λ. Let $\Theta = (\theta, \rho, \lambda)^T$ be the parameter vector. The log-likelihood function is given by

$$\log L = n\left[\log(1-\theta) + \log(1-\rho\theta) + \log\lambda\right] - \lambda\sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\log(1-\rho\theta^2 e^{-2\lambda x_i}) - 2\sum_{i=1}^{n}\log(1-\theta e^{-\lambda x_i}) - 2\sum_{i=1}^{n}\log(1-\rho\theta e^{-\lambda x_i}).$$

Partial derivatives of the log likelihood function with respect to the parameters are

$$\frac{\partial \log L}{\partial \theta} = -\frac{n}{1-\theta} - \frac{n\rho}{1-\rho\theta} + \sum_{i=1}^{n} -\frac{2\rho\theta e^{-2\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})} + 2\sum_{i=1}^{n} \frac{e^{-\lambda x_i}}{(1-\theta e^{-\lambda x_i})} + 2\sum_{i=1}^{n} \frac{\lambda x_i}{(1-\rho\theta e^{-\lambda x_i})},$$

$$\frac{\partial \log L}{\partial \rho} = -\frac{n\theta}{1-\rho\theta} + \sum_{i=1}^{n} -\frac{\theta^2 e^{-2\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})} + 2\sum_{i=1}^{n} \frac{\theta e^{\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})},$$

$$\frac{\partial \log L}{\partial \lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \frac{2\rho\theta^2 x_i e^{-2\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})} - 2\sum_{i=1}^{n} \frac{\theta x_i e^{-\lambda x_i}}{(1-\theta e^{-\lambda x_i})} - 2\sum_{i=1}^{n} \frac{\rho\theta x_i e^{-\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})}.$$

The MLEs of (θ, ρ, λ), say $(\hat{\theta}, \hat{\rho}, \hat{\lambda})$ are the solutions of simultaneous equations $\frac{\partial \log L}{\partial \theta} = 0$, $\frac{\partial \log L}{\partial \rho} = 0$ and $\frac{\partial \log L}{\partial \lambda} = 0$. The solutions of the three equations above has not a closed form. So a numerical technique such as Newton Raphson method can be employed to get the MLEs. It deserves mentioning that the maximization of log likelihood equation may be performed by using $maxLik$ package in R language, see Henningsen and Toomet (2011).

Now, we can study the existence and uniqueness of the MLEs when the other parameters are known.

Theorem 20 *Let $g_1(\theta; \lambda, \rho, x)$ denote the function $\frac{\partial \log L}{\partial \theta}$ where λ and ρ are the true values of the parameters. Then there exist at least one root solution for $g_1(\theta; \lambda, \rho, x) = 0$ for $\theta \epsilon (0, 1)$ when $\sum_{i=1}^{n} 2e^{-\lambda x_i} > n$ and the solution is unique if*

$$\sum_{i=1}^{n} \frac{2e^{-2\lambda x_i}}{(1-\theta e^{-\lambda x_i})^2} + \sum_{i=1}^{n} \frac{2\rho^2 e^{-2\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})^2} < \frac{n}{(1-\theta)^2} + \frac{n\rho^2}{(1-\rho\theta)^2} + \sum_{i=1}^{n} \frac{2\rho e^{-2\lambda x_i}\left(\frac{1}{\theta^2} + \rho e^{-2\lambda x_i}\right)}{\left(\frac{1}{\theta} - \rho\theta e^{-2\lambda x_i}\right)^2}.$$

Proof:
We have

$$g_1(\theta; \lambda, \rho, x) = -\frac{n}{1-\theta} - \frac{n\rho}{1-\rho\theta} + \sum_{i=1}^{n} -\frac{2\rho\theta e^{-2\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})} + 2\sum_{i=1}^{n} \frac{e^{-\lambda x_i}}{(1-\theta e^{-\lambda x_i})} + 2\sum_{i=1}^{n} \frac{\lambda x_i}{(1-\rho\theta e^{-\lambda x_i})}.$$

Now, $\lim_{\theta \to 0} g_1(\theta; \lambda, \rho, x) = (1+\rho)\left[-n + \sum_{i=1}^{n} 2e^{-\lambda x_i}\right]$.

Also

$$\lim_{\theta \to 1} g_1(\theta; \lambda, \rho, x) = -\infty - \frac{n\rho}{1-\rho} - \sum_{i=1}^{n} \frac{2\rho e^{-2\lambda x_i}}{(1-\rho e^{-2\lambda x_i})} + \sum_{i=1}^{n} \frac{2e^{-\lambda x_i}}{(1-e^{-\lambda x_i})} + \sum_{i=1}^{n} \frac{2\rho e^{-\lambda x_i}}{(1-\rho e^{-\lambda x_i})} < 0.$$

Hence there exist at least one root say, $\hat{\theta}\epsilon(0, 1)$ when $\sum_{i=1}^{n} 2e^{-\lambda x_i} > n$. The root is unique when $\frac{\partial g_1(\theta; \lambda, \rho, x)}{\partial \theta} < 0$, where

$$\sum_{i=1}^{n} \frac{2e^{-2\lambda x_i}}{(1-\theta e^{-\lambda x_i})^2} + \sum_{i=1}^{n} \frac{2\rho^2 e^{-2\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})^2} < \frac{n}{(1-\theta)^2} + \frac{n\rho^2}{(1-\rho\theta)^2} + \sum_{i=1}^{n} \frac{2\rho e^{-2\lambda x_i}\left(\frac{1}{\theta^2} + \rho e^{-2\lambda x_i}\right)}{\left(\frac{1}{\theta} - \rho\theta e^{-2\lambda x_i}\right)^2}.$$

Theorem 21 *Let $g_2(\rho; \lambda, \theta, x)$ denote the function $\frac{\partial \log L}{\partial \rho}$ where λ and θ are the true values of the parameters. Then there exist at least one root solution for $g_2(\rho; \lambda, \theta, x) = 0$ for $\rho\epsilon(0, \frac{1}{\theta})$ when $\sum_{i=1}^{n} e^{-\lambda x_i}(2 - \theta e^{-\lambda x_i}) > n$ and the solution is unique if*

$$\sum_{i=1}^{n} \frac{2e^{-2\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})^2} < \frac{n}{(1-\rho\theta)^2} + \sum_{i=1}^{n} \frac{\theta^2 e^{-4\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})^2}.$$

Proof:

We have $g_2(\rho;\lambda,\theta,x) = -\dfrac{n\theta}{1-\rho\theta} + \sum_{i=1}^{n} -\dfrac{\theta^2 e^{-2\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})} + 2\sum_{i=1}^{n}\dfrac{\theta e^{\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})}.$

Now $\lim_{\rho\to 0} g_2(\rho;\lambda,\theta,x) = \theta\left[-n + \sum_{i=1}^{n} e^{-\lambda x_i}(2-\theta e^{-\lambda x_i})\right].$

Also $\lim_{\rho\to\frac{1}{\theta}} g_2(\rho;\lambda,\theta,x) = -\infty - \sum_{i=1}^{n}\dfrac{\theta^2 e^{-2\lambda x_i}}{(1-\theta e^{-2\lambda x_i})} + \sum_{i=1}^{n}\dfrac{2\theta e^{\lambda x_i}}{(1-e^{-\lambda x_i})} < 0.$

Hence there exist at least one root say, $\hat{\rho}\epsilon(0,\frac{1}{\theta})$ when $\sum_{i=1}^{n} e^{-\lambda x_i}(2-\theta e^{-\lambda x_i}) > n$. The root is unique when $\frac{\partial g_2(\rho;\lambda,\theta,x)}{\partial\rho} < 0$, where

$$\sum_{i=1}^{n}\frac{2e^{-2\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})^2} < \frac{n}{(1-\rho\theta)^2} + \sum_{i=1}^{n}\frac{\theta^2 e^{-4\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})^2}.$$

Theorem 22 *Let $g_3(\lambda;\rho,\theta,x)$ denote the function $\frac{\partial \log L}{\partial\lambda}$ where ρ and θ are the true values of the parameters. Then there exist at least one root solution for $g_3(\lambda;\rho,\theta,x) = 0$ for $\lambda\epsilon(0,\infty)$ and the solution is unique if*

$$\sum_{i=1}^{n}\frac{2\theta x_i^2 e^{\lambda x_i}}{(e^{\lambda x_i}-\theta)^2} + \sum_{i=1}^{n}\frac{2\rho\theta x_i e^{\lambda x_i}}{(e^{\lambda x_i}-\rho\theta)^2} < \frac{n}{\lambda^2} + \sum_{i=1}^{n}\frac{4\rho\theta^2 x_i^2 e^{2\lambda x_i}}{(e^{2\lambda x_i}-\rho\theta^2)^2}.$$

Proof:

$$g_3(\lambda;\rho,\theta,x) = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\frac{2\rho\theta^2 x_i e^{-2\lambda x_i}}{(1-\rho\theta^2 e^{-2\lambda x_i})} - 2\sum_{i=1}^{n}\frac{\theta x_i e^{-\lambda x_i}}{(1-\theta e^{-\lambda x_i})} - 2\sum_{i=1}^{n}\frac{\rho\theta x_i e^{-\lambda x_i}}{(1-\rho\theta e^{-\lambda x_i})}.$$

We have

Now $\lim_{\lambda\to 0} g_3(\lambda;\rho,\theta,x) = \infty - \sum_{i=1}^{n} x_i + \sum_{i=1}^{n}\dfrac{2\rho\theta^2 x_i}{1-\rho\theta^2} - \sum_{i=1}^{n}\dfrac{2\theta x_i}{1-\theta} - \sum_{i=1}^{n}\dfrac{2\rho\theta x_i}{1-\rho\theta} = \infty.$

Also $\lim_{\rho\to\infty} g_3(\lambda;\rho,\theta,x) = -\sum_{i=1}^{n} x_i < 0.$

Hence there exist at least one root say, $\hat{\lambda}\epsilon(0,\infty)$ such that $g_3(\hat{\lambda};\rho,\theta,x) = 0$. The root is unique when $\frac{\partial g_3(\lambda;\rho,\theta,x)}{\partial\lambda} < 0$, where

$$\sum_{i=1}^{n}\frac{2\theta x_i^2 e^{\lambda x_i}}{(e^{\lambda x_i}-\theta)^2} + \sum_{i=1}^{n}\frac{2\rho\theta x_i e^{\lambda x_i}}{(e^{\lambda x_i}-\rho\theta)^2} < \frac{n}{\lambda^2} + \sum_{i=1}^{n}\frac{4\rho\theta^2 x_i^2 e^{2\lambda x_i}}{(e^{2\lambda x_i}-\rho\theta^2)^2}.$$

11. Simulation

We have conducted simulation studies to verify the performance of MLEs for different sample sizes and different parameter values for the proposed EIG distribution. We can apply inverse transformation method to simulate EIG random sample.

Different sample sizes considered in the simulation are $n = 50, 100, 150$ and 200. We have used $maxLik$ package in R language to find the estimate. We replicated the process 1000 times and report the average estimates and the associated mean squared errors (MSE) in Table 4. As the sample size increases the average bias and the mean squared errors decreases which indicates the consistency property of the MLEs.

Table 4 Simulation results for different values of the parameters θ, ρ and λ

(θ,ρ,λ)	n	$\hat{\theta}, MSE(\hat{\theta})$	$\hat{\rho}, MSE(\hat{\rho})$	$\hat{\lambda}, MSE(\hat{\lambda})$
$(.5,1,1)$	50	0.5052(0.0079)	1.0259(0.3259)	0.9985(0.2573)
$(.5,1,1)$	100	0.4931(0.0052)	1.0123(0.2748)	1.0073(0.2629)
$(.5,1,1)$	150	0.4944(0.0035)	1.0069(0.2657)	1.0059(0.2598)
$(.5,1,1)$	200	0.4967(0.0027)	1.0067(0.2634)	1.0039(0.2570)
$(.5,1.5,1)$	50	0.4964(0.0025)	1.4954(1.0015)	1.0063(0.2574)
$(.5,1.5,1)$	100	0.4969(0.0012)	1.4963(0.9981)	1.0052(0.2559)
$(.5,1.5,1)$	150	0.49903(0.0008)	1.4974(0.9965)	1.0025(0.2528)
$(.5,1.5,1)$	200	0.4988(0.0005)	1.4995(0.9944)	1.0025(0.2527)
$(.8,.5,2)$	50	0.78814(0.0067)	0.5561(0.1843)	1.9673(1.4987)
$(.8,.5,2)$	100	0.7860(0.00470)	0.5275(0.1058)	1.9914(1.464)
$(.8,.5,2)$	150	0.7914(0.0025)	0.5170(0.0911)	1.9971(1.4634)
$(.8,.5,2)$	200	0.7944(0.0014)	0.5118(0.0909)	2.0005(1.4623)
$(.8,1,2)$	50	0.7979(0.0014)	0.9947(0.0399)	2.0108(1.469)
$(.8,1,2)$	100	0.7989(0.00075)	0.9972(0.0399)	2.0038(1.4501)
$(.8,1,2)$	150	0.7994(0.00045)	0.9983(0.0399)	2.0011(1.4429)
$(.8,1,2)$	200	0.8001(0.00034)	0.9992(0.0201)	2.0001(1.4359)

12. Application to Real Data

Here we present applications to real data sets for illustrating the potentiality of the new distribution. We compare the fit of the distribution with the following continuous lifetime distributions.

- Exponential (E) distribution with cdf

$$F(x) = 1 - e^{-\alpha x};\ \alpha > 0.$$

- Gamma (G) distribution with cdf

$$F(x) = \int_0^x \frac{\theta^p}{\Gamma p} x^{p-1} e^{-\theta x} dx;\ \theta,\ p > 0.$$

- Weibull (W) distribution with cdf

$$F(x) = 1 - e^{-(\theta x)^\alpha};\ \alpha,\ \theta > 0.$$

- Generalized Exponential (GE) distribution with cdf

$$F(x) = (1 - e^{-\theta x})^\alpha;\ \alpha > 0.$$

- Exponential Geometric (EG) distribution with cdf

$$F(x) = \left[\frac{1 - e^{-\theta x}}{1 - pe^{-\theta x}}\right];\ \theta,\ p > 0.$$

- Exponential Logarithmic (EL) distribution with cdf

$$F(x) = 1 - \frac{\log\left[1 - (1-p)e^{-\theta x}\right]}{\log p};\ \theta > 0,\ o < p < 1.$$

- Marshall Olkin Exponential (MOE) distribution with cdf

$$F(x) = \frac{1 - e^{-(\theta x)}}{p + (1-p)(1 - e^{-(\theta x)})};\ \theta,\ p > 0.$$

- Marshall Olkin Weibull (MOW) distribution with cdf

$$F(x) = \frac{1 - e^{-(\theta x)^{\beta}}}{p + (1-p)(1 - e^{-(\theta x)^{\beta}})};\ \theta,\ \beta,\ p > 0.$$

- Weibull Geometric (WG) distribution with cdf

$$F(x) = \frac{1 - e^{-(\beta x)^{\alpha}}}{1 - pe^{-(\beta x)^{\alpha}}};\ \alpha,\ \beta > 0,\ 0 < p < 1.$$

- New Extended Weibull (NEW) distribution with cdf

$$F(x) = 1 - e^{-(ax^{b}e^{-cx})};\ a \geq 0,\ b > 0,\ c \geq 0.$$

In order to identify the shape of the hazard rate function of the data, we consider a graphical method based on the Total Time on Test (TTT) plot. As we know, the empirical TTT plot is given by

$$G(r/n) = \frac{\left(\sum_{i=1}^{r} X_{(i)} + (n-r)X_{(r)}\right)}{\sum_{i=1}^{n} X_{(i)}}, r = 1, 2, ..., n,$$

where $X_{(i)}$ denote the i^{th} order statistic of the sample. If the empirical TTT transform is convex, concave, convex then concave and concave then convex, the shape of the corresponding hazard rate function is respectively, decreasing, increasing, bathtub-shaped and upside-down bathtub.

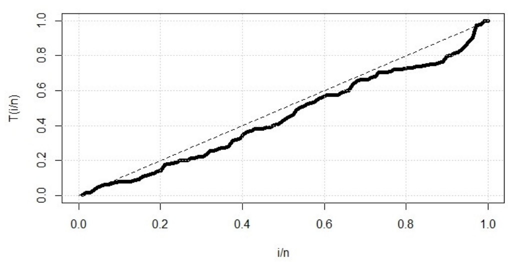

Figure 1 The empirical TTT plot of coal mining data

The following real data set represents intervals in days between 109 successive coal mining disasters in Great Britain, for the period 1875-1951 taken from Maguire et al. (1952). This data set was used by Adamidis and Loukas (1998), Kus (2007), Madhavi and Kundu (2017). The data set is given in Table 5.

Table 5 Coal Mining Data set

1	4	4	7	11	13	15	15	17	18
19	19	20	20	22	23	28	29	31	32
36	37	47	48	49	50	54	54	55	59
59	61	61	66	72	72	75	78	78	81
93	96	99	108	113	114	120	120	120	123
124	129	131	137	145	151	156	171	176	182
188	189	195	203	208	215	217	217	217	224
228	233	255	271	275	275	275	286	291	312
312	312	315	326	326	329	330	336	338	345
348	354	361	364	369	378	390	457	467	498
517	566	644	745	871	1312	1357	1613	1630	

Table 6 Descriptive statistics of Coal mining data set

Min	Median	Mean	Max	SD	Skewness	Kurtosis
1	145	233.3	1630	296.43	2.957	12.998

From Figure 1, we can see that, the hazard rate of the data set is decreasing. Also from Table 6, the distribution is positively skewed and leptokurtic. Hence we fit EIG distribution for the data. For this data set, we estimate the unknown parameters of each distribution by the maximum likelihood method. To compare the models, we used three other criterions:

- Kolmogorov Smirnov test statistics (K-S) - small value is good;
- The p-value from the chi-square goodness-of-fit test - large value is good;
- Negative log-likelihood (-Log L) - small value is good.

The values of estimates, -log L, K-S and P- value for all the models are listed in Table 7.
From Table 7 we can see that EIG distribution fits better to coal mining data set.

Table 7 Parameter estimates and goodness of fit statistics for various models fitted to coal mining data

Model	Estimates	-log L	K-S	p-value
E	$\alpha = 0.0042$	703.3133	0.0786	0.5107
G	$(p = 0.8555, \theta = 0.0037)$	702.4007	0.0823	0.4517
W	$(\alpha = 0.8848, \theta = 0.0046)$	701.7724	0.0784	0.5135
GE	$(\alpha = 0.8598, \theta = 0.0039)$	702.5523	0.0833	0.4364
EG	$(\theta = 0.0030, p = 0.4927)$	701.3731	0.0791	0.5033
EL	$(\theta = 0.0032, p = 0.3255)$	701.5532	0.0810	0.4717
MOE	$(\theta = 0.0028, p = 0.4836)$	701.3831	0.0778	0.5252
MOW	$(\beta = 1.0945, \theta = 0.0023, p = 0.3169)$	701.2538	0.0761	0.5527
WG	$(\alpha = 1.0901, \beta = .0023, p = .6783)$	701.2543	.0766	.5452
NEW	$(a = 0.0098, b = 0.8627, c = 1.4698)$	701.2478	0.078	0.5223
EIG	$(\lambda = 0.0026, \rho = 1.0078, \theta = 0.3751)$	701.1524	0.0760	0.5544

We now follow the approach used by Balakrishnan and Ristic (2016). We derive the MLEs of the parameters of EIG distribution. We obtain MLEs of EIG distribution as $\hat{\theta} = 0.3751, \hat{\rho} = 1.0078, \hat{\lambda} = 0.0026$. Now we use the obtained estimates to derive the 95 percent bootstrap confidence intervals for the parameters θ, ρ and λ. We simulate 10000 samples of size 109 from EIG distribution with true values of the parameters taken as $\hat{\theta} = 0.3751, \hat{\rho} = 1.0078, \hat{\lambda} = 0.0026$. For each obtained sample, we have estimated the MLEs $\hat{\theta_i}^*, \hat{\rho_i}^*$ and $\hat{\lambda_i}^*$, where $i\epsilon 1, 2, ..., 10000$ and we used true values of estimates as starting values for the MLE. For the 95 percent bootstrap confidence interval we took the 250^{th} and 9750^{th} ordered estimates and obtained the 95 percent bootstrap confidence interval for parameters θ, ρ and λ as $[.0489, .5171]$, $[.6806, 1.3710]$ and $[.0019, .0050]$ respectively.

For testing the adequacy of the model, we use Kolmogrov Smirnov test. First we obtain the value of the Kolmogrov Smirnov statistic $D_x = max_{x_i}|G_n(x_i) - G^*_x(x_i)|$ for the random variable X based on the sample $(x_1, x_2, ..., x_{109})$. The function G_n is the empirical cumulative distribution function and G^*_X is the cumulative distribution function whose true parameters are the estimates $\hat{\theta}, \hat{\rho}$ and $\hat{\lambda}$ obtained as MLEs based on sample data $(x_1, x_2, ..., x_{109})$. For deriving the p value of the statistic, we simulated 10000 samples $\acute{x_1}, \acute{x_2}, ...\acute{x_{109}}$ of size 109 having EIG distribution with true values $\hat{\theta} = 0.3751, \hat{\rho} = 1.0078$ and $\hat{\lambda} = 0.0026$. For each simulated sample, we obtain the MLEs $\tilde{\theta}, \tilde{\rho}, \tilde{\lambda}$ and then obtain the values of Kolmogrov Smirnov statistic $\acute{D_{x,j}} = max_{x_i}|\acute{G}_n(x_i) - \acute{G}_X(x_i)|, j = 1, 2, ..., 10000$, where $\acute{G}_n$ is the empirical distribution function based on the simulated sample and $\acute{G}_X$ is the cumulative distribution function whose true parameters are the estimates $\tilde{\theta}, \tilde{\rho}, \tilde{\lambda}$ respectively. The p value is calculated as $p \approx \frac{[j : \acute{D}_{x,j} \geq D_x]}{10000}$. We have obtained the p value as .5922.

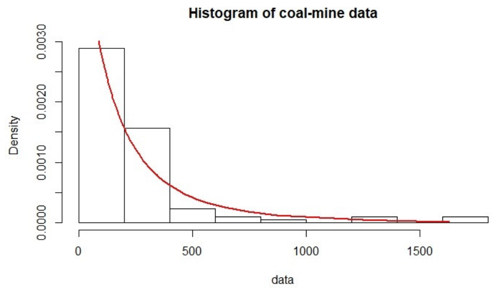

Figure 2 Histogram with fitted pdf's

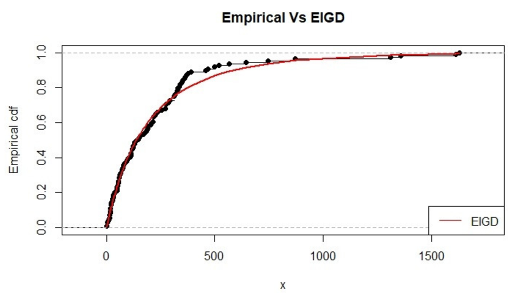

Figure 3 Empirical cdf with fitted cdf's for the coal mining data set

The fitted density and the empirical cdf plot of the EIG distribution are presented in Figure 2 and Figure 3 respectively. It indicates a satisfactory fit for the data.

13. Conclusion

In this paper, a new distribution called EIG distribution is developed using intervened Geometric distribution. The shape properties of the density function and hazard rate function are studied. Expression for moment generating function, moments, quantile function, mean deviation, Bonferroni curve and Lorenz curve are derived. Also some reliability properties of EIG distribution are studied. Distribution of order statistics are derived, various characterization results and stochastic ordering properties are proved. Also limiting distribution of sample extremes are obtained. To understand the performance of MLE, simulation studies are carried out. The obtained result are validated using a real life data set, which shows EIG distribution gives better fit to the data than other competitive models. Using parametric bootstrap method the adequacy of the model is established.

Acknowledgements

The authors are thankful to the the Editor-in-Chief and anonymous reviewers for their valuable suggestions and comments. The second author would like to thank to University Grants Commission, New Delhi, India for the financial assistance in the form of Junior Research Fellowship.

References

Adamidis K, Loukas S. A lifetime distribution with decreasing failure rate. Stat Probabil Lett. 1998; 39: 35-42.

Arnold BC, Balakrishnan N, Nagaraja HN. A first course in order statistics. New York: Wiley; 1992.

Balakrishnan N, Ristic MM. Multivariate families of gamma-generated distributions with finite or infinite support above or below the diagonal. J Multivarate Anal. 2016; 143: 194-207.

Dey S, Nassar M, Kumar D. Alpha power transformed inverse Lindley distribution: A distribution with an upside-down bathtub-shaped hazard function. J Comput Appl Math. 2019; 348: 130-145.

Dhanavanthan P. Compound intervened Poisson distribution. Biometrical J. 1998; 40: 641-646.

Dhanavanthan P. Estimation of the parameters of compound intervened Poisson distribution. Biometrical J. 2000; 42: 315-320.

Glanzel W, Hamedani GG. Characterizations of univariate continuous distributions. Stud Sci Math Hung. 2001; 37: 83-118.

Gradshteyn IS, Ryzhik IM. Table of integrals, series, and products- Seventh Edition. New York: Academic Press; 2007.

Hamedani GG, Safavimanesh F. Characterizations and infinite divisibility of certain 2016 univariate continuous distributions. Int. Math. Forum. 2017; 12: 195-228.

Henningsen A, Toomet O. maxLik: A package for maximum likelihood estimation in R. Comput Stat. 2011; 26: 443-458.

Huang M, Fung KY. Intervened truncated Poisson distribution. Sankhya B (2008). 1989; 51: 302-310.

Jayakumar K, Sankaran KK. Exponential intervened Poisson distribution. Commun Stat Theory M. 2019; 50: 3063-3093.

Kotz S, Nadarajah S. Extreme value distributions: Theory and applications. London: Imperial College Press; 2000.

Kumar CS, Sreejakumari S. Extended intervened Geometric distribution. J Stat Distrib Appl. 2016; 2: 8-13.

Kus C. A new lifetime distribution. Comput Stat Data Anal. 2007; 51: 4497- 509.

Lemonte AJ. A new exponential-type distribution with constant, decreasing, increasing, upside-down bathtub and bathtub-shaped failure rate function. Comput Stat Data Anal. 2013; 62: 149-170.

Lemonte AJ. The beta log-logistic distribution. Braz J Probab Stat. 2014; 28: 313-332.

Madhavi A, Kundu D. A new method for generating distributions with an application to exponential distribution. Commun Stat Theory. 2017; 46 : 6543-57.

Maguire BA, Pearson ES, Wynn AHA. The time intervals between industrial accidents. Biometrika. 1952; 39: 16-80.

Marshall AW, Olkin I. A new method for adding a parameter to a family of distributions with application to the exponential and Weibull families. Biometrika. 1997; 84: 641-652.

Marshall AW, Olkin I. Life distributions: Structure of nonparametric, semiparametric, and parametric families. New York: Springer; 2007.

Nadarajah S, Jayakumar K, Ristic MM. A new family of lifetime models. J Stat Comput Simul. 2013; 83: 1389-1404.

Scollinik DPM. On the intervened generalized Poisson distribution. Commun Stat Theory. 2006; 35: 953-963.

Shanmugam R. An inferential procedure for the Poisson intervention parameter. Biometrics. 1992; 48: 559-565.

Shanmugam R. An intervened Poisson distribution and its medical application. Biometrics. 1985; 41: 1025-29.